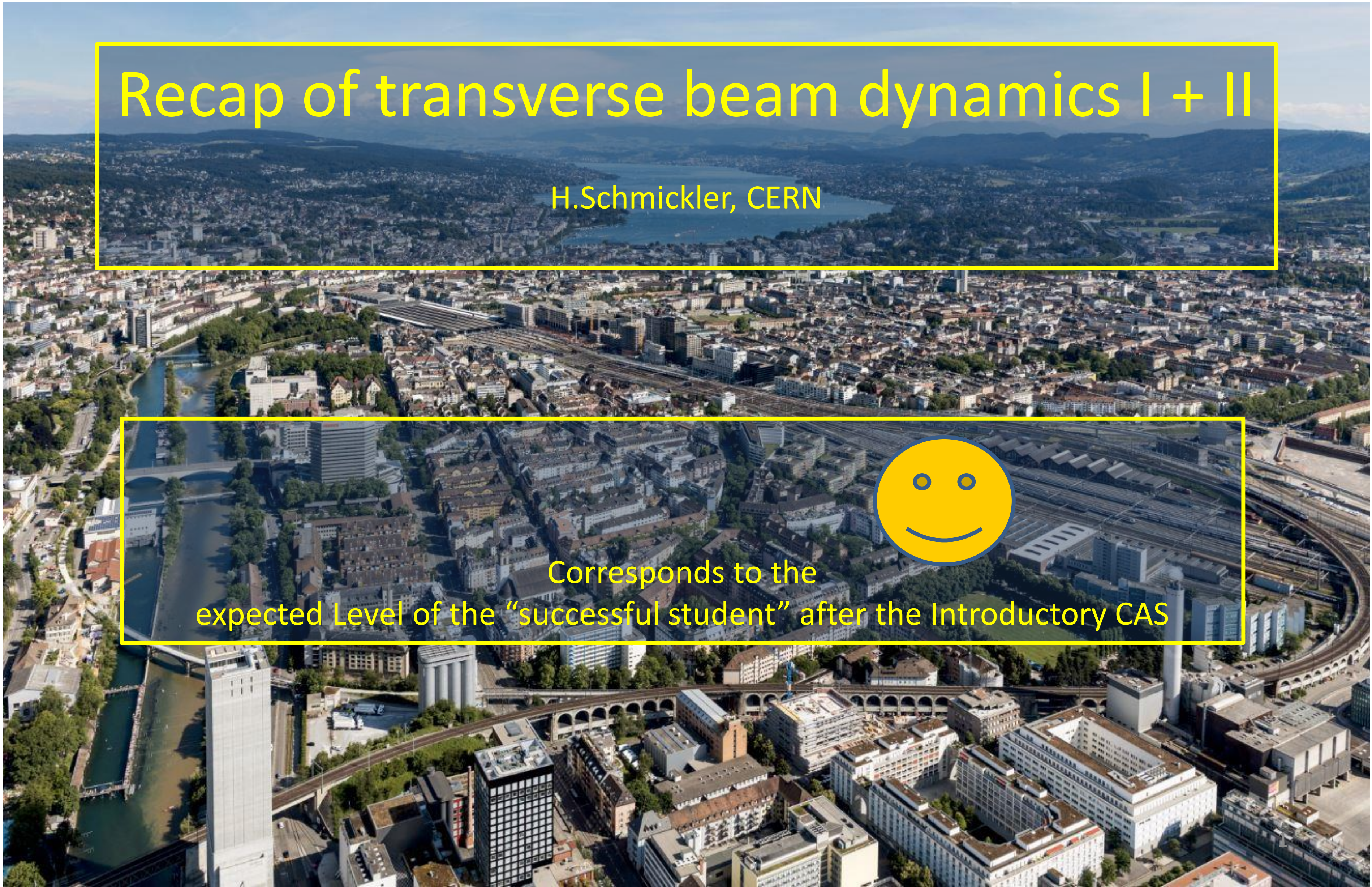
The CERN Accelerator School

Future High Energy Colliders, Zürich, 21.2-5.3.2018



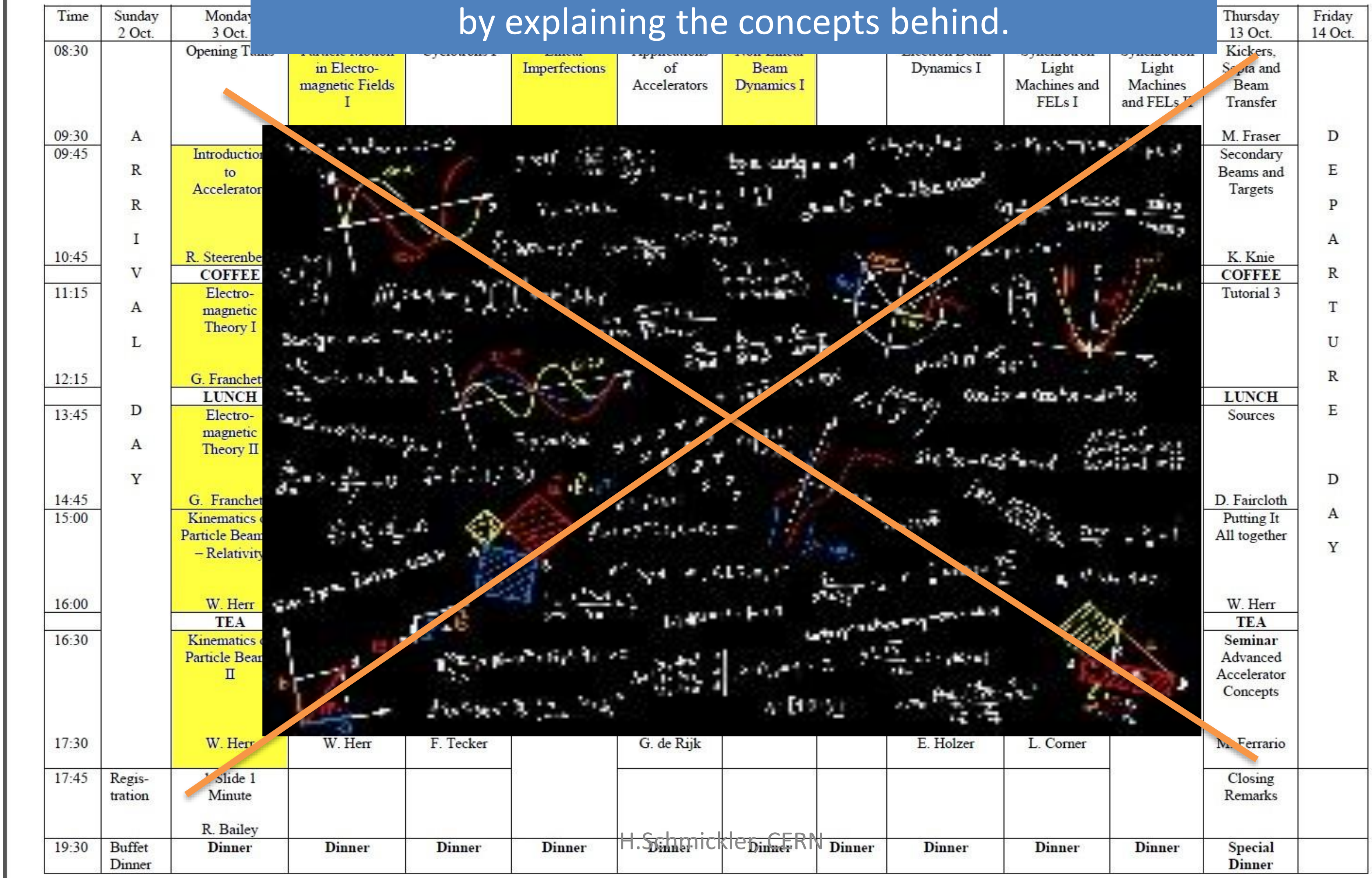
Recap of transverse beam dynamics I + II

H.Schmickler, CERN

Corresponds to the
expected Level of the “successful student” after the Introductory CAS

16 hours of compact lectures summarized in 2 hours.

Only possible by leaving out most of the mathematics and by explaining the concepts behind.



Time	Sunday 2 Oct.	Monday 3 Oct.										Thursday 13 Oct.	Friday 14 Oct.
08:30		Opening T	in Electro-magnetic Fields I		Imperfections	of Accelerators	Beam Dynamics I		Dynamics I	Light Machines and FELs I	Light Machines and FELs II	Kickers, Septa and Beam Transfer	
09:30	A R R I V A L D A Y											M. Fraser	D E P A R T U R E D A Y
09:45		Introduction to Accelerator										Secondary Beams and Targets	
10:45		R. Steerenbe										K. Knie	
		COFFEE										**COFFEE**	
11:15		Electro-magnetic Theory I										Tutorial 3	
12:15		G. Franchet											
		LUNCH										**LUNCH**	
13:45		Electro-magnetic Theory II										Sources	
14:45		G. Franchet										D. Faircloth	
15:00		Kinematics Particle Beam – Relativity										Putting It All together	
16:00		W. Herr										W. Herr	
		TEA										**TEA**	
16:30		Kinematics Particle Bean II										Seminar Advanced Accelerator Concepts	
17:30		W. Herr	W. Herr	F. Tecker		G. de Rijk			E. Holzer	L. Comer		M. Ferrario	
17:45	Regis-tration	Slide 1 Minute R. Bailey										Closing Remarks	
19:30	Buffet Dinner	**Dinner**	**Dinner**	**Dinner**	**Dinner**	**Dinner**	**Dinner**	**Dinner**	**Dinner**	**Dinner**	**Dinner**	**Special Dinner**	

Basics (only 5 minutes):

- Phenomenology of Special relativity, formulae for relativistic beams
- simple examples of E-fields and B-fields, multipole expansion of B-fields

Linear Optics:

- Hamiltonian formalism→ derivative of Hill's equation from Hamiltonian
  Hamiltonian in different Coordinate Systems, weak focusing

- linear optics: motion of single particle in a lattice, phase space plots
  - trajectory, closed orbit, dispersion, weak focusing
  - strong focusing, tune, chromaticity
  - linear Imperfections, down-feed, coupling
- "A taste" of non-linear dynamics

Liouville's Theorem:

- Definition of emittance
- emittance preservation in conservative systems
- filamentation due to non-linearities

Phenomenology of Collective Effects:

- Space Charge
- Touschek and Intrabeam Scattering
- Wakefields

Slides partially or fully taken from the lecturers in Budapest:
S. Sheehy
W. Herr
B. Holzer
G. Franchetti
A. Wolski
R. Tomas
F. Tecker
V. Kain (Erice 2017)
A. Cianchi (Egham 2017)

1: Relativistic particles

Conservation of transverse momentum

→ A moving object in its frame S' has a mass m' = $^{m}/_{\gamma}$

Or $m = \gamma m_0 = \frac{m_0}{\sqrt{1-(\frac{v}{c})^2}} \cong m_0 + \frac{1}{2} m_0 v^2 (\frac{1}{c^2})$ (approximation for small v)

Multiplied by c^2:

$$\boxed{mc^2 \cong m_0c^2 + \frac{1}{2} m_0 v^2 = m_0 c^2 + T}$$

Interpretation:

→ Total energy $E\ is \quad E = m \cdot c^2$

→ For small velocities the total energy is the sum of the kinetic energy plus the rest energy

→ Particle at rest has rest energy $E_0 = m_0 \cdot c^2$

→ Always true (Einstein): $\boldsymbol{E = m \cdot c^2 = \gamma m_0 \cdot c^2}$

Relativistic momentum $p = mv = \gamma m_0 v = \gamma m_0 \beta c$

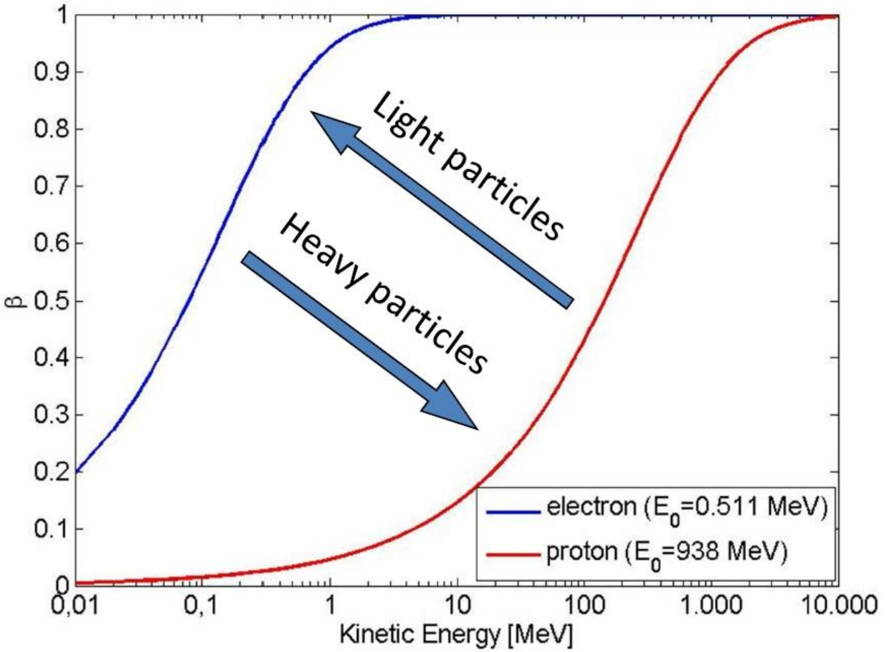
From page before (squared):

$$E^2 = m^2c^4 = \gamma^2 {m_0}^2 c^4 = \left(\frac{1}{1-\beta^2}\right){m_0}^2c^4 = \left(\frac{1-\beta^2+\beta^2}{1-\beta^2}\right){m_0}^2c^4 = (1+\gamma^2\beta^2){m_0}^2c^4$$

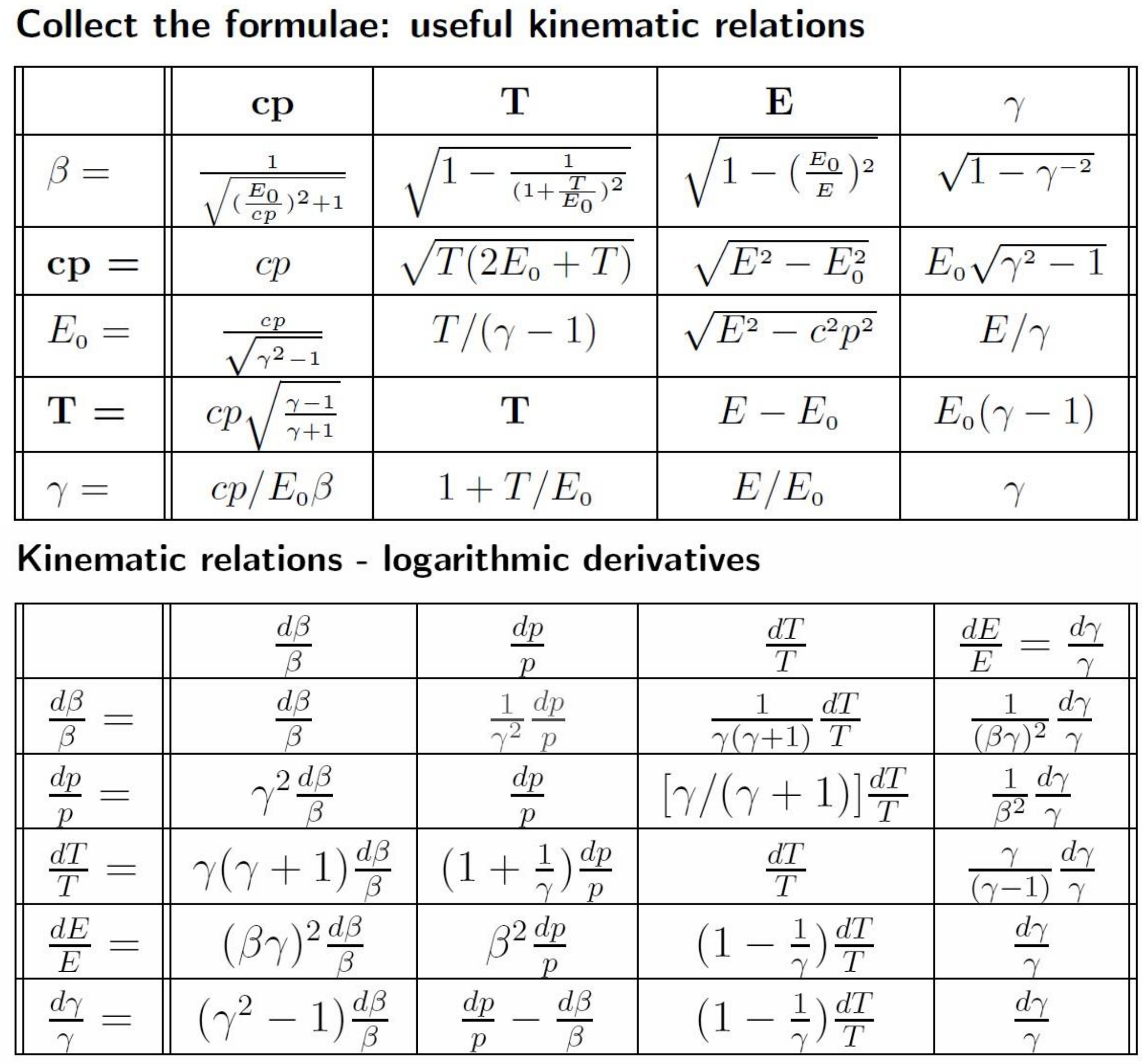
$$E^2 = (m_0c^2)^2 + (pc)^2 \Rightarrow \boxed{\frac{E}{c} = \sqrt{(m_0c)^2 + p^2}}$$

Or by introducing new units [E] = eV ; [p] =eV/c ; [m] = eV/c^2 $\boxed{E^2 = {m_0}^2 + p^2}$

Due to the small rest mass electrons reach already the speed of light with relatively low kinetic energy, but protons only in the GeV range



For those, who really want to calculate…



Collect the formulae: useful kinematic relations

	cp	**T**	**E**	γ
$\beta =$	$\frac{1}{\sqrt{(\frac{E_0}{cp})^2+1}}$	$\sqrt{1-\frac{1}{(1+\frac{T}{E_0})^2}}$	$\sqrt{1-(\frac{E_0}{E})^2}$	$\sqrt{1-\gamma^{-2}}$
cp =	cp	$\sqrt{T(2E_0+T)}$	$\sqrt{E^2-E_0^2}$	$E_0\sqrt{\gamma^2-1}$
$E_0 =$	$\frac{cp}{\sqrt{\gamma^2-1}}$	$T/(\gamma-1)$	$\sqrt{E^2-c^2p^2}$	E/γ
T =	$cp\sqrt{\frac{\gamma-1}{\gamma+1}}$	**T**	$E-E_0$	$E_0(\gamma-1)$
$\gamma =$	$cp/E_0\beta$	$1+T/E_0$	E/E_0	γ

Kinematic relations - logarithmic derivatives

	$\frac{d\beta}{\beta}$	$\frac{dp}{p}$	$\frac{dT}{T}$	$\frac{dE}{E}=\frac{d\gamma}{\gamma}$
$\frac{d\beta}{\beta}=$	$\frac{d\beta}{\beta}$	$\frac{1}{\gamma^2}\frac{dp}{p}$	$\frac{1}{\gamma(\gamma+1)}\frac{dT}{T}$	$\frac{1}{(\beta\gamma)^2}\frac{d\gamma}{\gamma}$
$\frac{dp}{p}=$	$\gamma^2\frac{d\beta}{\beta}$	$\frac{dp}{p}$	$[\gamma/(\gamma+1)]\frac{dT}{T}$	$\frac{1}{\beta^2}\frac{d\gamma}{\gamma}$
$\frac{dT}{T}=$	$\gamma(\gamma+1)\frac{d\beta}{\beta}$	$(1+\frac{1}{\gamma})\frac{dp}{p}$	$\frac{dT}{T}$	$\frac{\gamma}{(\gamma-1)}\frac{d\gamma}{\gamma}$
$\frac{dE}{E}=$	$(\beta\gamma)^2\frac{d\beta}{\beta}$	$\beta^2\frac{dp}{p}$	$(1-\frac{1}{\gamma})\frac{dT}{T}$	$\frac{d\gamma}{\gamma}$
$\frac{d\gamma}{\gamma}=$	$(\gamma^2-1)\frac{d\beta}{\beta}$	$\frac{dp}{p}-\frac{d\beta}{\beta}$	$(1-\frac{1}{\gamma})\frac{dT}{T}$	$\frac{d\gamma}{\gamma}$

Electromagnetic Fields and forces onto charged particles

- Described by Maxwell's equations and by the Lorentz-force
- Lots of mathematics, we will only "look" at the equations
- Only electric fields can transfer momentum to charged particles
  → EM cavities for acceleration → F. Tecker
- Magnetic fields are used to bend or focus the trajectory of charged particles
  → construction of different types of accelerator magnets

- Also electrostatic forces can bend and focus beams; but since the forces are small we often neglect this part

Integral form	Differential form
$\int_S \vec{E}\cdot d\vec{A} = \frac{Q}{\epsilon_0}$	$\nabla\cdot\vec{E} = \frac{\rho}{\epsilon_0}$
$\int_S \vec{B}\cdot d\vec{A} = 0$	$\nabla\cdot\vec{B} = 0$
$\oint_\Gamma \vec{E}\cdot d\vec{l} = -\frac{d\Phi(\vec{B})}{dt}$	$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$
$\oint_\Gamma \vec{B}\cdot d\vec{l} = \mu_0\left(I + \epsilon_0\frac{\partial}{\partial t}\int_S \vec{E}\cdot d\vec{A}\right)$	$\vec{\nabla}\times\vec{B} = \mu_0\left(\vec{j} + \epsilon_0\frac{\partial}{\partial t}\vec{E}\right)$

Lorentz force $\boxed{\vec{F} = q*(\cancel{\vec{E}} + \vec{v}\times\vec{B})}$

typical velocity in high energy machines: $v \approx c \approx 3*10^8\,{}^{m}/_{s}$

Example:

$$B = 1T \quad\rightarrow\quad F = q*3*10^8\frac{m}{s}*1\frac{Vs}{m^2}$$

$$F = q*\underbrace{300\frac{MV}{m}}_{E}$$

equivalent el. field E

technical limit for el. field

$$E \le 1\frac{MV}{m}$$

But: for specific cases we also use electrostatic elements

quadrupole

Separators for electron and positron beams in the same vacuum chamber

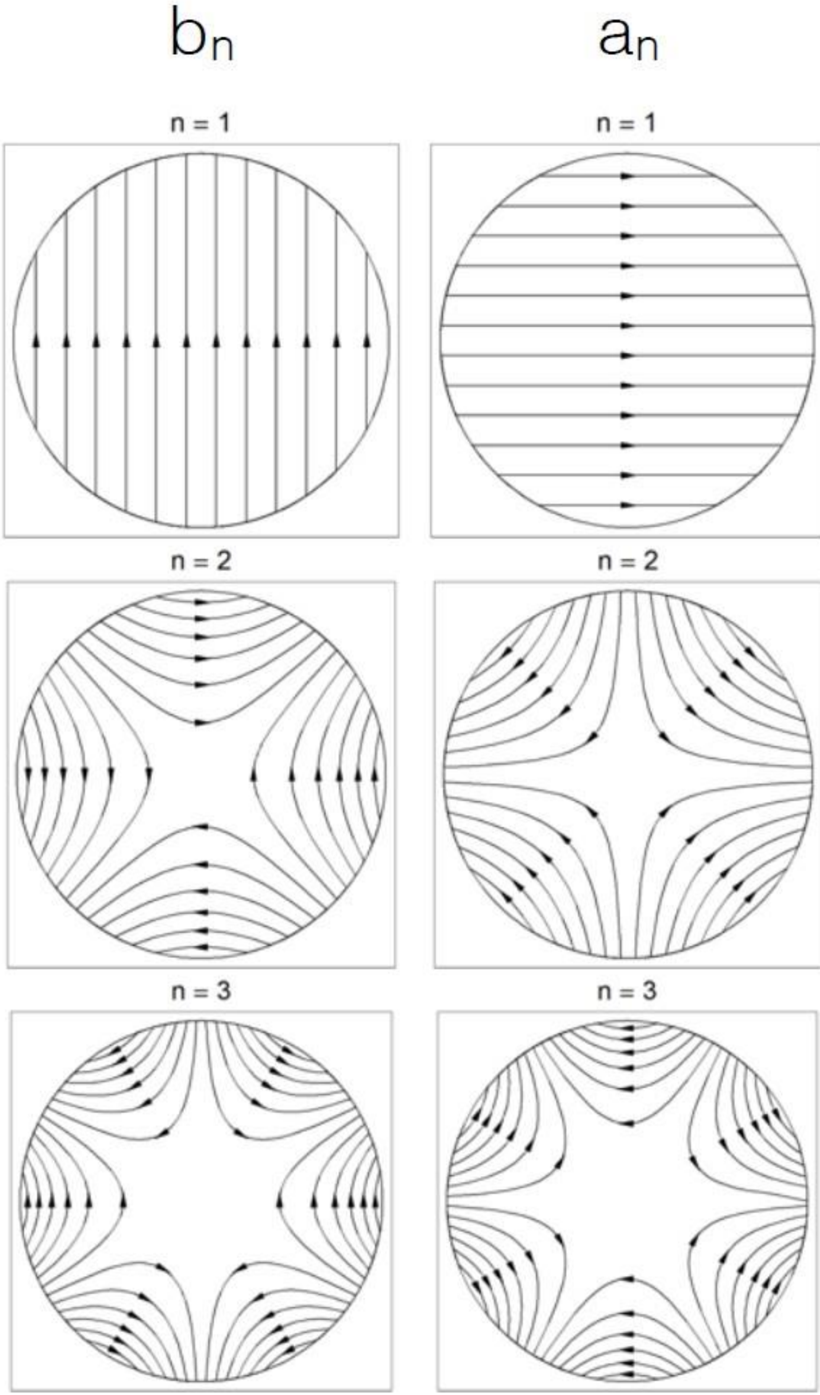
We need real magnets in an accelerator…not any arbitrary shapes of magnetic fields, but nicely classified field types by making reference to a multipole expansion of magnetic fields:

In the usual notation:

$$B_y + iB_x = B_{ref}\sum_{n=1}^{\infty}(b_n + ia_n)\left(\frac{x+iy}{R_{ref}}\right)^{n-1}$$

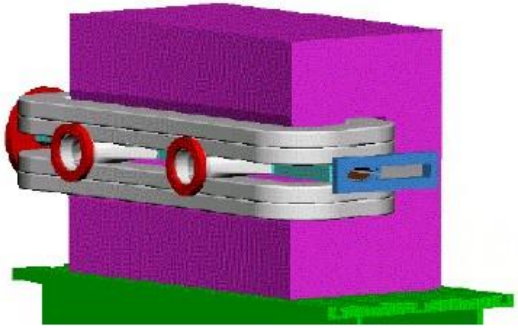
b_n are "normal multipole coefficients" (LEFT)
and a_n are "skew multipole coefficients" (RIGHT)
'ref' means some reference value

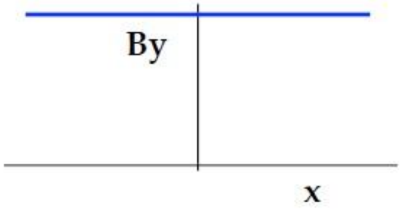
n=1, dipole field
n=2, quadrupole field
n=3, sextupole field

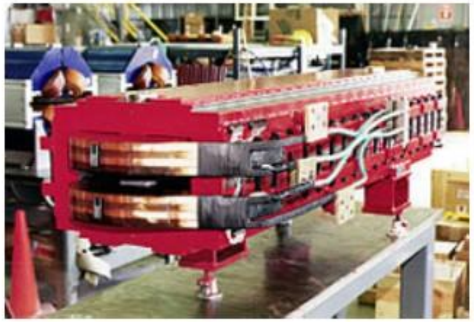


Images: A. Wolski, https://cds.cern.ch/record/1333874

Multipole Magnets







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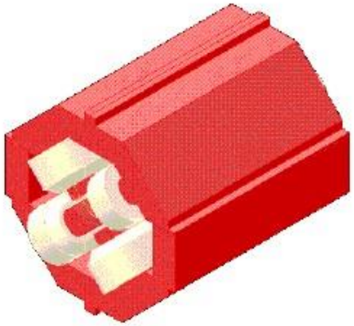

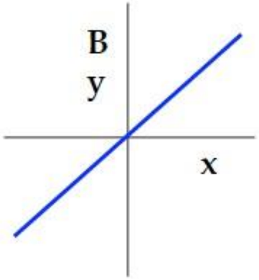

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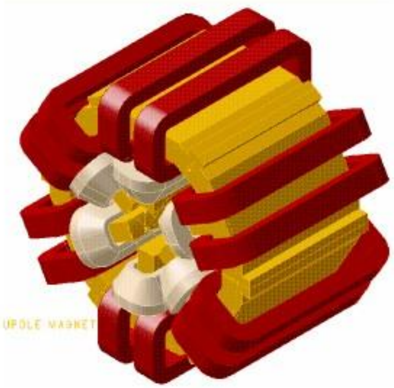
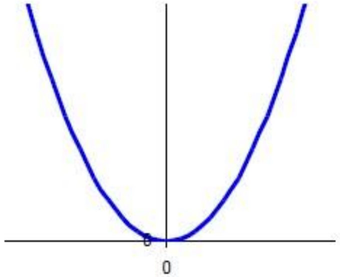




Images: Ted Wilson, JAI Course 2012

Image: Danfysik

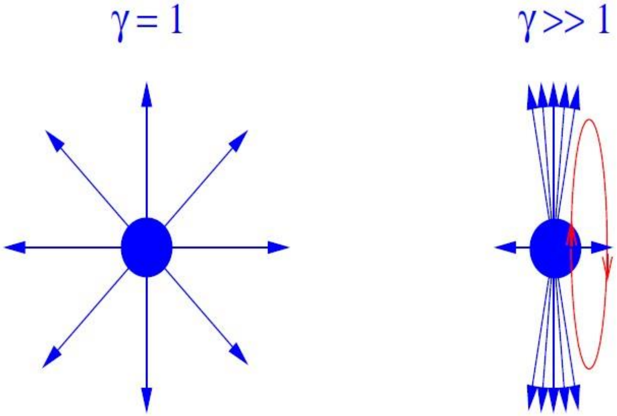
Back to relativity: transformation of fields into a moving frame

Use Lorentz transformation of $F^{\mu\nu}$ and write for components:

Lecture of
W. Herr

$$\begin{aligned} E'_x &= E_x & B'_x &= B_x \\ E'_y &= \gamma(E_y - v \cdot B_z) & B'_y &= \gamma(B_y + \tfrac{v}{c^2} \cdot E_z) \\ E'_z &= \gamma(E_z + v \cdot B_y) & B'_z &= \gamma(B_z - \tfrac{v}{c^2} \cdot E_y) \end{aligned}$$

Example Coulomb field: (a charge moving with constant speed)



- **In rest frame purely electrostatic forces**
- **In moving frame $\vec{E}$ transformed and $\vec{B}$ appears**

Transverse Beam Dynamics

??? high intensity beam described in 6D phase space??? No…

But:

Starting point:
- Single particle in single magnetic element
- complete decoupling of long., hor.& ver. motion
- particle with nominal momentum

My first accelerator:
- Single particle in many magnetic elements
- circular structure: synchrotron
- twiss parameters, orbit, tune…

Off-momentum particle:
- Dispersion
- Momentum compaction
- Chromaticity…a taste of non-linearities

Finally a beam of many particles (not too many!)
- emittance
- Liouville's theorem
- adiabatic damping and radiation damping

Linear Optics – Hamiltonian (1/3)

A little reminder of classical mechanics:

- Take a set of "canonical conjugate variables" (q, p in a single one dimensional case)
- q is called the generalized coordinate and p the generalized momentum
- Construct a function H, which satisfies the dynamical equations of the system:

$$\frac{\partial q}{\partial t} = \dot{q} = \frac{\partial H}{\partial p} \qquad \text{and} \qquad \frac{\partial p}{\partial t} = \dot{p} = -\frac{\partial H}{\partial q}$$

- H "= the Hamiltonian " of the system is a constant of motion
  (= H does not explicitly depend on t) .
- The Hamiltonian of a system is the total energy of the system: H = T +V
  (sum of potential and kinetic energy)

Proof:

$$\begin{aligned} \dot{H} &= \sum_{i=1}^{n} \frac{\partial H}{\partial x_i}\dot{x}_i + \sum_{i=1}^{n} \frac{\partial H}{\partial p_i}\dot{p}_i \\ &= \sum_{i=1}^{n} \frac{\partial H}{\partial x_i}\frac{\partial H}{\partial p_i} + \sum_{i=1}^{n} \frac{\partial H}{\partial p_i}\left(-\frac{\partial H}{\partial x_i}\right) = 0. \end{aligned}$$

Used x instead of q just to test your attention

Linear Optics – Hamiltonian (2/3)

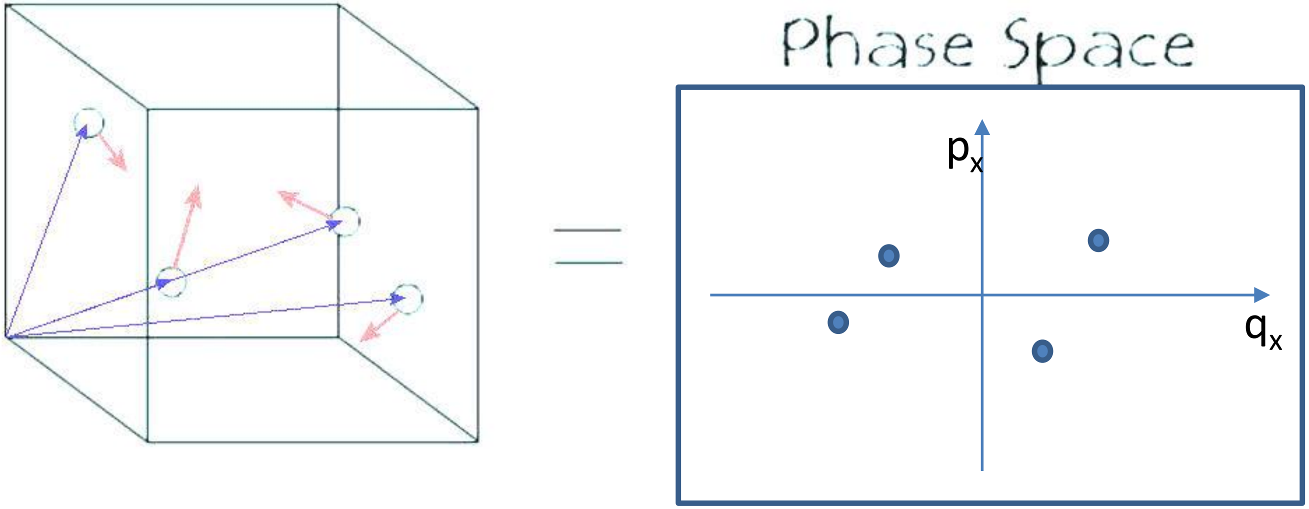
This leads immediately to the question:
What are canonically conjugate variables?
* Complete answer: Lecture of W.Herr later this course

Short answer:
Several combinations are possible, the most relevant for us are
- x (space) and p (momentum)
- E (energy) and t (time).

We can learn most of the physics, when we construct quantities from these canonical variables, which are constants of motion (energy, action…)

* Hint to a more complete answer:
- Describe the particle motion by a Lagrange function of **generalized coordinates** and **generalized velocities** and time.
  - define an action variable and assume that nature is made such that the action between any two points of particle motion is stationary
- This is fulfilled for Lagrange functions satisfying the Euler-Lagrange equation
- And this leads finally to the definition of **generalized momenta** instead of **generalized velocities**, the definition of the Hamiltonian function and then to the two equations of motion as shown on the last slide.

Recall: what is the “action” variable; what is phase space



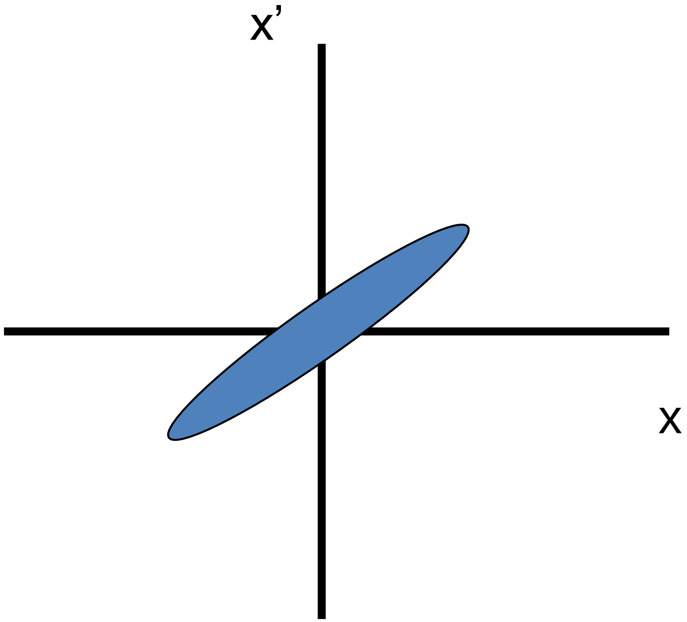
This shows only one of the three phase-spaces

Define action “S”:= $\int_{t_1}^{t_2} p\, dq$

“Stationary” action principle:=
Nature chooses path from t_1 to t_2 such that the action integral is a minimum and stationary
→ we have a new invariant, which we can use to study the dynamics of the system

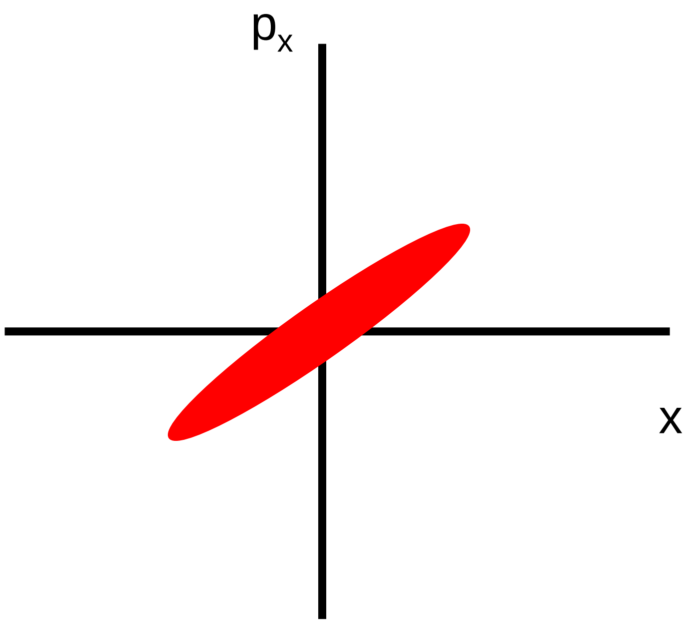
Warning: We often use the term phase space for the 6N dimensional space defined by x, x’ (space, angle), but this the “trace space” of the particles.
At constant energy phase space and trace space have similar physical interpretation

Trace space



$$x' = \frac{dx}{ds} = \frac{dx}{dt} \cdot \frac{dt}{ds} = \frac{\beta_x}{\beta_s}$$

Phase space

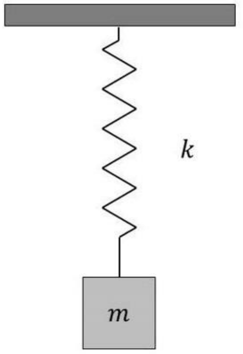


$$p_x = m_o \, c \, \gamma_{rel} \, \beta_x$$

Linear Optics – Hamiltonian (3/3)

Example: Mass-spring system

$$H = T + V = \frac{1}{2}\,\text{k}\,x^2 + \frac{p^2}{2m} = \text{E}$$



Hamiltonian formalism to obtain the equations of motion:

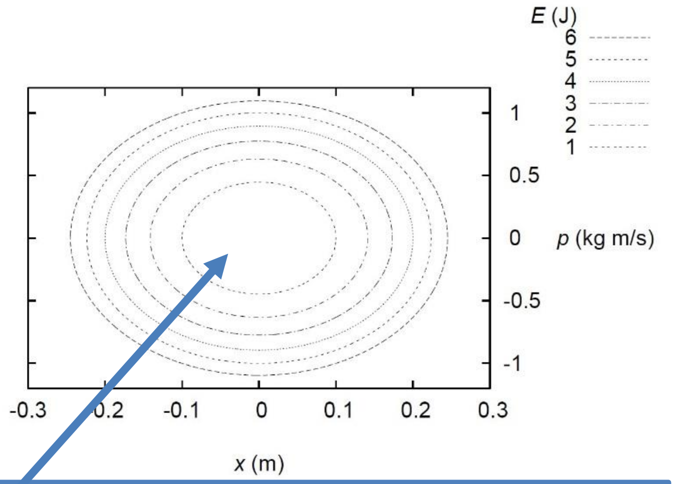
$$\frac{\delta x}{\delta t} = \dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m} \text{ or p=m}\dot{x} \text{ = mv}$$

$$\frac{\delta p}{\delta t} = \dot{p} = -\frac{\partial H}{\partial x} = \text{-kx}$$

We are used to start with the force equation:

$F = ma = m\ddot{x}$ = - kx

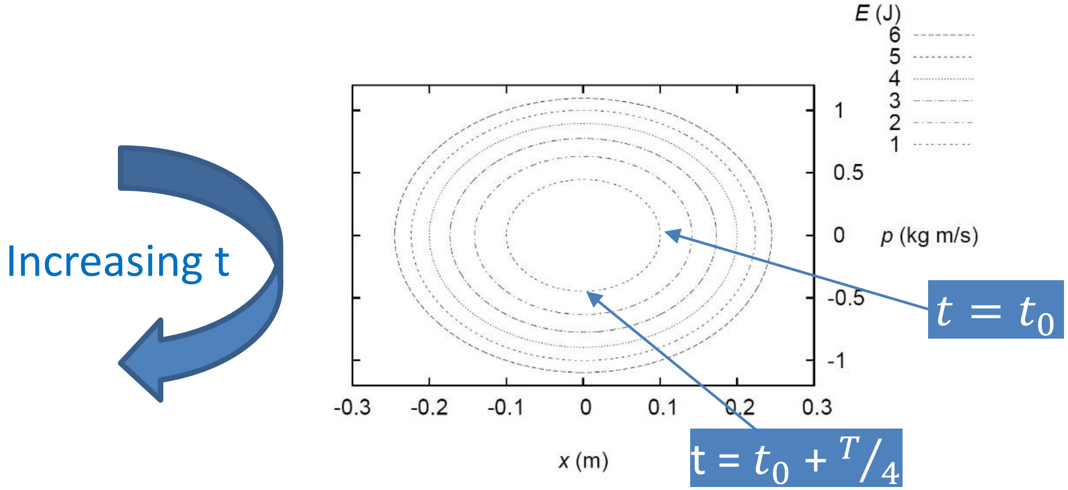
With the well known sinusoidal solution for x(t).



Instead we look at the trajectory of the system in a phase space.

In this simple case the Hamiltonian itself is the equation of the ellipse.

A further look at phase-space plots



- The particle follows in phase space a trajectory, which has an elliptic shape.
- In the example, the free parameter along the trajectory is time (we are used to express the space-coordinate and momentum as a function of time)
- This is fine for a linear one-dimensional pendulum, but it is not an adequate description for transverse particle motion in a circular accelerator
  → we will choose soon “s”, the path length along the particle trajectory as free parameter
- Any linear motion of the particle between two points in phase space can be written as a matrix transformation: $\binom{x}{x'}(s)=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\binom{x}{x'}(s_0)$
- In matrix annotation we define an action “J” as product J:= $\frac{1}{2}\binom{x}{x'}(s)\ \binom{x}{x'}(s_0)$.
- J is a motion invariant and describes an ellipse in phase space. The area of the ellipse is $2\pi J$

Why all this? Later we will define the emittance of a beam as the average action variable of all particles… but for the moment we stick to single particles … and we follow them through magnetic elements.

Why at all "Hamiltonians"?

- Why not just Newton's law and Lorentz force?
  Newton requires rectangular coordinates and time ; for curved trajectories one needs to introduce "reaction forces".
- Hamiltonian equations of motion are two systems of first order <-> Lagrangian treatment yields one equation of second order.
- Hamiltonian equations use the canonical variables p and q,
  Lagrangian description uses q and ${}^{\partial q}\!/_{\partial t}$ and t
  p, q are independent, the others not.

- Several people use Hill's equation as starting point:
  - always needs an "Ansatz" for a (periodic) solution:
  No real accelerator is built fully periodically
  - Hill's equation follows directly out of a simplified Hamiltonian description (later slide)
- Last not least: (material of the CAS advanced course:)
  The Poisson brackets of p or q with the Hamiltonian encode the time behaviour of p and q: Basis for any numerical integration
  (Werner Herr, Non Linear Dynamics II)

Particle Motion through accelerator components

Linear treatment: matrix multiplication $\begin{pmatrix}x\\x'\end{pmatrix}(s)= \begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}x\\x'\end{pmatrix}(s_0)$

More general treatment: application of a map: $\begin{pmatrix}x\\x'\end{pmatrix}(s)= \mathrm{M}\begin{pmatrix}x\\x'\end{pmatrix}(s_0)$

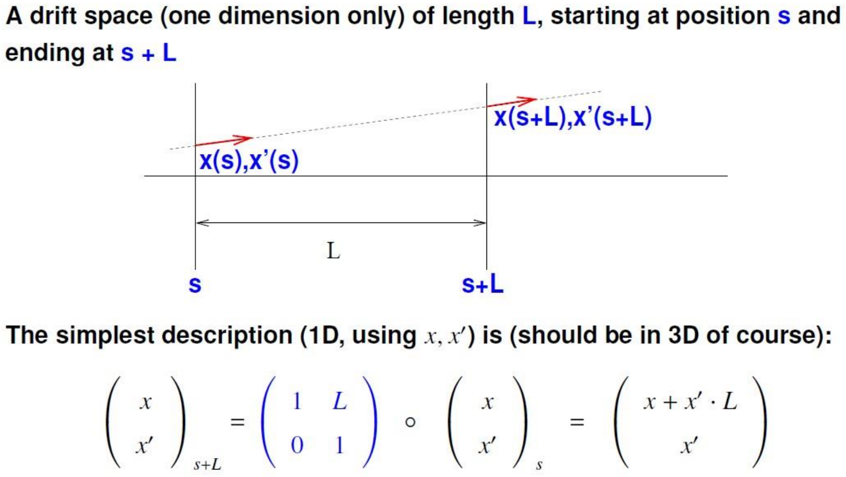
- the map can be any function of x and x’, but must not depend on the input parameters x (s_0) and x’(s_0);
- the map must be symplectic (→ more details: again W. Herr this course)

  (by the way: every matrix is a map, but not every map is a matrix)

  Symplecticity: ← energy conservation

- Following a particle through various elements is equivalent to multiplying the maps.

First (simple) case:



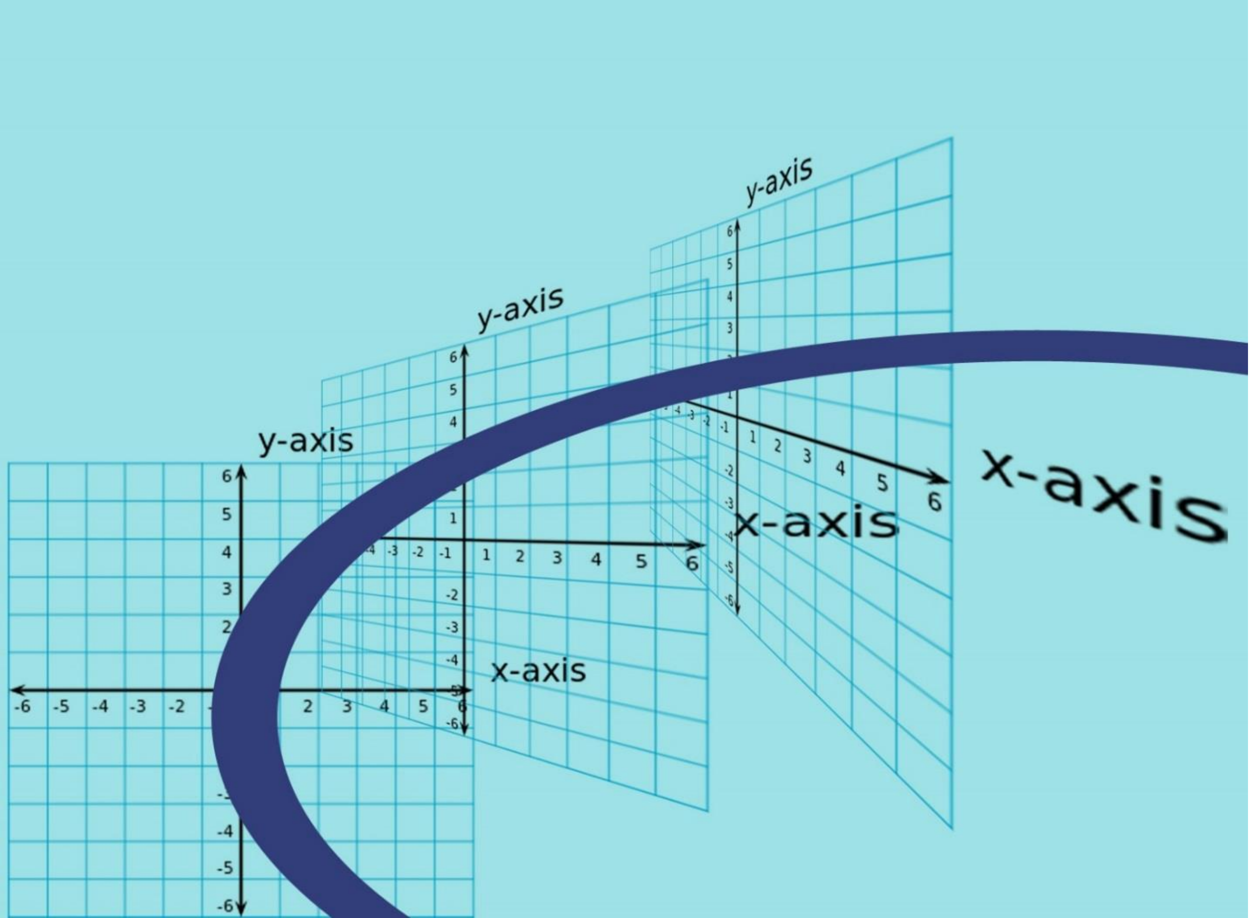
A drift space (one dimension only) of length L, starting at position s and ending at s + L

The simplest description (1D, using x, x') is (should be in 3D of course):

$$\begin{pmatrix}x\\x'\end{pmatrix}_{s+L} = \begin{pmatrix}1 & L\\0 & 1\end{pmatrix}\circ\begin{pmatrix}x\\x'\end{pmatrix}_{s} = \begin{pmatrix}x + x'\cdot L\\x'\end{pmatrix}$$

Back to the Hamiltonian for a moment:

So far we have been switching from time-dependent variables to s-dependent variables without paying attention to it:
In a linear 1 D motion this is a equivalent since s= vt
But if we want to describe motion transverse to a curved reference line,
we must use “s” as independent variable. At every moment we have perpendicular to the tangent vector of the particle trajectory a transverse Cartesian coordinate system.



Hamiltonian for a (ultra relativistic, i.e. $\gamma \gg 1,\ \beta \approx\ 1$) particle in an electro-magnetic field is given by (any textbook on Electrodynamics):

$$H(\vec{x},\vec{p},t) = c\sqrt{(\vec{p} - e\vec{A}(\vec{x},t))^2 + m_0^2c^2} + e\Phi(\vec{x},t) \qquad \text{(ugly...)}$$

where $\vec{A}(\vec{x},t)$, $\Phi(\vec{x},t)$ are the vector and scalar potentials (i.e. the V)

Using canonical variables (2D$^{*)}$) and the design path length s as independent variable (bending field B_0 in y-plane) and no electric fields:

$$H \;=\; \overbrace{-(1+\frac{x}{\rho})}^{\text{due to t} \rightarrow \text{s}} \cdot \overbrace{\sqrt{(1+\delta)^2 - p_x^2 - p_y^2}}^{kinematic} + \overbrace{\frac{x}{\rho} + \frac{x^2}{2\rho^2}}^{\text{due to t} \rightarrow s} - \overbrace{\frac{A_s(x,y)}{B_0\rho}}^{normalized}$$

where $p \;=\; \sqrt{E^2/c^2 \;-\; m^2c^2}$ total momentum, $\delta = (p - p_0)/p_0$ is relative momentum deviation and $A_s(x,y)$ (normalized) longitudinal (along s) component of the vector potential.

$^{*)}$ **Only transverse fields now, skipping several steps (see e.g. S. Sheehy, CAS Budapest 2016)..**

Where are we now?

- we describe every element in the trajectory of a particle with the corresponding Hamiltonian.
- we describe the particle motion through an element by a matrix (map) multiplication onto its phase-space vector.
- we generate more complex accelerator configurations by multiplying the maps of the induvial elements.
- we have changed the coordinate system and describe now the trajectory of a particle as a function of “s” and not of “t”.
- But: we are still treating single particles in a single passage through an accelerator component.

What comes next?

- We show that Hill’s equations come naturally out of the Hamiltonian formalism
- We look at transverse focusing…in particular a FODO lattice
- We look again and again at phase space diagrams.

A first application - the simplest possible:

Keeping only the lower orders (focusing) and $\delta = 0$ we have:

$$H = \frac{p_x^2 + p_y^2}{2} - \frac{x^2}{2\rho^2(s)} + \frac{k_1(s)}{2}(x^2 - y^2)$$

Putting it into Hamilton's equations (for x, ditto for y):

$$\frac{\partial H}{\partial x} = -\frac{dp_x}{ds} \qquad \frac{\partial H}{\partial p_x} = \frac{dx}{ds} = p_x$$

it follows immediately:

$$\frac{d^2x}{ds^2} + \left(\frac{1}{\rho(s)^2} - k_1(s)\right) x = 0 \qquad \frac{d^2y}{ds^2} + k_1(s)\, y = 0$$

Hill's equations are a direct consequence of Hamiltonian treatment of EM fields to lower orders

Hamiltonians of some machine elements (3D)

In general for multipole n:

$$H_n = \frac{1}{1+n} \mathcal{Re} \left[(k_n + \mathrm{i}k_n^{(s)})(x+\mathrm{i}y)^{n+1} \right] + \frac{p_x^2 + p_y^2}{2(1+\delta)}$$

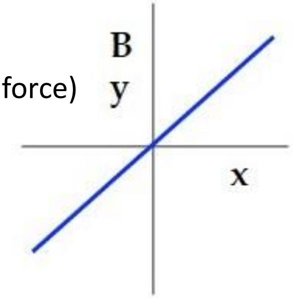
We get for some important types (normal components k_n only):

dipole: $$H = -\frac{-x\delta}{\rho} + \frac{x^2}{2\rho^2} + \frac{p_x^2 + p_y^2}{2(1+\delta)}$$

quadrupole: $$H = \frac{1}{2} k_1 (x^2 - y^2) + \frac{p_x^2 + p_y^2}{2(1+\delta)}$$

sextupole: $$H = \frac{1}{3} k_2 (x^3 - 3xy^2) + \frac{p_x^2 + p_y^2}{2(1+\delta)}$$

Such a field (force) we need for focusing



Weak focusing from dipoles

dipole: $$H = -\frac{-x\delta}{\rho} + \boxed{\frac{x^2}{2\rho^2}} + \frac{p_x^2 + p_y^2}{2(1+\delta)}$$

quadrupole: $$H = \frac{1}{2}\boxed{k_1(x^2} - y^2) + \frac{p_x^2 + p_y^2}{2(1+\delta)}$$

This means that we can construct a focusing circular accelerator based only on dipoles…

in particular when ρ is small.

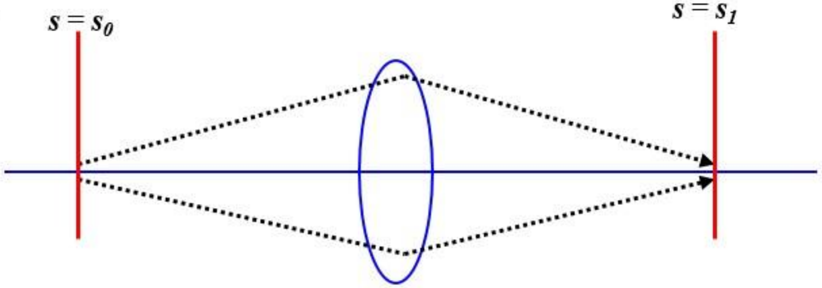
This has been done in the 1950’s and it was called “ a weak focusing synchrotron”

For this evening (with a cold beer):

How about the vertical plane? There are no dipoles. Or why do the particles not fall down?

We need stronger focusing….quadrupoles

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s1} = M_{foc} * \begin{pmatrix} x \\ x' \end{pmatrix}_{s0}$$



$$M_{foc} = \begin{pmatrix} \cos(\sqrt{|K|}s) & \frac{1}{\sqrt{|K|}}\sin(\sqrt{|K|}s \\ -\sqrt{|K|}\sin(\sqrt{|K|}s) & \cos(\sqrt{|K|}s) \end{pmatrix}_0$$

$$f = \frac{1}{kl_q} >> l_q$$ *… **focal length** of the lens is much bigger than the length of the magnet*

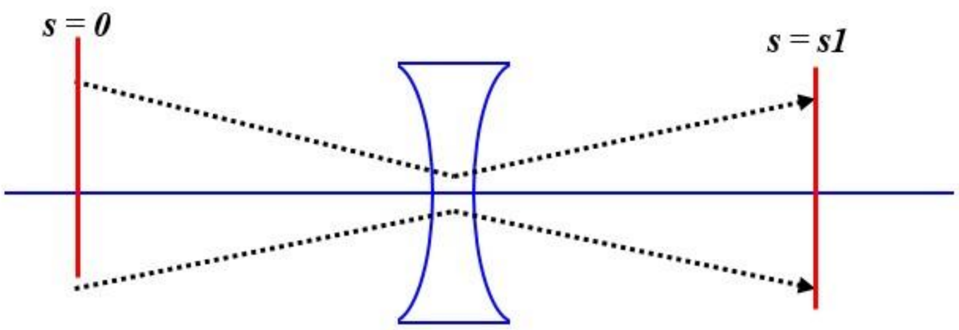
limes: $l_q \rightarrow 0$ *while keeping* $k\, l_q = const$

$$M_x = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$$

Negative = focusing

The negative sign in the Hamiltonian makes the same quadrupole defocusing the other plane.

$$M_{defoc} = \begin{pmatrix} \cosh\sqrt{|K|}l & \frac{1}{\sqrt{|K|}}\sinh\sqrt{|K|}l \\ \sqrt{|K|}\sinh\sqrt{|K|}l & \cosh\sqrt{|K|}l \end{pmatrix}$$



$$f = \frac{1}{kl_q} >> l_q$$

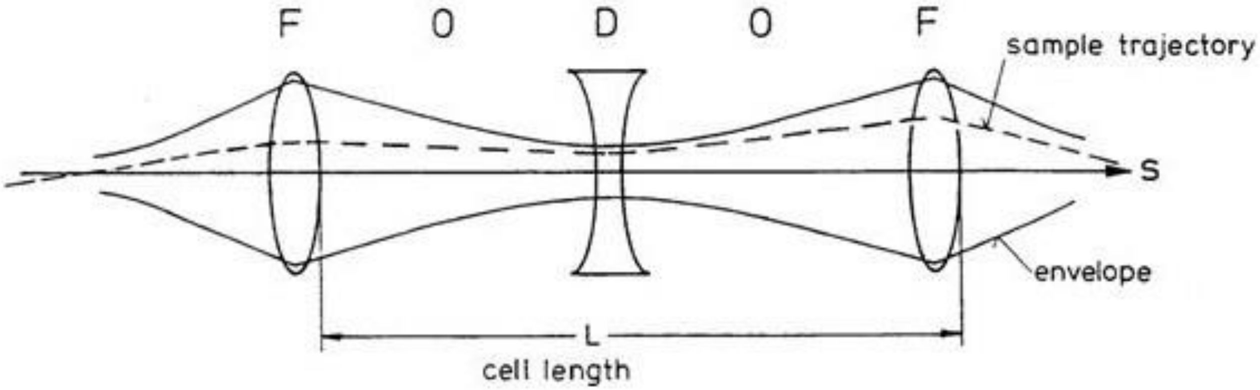
*... **focal length** of the lens is much bigger than the length of the magnet*

limes: $l_q \rightarrow 0$ *while keeping* $k\,l_q = const$

$$M_x = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}$$

Positive = defocusing

Consider an alternating sequence of focussing (F) and defocussing (D) quadrupoles separated by a drift (O)



The transfer matrix of the basic FODO cell reads

$$M = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{L}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{L}{2} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1+\frac{L}{2f} & L\left(1+\frac{L}{4f}\right) \\ -\frac{L}{2f^2} & 1-\frac{L}{2f}-\frac{L^2}{4f^2} \end{pmatrix}$$

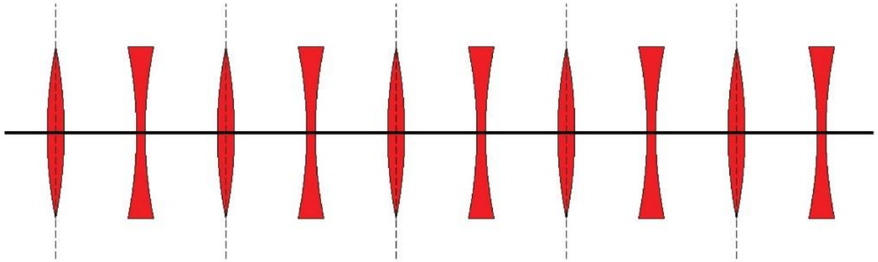
Transfer Matrix in 6-D

In order to calculate numbers one usually defines a FODO cell from the middle of the first F-quadrupole up to the middle of the last F-quadrupole.

Hence the resulting transfer matrix looks a little different:

$$M= M_Q(2f_0)\cdot M_D(L)\cdot\ M_Q(-f_0)\cdot M_D(L)\cdot\ M_Q(2f_0)$$

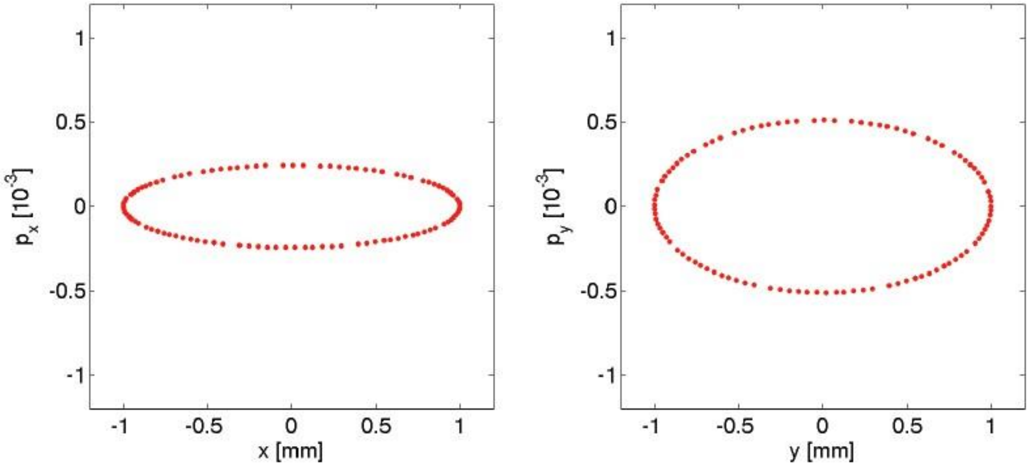
$$\begin{pmatrix} 1-\frac{L^2}{2f_0^2} & \frac{L}{f_0}(L+2f_0) & 0 & 0 & 0 & 0 \\ \frac{L}{4f_0^3}(L-2f_0) & 1-\frac{L^2}{2f_0^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-\frac{L^2}{2f_0^2} & -\frac{L}{f_0}(L-2f_0) & 0 & 0 \\ 0 & 0 & -\frac{L}{4f_0^3}(L+2f_0) & 1-\frac{L^2}{2f_0^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{2L}{\beta_0^2\gamma_0^2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$



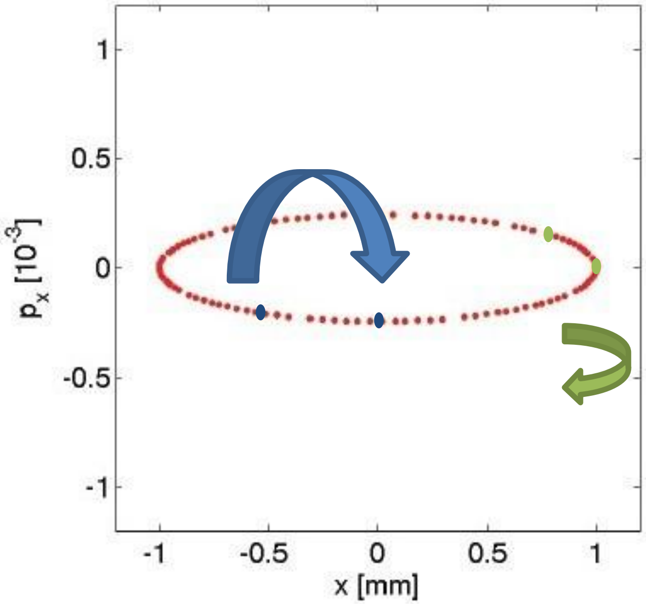
Let us consider the case $L = 1\,\text{m}$, $f_0 = \sqrt{2}\,\text{m}$. Take a particle with initial coordinates at the start of a FODO cell:

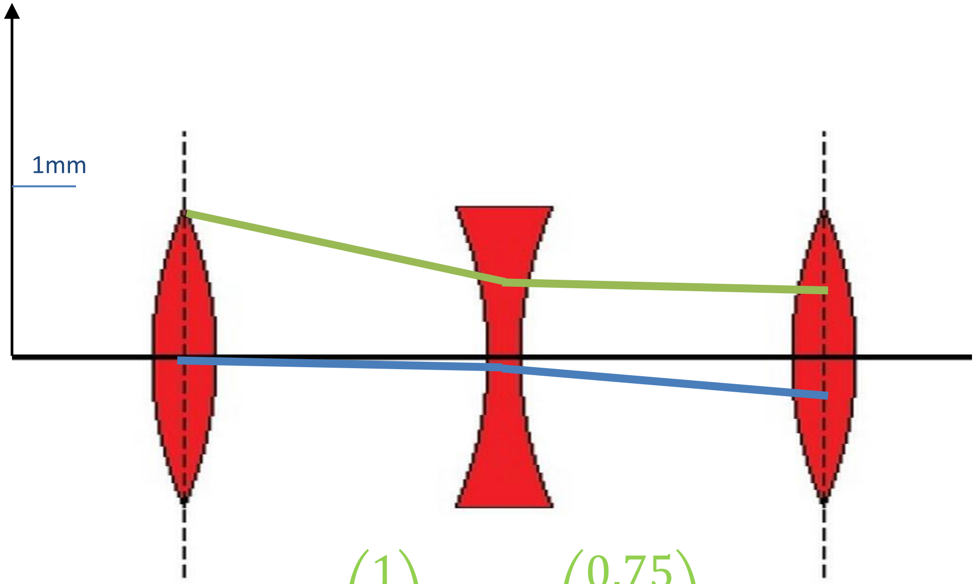
$$x = 1\,\text{mm}, \quad p_x = 0, \quad y = 1\,\text{mm}, \quad p_y = 0$$

Now track the particle through 100 FODO cells by applying the transfer matrix to the vector constructed from the coordinates, and plot p_x vs x, and p_y vs y:



More details on the Illustrating Example

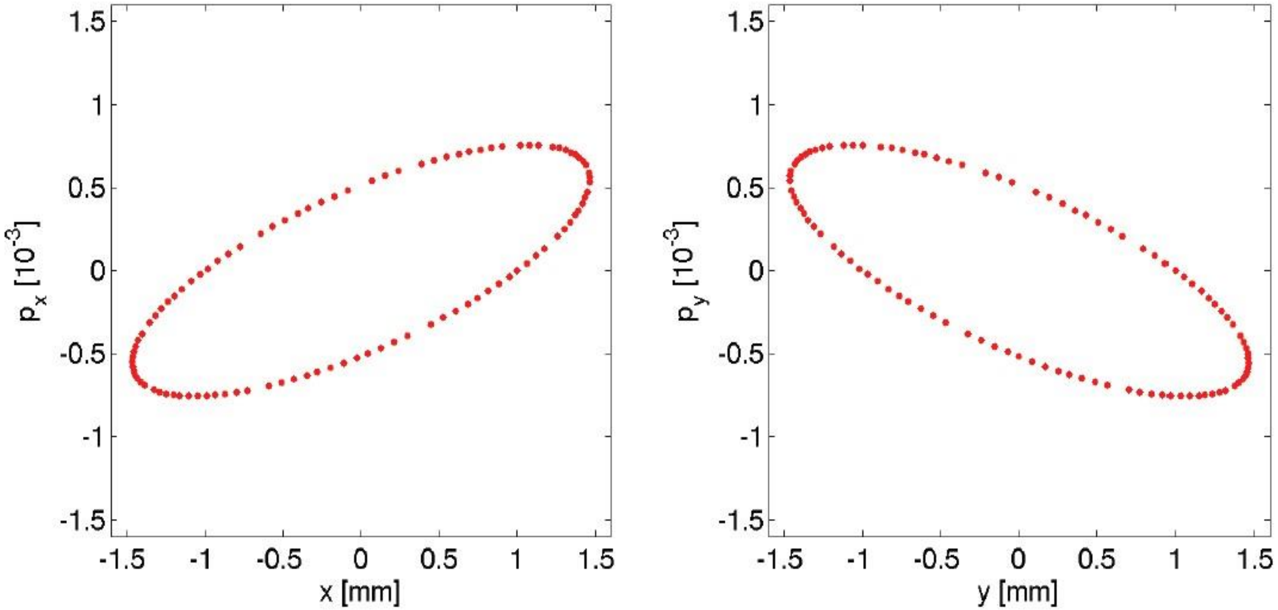




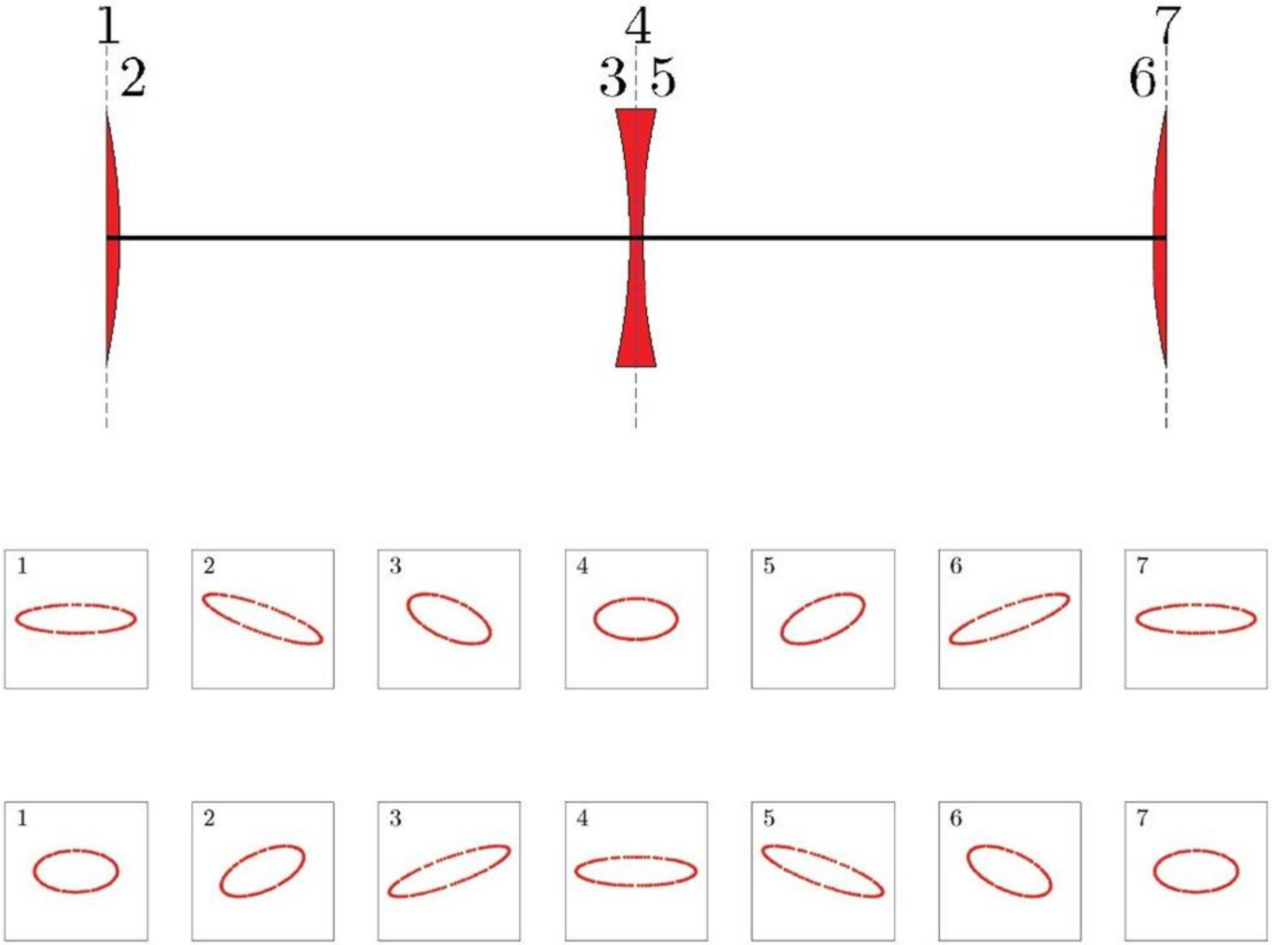
$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0.75 \\ 0.16 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ -.2 \end{pmatrix} \longrightarrow \begin{pmatrix} -.54 \\ -.15 \end{pmatrix}$$

What happens if we repeat the exercise, but starting the FODO cell at the center of the drift before the (horizontally) defocusing quadrupole? Again, we plot ellipses, but this time, they are tilted:



Evolution of the Phase Space Ellipse in a FODO Cell



Our first synchrotron

The previous example of 100 consecutive FODO cells describes very well a regular transport line or a linac (in which we have switched off the cavities).

If we add dipoles into the driftspaces, the situation for the transverse particle motion does not change (neglecting the weak focusing part).

So actually with the previous description we also describe a very simple regular synchrotron.

The phase space ellipse we can compute provided we know the total transfer map (matrix) M_{tot}:

$$\mathrm{J}=\frac{1}{2}\begin{pmatrix} x \\ x' \end{pmatrix}(s_0)\ \begin{pmatrix} x \\ x' \end{pmatrix}(s_0+C)\ =\frac{1}{2}\begin{pmatrix} x \\ x' \end{pmatrix}(s_0)\ \mathrm{Mtot}\begin{pmatrix} x \\ x' \end{pmatrix}(s_0\ )$$

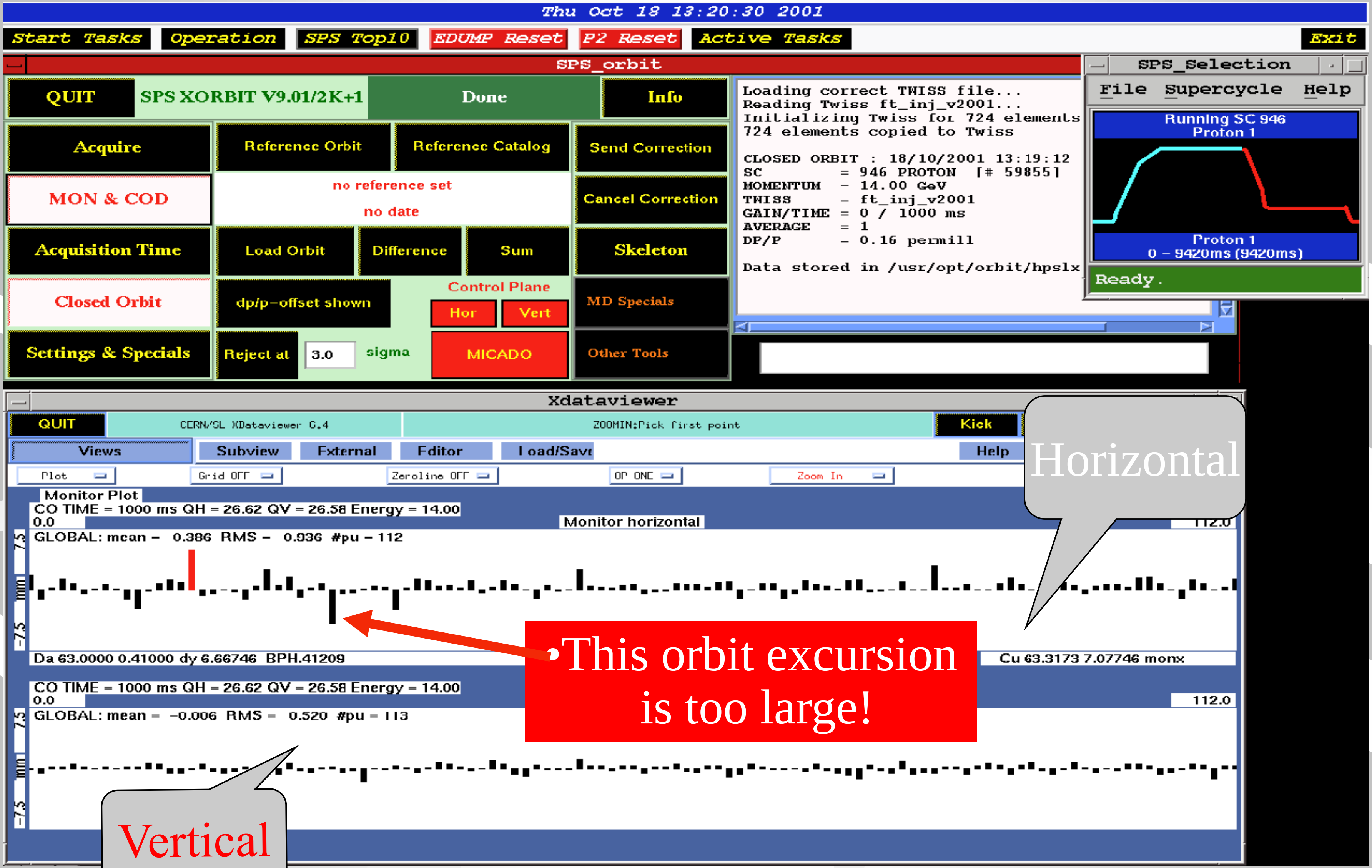
The phase space plots will look qualitatively the same as in the previous case.

Definition: trajectory (single passage) or closed orbit (multiple passages):

(1)

Fix point of the transfer matrix…in our cases so far the “0” centre of all ellipses.

Orbit Acquisition



```
Loading correct TWISS file...
Reading Twiss ft_inj_v2001...
Initializing Twiss for 724 elements
724 elements copied to Twiss

CLOSED ORBIT  :  18/10/2001 13:19:12
SC           = 946 PROTON  [# 59855]
MOMENTUM     - 14.00 GeV
TWISS        - ft_inj_v2001
GAIN/TIME    = 0 / 1000 ms
AVERAGE      = 1
DP/P         - 0.16 permill

Data stored in /usr/opt/orbit/hpslx
```

```
CO TIME = 1000 ms QH = 26.62 QV = 26.58 Energy = 14.00
0.0
GLOBAL: mean -   0.386  RMS -   0.936  #pu - 112
Da 63.0000 0.41000 dy 6.66746  BPH.41209
Cu 63.3173 7.07746 monx
```

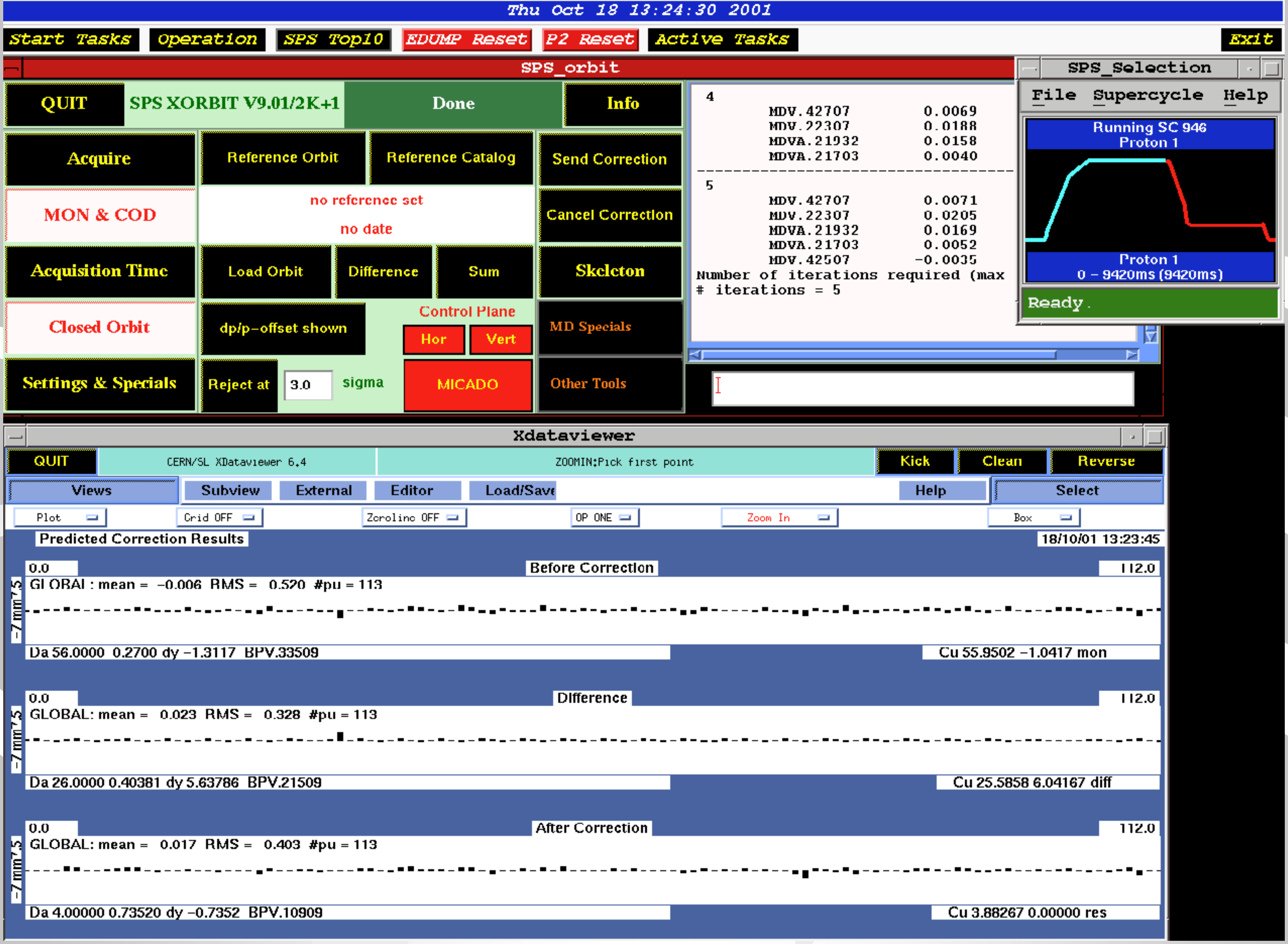
```
CO TIME = 1000 ms QH = 26.62 QV = 26.58 Energy = 14.00
0.0
GLOBAL: mean =  -0.006  RMS =   0.520  #pu = 113
```

Horizontal

- This orbit excursion is too large!

Vertical

Orbit Correction (Operator Panel)



no reference set

no date

Reject at 3.0 sigma

```
 4
          MDV.42707          0.0069
          MDV.22307          0.0188
          MDVA.21932         0.0158
          MDVA.21703         0.0040
----------------------------------------
 5
          MDV.42707          0.0071
          MDV.22307          0.0205
          MDVA.21932         0.0169
          MDVA.21703         0.0052
          MDV.42507         -0.0035
Number of iterations required (max
# iterations = 5
```

Predicted Correction Results 18/10/01 13:23:45

Before Correction

GLOBAL : mean = –0.006 RMS = 0.520 #pu = 113

Da 56.0000 0.2700 dy –1.3117 BPV.33509

Cu 55.9502 –1.0417 mon

Difference

GLOBAL: mean = 0.023 RMS = 0.328 #pu = 113

Da 26.0000 0.40381 dy 5.63786 BPV.21509

Cu 25.5858 6.04167 diff

After Correction

GLOBAL: mean = 0.017 RMS = 0.403 #pu = 113

Da 4.00000 0.73520 dy –0.7352 BPV.10909

Cu 3.88267 0.00000 res

Orbit Correction (Detail)

```
CERN/SL XDataviewer 6.4          ZOOMIN;Pick first point

Predicted Correction Results                                18/10/01 13:23:45

Before Correction
GLOBAL: mean =  -0.006  RMS =  0.520  #pu = 113
Da 51.0000 1.13000 dy 4.38471  BPV.32509              Cu 50.9724 5.51471 mon

Difference
GLOBAL: mean =  0.023  RMS =  0.328  #pu = 1 3
Da 51.0000 -0.4302 dy 4.25372  BPV.32509              Cu 50.8729  3.8235 diff

After Correction
GLOBAL: mean =  0.017  RMS =  0.403  #pu = 113
Da 51.0000 0.69981 dy 6.80019  BPV.32509              Cu 51.1716 7.50000 res
```

Courant – Snyder formalism / Twiss parameters

- Same beam dynamics
- Introduced in the late 50's by
- The classical way to parametrize the evolution of the phase space ellipse along the accelerator

Basic concept of this formalism:

1) Write the transfer matrix in this form (2 dimensional case):

$$M = I\ cos\mu + S\ \cdot A\ sin\mu$$

I $=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$; S $= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$; A$= \begin{pmatrix} \gamma & \alpha \\ \alpha & \beta \end{pmatrix}$

2) M must be symplectic → $\beta\gamma\ -\ \alpha^2$ = 1

3) Four parameters: $\alpha(s);\ \beta(s);\ \gamma(s) and\ \mu(s)$, with one interrelation (2)

→ Three independent variables

4) Again, the preserved action variable J describes an ellipse in phase-space:

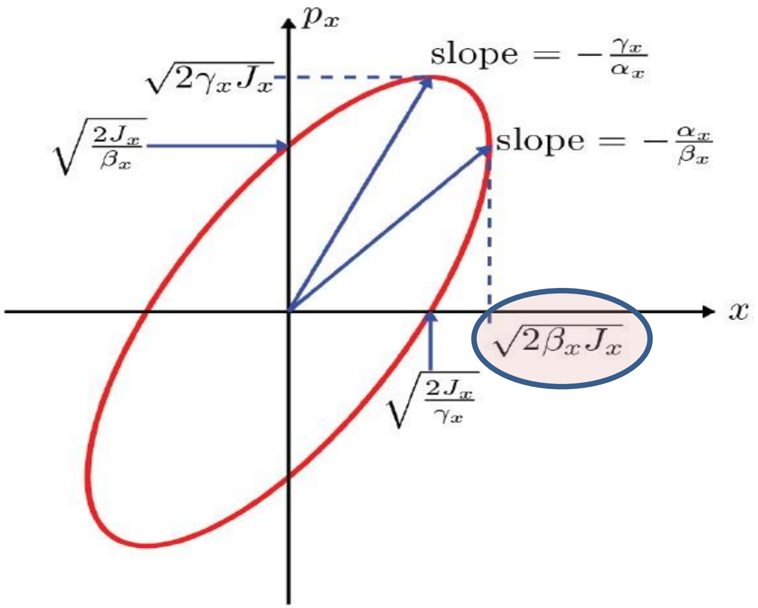
$$J = \frac{1}{2}(\gamma x^2 + 2\alpha x\text{p} + \beta p^2)$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M * \begin{pmatrix} x \\ x' \end{pmatrix}_{s0} \quad M = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix}$$

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_s = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC'+CS' & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} \cdot \begin{pmatrix} \beta_0 \\ \alpha_0 \\ \gamma_0 \end{pmatrix}$$

The Phase Space Ellipse

$$J_x = \frac{1}{2}\left(\gamma_x x^2 + 2\alpha_x x p_x + \beta_x p_x^2\right) \qquad \text{Area} = 2\pi J_x$$



Example: Propagation of twiss parameters along s between two focusing quadrupoles

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M * \begin{pmatrix} x \\ x' \end{pmatrix}_{s0} \qquad M = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix}$$

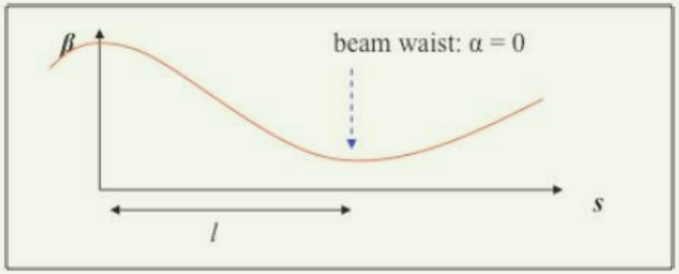
$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_s = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC'+CS' & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} \cdot \begin{pmatrix} \beta_0 \\ \alpha_0 \\ \gamma_0 \end{pmatrix}$$

And in Matrix-Annotation:

$$A_{s_0} = \begin{pmatrix} \gamma & \alpha \\ \alpha & \beta \end{pmatrix} \rightarrow \quad A_s = M^T A_{s_0} \; M$$

$$\beta_s = C^2\beta_0 - \cancel{2SC}\,\alpha_0 + S^2\gamma_0 = \beta_0 + {s^2}/{\beta_0}$$

Example: Beta function between two strong focusing quadrupoles



Drift $M = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$

$$A_{s_0} = \begin{pmatrix} \gamma_0 & \alpha_0 \\ \alpha_0 & \beta_0 \end{pmatrix} = \begin{pmatrix} \gamma_0 & 0 \\ 0 & \beta_0 \end{pmatrix} = \begin{pmatrix} {1}/{\beta_0} & 0 \\ 0 & \beta_0 \end{pmatrix}$$

Starting from waist $\alpha = 0$

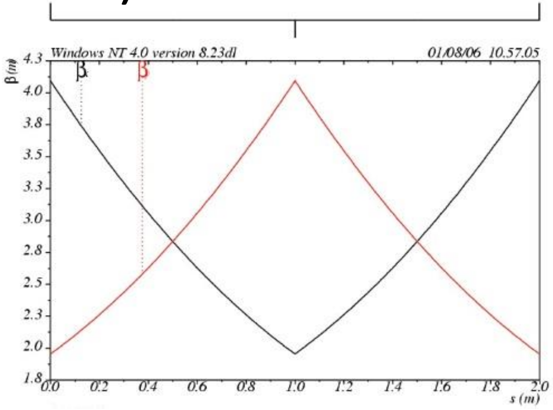
Using: $\beta\gamma - \alpha^2 = 1$

$$A_s = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} \cdot \begin{pmatrix} {1}/{\beta_0} & 0 \\ 0 & \beta_0 \end{pmatrix} \cdot \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} {1}/{\beta_0} & {s}/{\beta_0} \\ {s}/{\beta_0} & \beta_0 + {s^2}/{\beta_0} \end{pmatrix}$$

$$\beta_s = \beta_0 + {s^2}/{\beta_0}$$

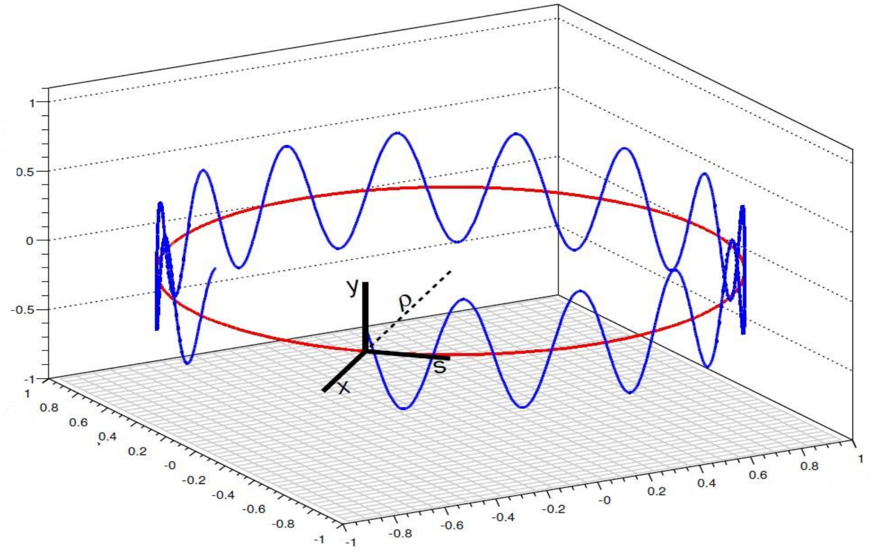
Interpretation of the Twiss parameters (1/2)

1) Horizontal and vertical beta function $\beta_{H,V}\ (s)$:

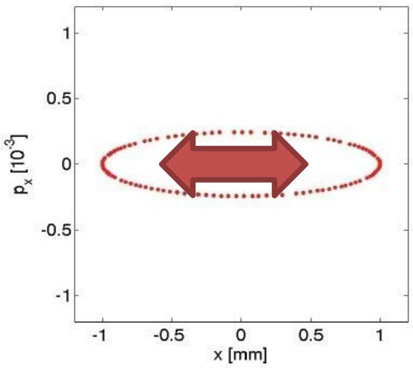


- Proportional to the square of the projection of the phase space ellipse onto the space coordinate
- Focusing quadrupole → low beta values

Although the shape of phase space changes along s, the rotation of the particle on the phase space ellipse projected onto the space co-ordinate looks like an harmonic oscillation with variable amplitude: called **BETATRON-Oscillation**



$$x(s) = const \cdot \sqrt{\beta(s)} \cdot cos\{\mu(s) + \varphi\}$$



H.Schmickler, CERN

Interpretation of the Twiss parameters (2/2)

2.) $\alpha = -\frac{1}{2}\frac{d\beta}{ds}$

α indicates the rate of change of β along s
α zero at the extremes of beta (waist)

3.) $\mu = \int_{s1}^{s2} \frac{1}{\beta}\,\mathrm{ds}$

Phase Advance: Indication how much a particle rotates in phase space when advancing in s

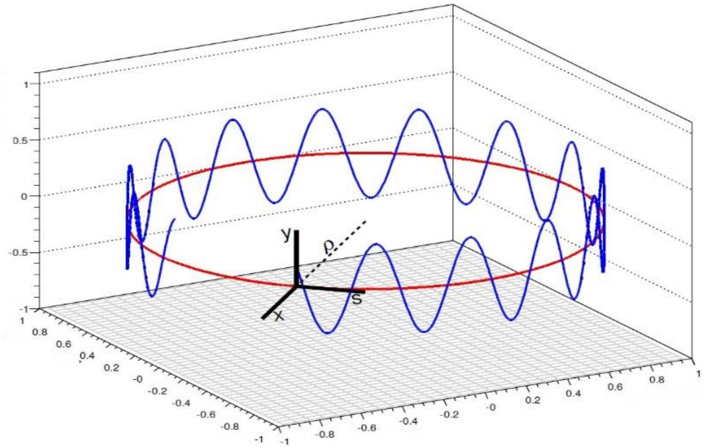
Of particular importance: Phase advance around a complete turn of a circular accelerator, called the betatron tune Q (H,V) of this accelerator

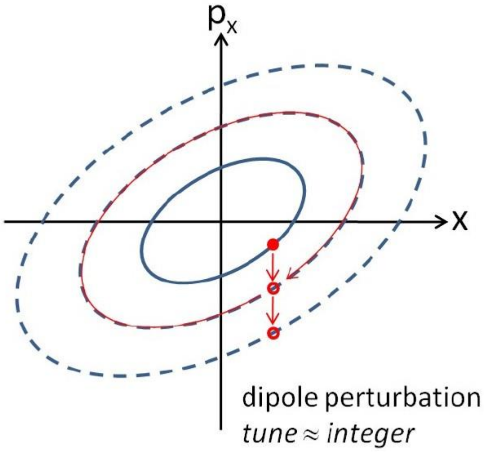
$$Q_{H,V} = \frac{1}{2\pi}\int_0^C \frac{1}{\beta_{H,V}}\,ds$$

The betatron tunes $Q_{H,V}$

- Part of the most important parameters of a circular accelerator
- The equivalent in a linac is called “phase advance per cell”
- For a circular accelerator it is the phase advance over one turn in each respective plane.

- In large accelerators the betatron tunes are large numbers (LHC ˜ 65), i.e. the phase space ellipse turns about 65 times in one machine turn.
- We measure the tune by exciting transverse oscillations and by spectral analysis of the motion observed with one pickup.
  This way we measure the **fractional part of the tune; often called $q_{H,V}$**





- Integer tunes (fractional part= 0) lead to resonant infinite growth of particle motion even in case of only small disturbances.

Importance of betatron tunes

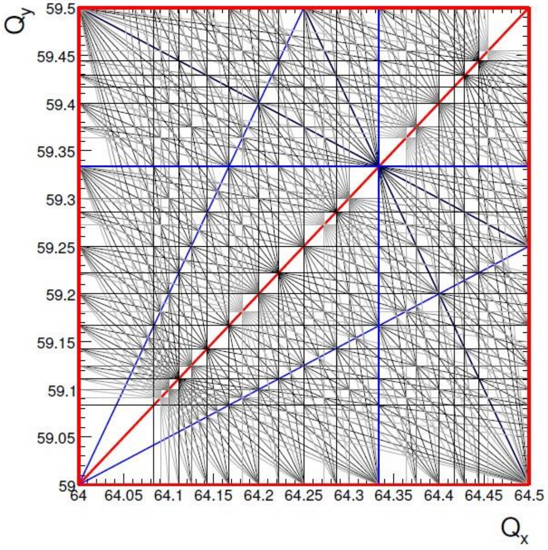
If we include vertical as well as horizontal motion, then we find that resonances occur when the tunes satisfy:

$$m_x \nu_x + m_y \nu_y = \ell,$$

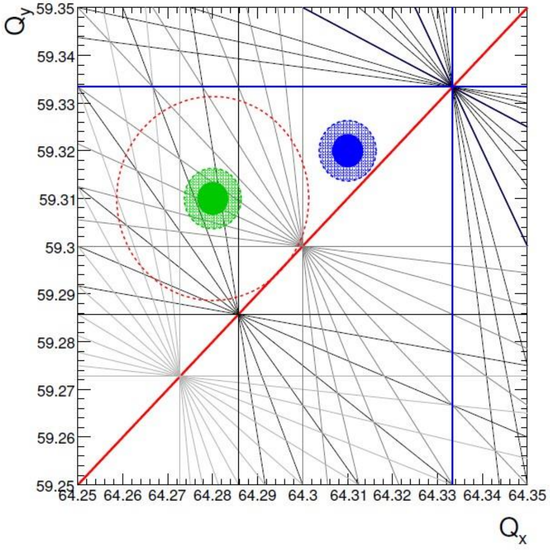
where m_x, m_y and ℓ are integers.

The order of the resonance is $|m_x| + |m_y|$.

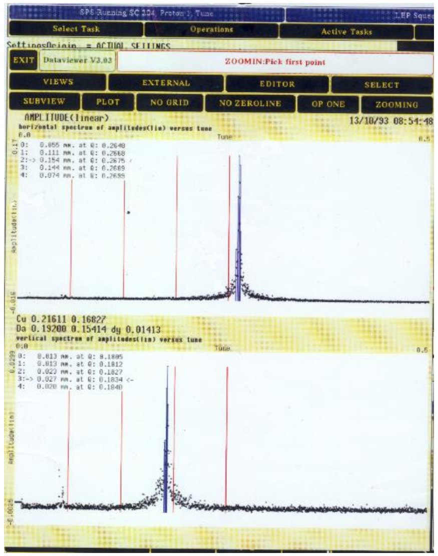
The couple (Q_H ,Q_V) is called the working point of the accelerator. Below: tune measurement example from LEP



(a) Full tune diagram

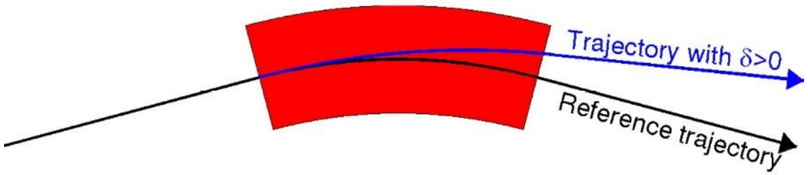


(b) Zoom around LHC Q working points

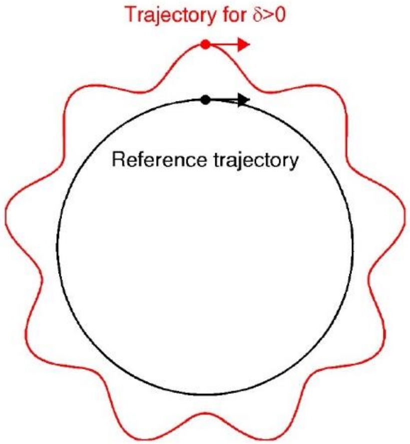


```
SettingsOrigin = ACTUAL SETTINGS
AMPLITUDE(linear)                                    13/10/93 08:54:48
horizontal spectrum of amplitudes(lin) versus tune
0:    0.055 mm. at Q: 0.2640
1:    0.111 mm. at Q: 0.2668
2:-> 0.154 mm. at Q: 0.2675 <
3:    0.144 mm. at Q: 0.2689
4:    0.074 mm. at Q: 0.2695
Cu 0.21611 0.16827
Da 0.19200 0.15414 dy 0.01413
vertical spectrum of amplitudes(lin) versus tune
0:    0.013 mm. at Q: 0.1805
1:    0.013 mm. at Q: 0.1812
2:    0.023 mm. at Q: 0.1827
3:-> 0.027 mm. at Q: 0.1834 <-
4:    0.020 mm. at Q: 0.1840
```

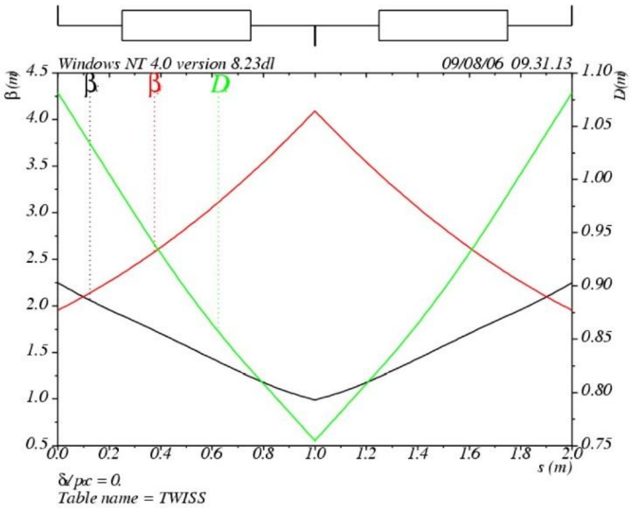
Slides on “off-momentum” particles in a synchrotron



What happens: A particle with a momentum deviation $\delta = \frac{\delta p}{p} > 0$ gets bent less in a dipole.



- In a weakly focusing synchrotron it would just settle to another circular orbit with a bigger diameter
- In an alternate gradient synchrotron it is more complicated: The focusing/defocusing is also dependent on the momentum, so the resulting orbit follows the optics of the accelerator.

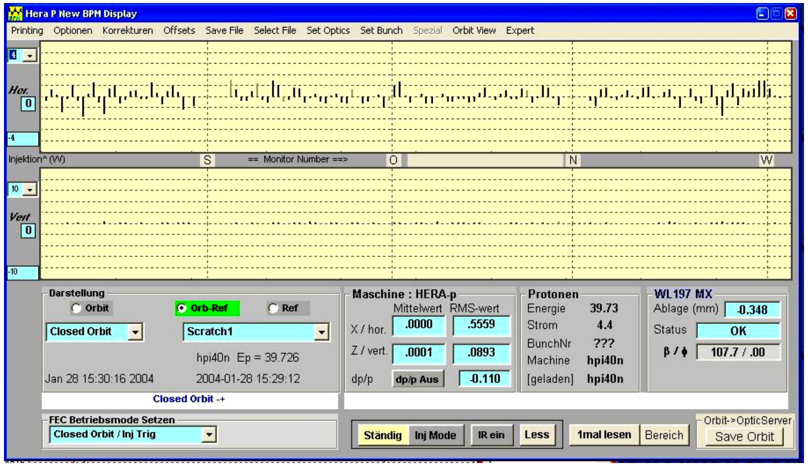


We describe the dispersion as a function of s as $D(s)$; the resulting position of a particle is thus simply:

$$x_{\delta p} = x_0 + D(s)\,\frac{\delta p}{p}$$

Typical values of D(s) are some meters, with $\frac{\delta p}{p} = 10^{-3}$ the orbit deviation becomes millimeters

Measurement example



Darstellung: ○ Orbit ● Orb-Ref ○ Ref

Closed Orbit / Scratch1

hpi40n Ep = 39.726

Jan 28 15:30:16 2004 2004-01-28 15:29:12

Closed Orbit -+

Maschine : HERA-p

	Mittelwert	RMS-wert
X / hor.	.0000	.5559
Z / vert.	.0001	.0893
dp/p	dp/p Aus	-0.110

Protonen
Energie 39.73
Strom 4.4
BunchNr ???
Machine hpi40n
[geladen] hpi40n

WL197 MX
Ablage (mm) -0.348
Status OK
β / ϕ 107.7 / .00

FEC Betriebsmode Setzen: Closed Orbit / Inj Trig

Orbit->OpticServer

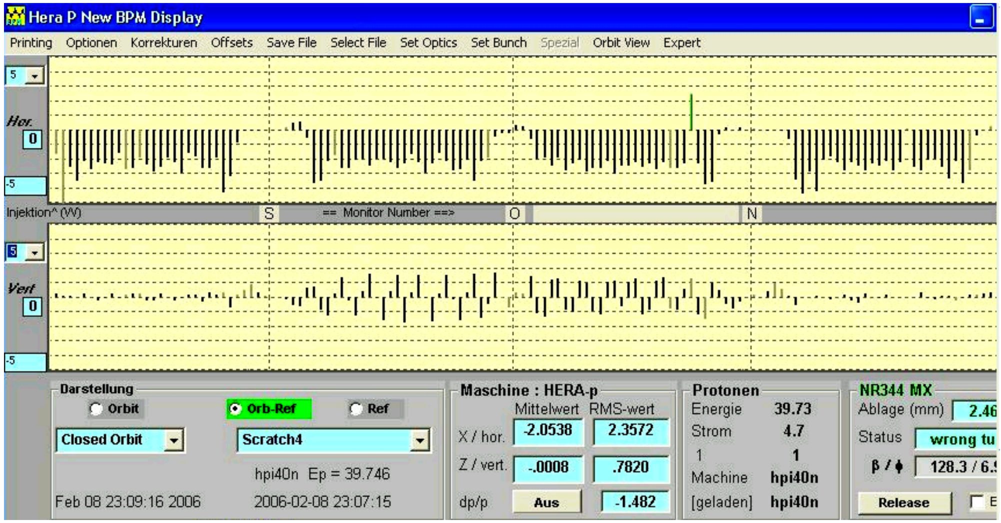
HERA Standard Orbit

This gives also an example of an orbit measurement.
More on this: again R.Jones (BI)

dedicated energy change of the stored beam
→ closed orbit is moved to a
dispersions trajectory

$$x_D = D(s) * \frac{\partial p}{p}$$

HERA Dispersion Orbit



Darstellung: ○ Orbit ● Orb-Ref ○ Ref

Closed Orbit / Scratch4

hpi40n Ep = 39.746

Feb 08 23:09:16 2006 2006-02-08 23:07:15

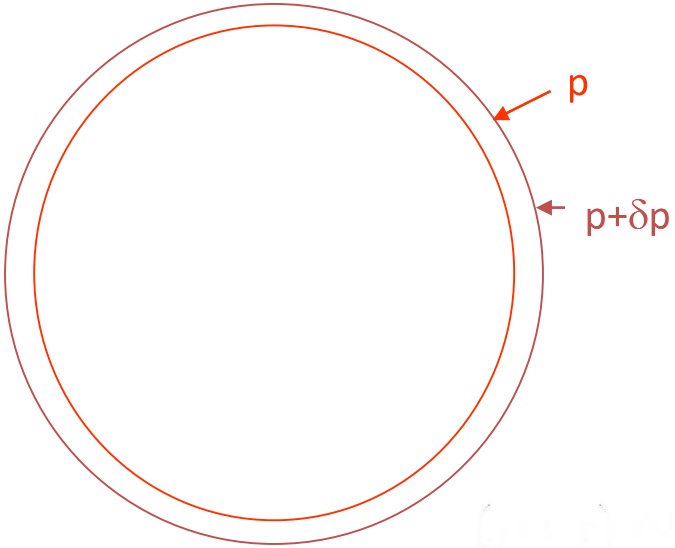
Maschine : HERA-p

	Mittelwert	RMS-wert
X / hor.	-2.0538	2.3572
Z / vert.	-.0008	.7820
dp/p	Aus	-1.482

Protonen
Energie 39.73
Strom 4.7
1 1
Machine hpi40n
[geladen] hpi40n

NR344 MX
Ablage (mm) 2.46
Status wrong tu
β / ϕ 128.3 / 6.

Momentum compaction factor



If a particle is slightly shifted in momentum it will have a different orbit and the orbit length is different.

The "momentum compaction factor" is defined as:

$$\alpha_c = \frac{dL/_{L}}{dp/_{p}}$$

$$\boxed{\alpha_c = \frac{p}{L}\frac{dL}{dp}}$$

$$\alpha_c = \frac{1}{L}\int_C \frac{D_x(s)}{\rho(s)}ds_0$$

With ρ=∞ in straight sections we get:

$$\alpha_c = \frac{\langle D_x \rangle_m}{R}$$

$< >_m$ means that the average is considered over the bending magnet only

Typical numbers: $\alpha_c \approx\ 10^{-3} \ldots 10^{-4};\ {}^{\Delta p}/_{p} \approx\ 10^{-3} \rightarrow\ {}^{\Delta L}/_{L} \approx 10^{-6} \ldots 10^{-7}$

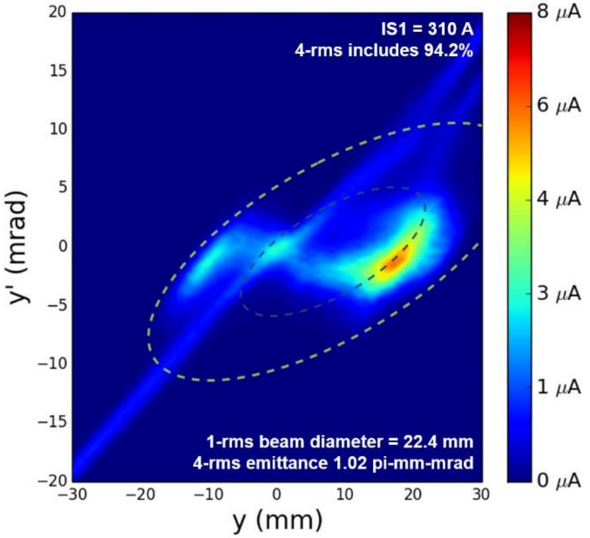
→ Much more on this in long. dynamics (F. Tecker).

Coffee
BREAK

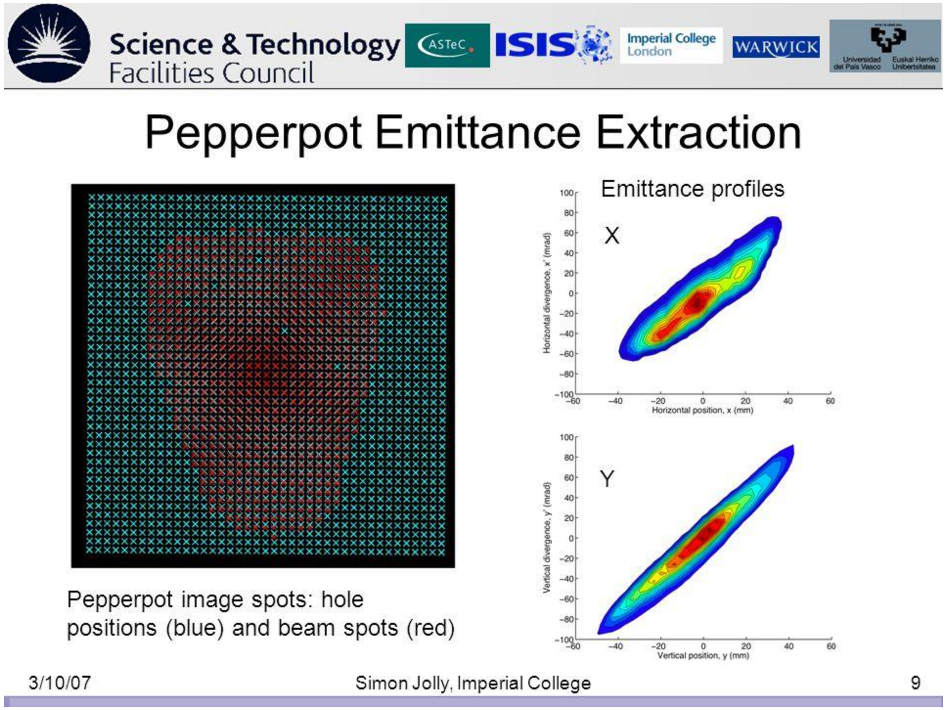
Finally: a beam

We focus on "bunched" beams, i.e. many (10^{11}) particles bunched together longitudinally (much more on this in the RF classes).

From the generation of the beams the particles have transversally a spread in their original position and momentum.



Source: ISODAR (Isotope at rest experiment)



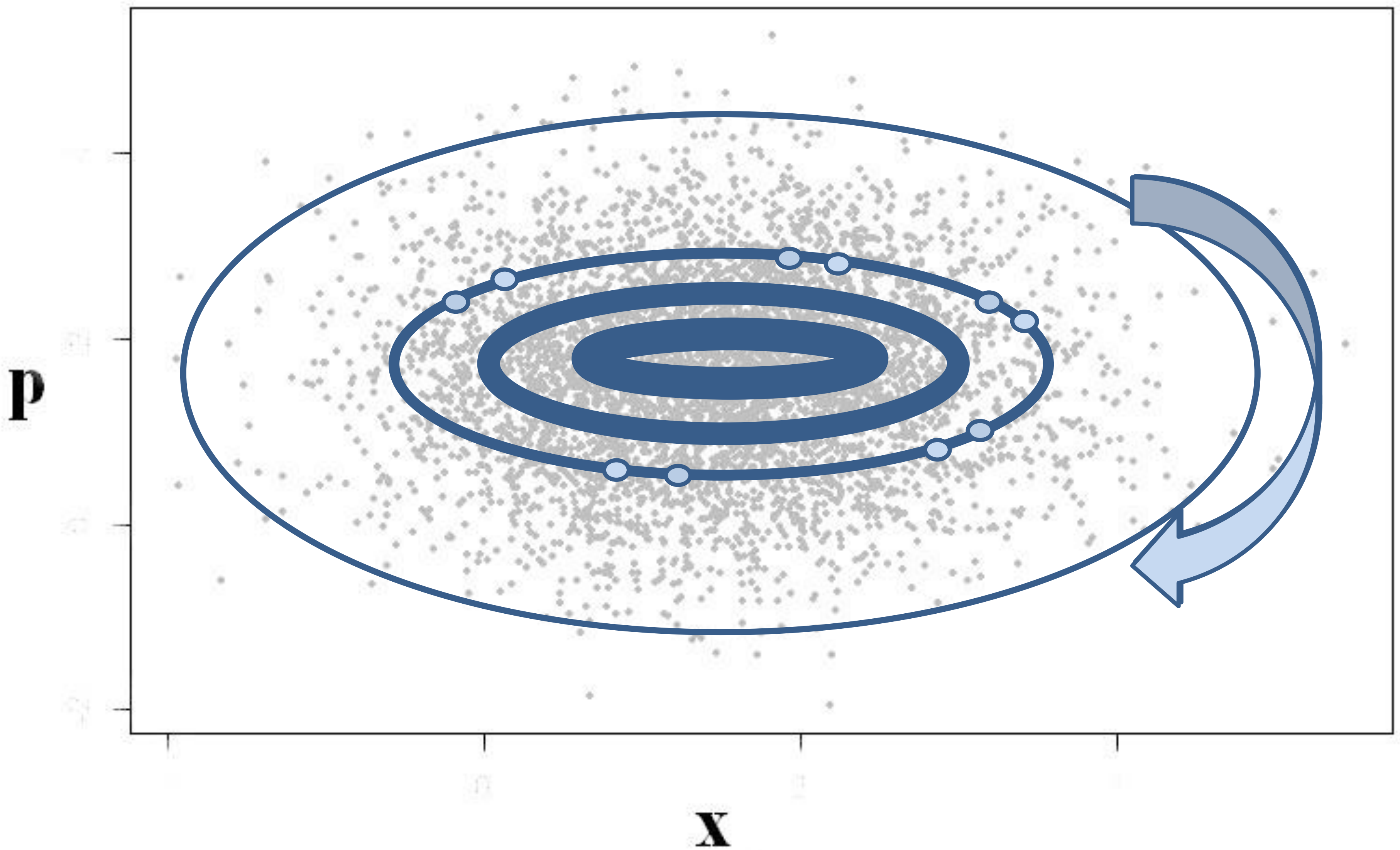
Pepperpot Emittance Extraction

Emittance profiles

Pepperpot image spots: hole positions (blue) and beam spots (red)

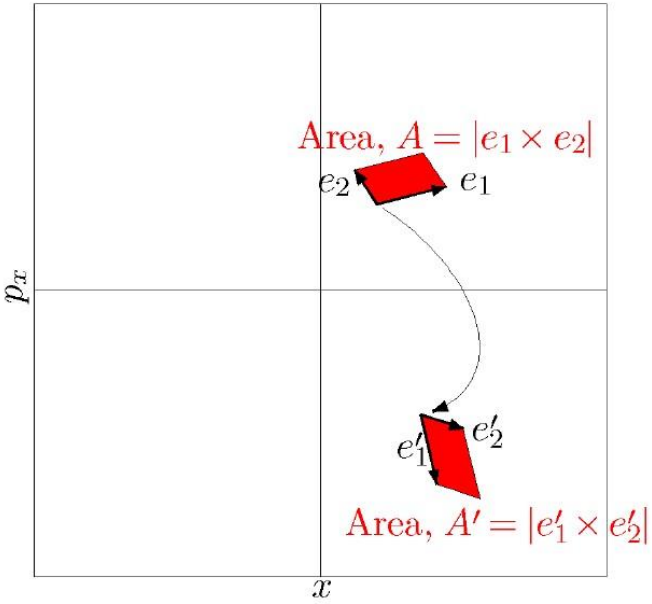
3/10/07 Simon Jolly, Imperial College 9

Gaussian beam profile in x and p



Liouville’s Theorem (1/2)

1. All particle rotate in phase space with the same angular velocity (in the linear case)
2. All particle advance on **their** ellipse of constant action
3. All constant action ellipses transform the same way by advancing in “s”



Physically, a symplectic transfer map conserves phase space volumes when the map is applied.

This is Liouville’s theorem, and is a property of charged particles moving in electromagnetic fields, in the absence of radiation.

→ Since volumes in phase space are preserved, (1)-(3) means That the whole beam phase space density distribution transforms the same way as the individual constant action ellipses of individual particles.

Liouville's Theorem (2/2)

We now define the emittance of a beam as the average action of all particles!

→ Since the action J of a particle is constant and the phase space area A covered by the action ellipse is $A = 2\pi J$, we can represent the whole beam in phase space by an ellipse with a surface = $2\pi\langle J\rangle$ *

→ all equations for the propagation of the phase space ellipse apply equally for the whole beam

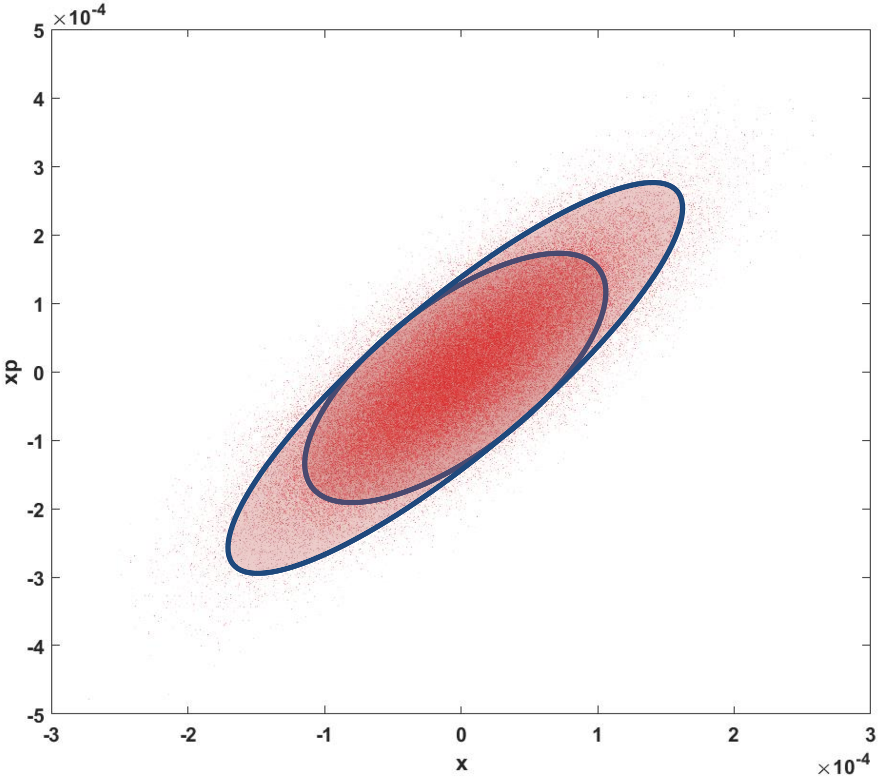
!!! In case we talk about a single particle, the ellipse we draw is "empty" and any particle moves from one point to another; if we consider a beam, the ellipse is full of particles!!!

- * There are several different definitions of the emittance ε, also different normalization factors. This depends on the accelerator type, but the above definition describes best the physics.
  - Another often used definition is called RMS emittance

$\varepsilon = const * \langle x^2\rangle\langle p^2\rangle - \langle xp\rangle^2$ or $\varepsilon = const * \langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2$

attention: the first definition describes well the physics, the second describes what we eventually can measure

RMS emittance



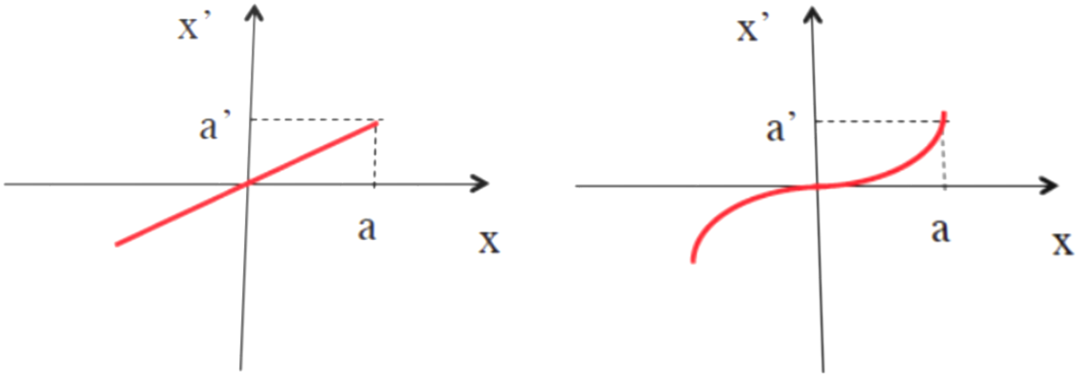
$$\sigma_x^2(z)=\langle x^2\rangle=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}x^2 f(x,x',z)\,dx\,dx'$$

$$\sigma_{x'}^2(z)=\langle x'^2\rangle=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}x'^2 f(x,x',z)\,dx\,dx'$$

$$\sigma_{xx'}(z)=\langle xx'\rangle=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}xx' f(x,x',z)\,dx\,dx'$$

$$\boxed{\varepsilon_{rms}=\sqrt{\sigma_x^2\sigma_{x'}^2-\sigma_{xx'}^2}=\sqrt{\left(\langle x^2\rangle\langle x'^2\rangle-\langle xx'\rangle^2\right)}}$$

Importance of RMS emittance



$$x' = Cx^n$$

$$\varepsilon_{rms} = C\sqrt{\langle x^2\rangle\langle x^{2n}\rangle - \langle x^{n+1}\rangle^2}$$

$$\begin{cases} n = 1 \Rightarrow \varepsilon_{rms} = 0 \\ n > 1 \Rightarrow \varepsilon_{rms} \neq 0 \end{cases}$$

Even when the phase-space area is zero, if the distribution lies on a curved line its rms emittance is not zero.

RMS emittance is not an invariant for Hamiltonian with non linear terms.

Remarks

1. We have already identified the action as a preserved quantity in a conservative system ← → the emittance of a particle beam is preserved in a conservative beam line.
2. The sentence above is often quoted as Liouville's theorem, but this is incorrect. Liouville's theorem describes the preservation of phase space volumes, the preservation of the phase space of a beam is then just results from the Hamiltonian description.

3. We can identify the constant in the previous equation:

$$x(s) = \sqrt{\varepsilon} \cdot \sqrt{\beta(s)} \cdot cos\{\mu(s) + \varphi\}$$

More on beam emittance

The reference momentum increases during acceleration

$P_0 = \beta_0\gamma_0 mc \rightarrow P_1 = \beta_1\gamma_1 mc$ $(\beta, \gamma\ relativistic\ parameters)$

we can show: $\beta_0\gamma_0\epsilon_0 = \beta_1\ \gamma_1\epsilon_1$

So the transverse emittances scale with the product $\beta\gamma$

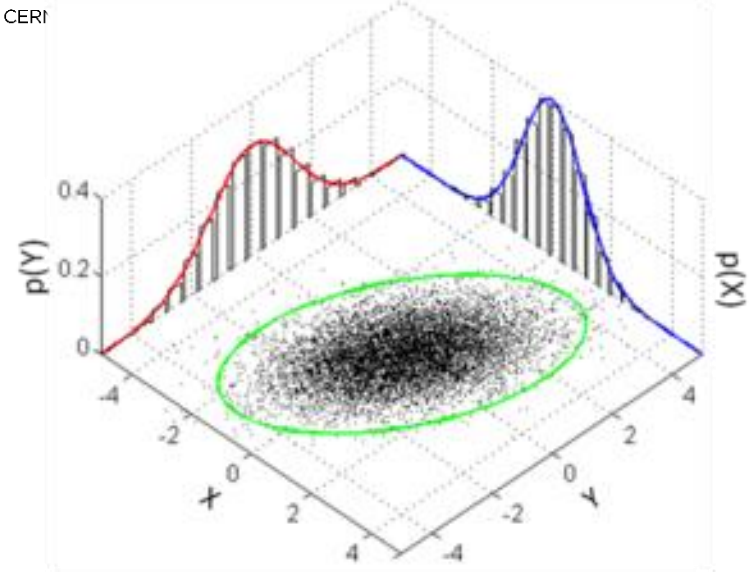
For this reason we define:

normalized emittance ε_N: = $\beta\gamma\varepsilon$ and we call ε the geometric emittance
The "shrinking" of the transverse emittance during acceleration is called "adiabatic damping" (only $\varepsilon = const * \langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2$ scales with energy)

Other ways to influence the emittance (advanced subjects):
- make it bigger by error (injection errors….)
- make it smaller by cooling (stochastic cooling; electron-cooling….)

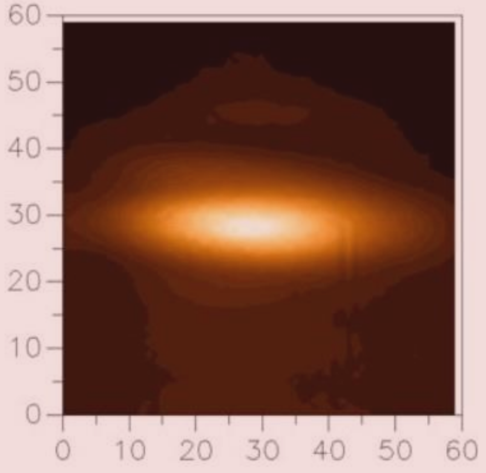
Not to be confused with:
Radiation damping = Reduction in emittance due to the emission of photons as synchrotron radiation

What do we normally measure from the phase-space ellipse?

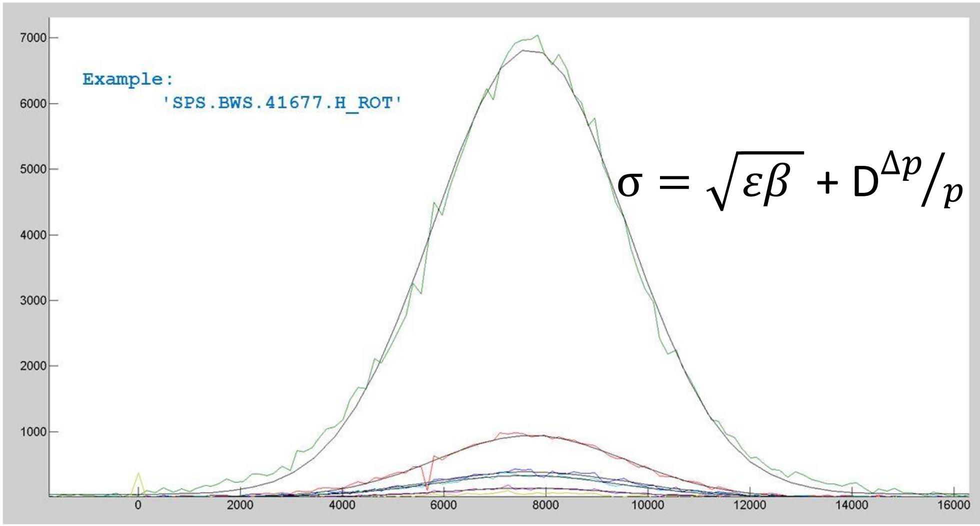


- At a given location in the accelerator we can measure the position of the particles, normally it is difficult to measure the angle…so we measure the projection of the phase space ellipse onto the space dimension:
→called a profile monitor

Attention! The standard 2 D image of a synchrotron light based beam image is NOT a phase space measurement



FITTING

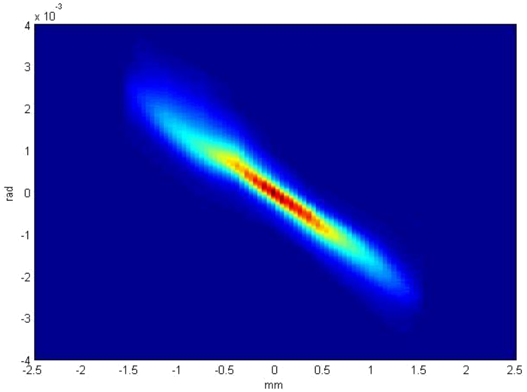


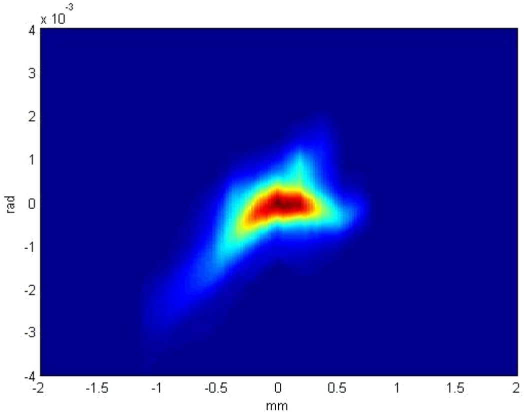
Example:
'SPS.BWS.41677.H_ROT'

$$\sigma = \sqrt{\varepsilon\beta} + \mathrm{D}^{\Delta p}/_{p}$$

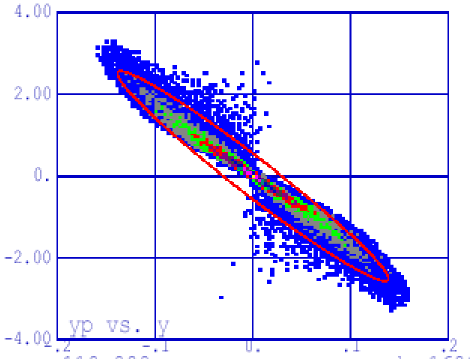
Phase space mapping

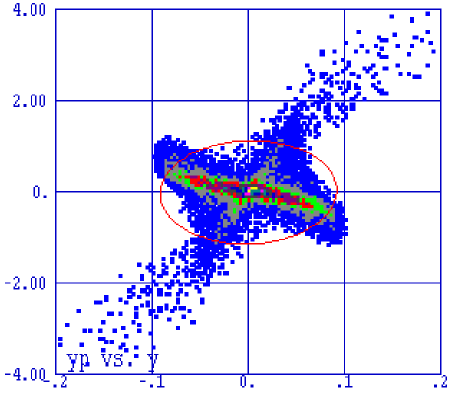
Measurements



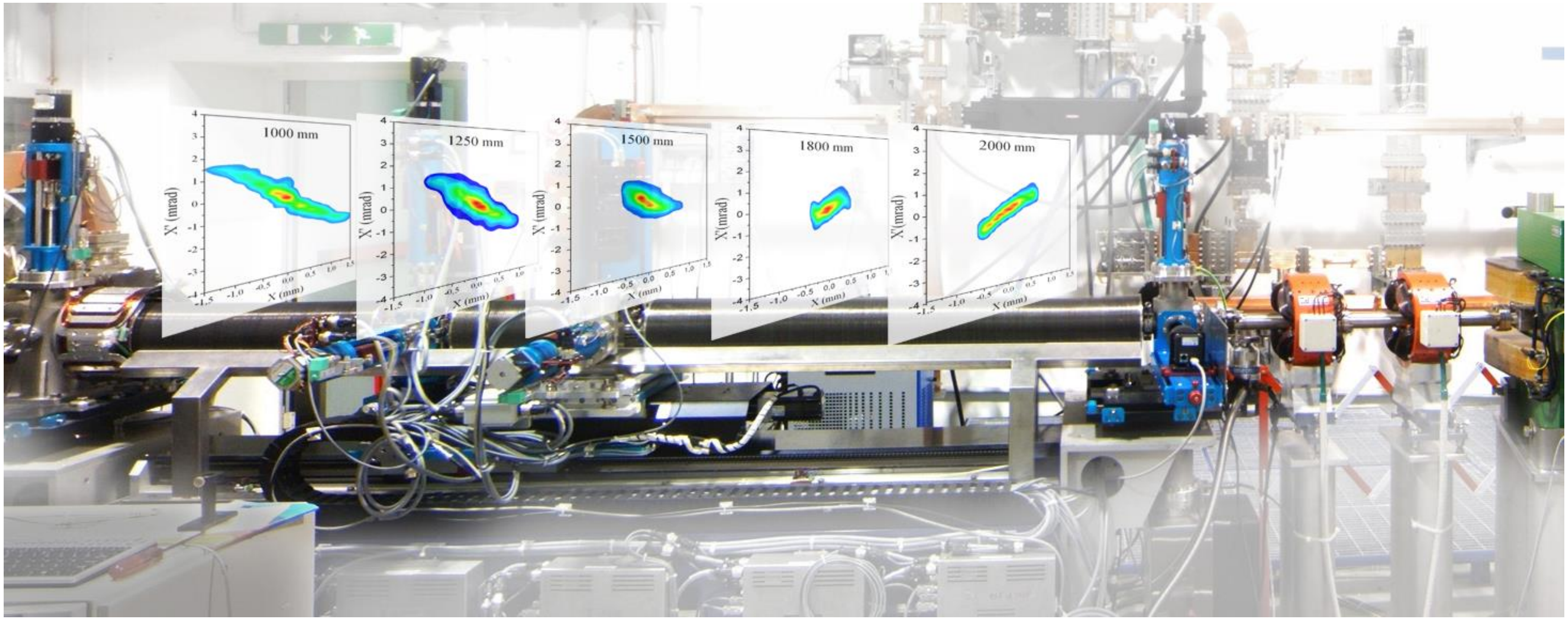


Simulations





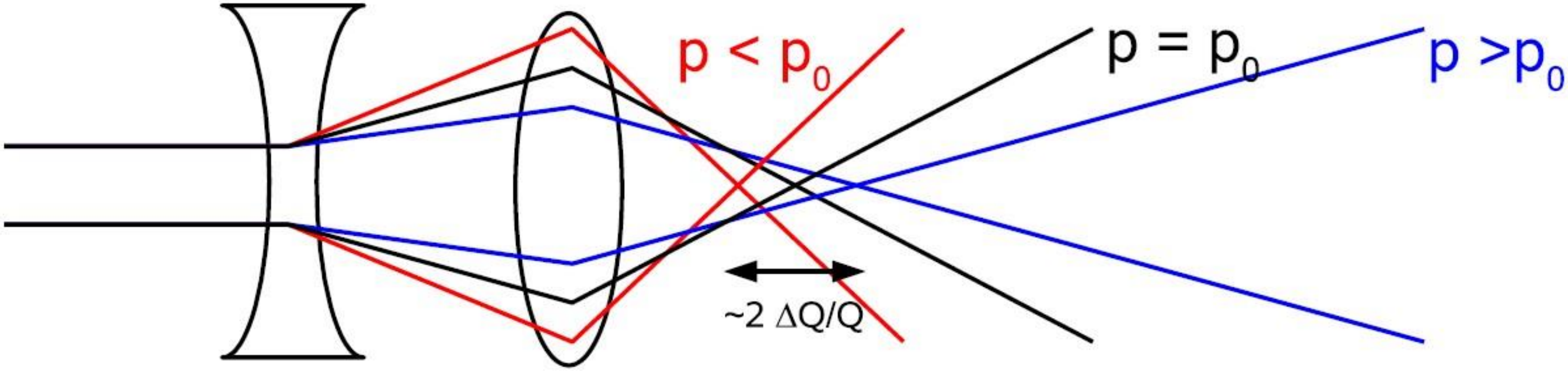
Phase space evolution



A. Cianchi et al., "High brightness electron beam emittance evolution measurements in an rf photoinjector", Physical Review Special Topics Accelerator and Beams 11, 032801,2008

A first taste of non-linearities (1/6)

- So far we have completely neglected the longitudinal plane
- Still, we will not couple the motion in the longitudinal and transverse plane (advanced course), but we need to consider "off momentum particles" with a longitudinal momentum $\frac{\Delta p}{p_0} \neq 0$.
- We already defined the Dispersion function, which describes the change in orbit
- Now we look at what happens to the focusing in the quadrupoles:



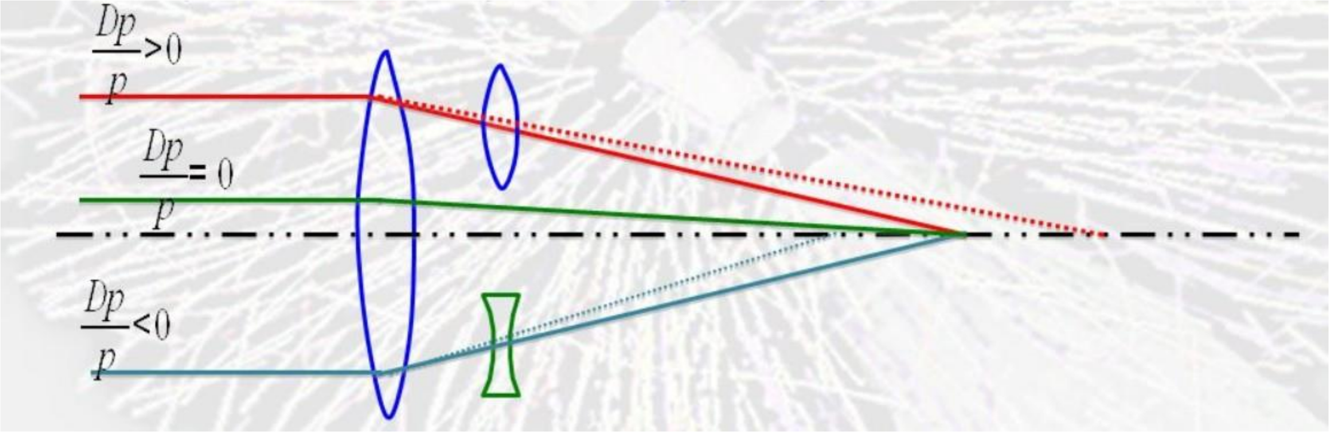
A first taste of non-linearities (2/6)

- Due to the change in focusing strength of the quadrupoles with varying momentum, particles have different betatron-tunes:

Definition: Chromaticity (H,V) := Dependence of tune on momentum
$\Delta Q = Q' \frac{\Delta p}{p}$ or relative chromaticity $\xi = \frac{Q'}{Q}$

- Is this bad? : Yes, the working point gets a “working blob”
- We need to correct. How?

i) Inserting a magnetic element where we have dispersion (this separates in space particles with lower and higher momenta
ii) Having there a “quadrupole”, for which the strength grows for larger distances from the centre: a sextupole



A first taste of non-linearities (3/6)

We will have a high price to pay for this chromaticity correction!
→ we have introduced the first non-linear element into our accelerator

The map M (no longer a matrix) of a single sextupole represents a “kick” in the transverse momentum:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M * \begin{pmatrix} x \\ x' \end{pmatrix}_{s0}$$

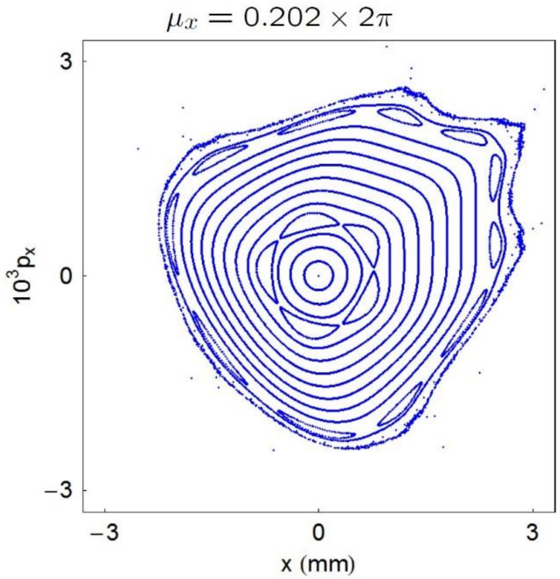
$$\begin{aligned} x &\mapsto x, \\ p_x &\mapsto p_x - \frac{1}{2} k_2 L x^2 \end{aligned}$$

We choose a fixed value k_2L = - 600 m^{-2} and we construct phase space portraits after repeated application of the map.

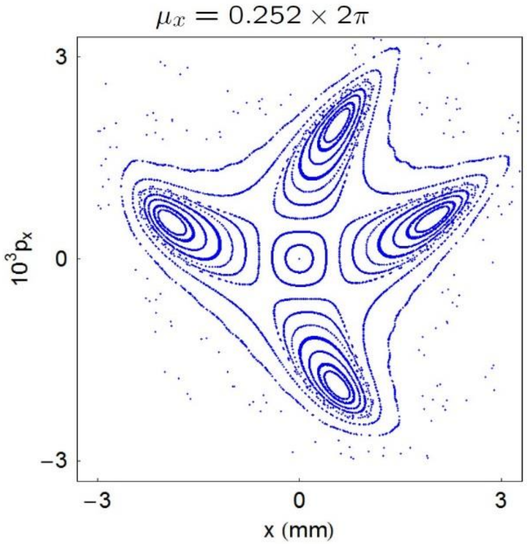
We vary the phase advance per turn (fractional part of the tune) from

$$0.2 \cdot 2\pi \ \ to \ 0.5 \cdot 2\,\pi$$

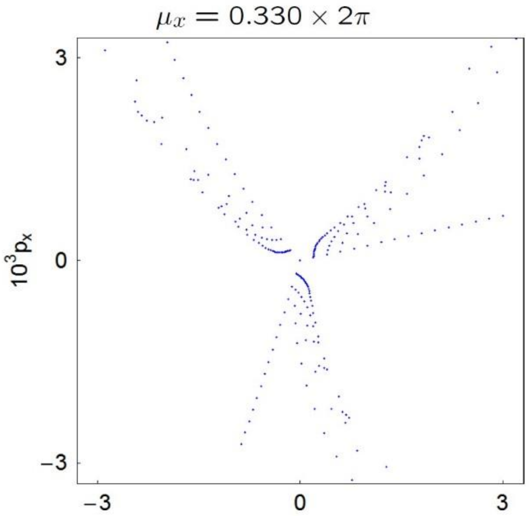
A first taste of non-linearities (4/6)



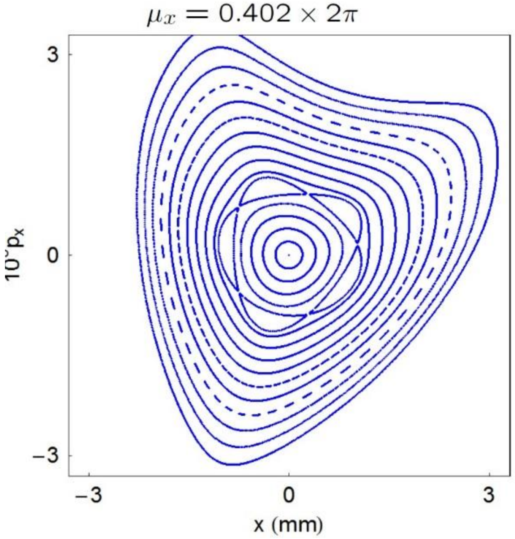
$\mu_x = 0.202 \times 2\pi$



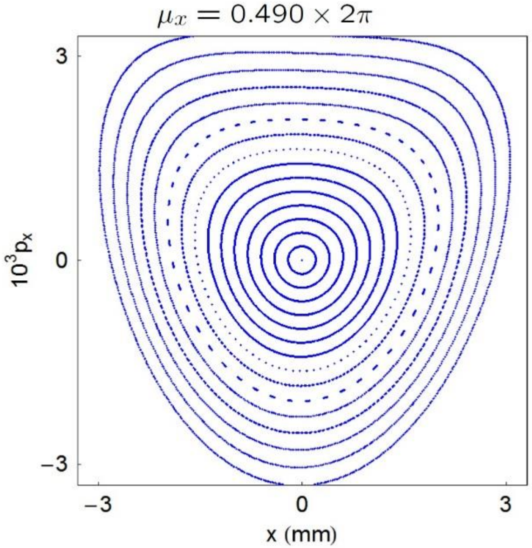
$\mu_x = 0.252 \times 2\pi$



$\mu_x = 0.330 \times 2\pi$

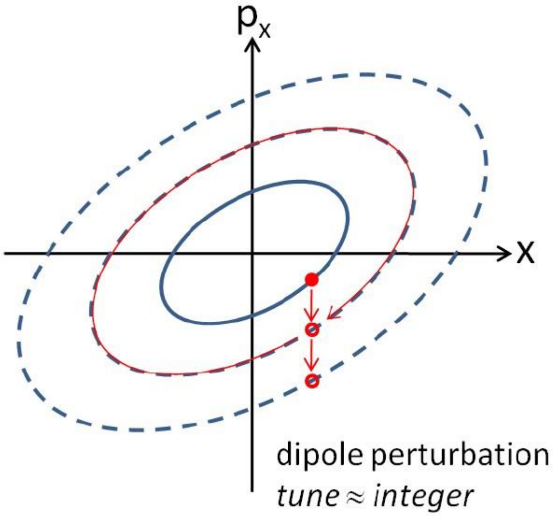


$\mu_x = 0.402 \times 2\pi$

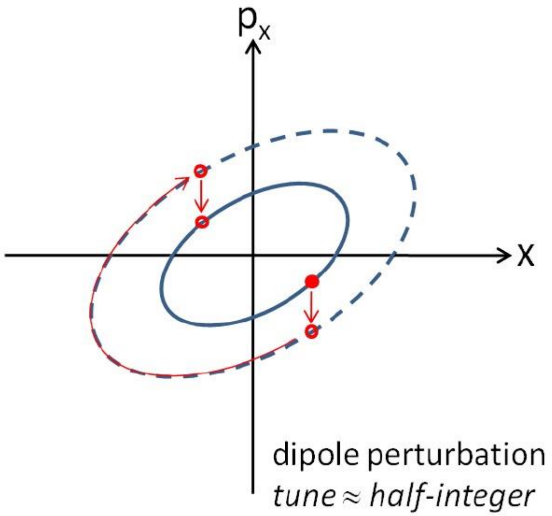


$\mu_x = 0.490 \times 2\pi$

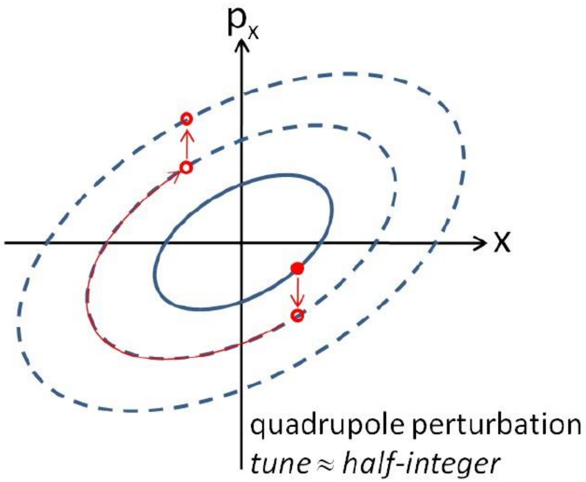
A first taste of non-linearities (6/6)



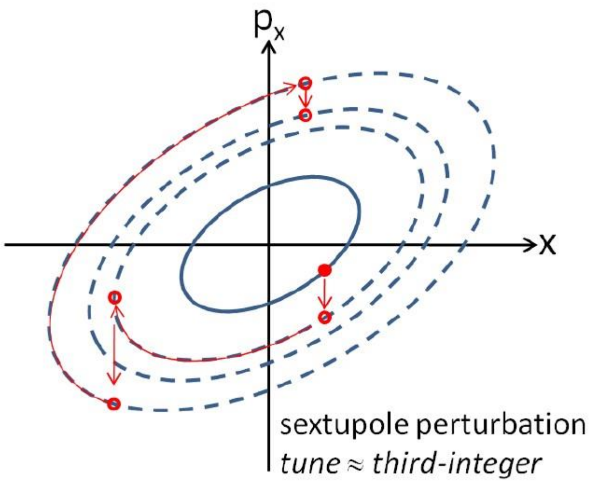
dipole perturbation
tune ≈ integer



dipole perturbation
tune ≈ half-integer



quadrupole perturbation
tune ≈ half-integer



sextupole perturbation
tune ≈ third-integer

Another useful example: Injection missteering

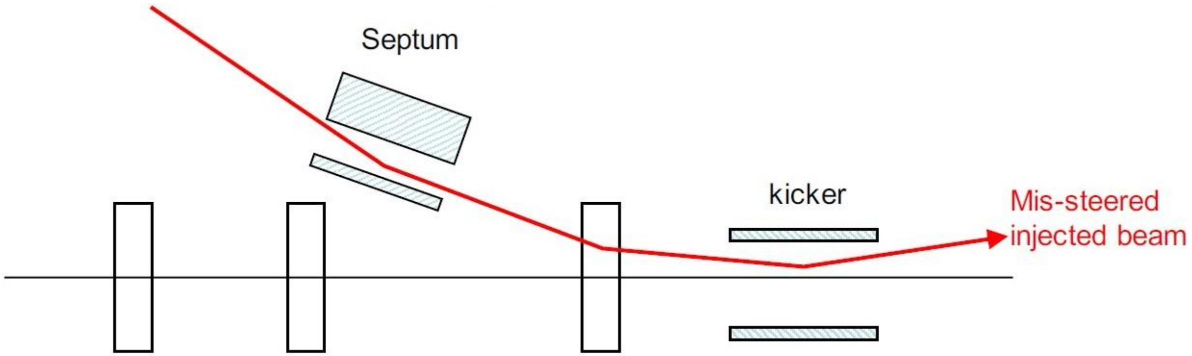
- The emittance is the average action of all particles in the beam:

$$\boxed{\varepsilon_x = \langle J_x \rangle}$$

- **RMS emittance:**

$$\boxed{\varepsilon = \sqrt{|\sigma|} = \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle x x' \rangle^2}}$$

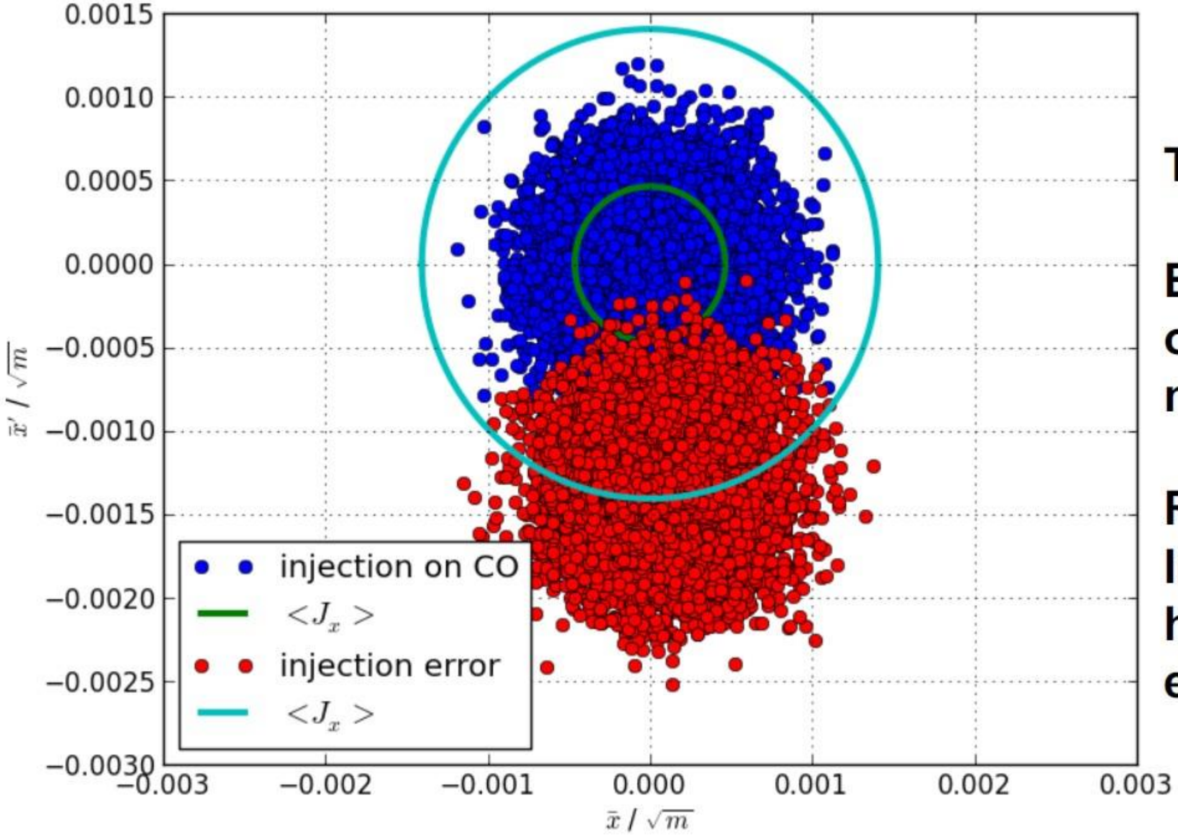
Example:



– Injection oscillations = if beam is not injected on the closed orbit, beam oscillates around closed orbit and eventually filaments (if not damped)

Steering error – linear machine

- What will happen to particle distribution and hence emittance?



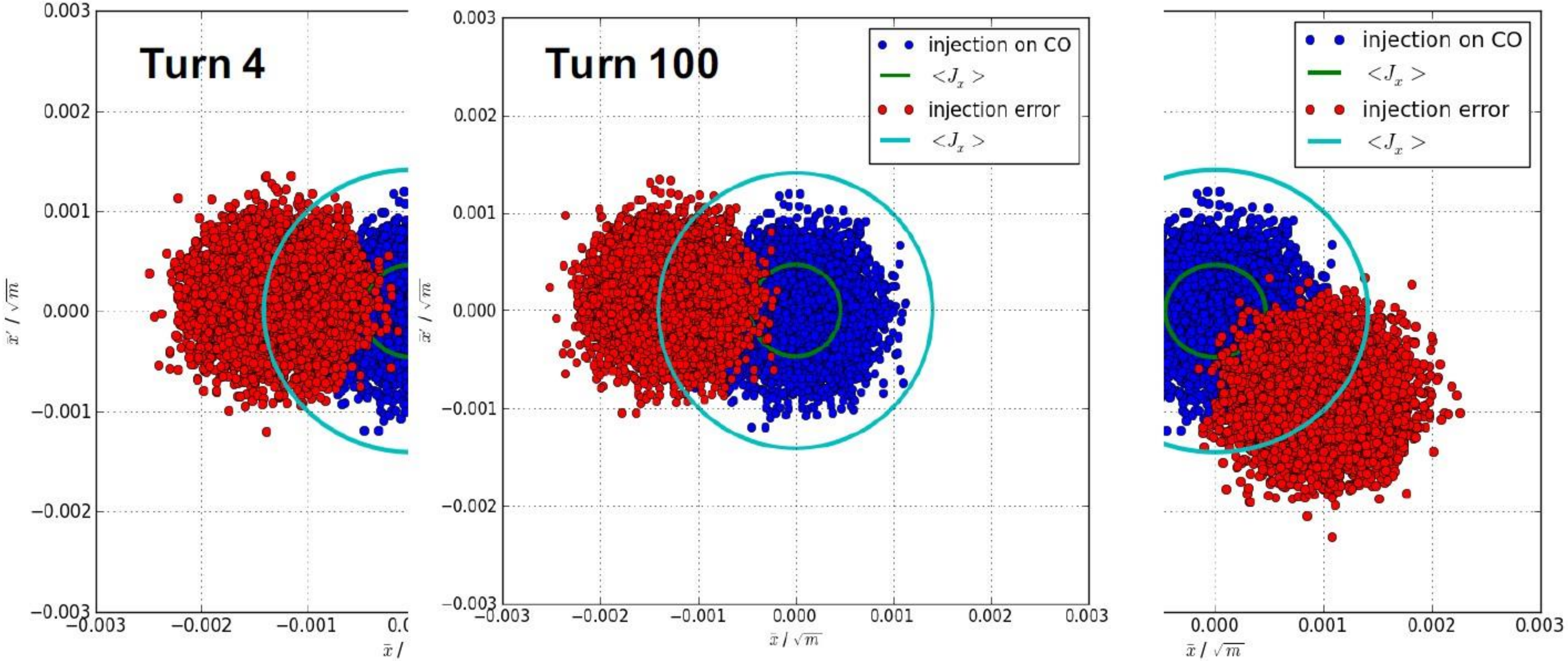
Turn 1:

Blue distribution: on axis injection – no error

Red distribution: Injection with horizontal injection error: mainly in x'

Steering error – linear machine

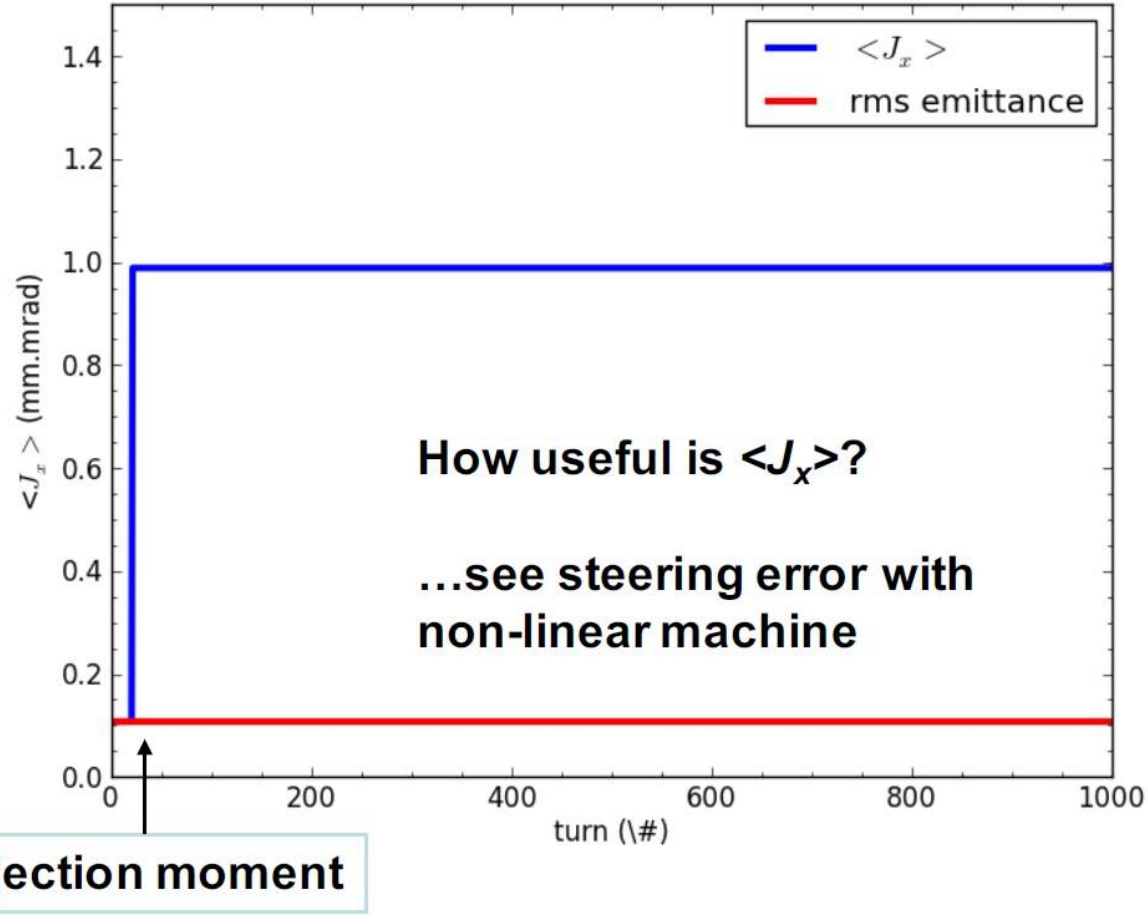
- What will happen to particle distribution and hence emittance?



- The beam will keep oscillating. The centroid will keep oscillating.

Steering error – linear machine

- How does $<J_x>$ behave for steering error in linear machine?
- And what about the rms definition?



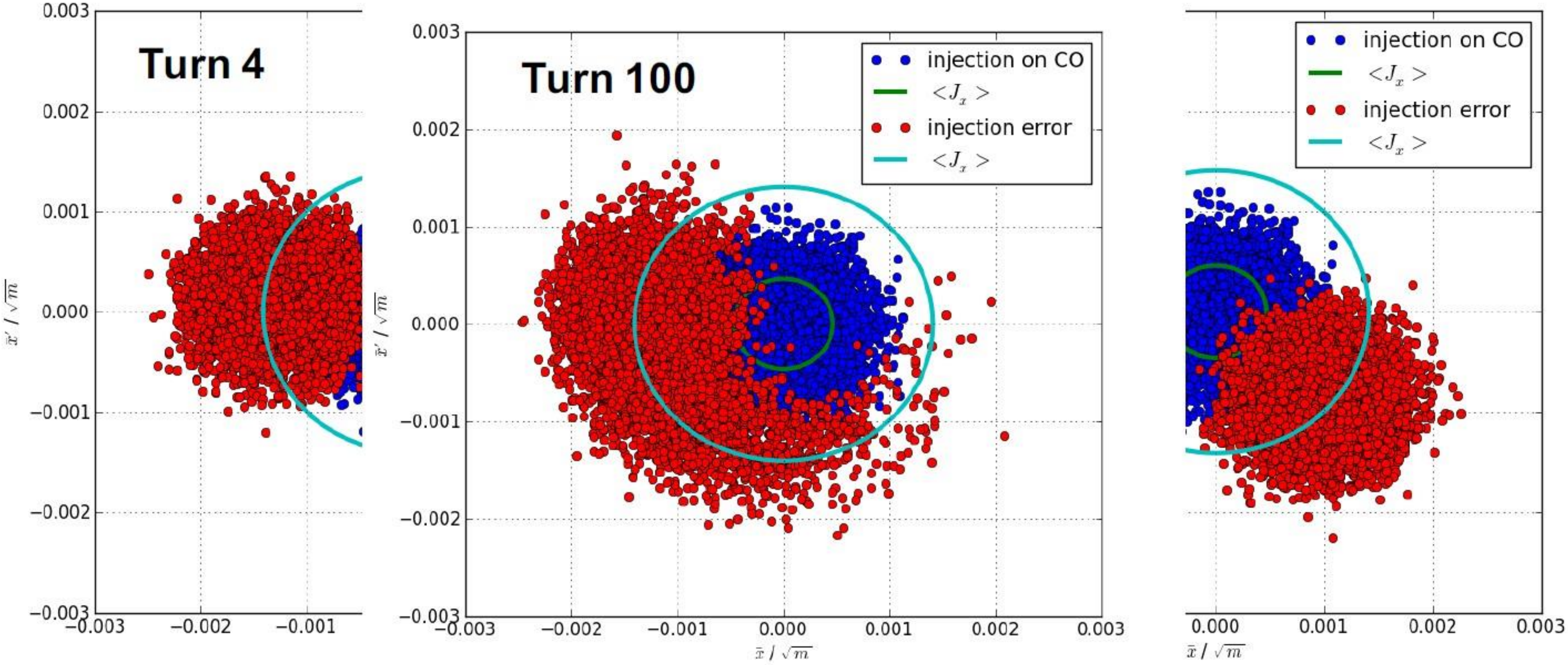
How useful is $<J_x>$?

…see steering error with non-linear machine

Injection moment

Steering error – non-linear machine

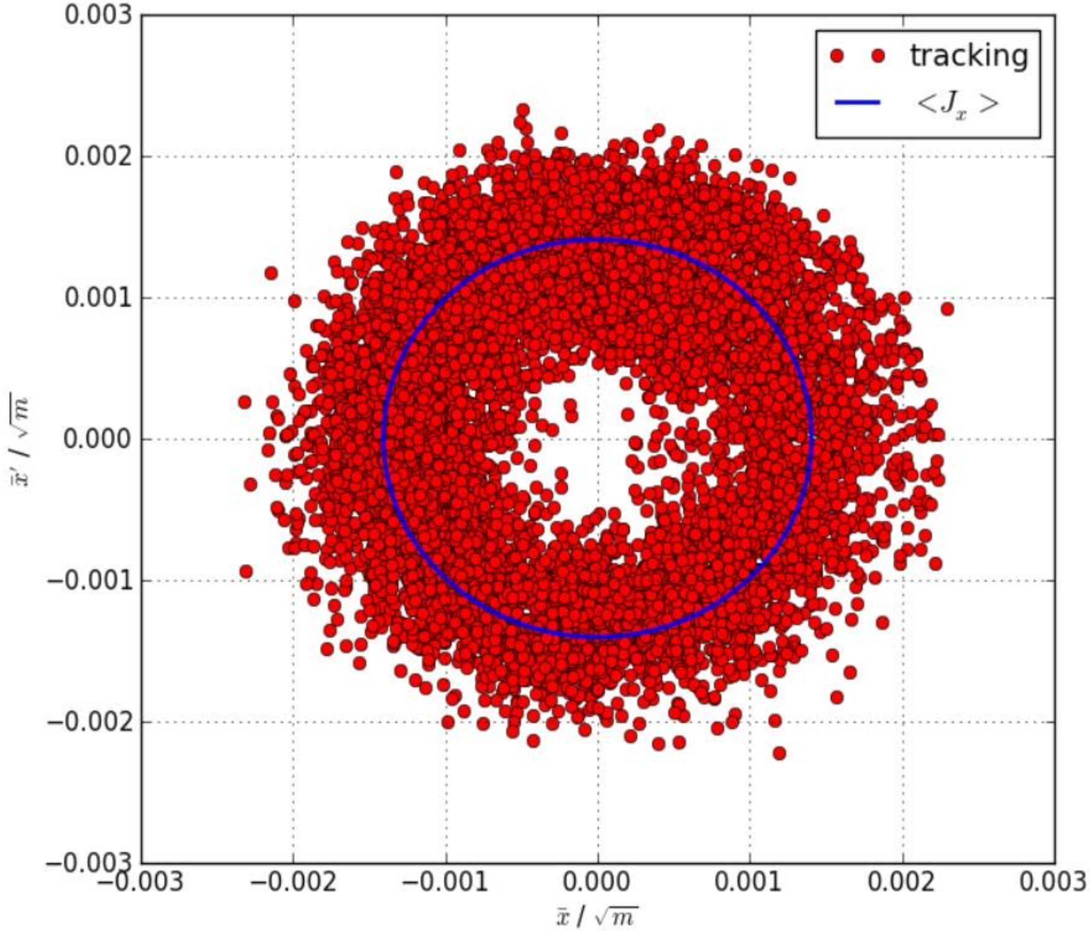
- What will happen to particle distribution and hence emittance?



- The beam is filamenting….

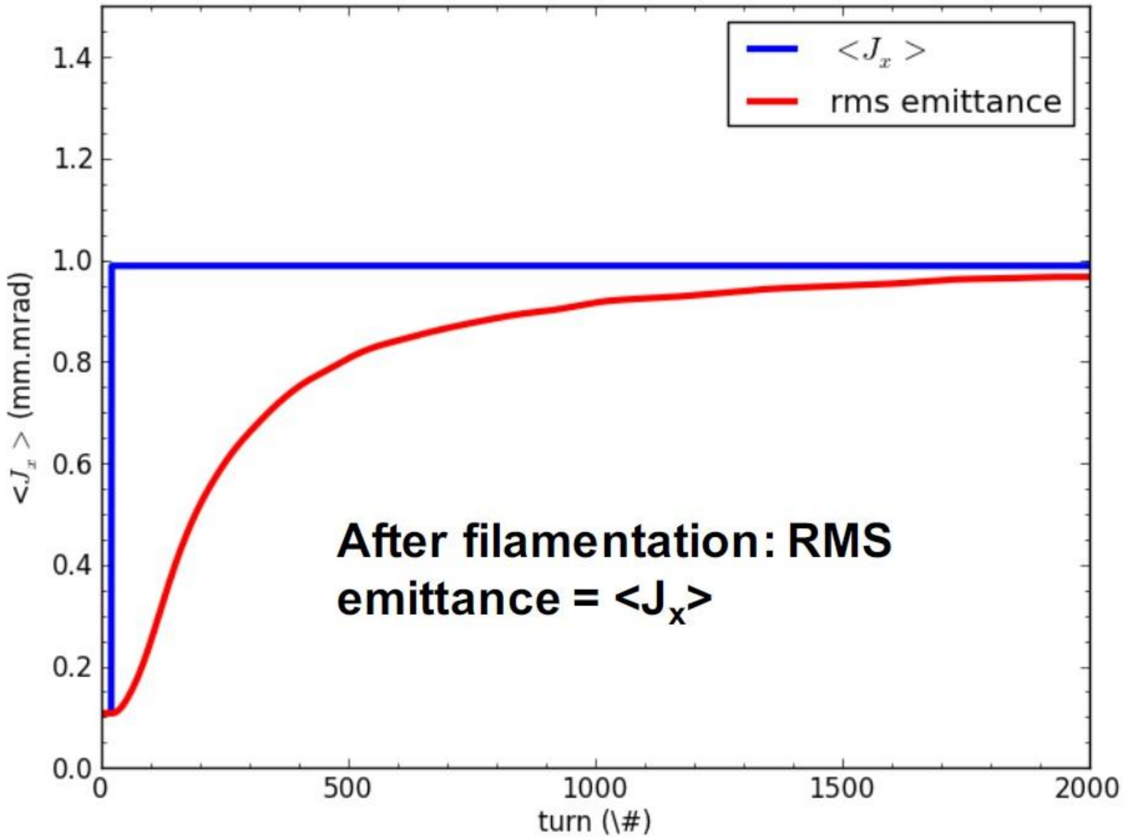
Steering error – non-linear machine

- Phase-space after an even longer time



Steering error – non-linear machine

- How does $\langle J_x \rangle$ behave for steering error in non-linear machine?
- And what about the rms emittance



After filamentation: RMS emittance = $\langle J_x \rangle$

Linear Imperfections

- Up to now we have constructed an alternate –gradient focusing synchrotron
- We have a well chosen working point
- We have corrected chromaticity
- (We still cannot accelerate! → see F. Tecker (long. Dynamics)
- We assume:

  - All magnetic elements have the calculated field strength and field quality

  - All magnetic elements are in the right place and powered with the right polarity
- Reality tells us:

  - Magnets have field errors, have other multipole components, have time varying fields due to ripple in the connected power converter

  - Magnets are wrongly mounted with horizontal and/or vertical offsets, rotations or tilts
- These effects influence:

  - the beta functions and phase advance around the ring (implicitly the tunes)

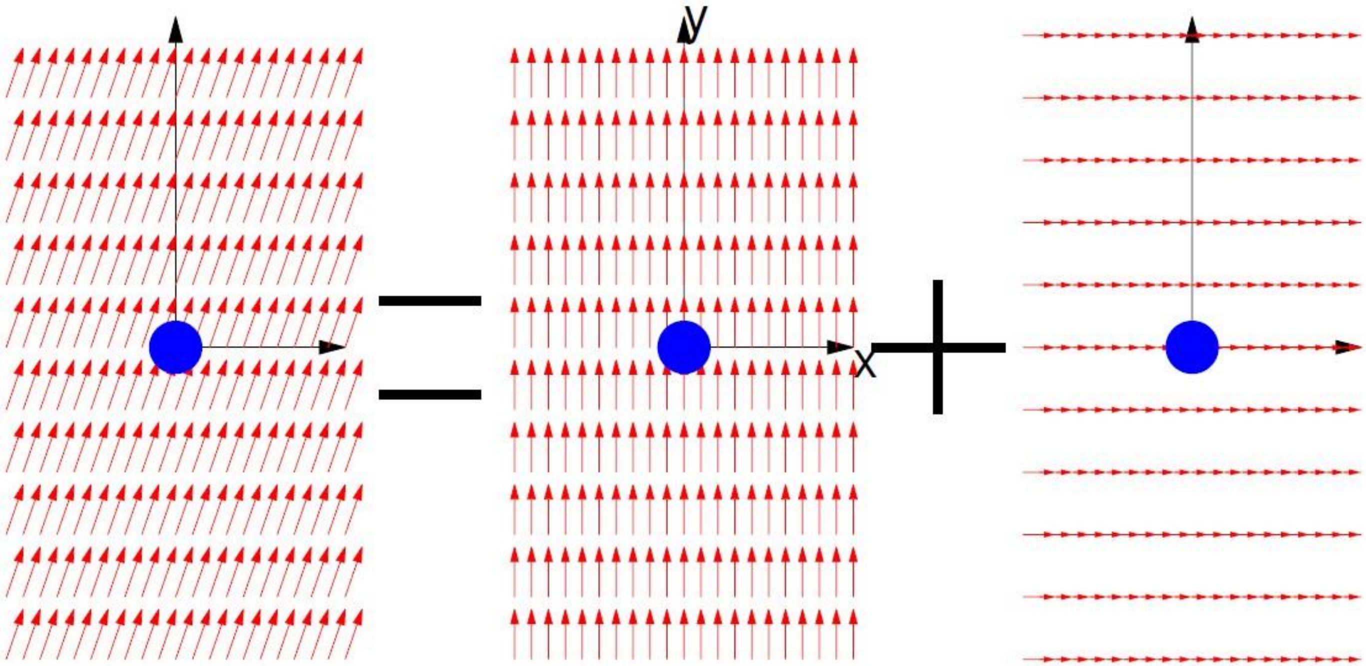
  - the closed orbit

  - the coupling between horizontal and vertical motion

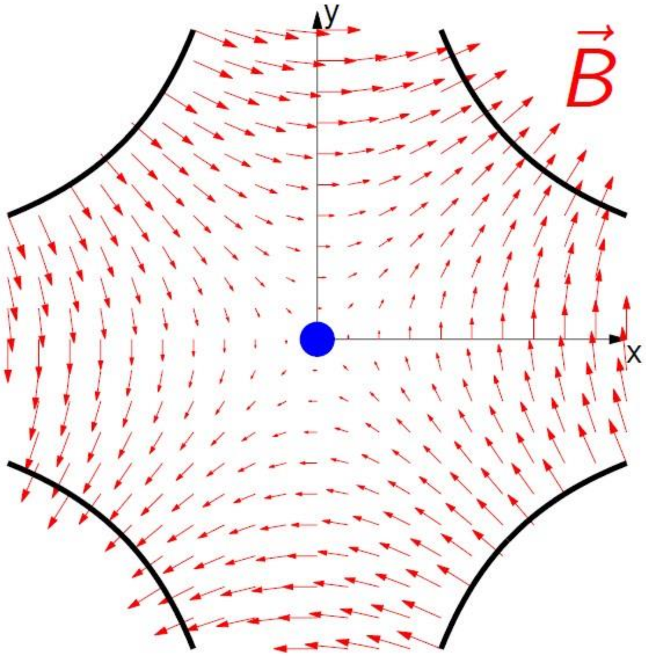
  …
- We need to diagnose and correct: Strong interaction between beam measurements and corrections (see also R.Jones BDI talks)

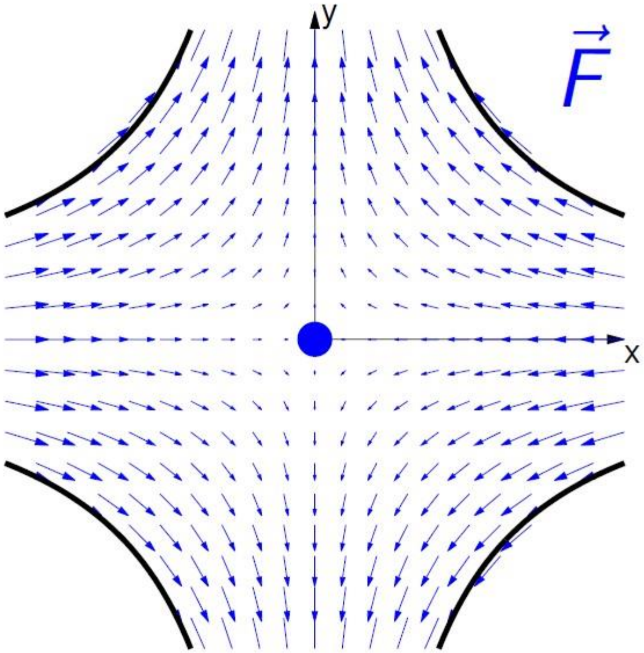
Dipole Errors

error	effect	correction
strength (k)	change in deflection	change excitation current, replace magnet
lateral shift	none	
tilt	additional vertical deflection	corrector dipole magnet



Quadrupole Errors (1/2)

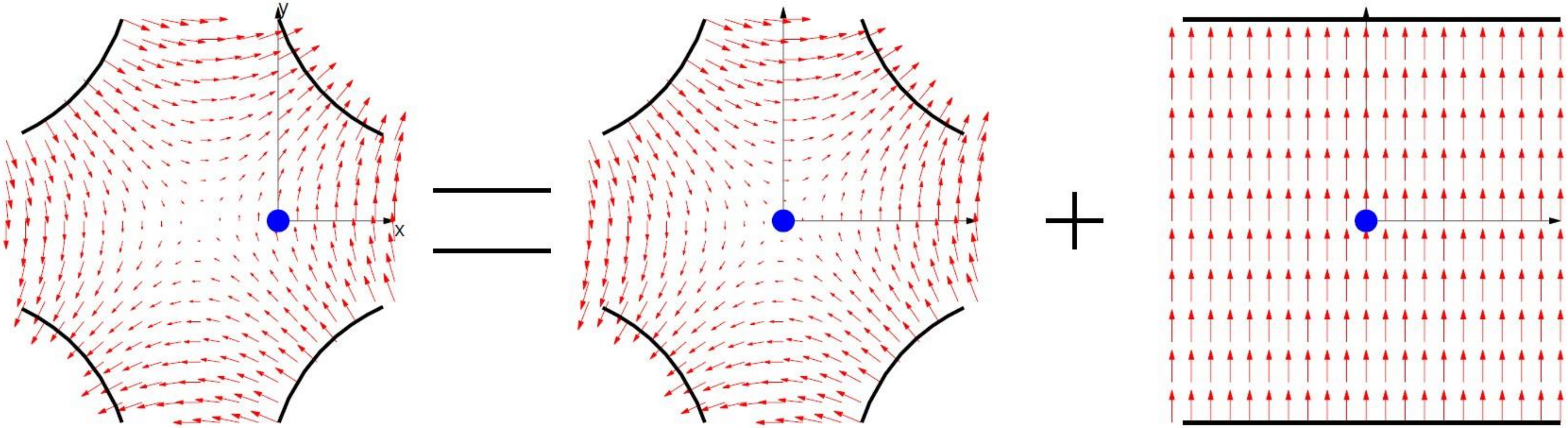




Note that $F_x = -kx$ and $F_y = ky$ making horizontal dynamics totally decoupled from vertical.

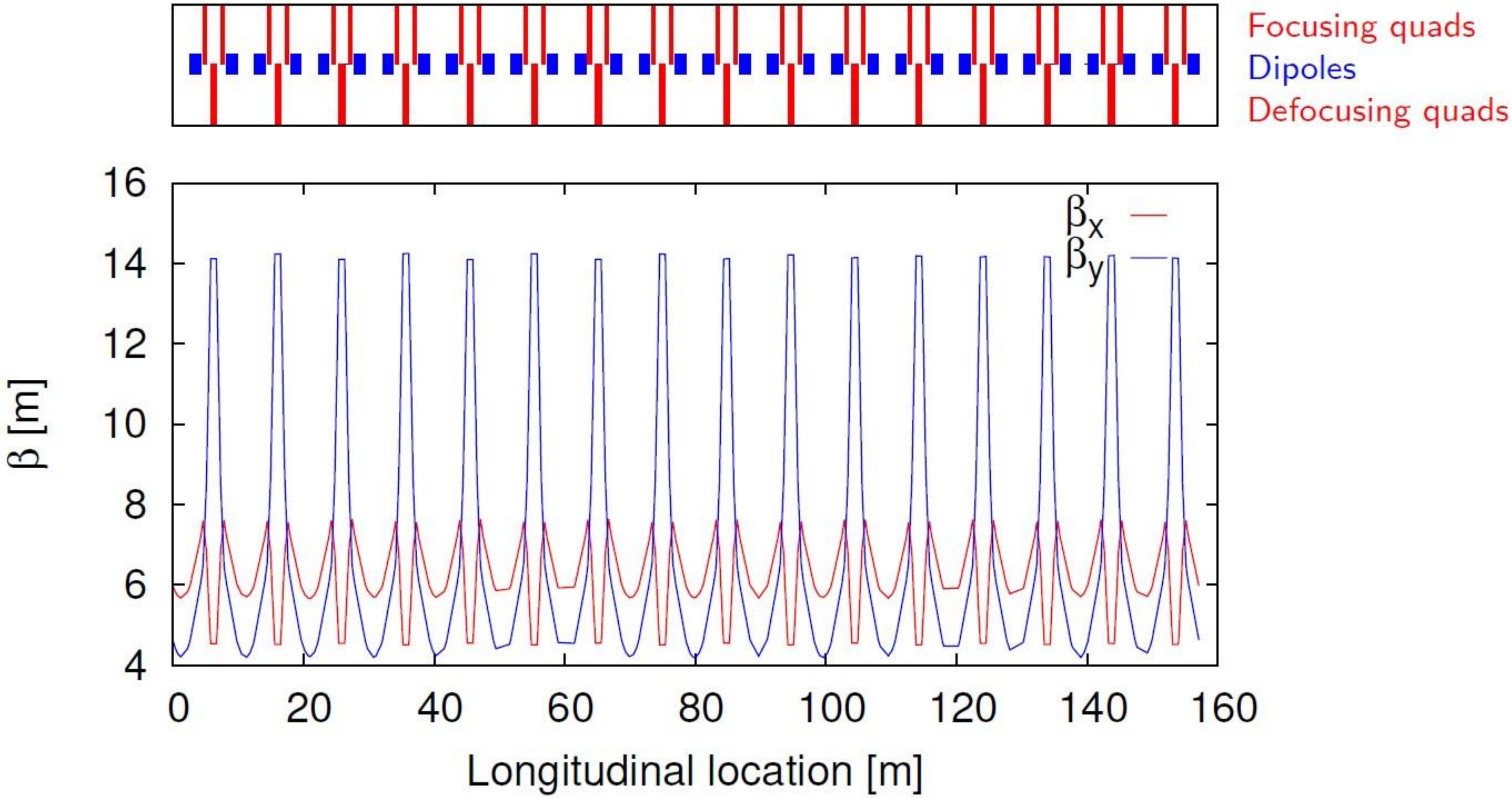
Quadrupole Errors 2/2

Error type	effect on beam	correction(s)
strength	Change in focusing, “beta-beating”	Change excitation current, Repair/Replace magnet
Lateral shift	Extra dipole kick	Excitation of a corrector dipole magnet
tilt	Coupling of the beam motion in the two planes	Excitation of a additional “skewed quadrupoles (45^0)

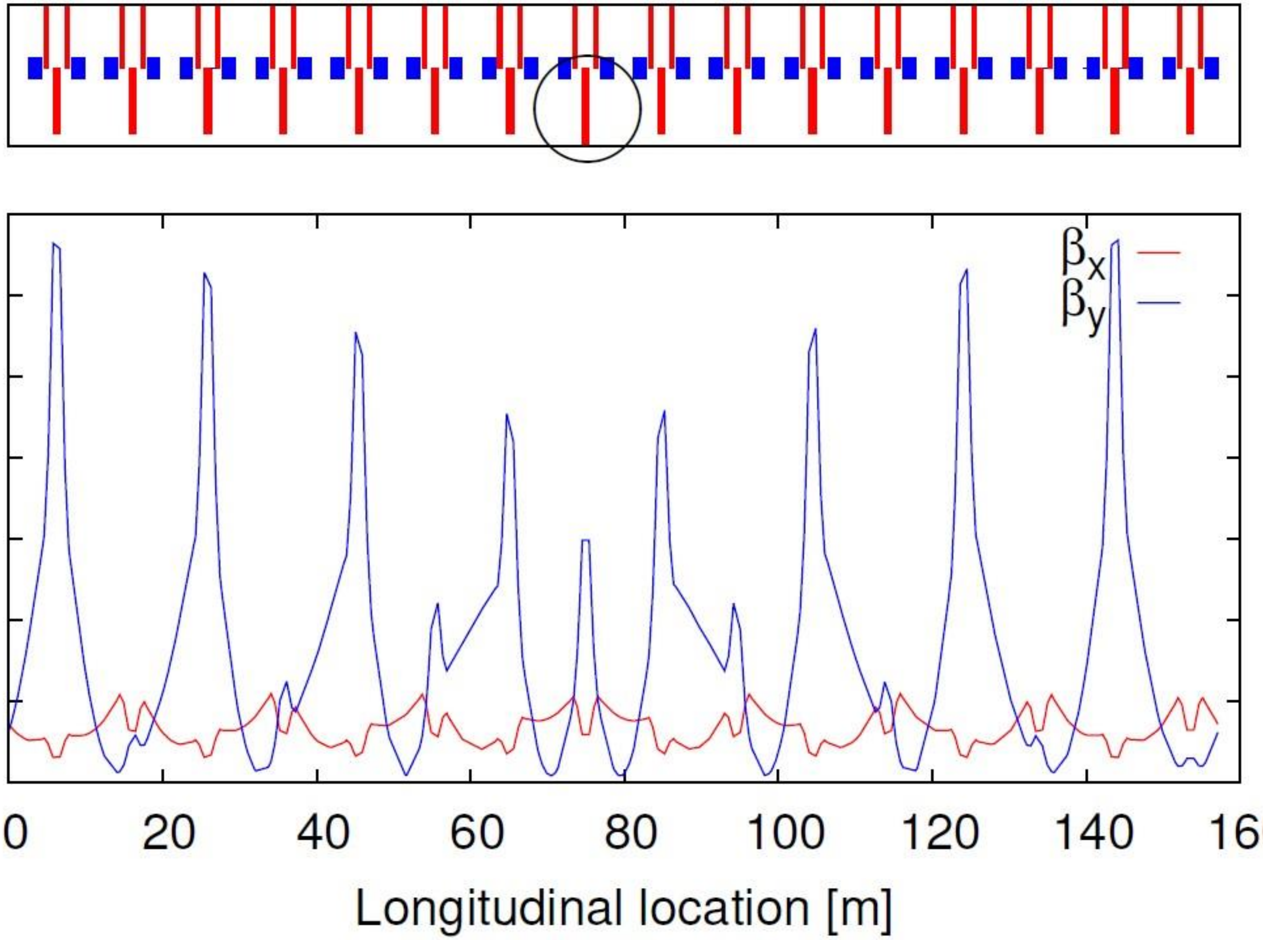


An offset quadrupole is seen as a centered quadrupole plus a dipole. This is called feed down.

Beta-beating (1/2)

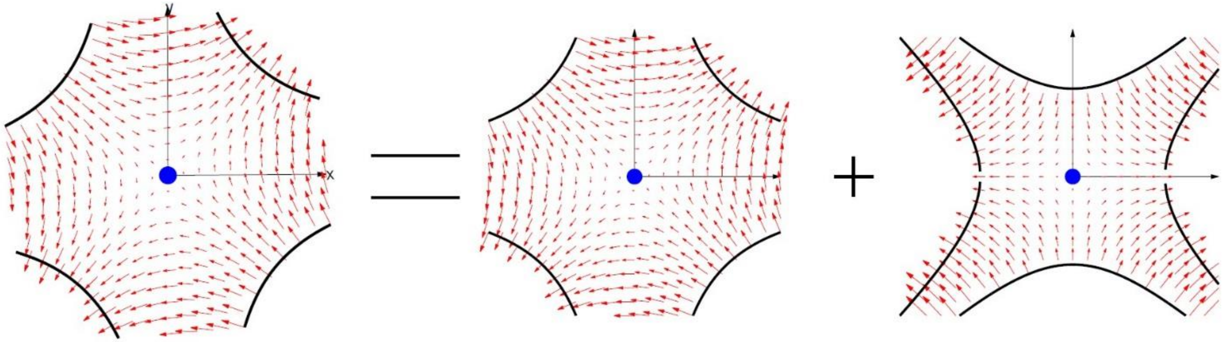


Beta-beating (2/2)

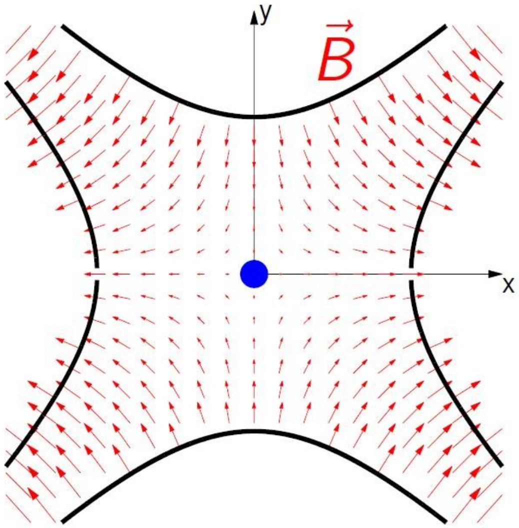


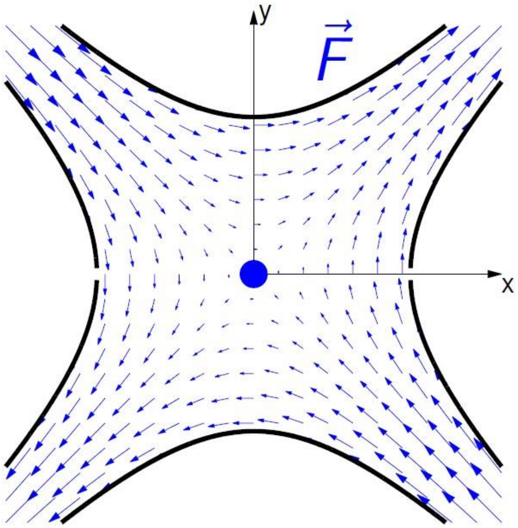
β functions change (β-beating$=\frac{\Delta\beta}{\beta} = \frac{\beta_{pert}-\beta_0}{\beta_0}$).

Quadrupole Errors 3/3



Any tilted quadrupole is seen as a normal quadrupole plus another quadrupole tilted by 45^0. (skew quad)



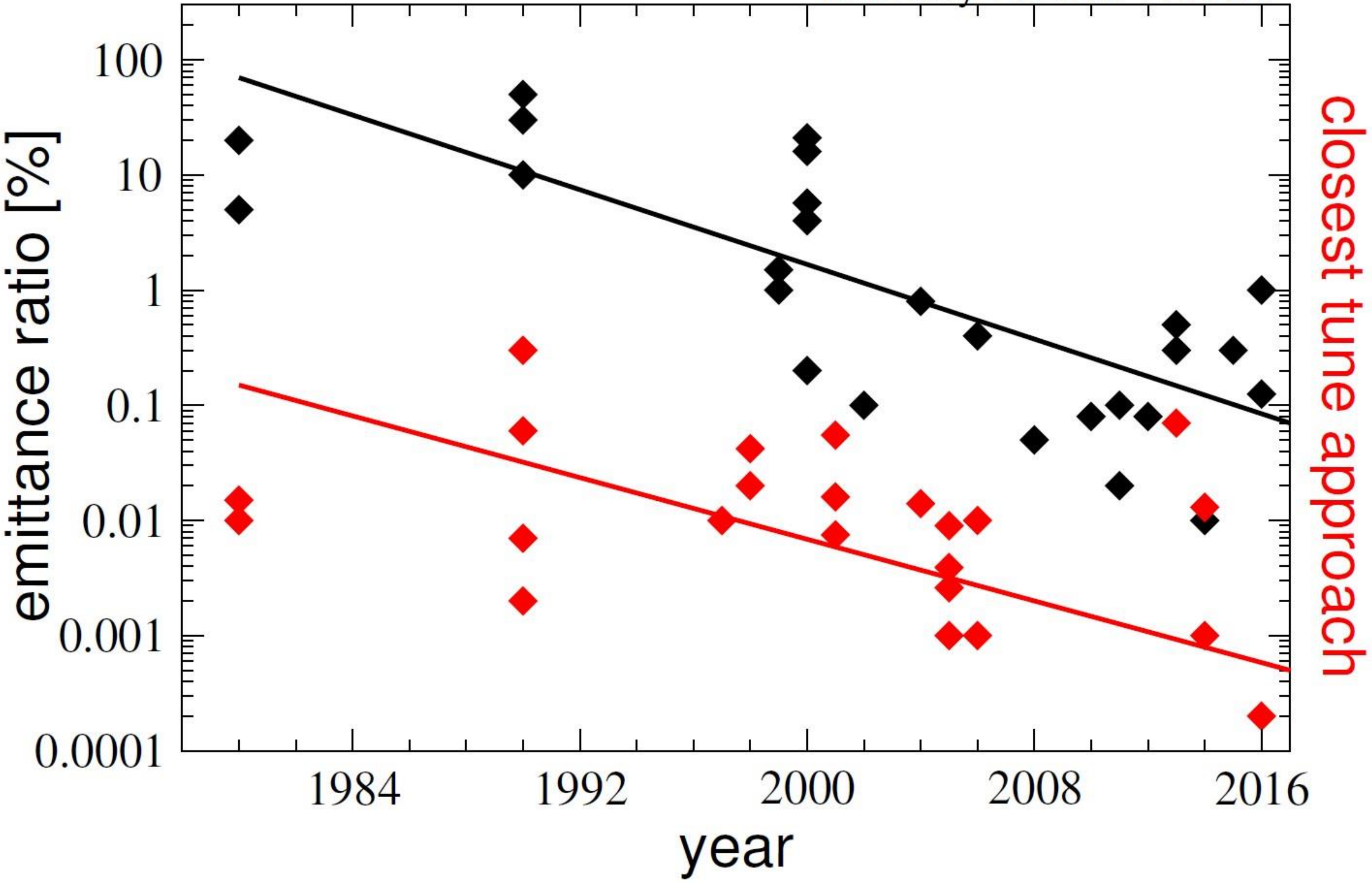


Note that in a skew quad $F_x = k_s y$ and $F_y = k_s x$ produce coupling between the x and y planes

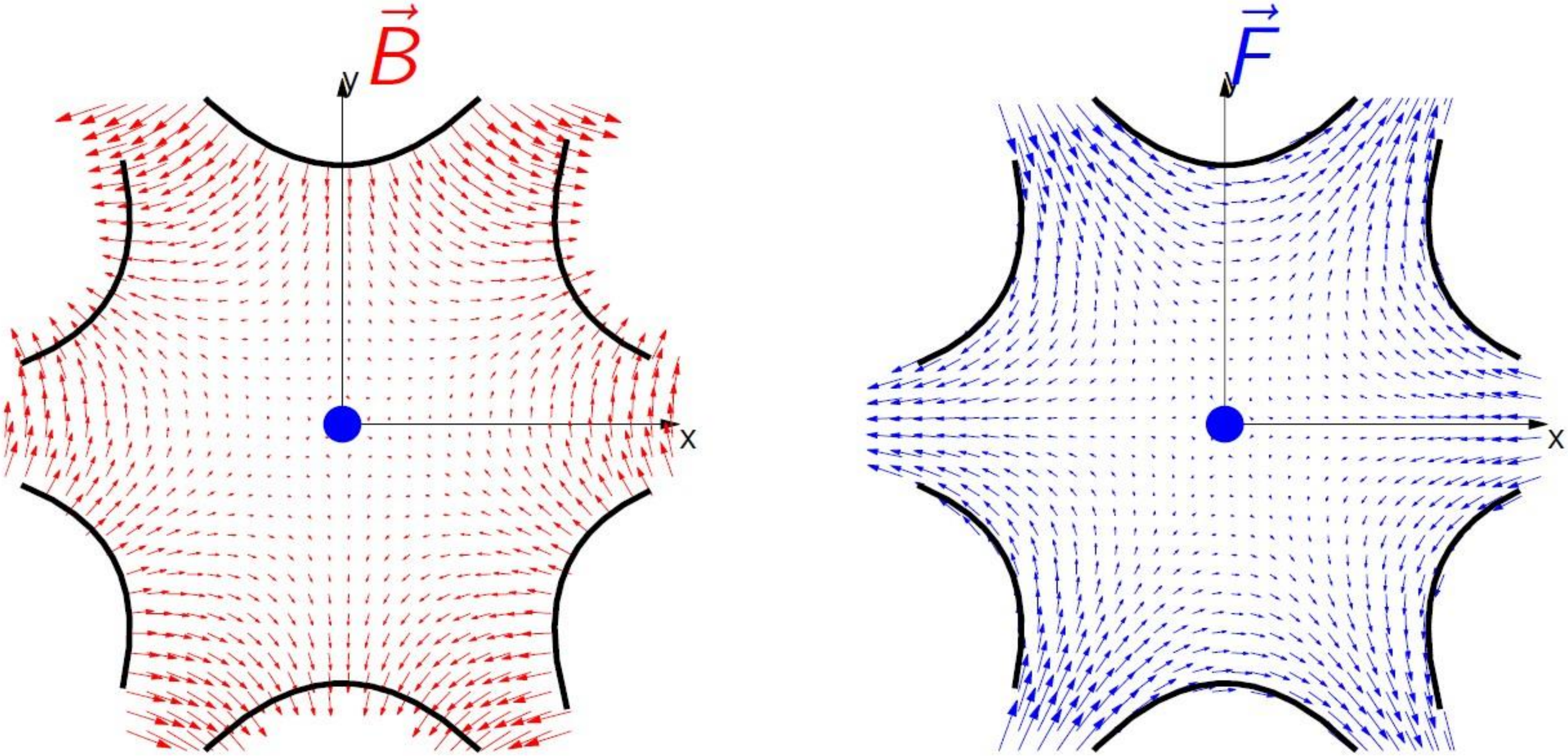
Additional skew quads in an accelerator are used to compensate coupling

Coupling control is most important in synchrotron light sources, since small vertical emittance (yielding high brightness of the photon beams) is predominantly achieved by decoupling the x and y planes.

Courtesy Andrea Franchi



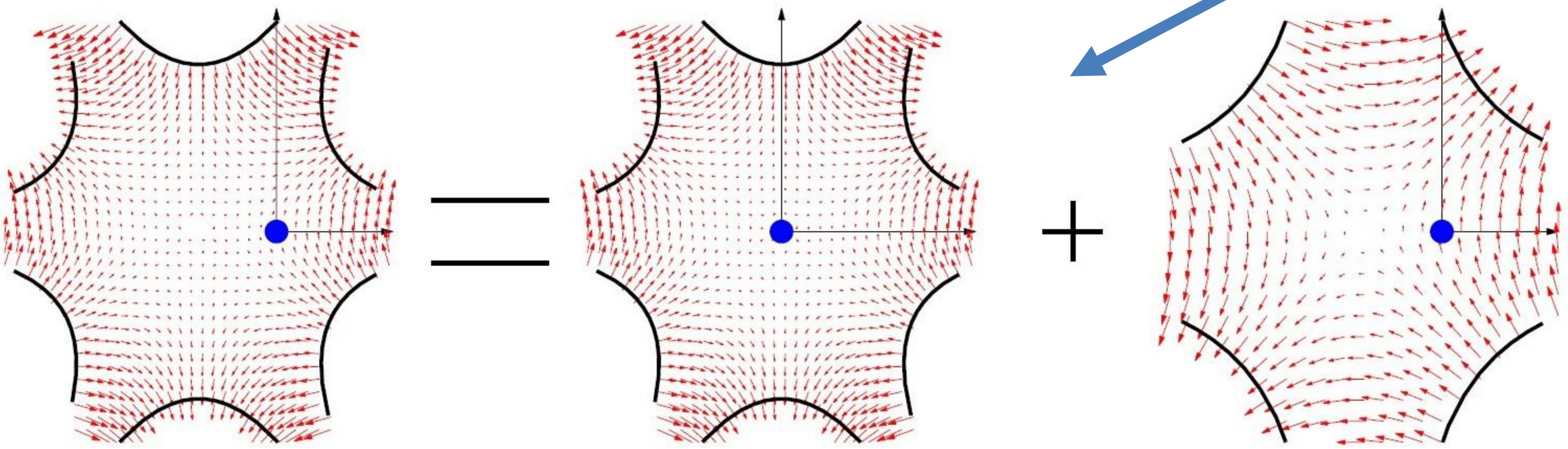
Last not least: Sextupole errors (1/2)



$$F_x = \frac{1}{2} K_2 (x^2 - y^2) \; , \quad F_y = -K_2 xy$$

Last not least: Sextupole errors (2/2)

Error type	effect on beam	correction(s)
strength	Change in chromaticity correction, beta-beating	Change excitation current, Repair/Replace magnet
Lateral shift	Extra quadrupole and skew quadrupole, beat-beating, tune change, coupling	Compensation with quadrupoles and skew quadrupoles, realignment
tilt	Error in the chromaticity correction	Excitation of a additional “skewed sextupoles (45^0)



A horizontally (vertically) displaced sextupole is seen as a centred sextupole plus an offset quadrupole (skew quadrupole)

Correction summary

Effect of dipole kicks (θ_i ; Φ_i) on closed orbit (CO)

$$CO(s) = \frac{\sqrt{\beta(s)}}{\boxed{2\sin\pi Q}} \sum_i \sqrt{\beta_i}\theta_i \cos(\pi Q - |\phi(s) - \phi_i|)$$

Effects of strength error in quadrupoles

$$\Delta Q_x \approx \frac{1}{4\pi}\overline{\beta_x}\Delta k_i L_i, \quad \Delta Q_y \approx -\frac{1}{4\pi}\overline{\beta_y}\Delta k_i L_i$$

β-beating from many sources:

$$\frac{\Delta\beta}{\beta}(s) \approx \pm \sum_i \frac{\Delta k_i L_i \overline{\beta_i}}{\boxed{2\sin(2\pi Q)}} \cos(2\pi Q - 2|\phi(s) - \phi_i|)$$

- Best correction: identify error source and repair(realignment; coil repair…)
- If not: Typically close to every quadrupole small dipole correctors are installed. So by measurement campaigns and data analyses corrections strength for these small dipoles and to (skew) quadrupoles are applied.
- More on this in the diagnostics lecture and the advanced part.

Last not least: Collective effects

Collective effects:
= Summary term for all effects when the coulomb force of the particles in a bunch can no longer be neglected; in other words when there are too many particles…

We distinguish:
i) self interaction of the particles within a bunch:

  1) space charge effects
  2) Intra beam scattering
  3) Touschek scattering

leads to emittance growth and particle loss
ii) Interaction of the particles with the vacuum wall
  →concept of impedance of vacuum system
leads to instabilities of single bunches and multiple bunches
iii) Interaction of with particles from other counter-rotating beam
  → beam-beam effects (→ more later this school)

Most is very advanced matter → here only concepts and buzz-words

Space-charge Forces

In the rest frame of a bunch of charged particles, the bunch will expand rapidly (in the absence of external forces) because of the Coulomb repulsion between the particles.

The electric field around a single particle of charge q at rest is a radial field:

$$E_r = \frac{q}{4\pi\varepsilon_0}\frac{1}{r^2}$$

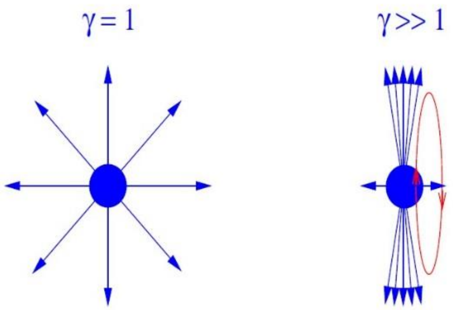
Applying a Lorentz boost along the z axis, with relativistic factor γ, the field becomes:

$${}_x = \frac{q}{4\pi\varepsilon_0}\frac{\gamma x}{\left(x^2 + y^2 + \gamma^2 z^2\right)^{3/2}} \qquad E_y = \frac{q}{4\pi\varepsilon_0}\frac{\gamma y}{\left(x^2 + y^2 + \gamma^2 z^2\right)^{3/2}} \qquad E_z = \frac{q}{4\pi\varepsilon_0}\frac{\gamma z}{\left(x^2 + y^2 + \gamma^2 z^2\right)^{3/2}}$$

For large γ, the field is strongly suppressed, and falls rapidly away from $z = 0$. In other words, the electric field exists only in a plane perpendicular to the direction of the particle.

Space Charge: Scaling with energy

Example Coulomb field: (a charge moving with constant speed)



Recall from relativity

- **In rest frame purely electrostatic forces**
- **In moving frame $\vec{E}$ transformed and $\vec{B}$ appears**

Electrical field : repulsive force between two charges of equal polarity
Magnetic field: attractive force between two parallel currents
after some work:

$$F_\mathrm{r} = \frac{eI}{2\pi\varepsilon_0\beta c}(1 - \beta^2)\frac{r}{a^2} = \frac{eI}{2\pi\varepsilon_0\beta c}\frac{1}{\gamma^2}\frac{r}{a^2}$$

→ space charge diminishes with ${}^1/_{\gamma^2}$ scaling
→ each particle source immediately followed by a linac or RFQ for acceleration

Space Charge Tune Shift

The tune spread from space-charge forces for particles in a Gaussian bunch of N_0 particles and rms bunch length σ_z is given by:

$$\Delta\nu_y = -\frac{2r_e N_0}{(2\pi)^{3/2}\sigma_z\beta^2\gamma^3}\oint\frac{\beta_y}{\sigma_y(\sigma_x+\sigma_y)}ds$$

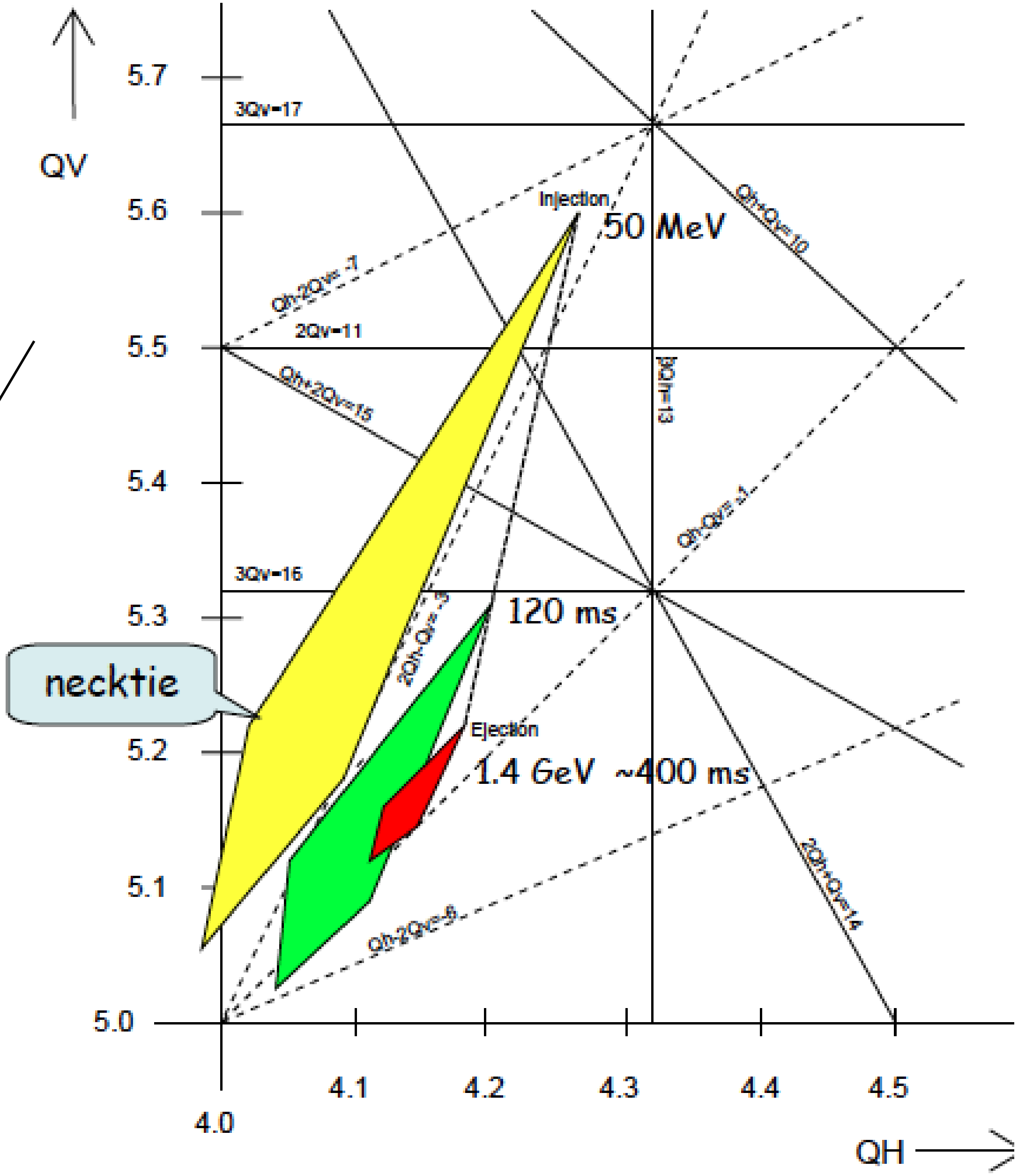
where the integral extends around the entire circumference of the ring.

Since every particle in the bunch experiences a different tune shift, it is not possible to compensate the tune spread as one could for a *coherent* tune shift (for example, by adjusting quadrupole strengths).

Note that the tune spread gets larger for:

- larger bunch charges
- shorter bunches
- larger beta functions
- lower beam energy (very strong scaling!)
- larger circumference
- smaller beam sizes

Space charge always defocusing



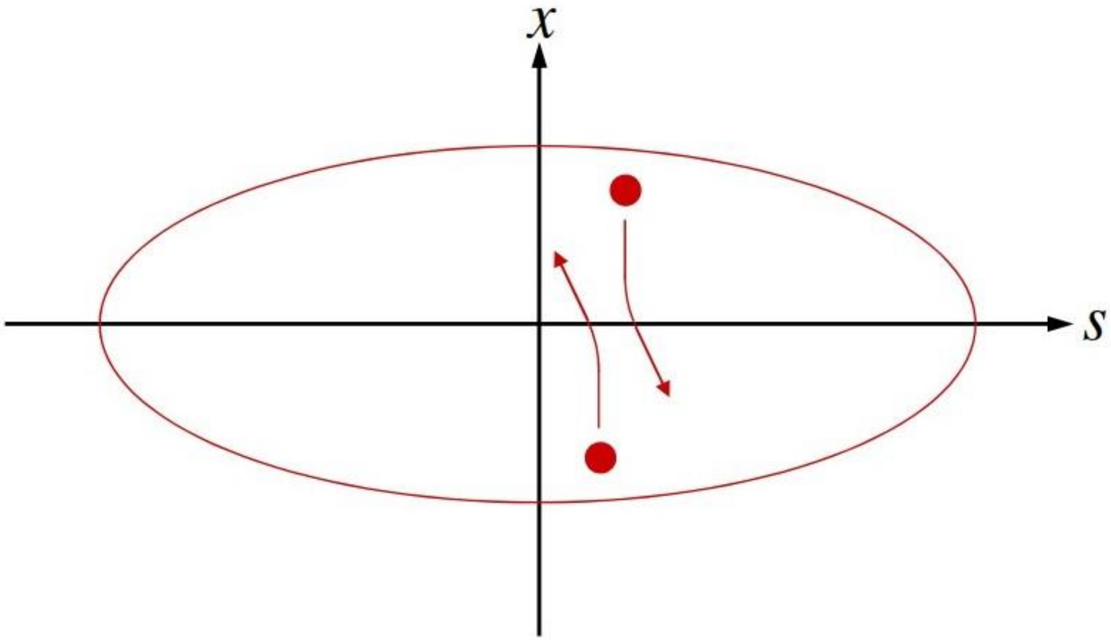
“footprint” of particles with space charge tune shift.

The effect dramatically reduces at higher energies

Intrabeam Scattering

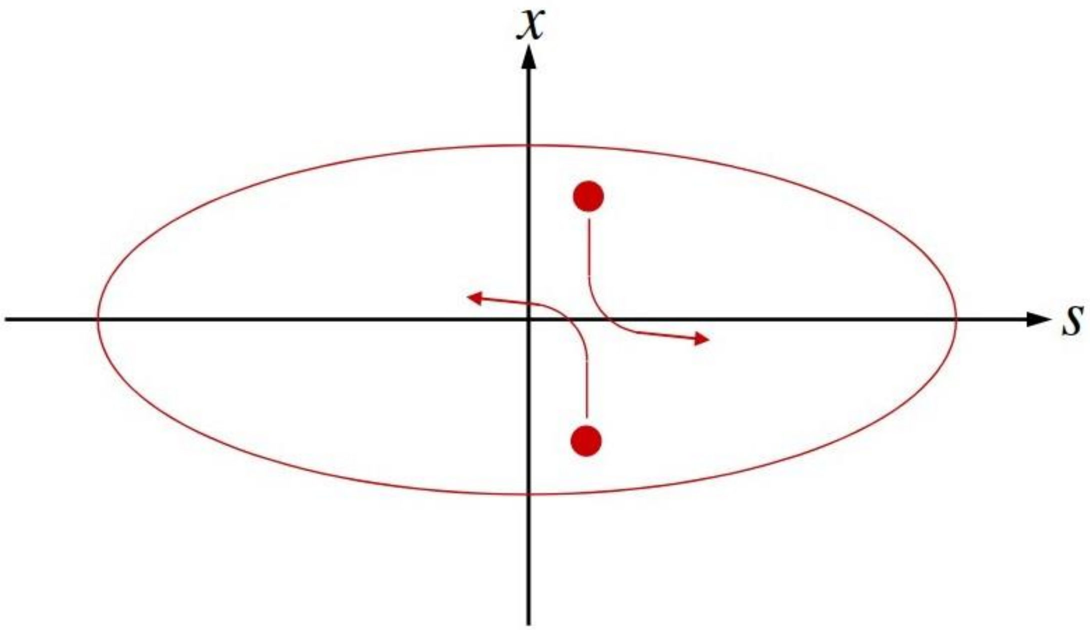
Particles within a bunch can collide with each other as they perform betatron and synchrotron oscillations. The collisions lead to a redistribution of the momenta within the bunch, and hence to a change in the emittances.

If a collision results in the transfer of transverse to longitudinal momentum at a location where the dispersion is non-zero, the result (after many scattering events) can be an increase in both transverse and longitudinal emittance.



Touscheck effect

The Touschek effect is related to intrabeam scattering, but refers to scattering events in which there is a large transfer of momentum from the transverse to the longitudinal planes. IBS refers to multiple small-angle scattering; the Touschek effect refers to single large-angle scattering events.

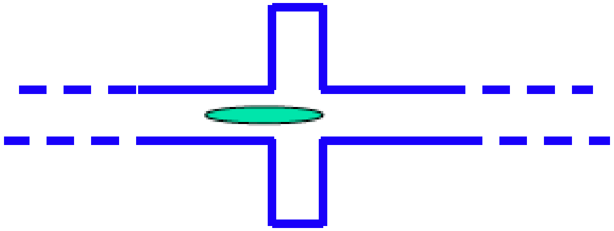


If the change in longitudinal momentum is large enough, the energy deviation of one or both particles can be outside the energy acceptance of the ring, and the particles will be lost from the beam.

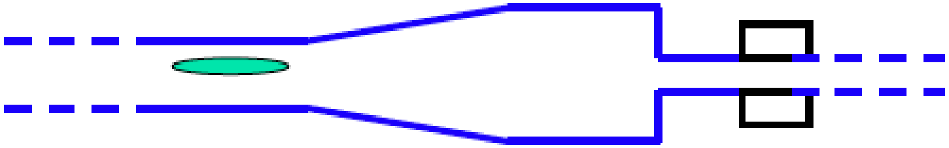
Interaction of beam with vacuum chamber

Resistive wall effect:
Finite conductivity

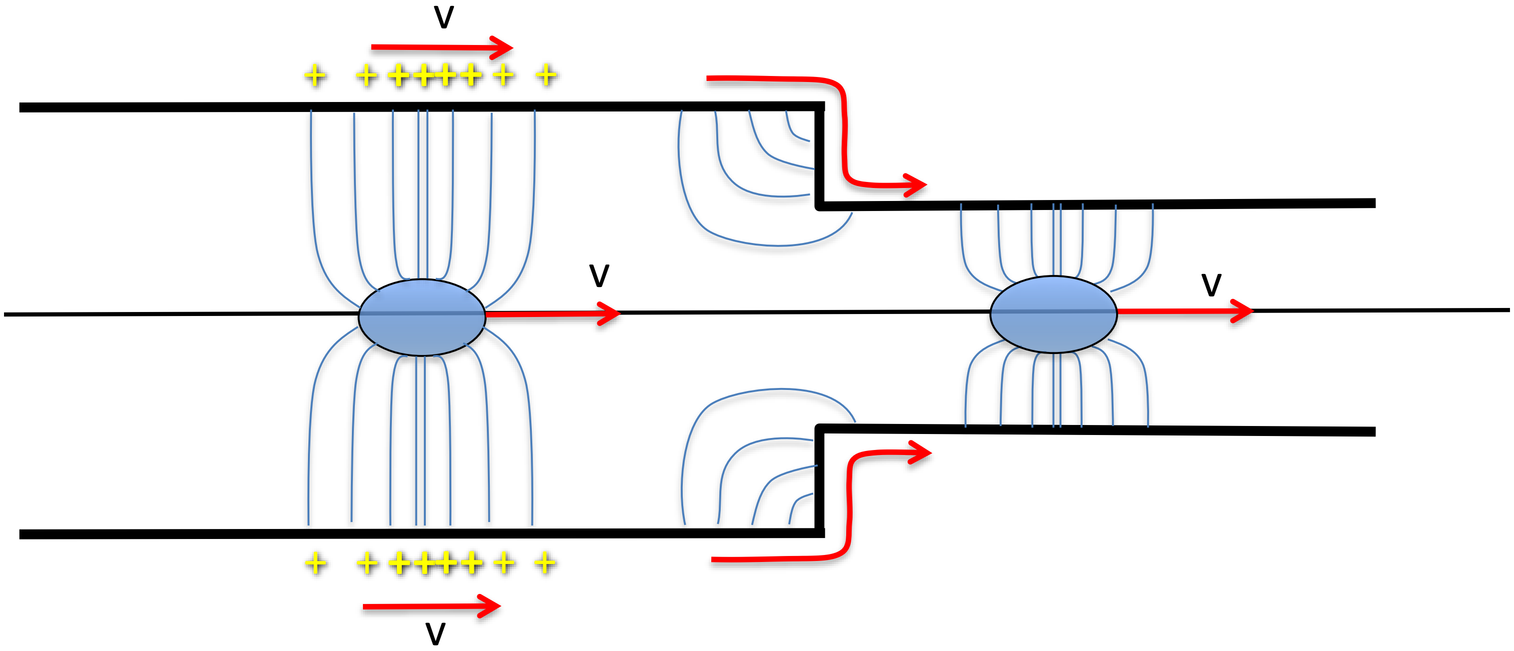
Narrow-band resonators:
Cavity-like objects



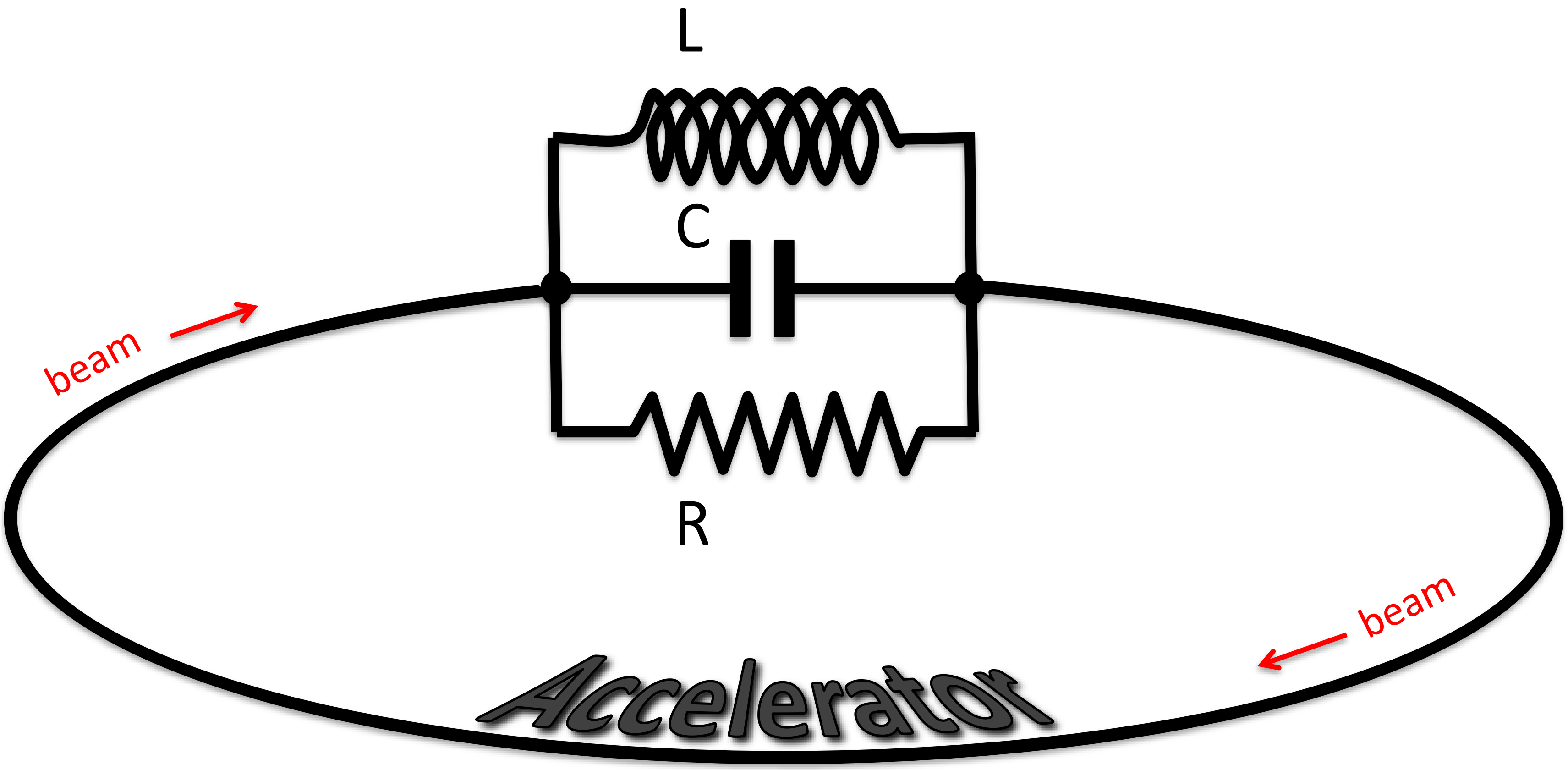
Broad-band resonators:
Tapers, other non-resonant structures

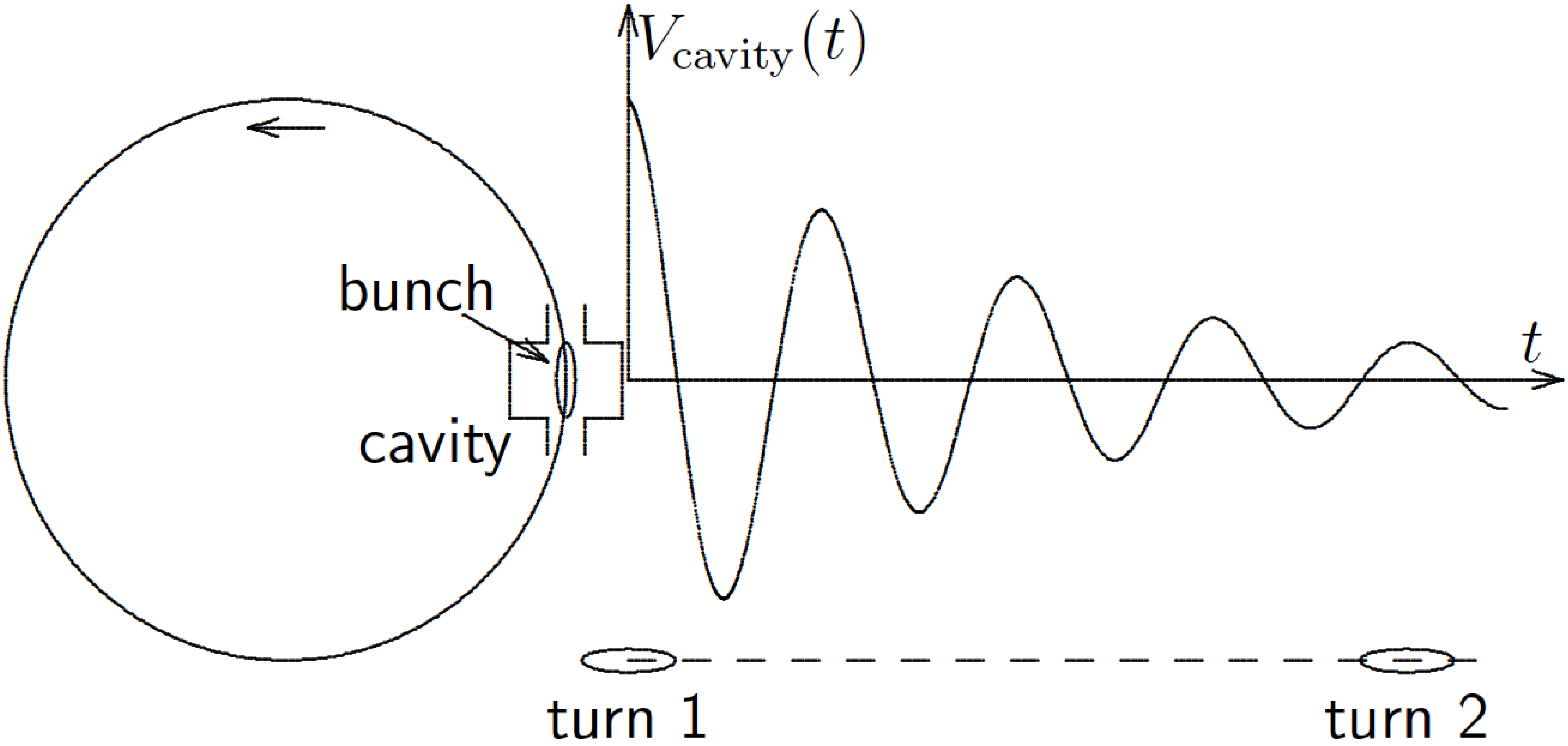


Bunch in a conducting pipe with sudden change

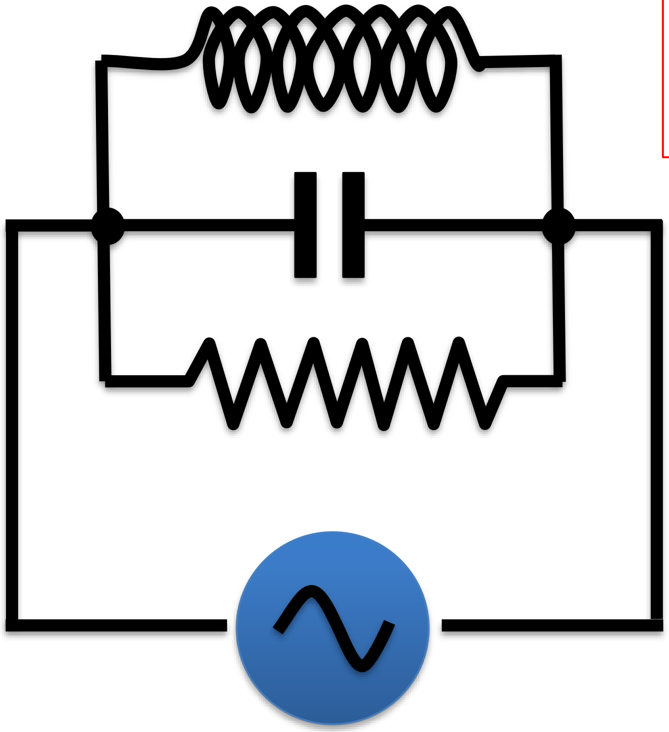


All together




$V_{\text{cavity}}(t)$
bunch
cavity
t
turn 1
turn 2

Impedance



$$I = \hat{I}\cos(\omega t)$$

Impedance

$$V(t) = Z_r(\omega)\hat{I}\cos(\omega t) - Z_i(\omega)\hat{I}\sin(\omega t)$$

$$Z_r(\omega) = R\frac{1}{1+Q^2\left(\frac{\omega^2-\omega_r^2}{\omega_r\omega}\right)^2}$$

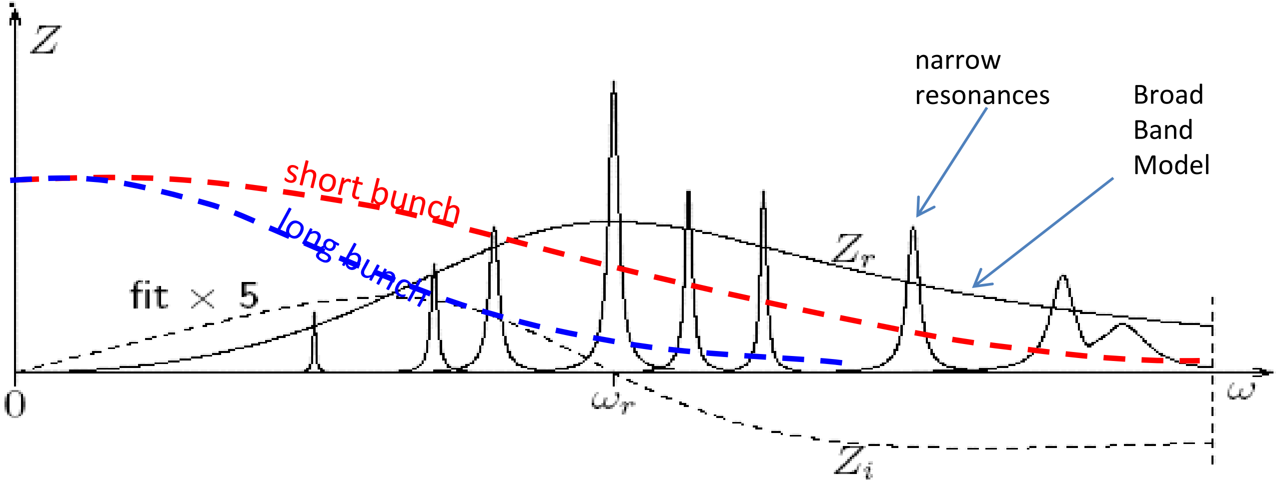
$$Z_i(\omega) = -R\frac{Q\frac{\omega^2-\omega_r^2}{\omega_r\omega}}{1+Q^2\left(\frac{\omega^2-\omega_r^2}{\omega_r\omega}\right)^2}$$

The real (resistive) part dissipates energy, the imaginary part creates instabilities

Consequences of impedances

Energy loss on pipes → heating (important in a superconducting accelerator)

Tune shift

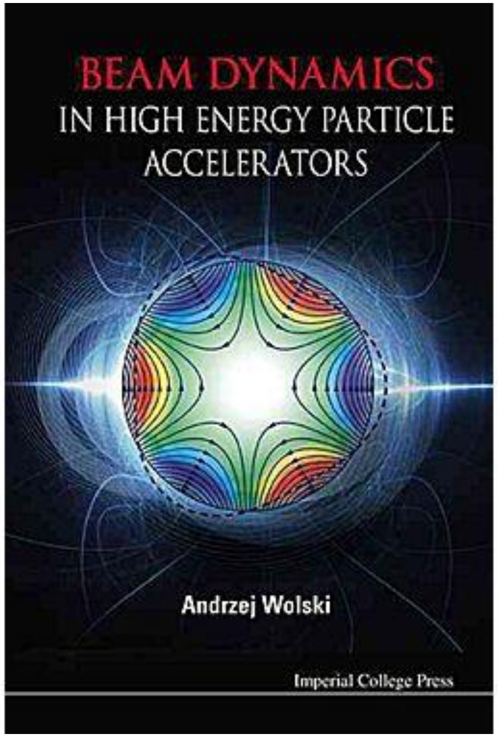


Single bunch instabilities (head-tail)

Multibunch instabilities

Summary

1) Back to school: relativity, EM fields, magnets…
2) Hamiltonian and canonical variables → equations of motion + invariants; map-approach
3) Single particle in various magnetic elements…action as invariant
4) multiple elements; circular accelerator
5) Twiss parameters
6) Finally a beam: emittance and emittance preservation
7) A taste of non-linearities
8) Linear imperfections (and some corrections)
9) Collective effects



Recommended reading:

- A. Wolski, Beam Dynamics in high energy particle accelerators, Imperial College Press, ISBN 978-1-78326-277-9
- CAS proceedings and references therein