A REVIEW OF SPECIAL RELATIVITY

Contents:

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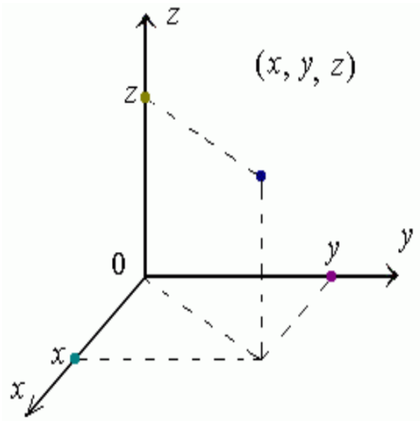
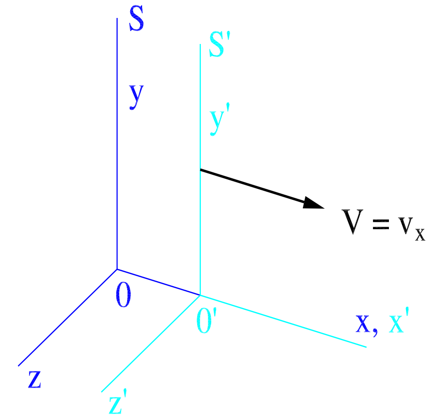
Preamble

In the following we will use the notion of frame of reference where coordinates are specified.

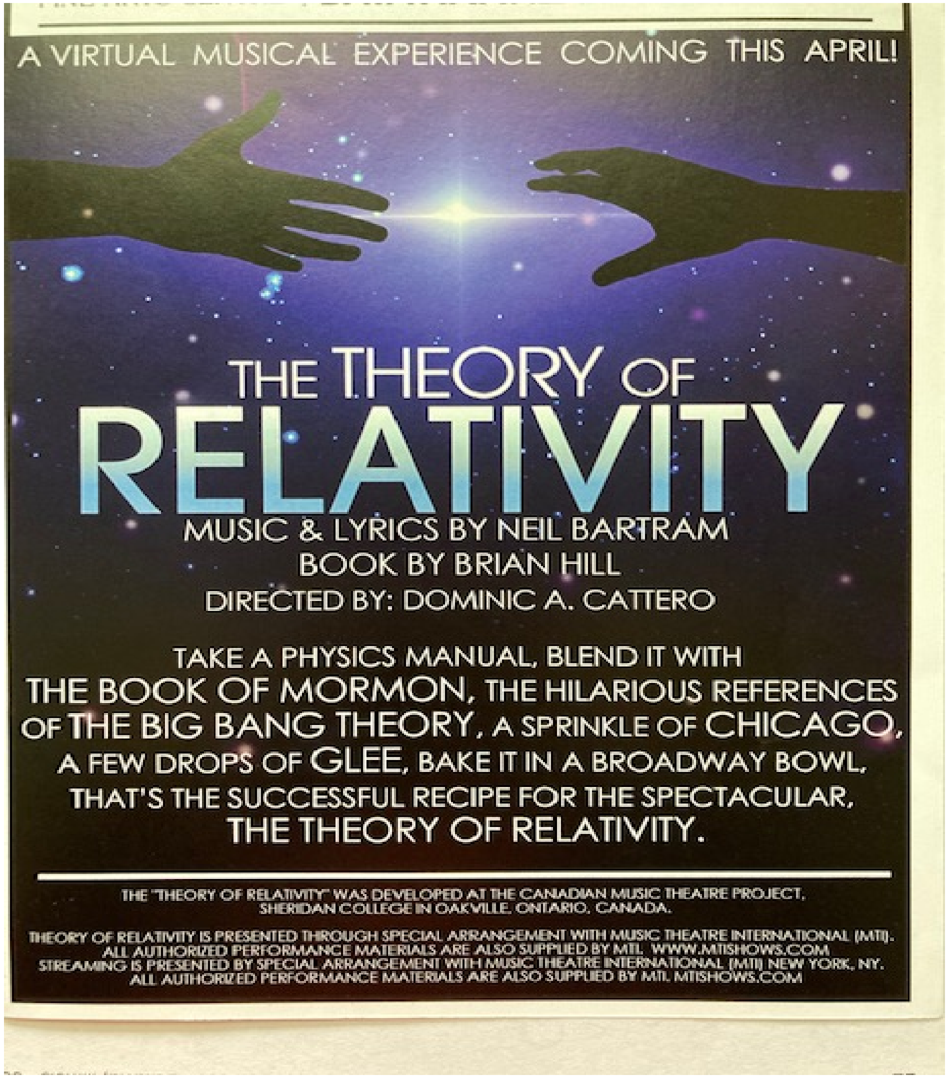
- We will use orthogonal frames (3 axes at 90^{0}) where the cartesian coordinates of a point in space are specified.



- Any frame may be made to coincide with any other by translations and rotations.
  - For this reason when considering frames attached to moving observers we will just consider translational motion along one common axis.
    This simplifies the math.



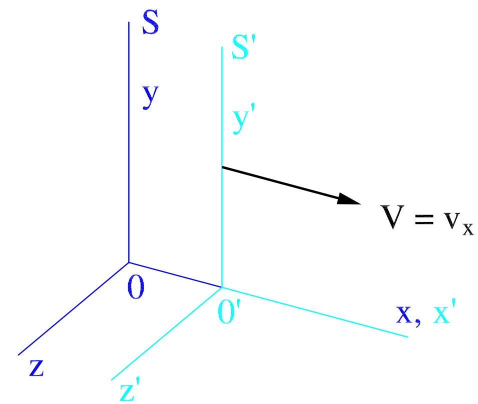
Running in Batavia last April...



WHY THE “THEORY OF RELATIVITY”?

Quantitive description of physical events needs a *frame of reference*, where the coordinates of the observed object are specified. Euclidean geometry specifies how coordinates of points in different frames are related. For instance, if S' is *translated* by x_0 wrt S along the common $\hat{x}$-axis it is

$$x' = x - x_0 \qquad y' = y \qquad z' = z \tag{1}$$

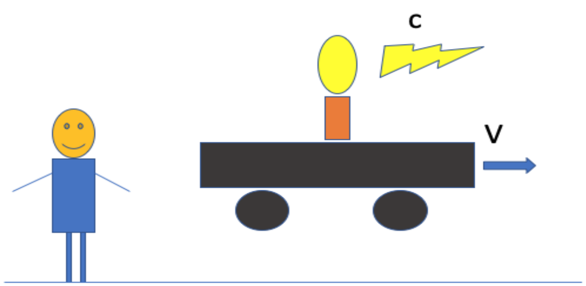


Suppose S' is *moving* wrt S along the $\hat{x}$-axis with velocity $\vec{V}$. Thus $x_0 = Vt$ (assuming O and O' coincide at $t=0$)and making the first and second derivates wrt time

$$\dot{x}' = \dot{x} - V \qquad \dot{y}' = \dot{y} \qquad \dot{z}' = \dot{z} \tag{2}$$

$$\ddot{x}' = \ddot{x} \qquad \ddot{y}' = \ddot{y} \qquad \ddot{z}' = \ddot{z} \tag{3}$$

Eqs. 1, 2 and 3 are the Galilean transformations for coordinates, velocity and acceleration. We implicitly assumed that $t' = t$ and that the lengths were invariant in the two frames.



- Eq.2 means that velocities add.
- Eq.3 says that the acceleration is invariant.

The basis of the classical mechanics are the three laws[a] of dynamics.

The first dynamics law is the principle of inertia (Galileo) which states

"A free body remains in a state of rest or of uniform motion"

A reference system where a free body is at rest or it moves with uniform velocity is said to be inertial.

Because of the Galileian transformations, any frame in uniform motion wrt an inertial one is inertial too.

The second law (Newton) states

"In an inertial system holds good $\vec{F} = m\vec{a}$ **"**

The variation of velocity with time (acceleration), $\vec{a}$, is proportional to the applied force, $\vec{F}$, through a constant, m ("inertial mass").

Implicitly it is assumed that m is a *characteristic* of the body which doesn't depend upon its motion.

The third Newton law states

"Whenever two bodies interact the force that body 1 exerts on body 2 is equal and opposite to the force that body 2 exerts on body 1"

Third law combined with the second one gives the momentum conservation law[b] for a closed system.

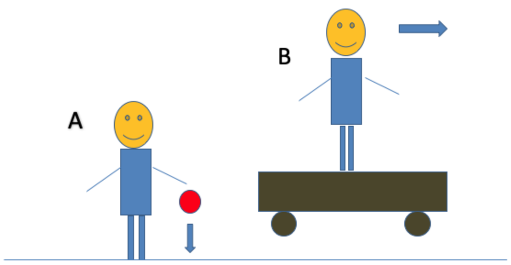
[a] Physics laws are not mathematical axioms but statements based on reproducible observations.
[b] Momentum: $\vec{p} \equiv m\vec{v}$

The three laws of dynamics hold good in inertial frames. As they are all equivalent it is reasonable to assume that all mechanics laws are the same for inertial observers.

This is expressed by the principle of relativity:

"Mechanics laws are the same for all inertial observers".

Suppose that Alex ($\boldsymbol{A}$) is studying the motion of a ball let to fall under the earth gravitation force. $\boldsymbol{A}$ measures that the object is subject to a constant acceleration of $\boldsymbol{a} \approx$9.8 ms^{-2}. By using different balls he finds that the acceleration is always the same, $\boldsymbol{g}$. $\boldsymbol{A}$ concludes that there must be a force acting on the balls which is directed towards the center of the earth and has magnitude $\boldsymbol{mg}$.



Assuming that Galileian transformations hold good, observer Beth ($\boldsymbol{B}$) on a train moving with uniform velocity $\vec{\boldsymbol{V}}=\hat{\boldsymbol{x}}\boldsymbol{V}$ wrt $\boldsymbol{A}$ will describe the ball motion as

$$\dot{x}' = \dot{x} - V = -V \qquad \dot{y}' = \dot{y}$$
$$\ddot{x}' = \ddot{x} = 0 \qquad \ddot{y}' = \ddot{y}$$

and as the mass, $\boldsymbol{m}$ is a constant, will agree with $\boldsymbol{A}$ on magnitude and direction of the force. Galileian transformations satisfy the the principle of relativity!

The relativity principle allows us to chose the most convenient frame for describing an event.

Is EM invariant under Galilean transformations?

In the second half of the XIX century Maxwell had summarized the whole EM phenomena into 4 differential equations containing the constant c, from which one finds the wave equation for fields and potentials.

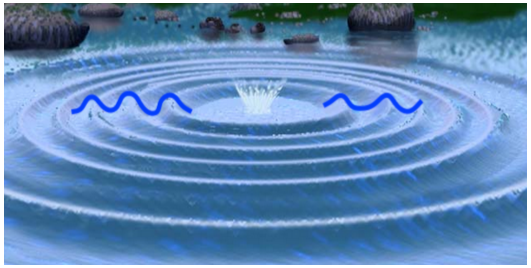
Simplest case: $$\left[\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\Phi = 0$$

The constant c is the velocity of propagation of the wave and is numerically equal to the speed of light in vacuum.

- Because of the addition of the velocities it is weird that it is a constant, unless we assume it is the velocity wrt a propagation medium. In Maxwell own words:

  "We can scarcely avoid the inference that light consists in the transverse undulations of the same <u>medium</u> which is the cause of electric and magnetic phenomena."

  The supporting medium was named "*ether*".



- This would mean also that Maxwell equations hold good only in that frame. Indeed s the wave equation

$$\left[\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\Phi = 0$$

becomes under Galileian transformation

$$\left[\frac{\partial^2}{\partial x'^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2} - \frac{V^2}{c^2}\frac{\partial^2}{\partial x'^2} - 2\frac{V}{c^2}\frac{\partial^2}{\partial x'\partial t'}\right]\Phi = 0$$

  – The equation is <u>not</u> invariant. EM laws are written in a frame connected to the ether!

Hypotheses

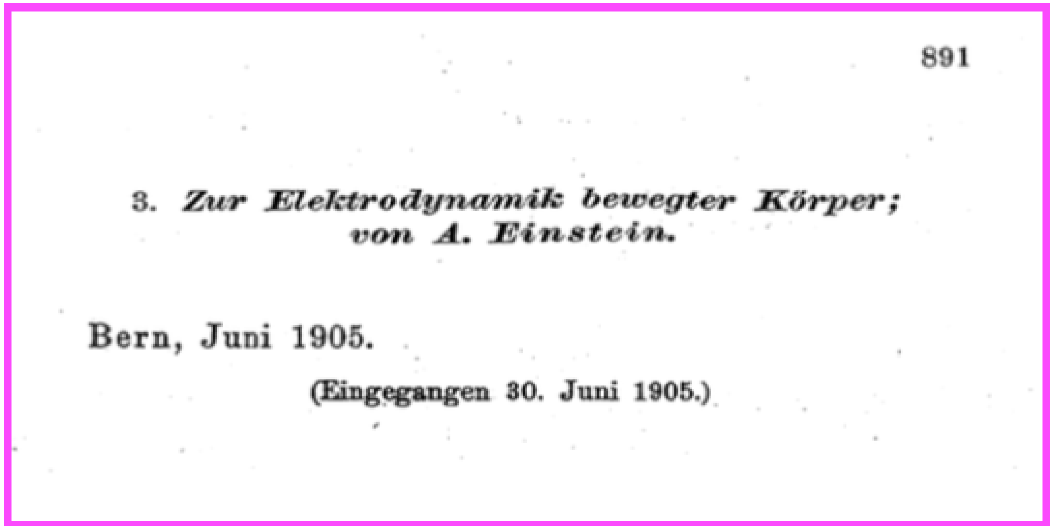
1. The relativity principle holds good only for the mechanics; for the EM exists a preferred frame of reference where the speed of the light is c (the reference system where the ether is at rest).
   - A bunch of experiments (starting with the famous Michelson-Morley experiment in 1887) aiming to prove the existence of the ether failed. Their results suggested instead that the speed of the light was a constant non dependent upon the status of motion of source or observer.
2. The at the time relatively young EM laws are wrong.
   - Attempts at modifying the EM in such a way that it would be invariant under Galilean transformations led to predictions of new phenomena which couldn't be proven by experiments.
3. EM laws are correct, but the Galilean transformations (and mechanics laws) must be modified.

Because of experimental evidence, only the third hypotheses was left.

If the Galilean transformations which look so self-evident were wrong a rethinking of physics basics was necessary.

Relativity of time

In the "Annus Mirabilis" 1905 Einstein published 4 fundamental papers. The third of them contained the idea of relativity of time and the basis of the theory of special relativity.



891

3. *Zur Elektrodynamik bewegter Körper;*
von A. Einstein.

Bern, Juni 1905.

(Eingegangen 30. Juni 1905.)

The paper starts, on the basis of the exerimental evidence, by giving up the existence of ether and introducing instead a "Principle of Relativity" based on two postulates

1) Physics laws are the same in all inertial reference systems, there is no preferred reference system.

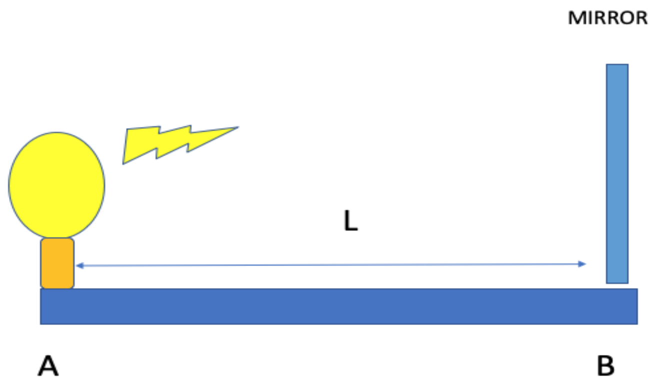
2) The speed of the light in the empty space has the same finite value c in all inertial reference systems.

The paper goes on demonstrating that it is not possible to synchronize clocks attached to frames in relative motion.

To find out whether two clocks at rest in different locations of an inertial frame are synchronized we proceed as follows. The observer A has a clock and sends a light ray at time t_A to observer B which receives it at t_B.

A mirror reflects the light back to A which receive it at t'_A.

Because of the second postulates, the clocks are synchronized if

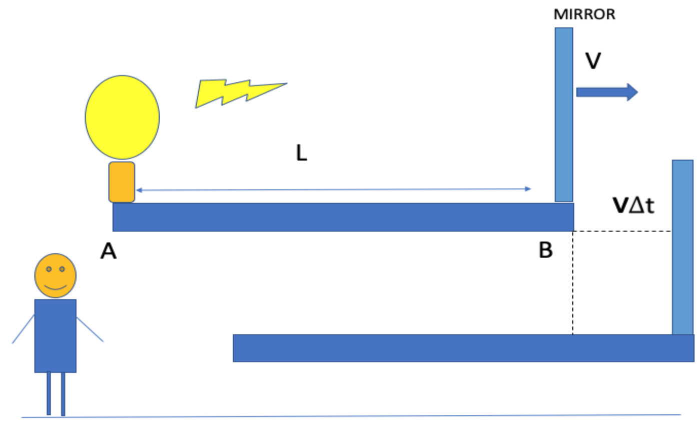


$$t_B - t_A = t'_A - t_B \quad \text{or} \quad \Delta t_{A\to B} = \Delta t_{B\to A}$$

If the clocks are identical they stay synchronized.

Any inertial observer can synchronize its clocks by the same procedure. How the time measured by observers in relative motion are related?

Let's look for instance to the synchronizing operation for two clocks attached to the ends, $\boldsymbol{A}$ and $\boldsymbol{B}$, of a rod moving along the $\boldsymbol{x}$-axis as seen by a stationary observer.
While the light moves to $\boldsymbol{B}$, $\boldsymbol{B}$ moves further and once reflected back to $\boldsymbol{A}$, $\boldsymbol{A}$ moves toward the light.



Therefore for the resting observer the time needed to reach $\boldsymbol{B}$, $\boldsymbol{t_B}$, is obtained by setting

$$ct_B = L + Vt_B \quad \rightarrow \quad t_B = L/(c - V)$$

while the time needed to reach $\boldsymbol{A}$ is obtained from

$$ct_A = L - Vt_A \quad \rightarrow \quad t_A = L/(c + V)$$

$$\Delta t_{B\rightarrow A} - \Delta t_{A\rightarrow B} = \frac{2VL}{c^2[1 - (V/c)^2]} \neq 0$$ consequence of c being finite!

If the clocks in the moving frame would be synchronous with the stationary ones they wouldn't be synchronous in their own frame. The "stationary" frame would dictate the timing. However stationarity is relative, the inertial frames are all equivalent: if there exist no privileged frame, we must abandon the idea of universal time.

Lorentz transformations “abridged”

By assuming the speed of light constant in all reference systems, the Galilean transformations, implying the velocity addition rule, must be modified. The new transformations must reduce to the Galileian ones when the relative motion is slow ($V \ll c$). According to the first Einstein postulate, the empty space is **isotrope** (all direction are equivalent) and **homogeneous** (all points are equivalent); it would make no sense to postulate that the laws are invariant in a space which is not homogeneous and isotrope. Homogeneity implies **linearity**:

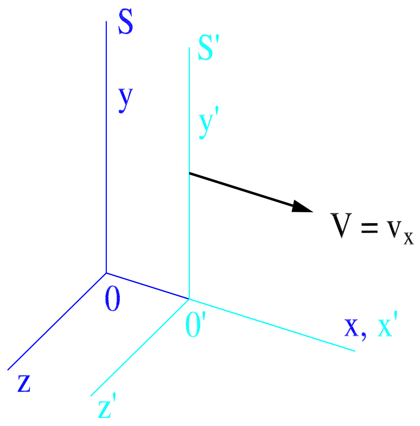
$$
\begin{aligned}
x' &= a_{11}x + a_{12}y + a_{13}z + a_{14}t \\
y' &= a_{21}x + a_{22}y + a_{23}z + a_{24}t \\
z' &= a_{31}x + a_{32}y + a_{33}z + a_{34}t \\
t' &= a_{41}x + a_{42}y + a_{43}z + a_{44}t
\end{aligned}
$$

wow:

16 unknowns!

For the case we are considering of motion along the common x-axis the coordinates y and z do not play a role and therefore it is reasonable to write

$$
\begin{aligned}
x' &= a_{11}x + a_{14}t \\
y' &= y \\
z' &= z \\
t' &= a_{41}x + a_{44}t
\end{aligned}
$$



The origin of the S' frame is described in S by $x_0 = Vt$ and by definition it is x'_0=0 at any time. Therefore

$$0 = x'_0 = a_{11}x_0 + a_{14}t = a_{11}Vt + a_{14}t$$

that is $a_{14}/a_{11} = -V$ and

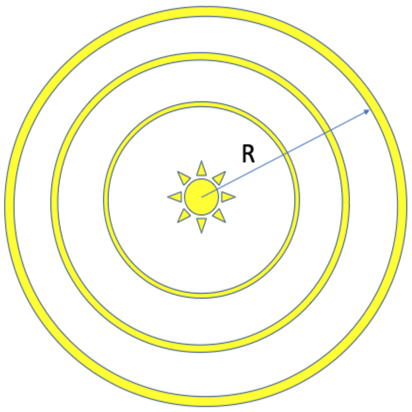
$$x' = a_{11}(x + a_{14}t/a_{11}) = a_{11}(x - Vt)$$

For finding the values of the remaining 3 coefficients we resort to the fact that, according to the two postulates, the speed of light is the same in S and S' and that the wave equation is invariant in form.

Suppose an EM spherical wave leaves the origin of the frame S at t=0.

The propagation is described in S by the equation of a sphere which radius increases with time as

$$R^2 = x^2 + y^2 + z^2 = c^2t^2 \qquad (4)$$



In S' the wave propagates with the same speed c and therefore

$$x'^2 + y'^2 + z'^2 = c^2t'^2$$

which expressing the primed coordinates in terms of those in S becomes

$$a_{11}^2x^2 + a_{11}^2V^2t^2 - 2a_{11}xVt + y^2 + z^2 = c^2a_{41}^2x^2 + c^2a_{44}^2t^2 + 2a_{41}a_{44}xt \qquad (5)$$

Comparing Eqs. 4 and 5 we get a system of 3 equations in the 3 unknown a_{11}, a_{41} and a_{44}.

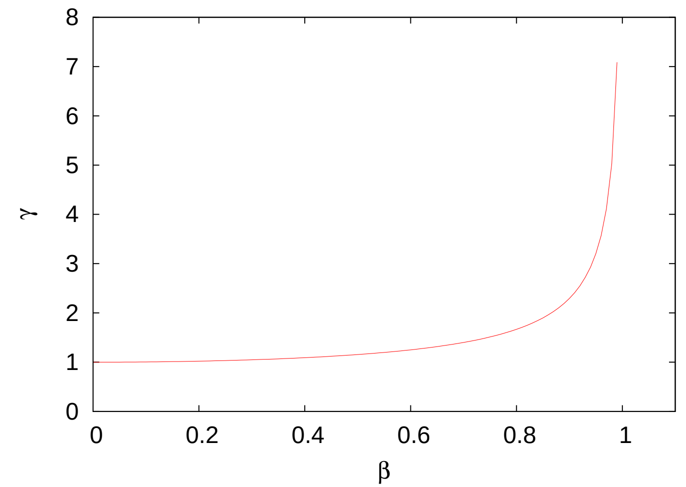
Solution

$$a_{11} = a_{44} = \gamma$$

$$a_{41} = -\gamma\beta/c$$

with

$$\beta \equiv V/c \quad (0 \div 1) \qquad \text{and} \qquad \gamma \equiv \frac{1}{\sqrt{1-\beta^2}} \quad (1 \div \infty)$$



The coordinate transformation for a translational motion along $\hat{x}$ with (constant) velocity $\vec{V} = \hat{x}V$ are (Lorentz transformations)

$$t' = \frac{t - xV/c^2}{\sqrt{1 - V^2/c^2}} \equiv \gamma(t - Vx/c^2)$$

$$x' = \frac{x - Vt}{\sqrt{1 - V^2/c^2}} \equiv \gamma(x - Vt)$$

$$y' = y \qquad\qquad z' = z$$

The inverse transformations are obtained replacing V with $-V$.

$$t' = \frac{t - xV/c^2}{\sqrt{1 - V^2/c^2}} \equiv \gamma(t - Vx/c^2)$$

$$x' = \frac{x - Vt}{\sqrt{1 - V^2/c^2}} \equiv \gamma(x - Vt)$$

$$y' = y \qquad\qquad z' = z$$

- $V \ll c$ the Lorentz transformations reduce to the Galilean ones. Good!
- For $V > c$ the transformations are meaningless because the argument of the square root, $1 - V^2/c^2$, becomes negative!
- The existence of a signal with $V > c$ would yield to a violation of the causality principle, as we will see.

Time is one of the 4 coordinates describing an event and as the spatial coordinates is subject to a (Lorentz) transformation between moving frames.

For spatial coordinates it is always possible if for instance $x_2 > x_1$ to find a new coordinates system such that $x'_2 < x'_1$.

Is it possible to find a Lorentz transformation which inverts the temporal order of events?

Assume an event happening at the time t_1 at the location x_1 in S and a second event happens at t_2 in x_2 with $t_2 > t_1$. Is it possible to find a Lorentz transformation such that $t'_2 < t'_1$? In S' it is

$$ct'_1 = \gamma(ct_1 - \beta x_1) \quad \text{and} \quad ct'_2 = \gamma(ct_2 - \beta x_2)$$

$$c(t'_2 - t'_1) = \gamma[c(t_2 - t_1) - \beta(x_2 - x_1)]$$

Therefore $t'_2 < t'_1$ if $\beta(x_2 - x_1) > c(t_2 - t_1)$, that is if $V(x_2 - x_1)/(t_2 - t_1) > c^2$. This may be possible depending on the values of $x_2 - x_1$ and $t_2 - t_1$. However if the first event in S *drives* the second one, x_2 and t_2 are not arbitrary.
If w is the speed of the signal which triggers the second event from the first one it is

$$x_2 - x_1 = w(t_2 - t_1)$$

$w \le c$ and $V < c$

$$c(t'_2 - t'_1) = \gamma[c(t_2 - t_1) - \beta w(t_2 - t_1)] = \gamma c(t_2 - t_1)\left(1 - \frac{Vw}{c^2}\right) > 0$$

Causality is not violated.

Some consequences of Lorentz transformations: time dilation and length contraction

Suppose a clock at rest in S measuring a time interval $t_2 - t_1$ between two events happening at the same location, $x_1 = x_2$. The time interval in the moving frame S' is measured by two different clocks because according to Lorentz transformations, the events happen in S' in different locations. The time difference in S' is

proper time (time measured by the same clock)

$$t'_2 - t'_1 = \gamma(t_2 - t_1) \geq t_2 - t_1$$

time dilation

Events happening at the same time, $t_1 = t_2$, but in different places in S, will be no more simultaneous in the moving frame S'

$$c(t'_2 - t'_1) = \gamma\beta(x_1 - x_2) \neq 0$$

Consider a rod of length L' along the x-axis and at rest in the moving frame S'.
The length in S is determined by the position of the rod ends at the same time $(t_1 = t_2)$ and therefore

length at rest

$$L' = x'_2 - x'_1 = \gamma(x_2 - x_1) = \gamma L \quad \rightarrow \quad L = L'/\gamma$$

length contraction

However the length of a rod aligned with one of the two axis perpendicular to the direction of motion is invariant. Angle are in general not invariant.

Warning!

- Speaking of time dilution: it is with respect to the frame where the time is measured by the same clock (proper time).
  - The statement "moving clock are slower" means moving with respect to the frame where a single clock is needed for the time measurement.
- Length contraction is with respect to the frame where the object is at rest (proper length).

The transformation for the components of the velocity, $\vec{u}$, are obtained from the coordinate transformations

$$u'_x \equiv \frac{dx'}{dt'} = \frac{dx - V\,dt}{dt - V\,dx/c^2} = \frac{u_x - V}{1 - u_x\beta/c}$$

$$u'_y \equiv \frac{dy'}{dt'} = \frac{dy}{\gamma(dt - V\,dx/c^2)} = \frac{u_y}{\gamma(1 - u_x\beta/c)}$$

$$u'_z \equiv \frac{dz'}{dt'} = \frac{dz}{\gamma(dt - V\,dx/c^2)} = \frac{u_z}{\gamma(1 - u_x\beta/c)}$$

with $u_x \equiv dx/dt$, $u_y \equiv dy/dt$ and $u_z \equiv dz/dt$.
Remember β (=V/c) refers to the motion of the frame.

- As time is not invariant, despite the lengths perpendicular to the motion direction being unchanged, the time needed to cover them is different.
  - Unlike classic kinematic, the velocity components perpendicular to the motion, unless vanishing, are affected by the motion of the frame.
- For u_x=c and u_y=u_z=0 it is

$$u'_x = \frac{c - V}{1 - V/c^2} = c\,\frac{c - V}{c - V} = c \quad \text{and} \quad u'_y = u'_z = 0$$

For u_y=c and u_x=u_z=0 it is u'_x=$-V$, u'_z=0 and

$$u'^2_x + u'^2_y = V^2 + c^2(1 - (V/c)^2 = c^2$$

In a similar way as for the velocity, it is possible to find the transformations for the acceleration $\vec{a}$

$$a'_x = \frac{a_x}{\gamma^3(1 - u_x\beta/c)^3}$$

$$a'_y = \frac{a_y}{\gamma^2(1 - u_x\beta/c)^2} + \frac{a_x u_y \beta/c}{\gamma^2(1 - u_x\beta/c)^3}$$

$$a'_z = \frac{a_z}{\gamma^2(1 - u_x\beta/c)^2} + \frac{a_x u_z \beta/c}{\gamma^2(1 - u_x\beta/c)^3}$$

- Acceleration in general is not invariant under Lorentz transformations.

A historical curiosity

The relativistic transformations were named by Poincaré after the dutch physicist Hendrik Lorentz who introduced them, before Einstein paper.

Lorentz had discovered that those transformations leave Maxwell equations invariant.

He had also introduced the notion of "local time" and of "contraction of bodies" for explaining the negative results of the Michelson-Morley experiment because he was convinced, as many other leading scientists, of the validity of the ether theory.

It seems that Einstein was not aware of Lorentz work... Anyway Einstein gave to the transformations a deep physical meaning making them extendable also to mechanics and causing a revolution of classic dynamics.

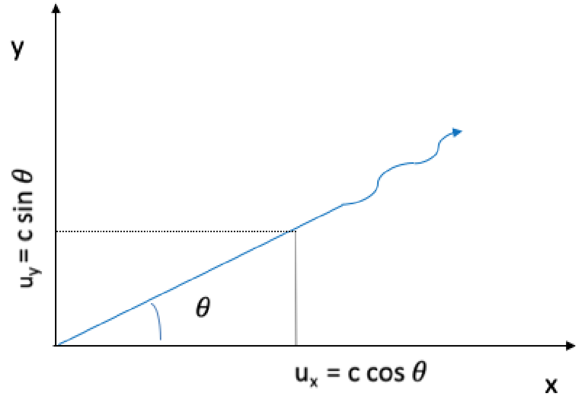
Experimental evidence of relativistic kinematics

Light aberration

Light aberration is the apparent motion of a light source due to the movement of the observer. It was first discovered in astronomy.
Source emitting photons at an angle θ wrt to the x-axis in the S frame where $u_y = c\sin\theta$ and $u_x = c\cos\theta$.
In S' it is $u'_y = c'\sin\theta'$ and $u_x = c'\cos\theta'$.
Using Galilean transformations for the velocity components

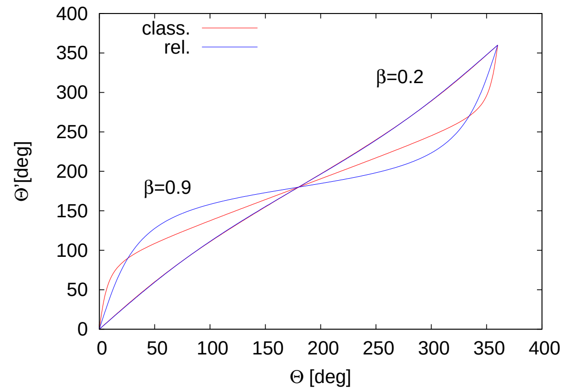
$$u'_y = u_y \text{ and } u'_x = u_x - V$$

$$\tan\theta' = u'_y/u'_x = u_y/(u_x - V)$$
$$\tan\theta' = \frac{\sin\theta}{(\cos\theta - \beta)}$$



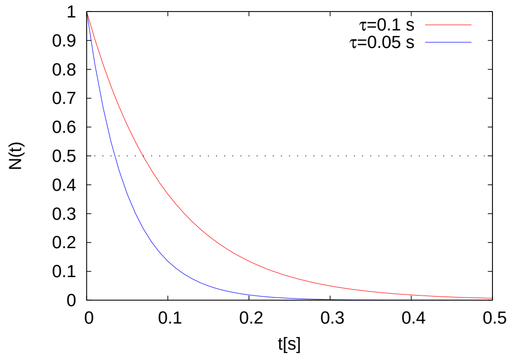
Using instead Lorentz transformations

$$\tan\theta' = \frac{\sin\theta}{\gamma(\cos\theta - \beta)}$$



High energy experiments involving emission of photons confirm the relativistic expression.

Lifetime of unstable particles



Beside e^-, p and n, in nature there are particles which are produced by scattering process and unlike e^+, $\bar{p}$ and $\bar{n}$, are "short-living". Their number decays in time as

$$N(t) = N_0 e^{-t/\tau}$$

The lifetime of charged pions at rest is τ_0=26×10^{-9} s. Time needed for the pions at rest to decay by half

$$N(t) = N_0 e^{-t/\tau} = \frac{N_0}{2} \quad \rightarrow \quad t = 18 \text{ ns}$$

They are produced by bombarding a proper target by high energy protons and leave the target with v ≈2.97×10^8 m/s that is β=0.99 and γ ≈7. It is observed that they are reduced to the half after 37 m from the target. If their lifetime would be as at rest they should become the half already after about 5 m.

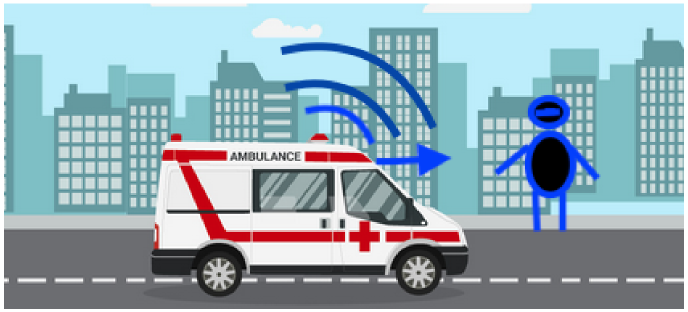
The experimental observation is explained if the pion lifetime in the laboratory frame is

$$\tau = \gamma\tau_0$$

as predicted by time dilation.
Time dilation may allow us realizing future colliders smashing muons!

Doppler effect

Doppler effect exists also classically: we experience it when we hear the siren of a police car or an ambulance. The frequency perceived by an observer at rest is higher when the car is approaching because the number of the acoustic wave knots per unit time is larger, while the frequency decreases when the source is moving away. Classically there is no "transverse" Doppler effect: in the moment the car is at the minimum distance it is $\Delta f = 0$.



Relativistically for a light wave the situation source (S) or receiver (R) in motion are identical. When the angle, θ, between wave propagation direction and motion is 0 the frequency is

$$f = f_0 \sqrt{\frac{1+\beta}{1-\beta}} \quad \text{with } \beta > 0 \text{ for } R \text{ and } S \text{ approaching, } \beta < 0 \text{ when they move away}$$

In addition because of the time dilation there is also a transverse ($\theta = 90^0$) Doppler effect

$$f = f_0/\gamma$$

This was predicted by Einstein who suggested an experiment using hydrogen ions for measuring it. The experiment realized for the first time by Ives and Stilwell in 1938 proved the correctness of Einstein prediction.

Relativistic Dynamics

Assuming $\vec{F}$ invariant and m constant, Newton law, $\vec{F} = m\vec{a}$, is not invariant under Lorentz transformations because we have seen that $\vec{a}$ is not invariant.

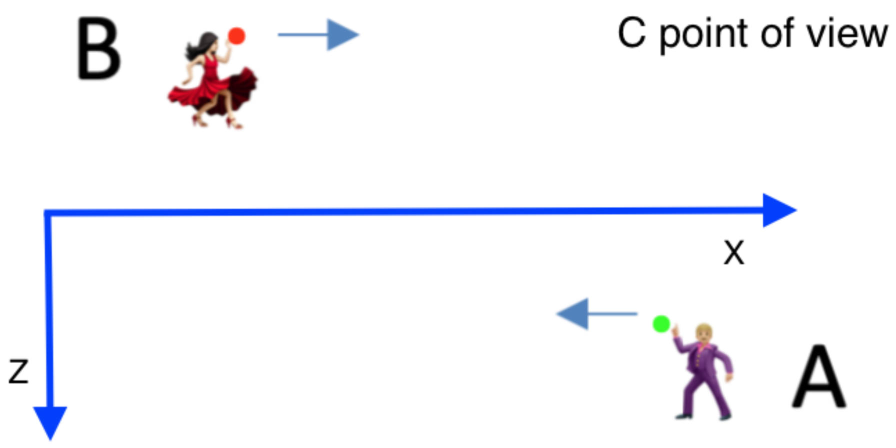
In addition mass can't be a constant because by applying a constant force to an object its speed would increase indefinitely becoming larger than c.

- Classical mechanics must be modified to achieve invariance under Lorentz transformations.
- The new expressions must reduce to the classical ones for $v/c \ll 1$.

In the 1905 paper, Einstein used the Lorentz force and the electro-magnetic field transformations to achieve the generalization of the definition of momentum and energy.

In 1909 two MIT professors of chemistry, Lewis and Tolman, suggested a different more straightforward approach involving purely mechanical arguments.

Let's assume there are two observers, Alex and Betty, moving towards each other with the same speed as seen by a third observer, Charlie.
Betty sits in S and Alex in S'.
Alex and Betty have identical elastic balls.



Betty releases the red ball with u_x^B=0 and u_z=$u_z^B \neq \mathbf{0}$, while Alex releases the green ball with speed $u_x'^A$=0 and $\bar{u}_z'^A$ numerically equal and opposite to the red ball velocity, that is

$$u_z'^A = -u_z^B$$

Green ball in S' Red ball in S

The experiment is set so up that the two balls collide and rebound. Now let's consider Betty point of view. For Betty it is

$$\Delta p_x^B = 0 \quad \Delta p_z^B = 2m_B u_z^B$$

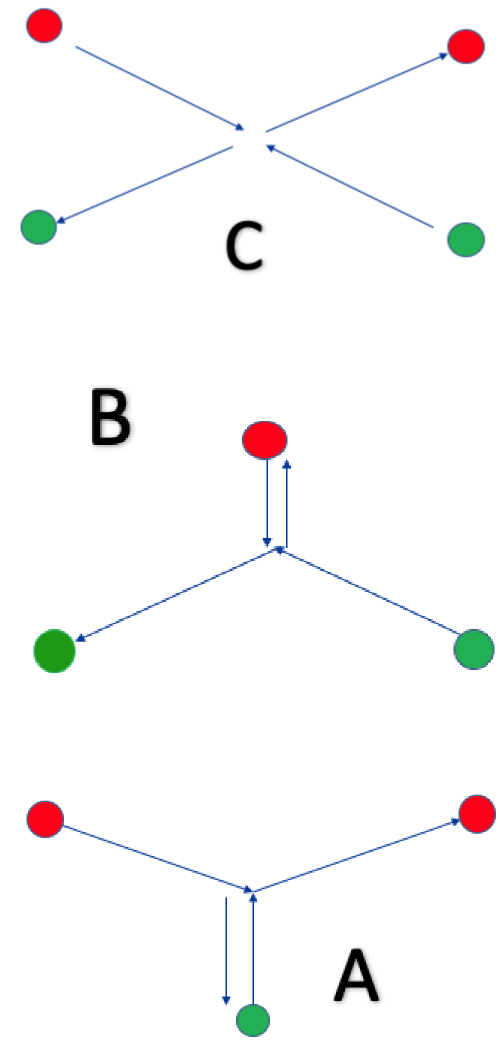
$$\Delta p_x^A = 0 \quad \Delta p_z^A = 2m_A u_z^A$$

We need here the inverse velocity transformation because we know the numerical value of the z direction component in the moving frame S'

$$u_z = \frac{u_z'}{\gamma(1 + u_x'\beta/c)}$$

In our case u_x'=0 and u_z'=$-u_z^B$ and therefore

$$u_z^A = u_z'^A/\gamma = -u_z^B/\gamma \qquad \text{with} \;\; \gamma = \frac{1}{\sqrt{1-(u_x^A/c)^2}}$$



Momentum, classically defined as $\vec{p}=m\vec{v}$, is conserved if

$$\Delta p_z^B = -\Delta p_z^A$$

that is

$$m_B u_z^B = -m_A u_z^A = \frac{1}{\gamma} m_A u_z^B \quad \rightarrow \quad m_A = \gamma m_B$$

We may assume that u_z^B is small so that m_B is the mass at rest, m_0, and $m_A=m(v)$.
So we have found that

$$m = \gamma m_0$$

We can keep the momentum definition from classic dynamic by giving up the invariance of mass. Relativistically mass is not conserved.
A clear example is the annihilation of a e^+e^- pair into 2 photons.

Let's try modifying the classic Newton law

$$\vec{F} = \frac{d\vec{p}}{dt} = m\frac{d\vec{v}}{dt}$$

into

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}(\gamma m_0 \vec{v}) = m_0 \vec{v}\frac{d\gamma}{dt} + m_0 \gamma \frac{d\vec{v}}{dt}$$

$\vec{F}$ and $\vec{a}$ are not parallel!

By scalar multiplication by $\vec{v}$ it is

$$\vec{F} \cdot \vec{v} = \vec{v} \cdot \frac{d\vec{p}}{dt}$$

work/unit time $= dE/dt$

$$\vec{v} \cdot \frac{d\vec{p}}{dt} = m_0 \gamma \vec{v} \cdot \frac{d\vec{v}}{dt} + m_0 \frac{v^2}{c^2} \gamma^3 v \frac{dv}{dt} = m_0 \gamma v \left(1 + \frac{v^2 \gamma^2}{c^2}\right) = m_0 \gamma^3 v \frac{dv}{dt}$$

that is

$$\frac{dE}{dt} = \vec{F} \cdot \vec{v} = m_0 \gamma^3 v \frac{dv}{dt}$$

It is easy to verify that this equation is satisfied by defining the energy as

$$E = mc^2 = \gamma m_0 c^2$$

For v=0 it is $E_0 = m_0 c^2$ which is the energy at rest.

The (relativistic) kinetic energy is obtained by subtracting the rest energy from the total energy

$$T = mc^2 - m_0 c^2 = m_0 c^2 (\gamma - 1) \neq \frac{1}{2} \gamma m_0 v^2$$

which gives the classical kinetic energy $T \simeq m_0 v^2/2$ for $v \ll c$.

Kinetic energy measurement

Experiments confirmed the validity of the relativistic relationship between $\vec{p}$ and $\vec{v}$.
Bertozzi experiment measured directly the velocity of e^- accelerated in a linear accelerator.

- e^- speed was measured through the time of flight.
- Kinetic energy relied on the knowledge of the accelerating field and on the measurement of the heat deposited at the aluminum target.

The results also show clearly the presence of a limit speed, c.

552 WILLIAM BERTOZZI

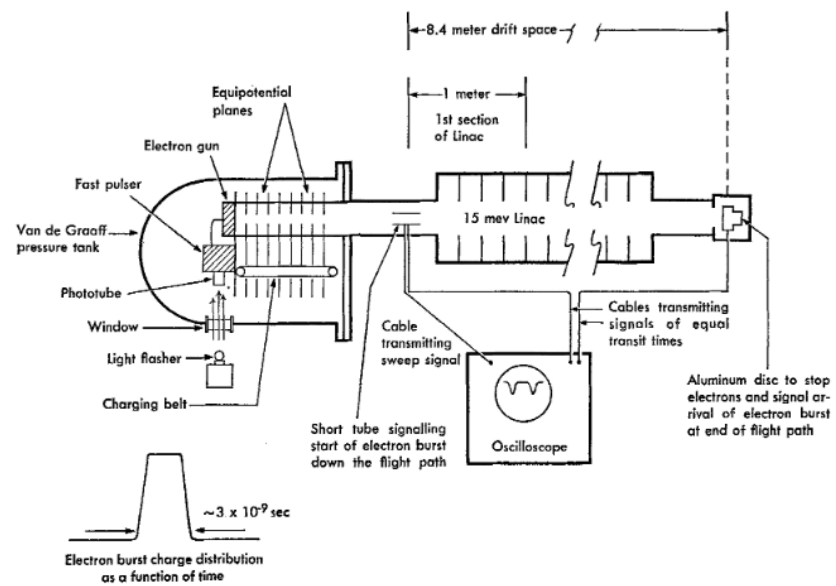
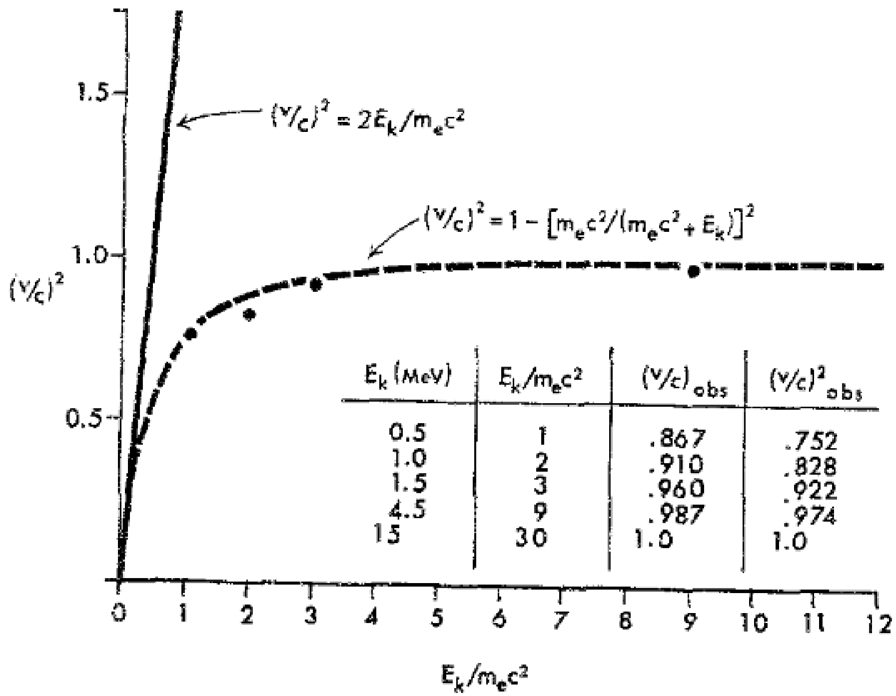


FIG. 1. Schematic diagram of the experiment set up for measuring the time of flight of the electron burst from the Van de Graaff.



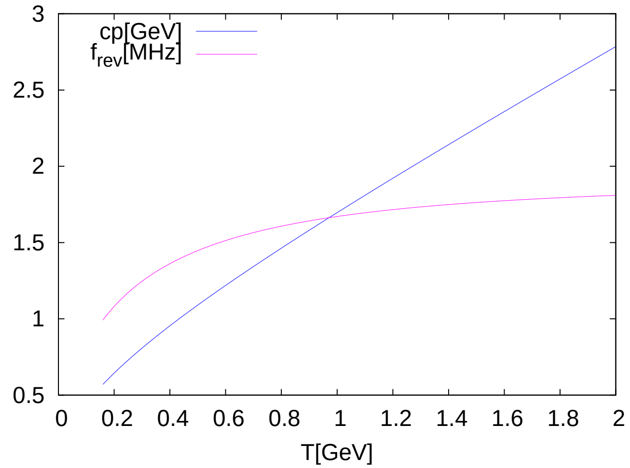
$m_e = e^-$ rest mass

Importance of relativity for accelerators

Example of CERN PS Booster.

Circumference: L=157 m.

Particles are injected from Linac4 with T=160 MeV and accelerated to T=2 GeV.



- The dipole field must be ramped up according to momentum for keeping the particles on the design orbit ($\rho = p/eB$).
- $f_{rf} = h f_{rev}$. For large γ it is $f_{rev} \approx h \frac{c}{L}(1 - \frac{1}{2\gamma^2})$
  $\rightarrow$ almost constant at high energy as the speed approaches c.
  - Particularly true for $e^{\pm}$ which have 1836 larger γ for the same energy.

Relativity has basic relevance for accelerators!

Transformations of momentum, energy and force

The transformations for the momentum follows from the definition $\vec{p}=m\vec{u}$ and from the transformations for the velocity. The result is

$$p'_x = \gamma_V (p_x - E\, V/c^2) \qquad \text{with} \qquad \gamma_V = \frac{1}{\sqrt{1 - V^2/c^2}}$$

$\gamma_u m_0 c^2$

frame speed

$$p'_y = p_y \qquad p'_z = p_z$$

$$E' = \frac{E - V\, p_x}{\sqrt{1 - V^2/c^2}}$$

- The transformations have the same form as for the coordinates transformations with

$$\vec{r} \rightarrow \vec{p} \qquad \text{and} \qquad t \rightarrow E/c^2$$

- Relativistic energy and momentum are closely connected.
  - The quantity $(E/c)^2 - (p_x^2 + p_y^2 + p_z^2)$ is invariant. Indeed from the definitions $(m_0\gamma)^2 c^2 - (m_0\gamma)^2(u_x^2 + u_y^2 + u_z^2) = m_0^2\gamma^2(c^2 - u^2) = m_0^2\gamma^2 c^2(1-\beta^2) = m_0^2 c^2 = \text{const.}$
- If energy and momentum are conserved in one inertial frame of reference they are conserved in all inertial frames.
- If momentum in conserved in two inertial frames, energy too is conserved in both frames.

The transformations for the force are

$$F'_x = F_x - \frac{u_y V}{c^2 - u_x V} F_y - \frac{u_z V}{c^2 - u_x V} F_z$$

$$F'_{y,z} = \frac{\sqrt{1 - V^2/c^2}}{1 - u_x V/c^2} F_{y,z}$$

For $V \ll c$ it is $\vec{F}'=\vec{F}$ which is the classic result.

If the force is acting on a particle which is instantaneously at rest in S (u=0), the transformations simplify

$$F'_x = F_x \qquad F'_y = \frac{1}{\gamma} F_y \qquad F'_z = \frac{1}{\gamma} F_z$$

If the particle is subject to a force, the frame where it is at rest can't be inertial!

However there is always one inertial frame where it is "instantaneously at rest".

Transformations of EM fields

The transformations are found by applying the force transformations to the force experienced by a charged particle moving with velocity $\vec{u}$ in an EM field

$$\vec{F} = q\vec{E} + q\vec{u} \times \vec{B} \qquad \text{(Lorentz force)}$$

In the moving frame S' it must have the same form

$$\vec{F}' = q\vec{E}' + q\vec{u}' \times \vec{B}'$$

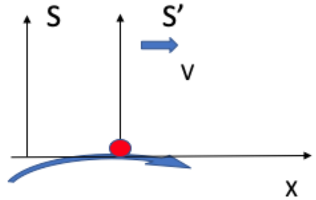
$q'=q$ in agreement with measurements

If the particle is instantaneously at rest in S' $(u'=0)$ it is

$F_x = F'_x \qquad F_y = F'_y/\gamma \qquad F_z = F'_z/\gamma$

In S' there is no effect from $\vec{B}'$ because $\vec{u}'=0 \;\rightarrow \vec{F}' = q\vec{E}'$

We choose for simplicity the frames orientation so that $V=u_x$ and $u_y=u_z=0$



$$\vec{u}\times\vec{B} = \hat{x}(u_yB_z - u_zB_y) + \hat{y}(-u_xB_z + u_zB_x) + \hat{z}(u_xB_y - u_yB_x) = -\hat{y}\,VB_z + \hat{z}\,VB_y$$

$\rightarrow$ The magnetic force has no x-component that is $F_x = qE_x$.

$$qE_x = F_x = F'_x = qE'_x \qquad \rightarrow \quad E'_x = E_x$$

$$\underset{\text{force transformation}}{\uparrow}$$

$$qE_y - qVB_z = F_y = \frac{1}{\gamma}F'_y = \frac{1}{\gamma}qE'_y \qquad \rightarrow \quad E'_y = \gamma(E_y - VB_z)$$

$$qE_z + qVB_y = F_z = \frac{1}{\gamma}F'_z = \frac{1}{\gamma}qE'_z \qquad \rightarrow \quad E'_z = \gamma(E_z + VB_y)$$

Finding the magnetic field transformations is more complicated because there is no frame where the electric force vanishes. The result is

$$B'_x = B_x \qquad B'_y = \gamma\Big(B_y + \frac{V}{c^2}E_z\Big) \qquad B'_z = \gamma\Big(B_z - \frac{V}{c^2}E_y\Big)$$

Denoting by "parallel" and "normal" the fields components wrt to direction of motion the field transformations can be written in the general form

$$E'_{\parallel} = E_{\parallel} \qquad B'_{\parallel} = B_{\parallel}$$

$$E'_{\perp} = \gamma(\vec{E} + \vec{V} \times \vec{B})_{\perp} \qquad B'_{\perp} = \gamma(\vec{B} - \vec{V} \times \vec{E}/c^2)_{\perp}$$

For the inverse transformations $\vec{V}$ mus be replaced by $-\vec{V}$.

Transformation of Source Distributions

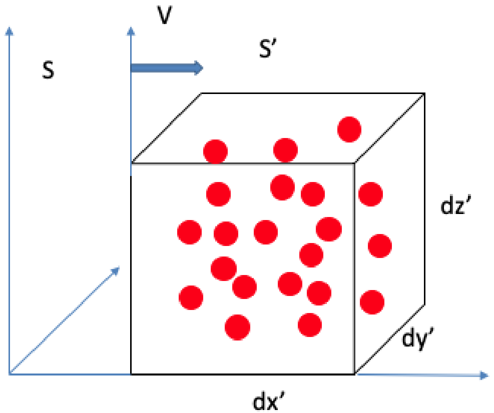
Let us consider a distribution of charges at rest in S'. The charge density is given by

$$\rho'(x', y', z', t') = \frac{qN}{dx'dy'dz'}$$

In S, moving with velocity $-V$ wrt S', the volume element is

$$dx\,dy\,dz = \frac{dx'}{\gamma}dy'dz'$$

length contraction



Charge density in S

$$\rho = \frac{qN}{dx\,dy\,dz} = \gamma\rho'$$

As the charge distribution moves in S with velocity $+\hat{x}V$, in S there is a current with density

$$j_x = \rho V = \gamma\rho' V \qquad \text{(in general: } \vec{j} = \rho\vec{V} = \gamma\rho'\vec{V}\text{)}$$

There is an analogy with E/c (or mc) and $\vec{p}$

$$\rho \;\rightarrow\; m \quad \text{and} \quad \vec{j} \;\rightarrow\; \vec{p}$$

The quantity $(\rho c, \vec{j})$ transforms according to the Lorentz transformations.

Invariance of Maxwell Equations

Knowing how fields and sources transform one can prove that Maxwell equations are invariant under Lorentz transformation. This was demonstrated by Lorentz before Einstein formulated the special relativity theory.

We want to show that if the Maxwell equations hold good in S, they hold with the same form also in S'.

For example let'us prove that

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \qquad \Rightarrow \qquad \nabla' \cdot \vec{E}' = \frac{\rho'}{\epsilon_0}$$

The partial derivatives in S' and in S are related by the cyclic rule

$$\frac{\partial}{\partial ct'} = \frac{\partial ct}{\partial ct'}\frac{\partial}{\partial ct} + \frac{\partial x}{\partial ct'}\frac{\partial}{\partial x} + \frac{\partial y}{\partial ct'}\frac{\partial}{\partial y} + \frac{\partial z}{\partial ct'}\frac{\partial}{\partial z} = \gamma\left(\frac{\partial}{\partial ct} + \beta\frac{\partial}{\partial x}\right)$$

$$\frac{\partial}{\partial x'} = \frac{\partial ct}{\partial x'}\frac{\partial}{\partial ct} + \frac{\partial x}{\partial x'}\frac{\partial}{\partial x} + \frac{\partial y}{\partial x'}\frac{\partial}{\partial y} + \frac{\partial z}{\partial x'}\frac{\partial}{\partial z} = \gamma\left(\beta\frac{\partial}{\partial ct} + \frac{\partial}{\partial x}\right)$$

$$\frac{\partial}{\partial y'} = \frac{\partial}{\partial y} \qquad \frac{\partial}{\partial z'} = \frac{\partial}{\partial z}$$

By using these expressions, the field transformations and the fact that Maxwell equation hold good in S, we find

$$
\begin{aligned}
\nabla' \cdot \vec{E}' &= \frac{\partial E'_x}{\partial x'} + \frac{\partial E'_y}{\partial y'} + \frac{\partial E'_z}{\partial z'} \\
&= \gamma \frac{\partial E'_x}{\partial x} + \frac{\partial E'_y}{\partial y} + \frac{\partial E'_z}{\partial z} + \gamma\beta \frac{\partial E'_x}{\partial ct} \\
&= \gamma \frac{\partial E_x}{\partial x} + \gamma \frac{\partial E_y}{\partial y} + \gamma \frac{\partial E_z}{\partial z} - \gamma V \frac{\partial B_z}{\partial y} + \gamma V \frac{\partial B_y}{\partial z} + \gamma\beta \frac{\partial E_x}{\partial ct} \\
&= \gamma \nabla \cdot \vec{E} - \gamma V \left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} \right) + \gamma\beta \frac{\partial E_x}{\partial ct} \\
&= \gamma \frac{\rho}{\epsilon_0} - \gamma V \left(\nabla \times \vec{B} - \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} \right)_x \\
&= \gamma \frac{\rho}{\epsilon_0} - \gamma V \frac{j_x}{\epsilon_0 c^2} \\
&= \frac{\gamma}{\epsilon_0 c} \left(\rho c - \beta j_x \right) \\
&= \frac{\rho'}{\epsilon_0}
\end{aligned}
$$

THE CM ENERGY

The center of momentum for an isolated ensemble of particles is defined as the inertial frame where it holds

$$\sum_i \vec{p}_i = \sum_i \frac{m_{0,i}\vec{v}_i}{\sqrt{1 - V^2/c^2}} = 0$$

frame velocity

We have seen that $(E/c)^2 - |\vec{p}|^2 = m_0^2 c^2$.
For the total energy and momentum of the ensemble

$$E = \sum_i E_i \quad \text{and} \quad \vec{P} = \sum_i \vec{p}_i$$

total energy

total momentum

the invariant is easily evaluated in the CM frame

energy in CM

$$\left(\sum_i E_i/c\right)^2 - \sum_i \vec{p}_i \cdot \sum_i \vec{p}_i = \left(\sum_i E'_i/c\right)^2$$

Let us consider two simple cases:

a) two ultra-relativistic particles colliding "head-on";
b) one ultra-relativistic particle hitting a particle at rest.

For the system of two particles it is

$$\frac{(E_1' + E_2')^2}{c^2} = \frac{(E_1 + E_2)^2}{c^2} - (\vec{p}_1 + \vec{p}_2) \cdot (\vec{p}_1 + \vec{p}_2)$$
$$= \frac{(E_1 + E_2)^2}{c^2} - p_1^2 - p_2^2 - 2\vec{p}_1 \cdot \vec{p}_2$$

Moreover for ultra-relativistic particles it is

$$p = mv \simeq mc = \frac{E}{c}$$

a) $\vec{p}_1/p_1 = -\vec{p}_2/p_2$

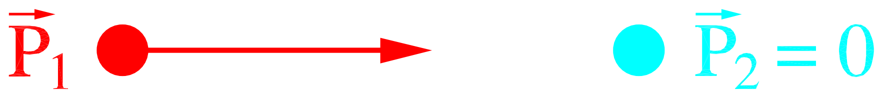
$$\frac{(E_1' + E_2')^2}{c^2} = \frac{E_1^2}{c^2} + \frac{E_2^2}{c^2} + 2\frac{E_1 E_2}{c^2} - \frac{E_1^2}{c^2} - \frac{E_2^2}{c^2} + 2\frac{E_1 E_2}{c^2} = 4\frac{E_1 E_2}{c^2}$$

and thus

$$E_1' + E_2' = 2\sqrt{E_1 E_2}$$

LHC (p/p): E_1=E_2=6.5 TeV $\rightarrow$ energy in the center of mass $E_1' + E_2'$=2×6.5=13 TeV.
HERA ($p/e^{\pm}$): E_1=920 GeV and E_2=27.5 GeV $\rightarrow$ $E_1' + E_2'$=318 GeV.

b) $\vec{p}_2 = 0$ and $E_2 = m_{0,2}c^2$



$$\frac{(E_1' + E_2')^2}{c^2} = \frac{(E_1 + E_2)^2}{c^2} - p_1^2 - p_2^2 - 2\vec{p}_1 \cdot \vec{p}_2$$

$$\frac{(E_1' + E_2')^2}{c^2} = \frac{E_1^2}{c^2} + \frac{E_2^2}{c^2} + 2\frac{E_1 E_2}{c^2} - \frac{E_1^2}{c^2} = \frac{E_2^2}{c^2} + 2\frac{E_1 E_2}{c^2}$$

and therefore

$$(E_1' + E_2') = \sqrt{E_2(E_2 + 2E_1)} = \sqrt{E_2(m_{0,2}c^2 + 2E_1)} \simeq \sqrt{2E_1 E_2}$$

For example, with $E_2 = 0.938$ GeV (proton rest mass) to get in the CM an energy of 318 GeV must be E_1 =54 TeV.

From this example we see the advantage of collider experiments wrt. fixed target ones (intensity permitting).

A summary of some useful (?) relationships

$$\gamma \equiv \frac{1}{\sqrt{1-(v/c)^2}} \qquad \beta \equiv \frac{v}{c} = \sqrt{1-\frac{1}{\gamma^2}}$$

$$m = \gamma m_0 \qquad \vec{p} = \gamma m_0 \vec{v} = \frac{m_0 \vec{v}}{\sqrt{1-(v/c)^2}} \qquad \left(\frac{v}{c}\right)^2 = \frac{p^2}{(m_0 c)^2 + p^2}$$

$$E = mc^2 \qquad E_0 = m_0 c^2 \qquad \frac{E}{E_0} = \frac{m_0 \gamma c^2}{m_0 c^2} = \gamma$$

$$T = E - E_0 = m_0 \gamma c^2 - m_0 c^2 = m_0 c^2 (\gamma - 1)$$

$$E^2 = (T + E_0)^2 = m^2 c^4 = m_0^2 \gamma^2 c^4 = \frac{m_0^2 c^4}{1-(v/c)^2} = \frac{m_0^2 c^4}{1 - p^2/(m_0^2 c^2 + p^2)}$$

$$= \frac{m_0^2 c^4}{m_0^2 c^2}(m_0^2 c^2 + p^2) = m_0^2 c^4 + c^2 p^2$$

$$cp = c\gamma m_0 v = \frac{E}{E_0} c m_0 v = \frac{E}{m_0 c^2} c m_0 v = \beta E \qquad cp \simeq E \quad \text{for} \quad \beta \to 1$$

References

[1] R. Resnick, "Introduction to Special Relativity", John Wiley & Sons, 1968.

[2] R. P. Feynman, "Lectures on Physics", vol. I, Addison-Wesley, 1963.