INSTABILITIES

Introduction to Instabilities

Longitudinal beam instabilities – basics
"Negative Mass" Instability – qualitative
- quantitative

Stability Diagram
Landau Damping
Longitudinal Stability Criterion
Impedance (resonator)
Bunched beam longitudinal instability:
one bunch; many bunches
Microwave instability

More on Longitudinal Instabilities

Line spectra: single particle, single bunch
Higher-order coupled-bunch modes
Cures

Transverse Instabilities

Fields and forces
Transverse coupling impedances
Spectrum of beam signals
Instability of un-bunched beam
Bunched beam: Head-Tail instability
- zero chromaticity
- non-zero chromaticity shifts beam line spectrum

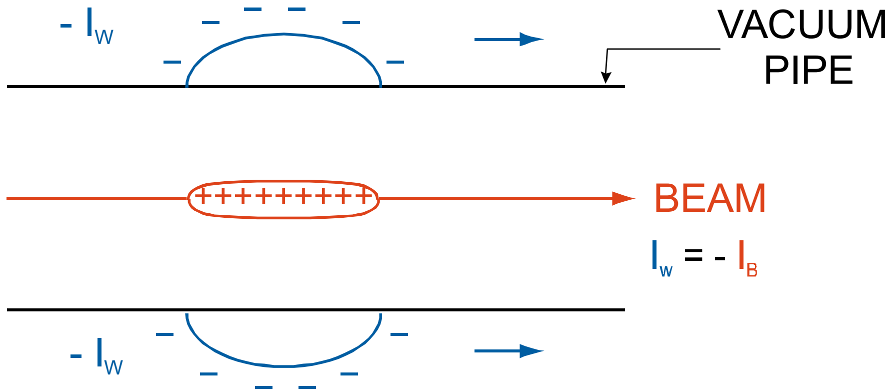
Many bunches – long and short
- growth rate
- stability vs. impedance

Resistive wall instability
Transverse wake fields
Cures

Further Reading:

A. Hofmann, Single beam collective phenomena – longitudinal, CAS Erice, 1976, CERN 77-13, p. 139
J.L. Laclare, Coasting beam instabilities, 1992 CAS Jyväskylä, CERN 94-01, pp. 349
J.L. Laclare, Bunched beam coherent instabilities, 1985 CAS Oxford, CERN 87-03, pp. 264
J. Gareyte, Observation and correction of instabilities in circular accelerators, CERN SL/91-09 (AP), Joint US-CERN Accelerator School, Hilton Head Island, USA, 1990
F. Pedersen, Multi-bunch instabilities, CERN PS 93-36 (RF), Joint US-CERN Accelerator School, Benaldamena, Spain 1992
A.W. Chao, Physics of collective beam instabilities in high energy accelerators, John Wiley&Sons, New York, 1993

Longitudinal Beam Instabilities – Basic Mechanism



Wall current I_w due to circulating bunch
Vacuum pipe not smooth, I_w sees an **IMPEDANCE** (resistive, capacitive, inductive)

Impedance $Z = Z_r + iZ_i$

Induced voltage $V \sim I_w Z = -I_B Z$

V may act back on the beam → INSTABILITIES INTENSITY DEPENDENT

General Scheme to investigate instabilities

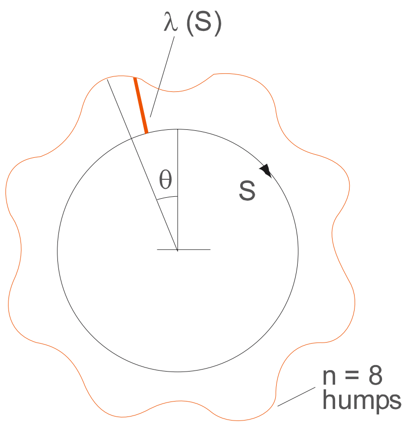
Step 1: Start with a nominal particle distribution (i.e. longitudinal position, density,…)

Step 2: Compute fields and induced wall currents with a small perturbation of this nominal distribution, and determine forces acting back on the beam

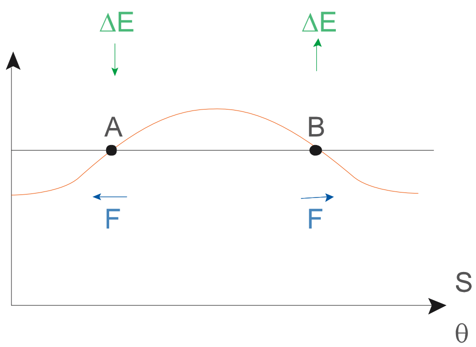
Step 3: Calculate change of distribution due to these forces:

If Initial Small Perturbation → INCREASED? INSTABILITY
→ DECREASED? STABILITY

“Negative Mass” Instability - Qualitative



Line density modulation



Zooming in one modulation

Un-bunched beam in a proton/ion ring
Line density $\lambda(s)$ [particles/m]
is modulated around the synchrotron

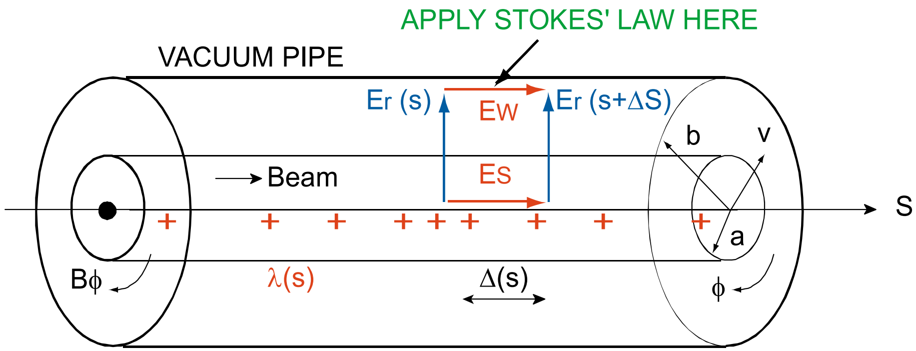
WILL THE HUMPS INCREASE
OR ERODE?

The self-force F (proportional to $-\partial\lambda/\partial s$) → Increases energy of particles in B
→ Decreases energy of particles in A

$\gamma < \gamma_t$: IF $\Delta E\uparrow$ $\omega_0\uparrow$ STABLE	A and B move away from the hump eroding the mountain
$\gamma > \gamma_t$: IF $\Delta E\uparrow$ $\omega_0\downarrow$ UNSTABLE	A and B move towards the hump, enhancing the mountain

It all depends on γ_t !

Negative Mass Instability: Fields Created by Beam



For small perturbations of $\lambda(s)$

$$E_r = \frac{e\lambda}{2\pi\varepsilon_0}\frac{1}{r} \qquad B\phi = \frac{\mu_0 e\lambda\beta c}{2\pi}\frac{1}{r} \qquad r \geq a$$

$$E_r = \frac{e\lambda}{2\pi\varepsilon_0}\frac{r}{a^2} \qquad B\phi = \frac{\mu_0 e\lambda\beta c}{2\pi}\frac{r}{a^2} \qquad r \leq a$$

STOKES'LAW: $$\underset{\text{LINE}}{\oint}\vec{E}d\vec{\ell} = -\frac{\partial}{\partial t}\underset{\text{SURFACE}}{\int}\vec{B}d\vec{\sigma} = -\frac{\partial}{\partial t}\Delta s\int_0^b B\phi\, dr$$

With $\frac{\partial\lambda}{\partial t} = -\frac{\partial\lambda}{\partial s}\frac{ds}{dt} = -\beta c\frac{\partial\lambda}{\partial s}$ and $g_0 = 1 + 2\ln(b/a)$, one gets $E_s = -\frac{eg_0}{4\pi\varepsilon_0}\frac{1}{\gamma^2}\frac{\partial\lambda}{\partial s} + E_W$

$E_W = 0$: perfectly conducting smooth wall

$$E_s = -\frac{eg_0}{4\pi\varepsilon_0}\frac{1}{\gamma^2}\frac{\partial\lambda}{\partial s}$$

Longitudinal "space charge" field

$E_W \neq 0$: Inductive wall

Inductance per m

$$E_w = -\frac{L}{2\pi R}\frac{dI_w}{dt} = -\frac{L}{2\pi R}e\beta c\frac{\partial\lambda}{\partial t} = \frac{L}{2\pi R}e\beta^2 c^2\frac{\partial\lambda}{\partial t}$$

Voltage per turn $U_s = e\beta c R\omega_0 L\frac{\partial\lambda}{\partial s}$

Negative Mass Instability: Field Acting Back on Beam

$\lambda(s)$ has n humps and rotates with Ω near $n\omega_0$

$\lambda = \lambda_0 + \lambda_1 e^{i(n\Theta-\Omega t)}$, $I + I_0 + I_1 e^{i(n\Theta-\Omega t)}$ instantaneous density and current

$$U_s = -I_1 e^{i(n\Theta-\Omega t)} \; Z(\Omega)$$

voltage per turn — (small) AC component — longitudinal impedance

U_s perturbs the motion of the pattern and leads to a complex frequency shift $\Delta\Omega = \Delta\Omega_r + i\Delta\Omega_i$

$\Omega = n\omega_0 + \Delta\Omega$ slightly perturbed frequency

A SHORTCUT TO CALCULATE $\Delta\Omega$

$$\underbrace{\left[\frac{E_0\beta^2\gamma}{2\pi\eta h f_0^2 e}\right]}_{\text{“m”}}\ddot{\varphi} + V_0\varphi = 0$$

equation of small-amplitude synchrotron oscillations in a stationary bucket

$$\ddot{\varphi} + \underbrace{\left[\frac{e\eta h V_0\omega_0^2}{2\pi E_0\beta^2\gamma}\right]}_{\omega_s^2}\varphi = 0$$

ω_s … synchrotron frequency

V_0 …voltage per turn
f_0 ….revolution frequency
η….$1/\gamma^2 - 1/\gamma_t^2$
E_0…particle rest energy

Negative Mass Instability: Shortcut to Compute $\Delta\Omega$

- Replace ω_s by $\Delta\Omega$
- Replace hV_0 by beam-induced voltage $\mathbf{i\,n\,Z\,I_0}$ with $\mathbf{Z = Z_r + i\,Z_i}$ complex impedance

$$\boxed{(\Delta\Omega)^2 = (\Omega - n\omega_0)^2 = -i\,\frac{e\eta\omega_0^2 n I_0}{2\pi\beta^2 E_0\gamma}\,(Z_r + iZ_i)}$$

Complex Frequency shift required to sustain **self-consistent** modulation

$$I(t,\Theta) = I_0 + I_1 \underbrace{e^{\Delta\Omega_i t}}_{\text{growth or damping of modulation pattern}} \, e^{i(n\Theta - (n\omega_0 + \Delta\Omega_r)t)}$$

$\Downarrow$ real frequency shift

Instantaneous current with $\Delta\Omega = \Delta\Omega_r + \mathbf{i}\Delta\Omega_i$

- $\mathbf{Z_r = 0}$

From $U_s = -I_1 e^{i(n\Theta - \Omega t)} Z$ and $Z_0 = 1/\varepsilon_0 c = 377\ \Omega$

$$Z_i = \frac{n g_0 Z_0}{2\beta\gamma^2}$$ **"space charge" impedance**

$$Z_i = -n\omega_0 L$$ **inductive impedance**

Z_i	$\gamma < \gamma_t$ ($\eta > 0$)	$\gamma > \gamma_t$ ($\eta < 0$) ($m < 0$)
> 0 (capacitive)	$\Delta\Omega_i = 0$ STABLE	$\Delta\Omega_i \neq 0$ UNSTABLE
< 0 (inductive)	$\Delta\Omega_i \neq 0$ UNSTABLE	$\Delta\Omega_i = 0$ STABLE

- $\mathbf{Z_r \neq 0}$ (more realistic)

$\Delta\Omega_i \neq 0$

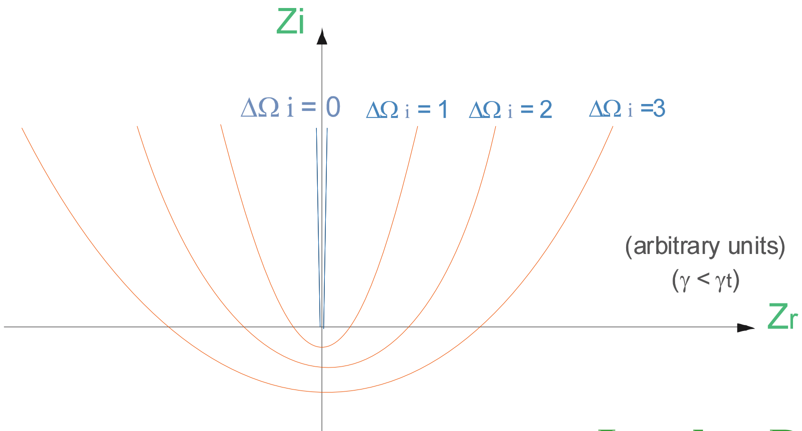
always one unstable solution

Stability Diagram

- Relates (complex) growth rate $\Delta\Omega$ to (complex) impedance Z

$$(\Delta\Omega)^2 = -i\,\xi\,(Z_r + iZ_i) = \xi\,(Z_i - iZ_r) = (\Delta\Omega_r + i\Delta\Omega_i)^2$$

- Plot contours $\Delta\Omega_i$ **= const** (= equal growth rate) into Z_r, Z_i plane. Equating real and imaginary parts yields **parabolae** for $\Delta\Omega_i$ **= const** $\Rightarrow Zr = 2\Delta\Omega_i\sqrt{Z_i/\xi + \Delta\Omega_i^2/\xi^2}$

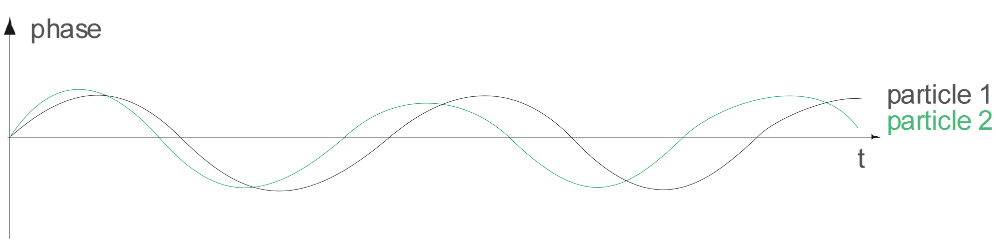


Stability Diagram

For any $Z_r \neq 0$ the beam is subject to the negative mass instability and is unstable

Is there a way out?

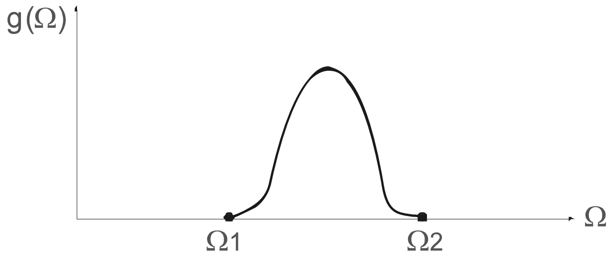
Landau Damping



Decoherence of two particles

In real machines the beam features a **frequency spread**, so individual particles **move with different speeds** around the ring → the **coherent motion becomes confused** and may **collapse faster** than the **rise time of the instability**

Landau Damping - Basic Idea



N particles (oscillators), each **resonating** at a frequency between Ω_1 and Ω_2 with a density $g(\Omega)$

$$\int_{\Omega_1}^{\Omega_2} g(\Omega)\, d\Omega = 1$$

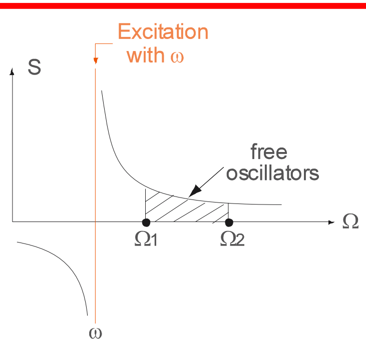
normalisation

$$X = \frac{1}{\Omega^2 - \omega^2} e^{i\omega t} = \frac{1}{(\Omega - \omega)\underbrace{(\Omega + \omega)}_{\sim 2\Omega_0}} e^{i\omega t}$$

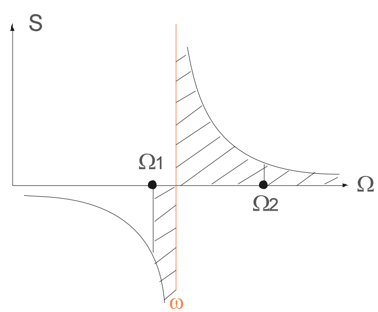
Single-particle response (**incoherent**) to an external excitation $\mathbf{e^{i\omega t}}$

$$S = \frac{N}{2\Omega_0} \int_{\Omega_1}^{\Omega_2} \frac{i\dfrac{dg(\Omega)}{d\Omega}}{\Omega - \omega}\, d\Omega \cdot e^{i\omega t}$$

Overall coherent response obtained by integrating the single-particle responses of the n oscillators



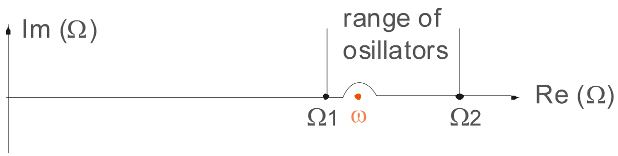
Coherent response of oscillators to excitation **outside** their frequency range



Coherent response of oscillators to excitation **inside** their frequency range

The integral S has a **pole at** $\mathbf{\Omega = \omega}$

Landau Damping and Stability Diagram



The trick to treat the **pole** in the integral: integrate **"around" it in the complex plane:**

$$S = i\frac{N}{2\Omega_0}\left[\int_{\Omega_1}^{\Omega_2}\frac{\frac{dg(\Omega)}{d\Omega}}{\Omega-\omega}\,d\Omega - i\Pi\right]e^{i\omega t} = \frac{N}{2\Omega_0}\left[-\Pi + i\underbrace{\int_{PV}\frac{\frac{dg(\Omega)}{d\Omega}}{\Omega-\omega}\,d\Omega}\right]e^{i\omega t}$$

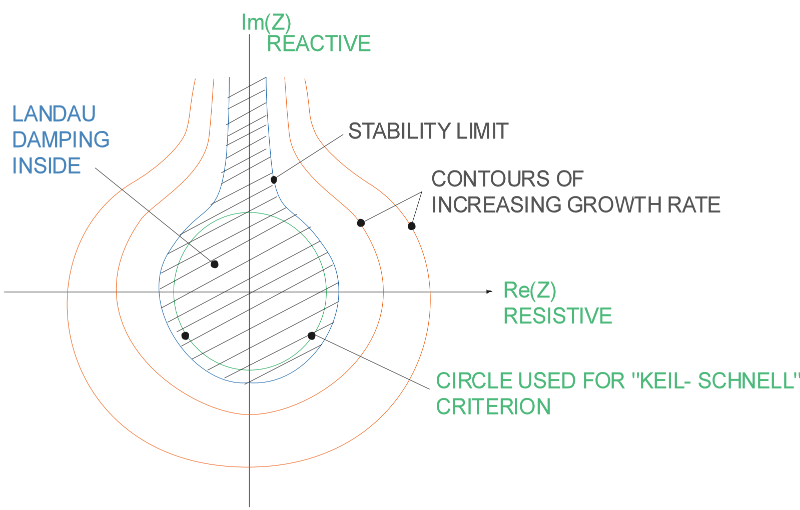
Principal value

"Residuum"

"Resistive" term absorbs energy (in phase with excitation)

Reactive term does not absorb energy (out of phase)

Stability Diagram with Landau Damping



The form of the "bottle" depends on **g(Ω);** for most distributions, a circle can be inscribed, giving a **handy approximation** for the **stability limit of un-bunched beams**

$$\left|\frac{Z}{n}\right| \le F\frac{m_0 c^2\beta^2\gamma|\eta|}{e}\frac{(\Delta p/p)^2}{I_0} \qquad \text{KEIL-SCHNELL CRITERION}$$

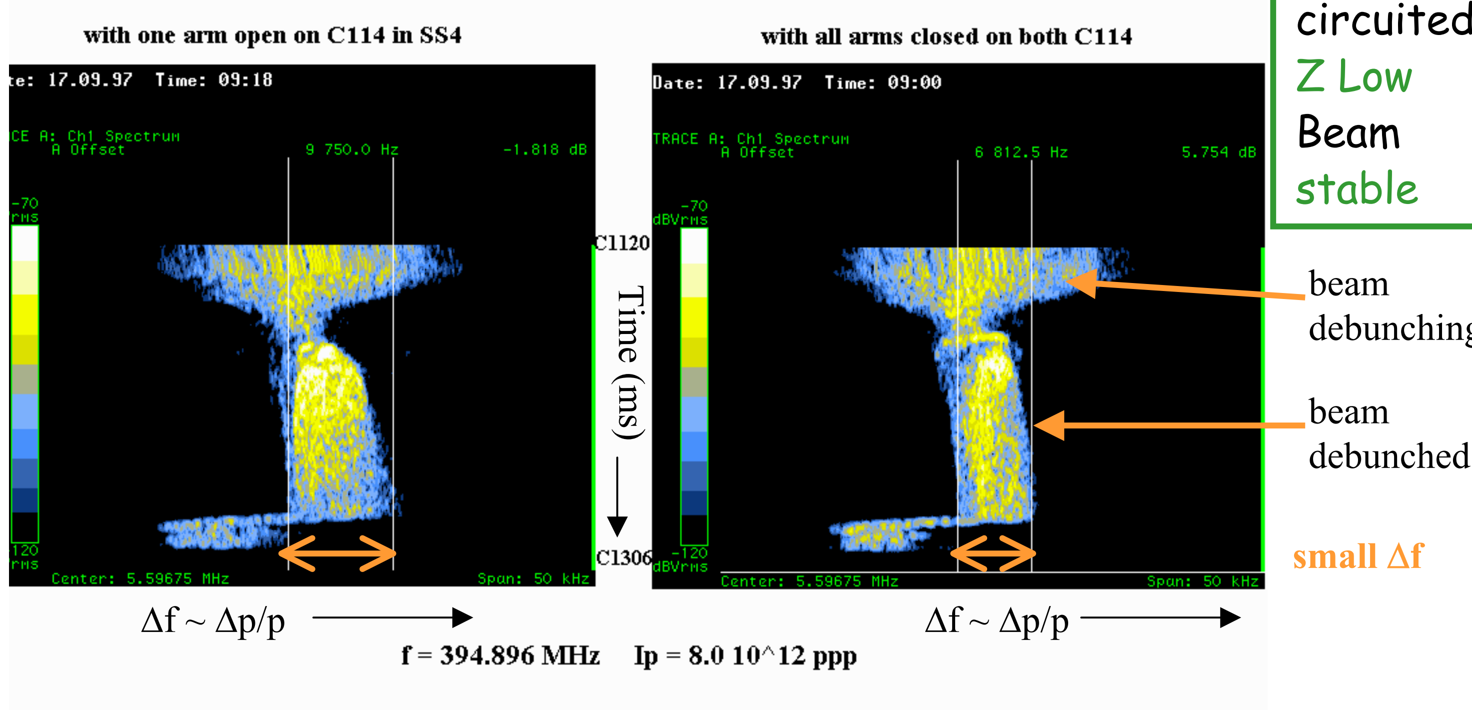
Landau Damping only works if ***coherent frequency*** lies ***inside the frequency spread*** of the oscillators

Coasting Beam Longitudinal Instability: Example

Open
Z High
Beam
Unstable

NARROW-BAND CAVITY

Short-circuited
Z Low
Beam
stable



large Δf

small Δf

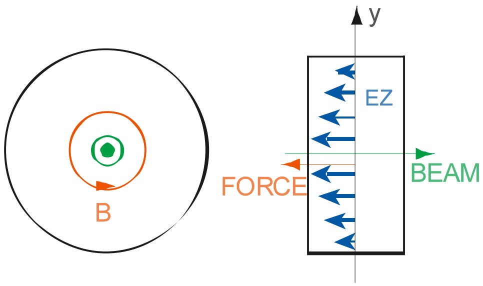
Increase in Δp/p due to **coasting beam longitudinal instability** in the CERN PS during de-bunching of protons.

Driving impedance: narrow-band cavity around 114 MHz.

Horizontal axis: **Δf** proportional to **Δp/p** measured via **"Schottky" scan** on a spectrum analyser

Vertical axis: time, circa 200 ms, moving downwards.

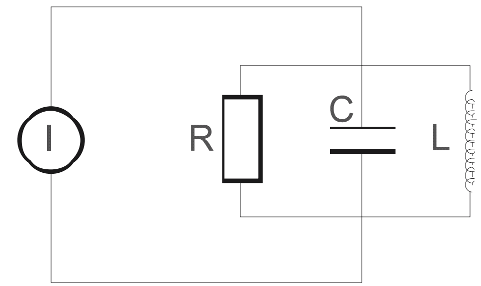
Impedance of a Resonator



Resonator equivalent to RLC circuit

$$\omega_r = \frac{1}{\sqrt{LC}} \quad \text{resonance frequency}$$

$$Q = R\sqrt{\frac{C}{L}} = \frac{R}{\omega_r L} \quad \text{quality factor}$$



$$\ddot{V} + \frac{\omega_r}{Q}\dot{V} + \omega_r^2 V = \omega_r \frac{R}{Q}\dot{I}$$

Differential equation of RLC circuit (I represents the beam)

$$V(t) = V_0 e^{-\alpha t}\cos\left[\omega_r\sqrt{1-1/4Q^2}\, t + \varphi\right]$$

Solution: **damped oscillation** with $\alpha = 1/\tau = \omega_r/2Q$

HOW TO COMPUTE IMPEDANCE?

- **Excite** RLC circuit with $\mathbf{I = I_0 e^{i\omega t}}$, any ω ($-\infty < \omega < \infty$)
- Look for solutions of the form $\mathbf{V(t) = V_0 e^{i\omega t}}$
- Insert these expressions into the differential equation above:

$$-\omega^2 V_0 e^{i\omega t} + i\frac{\omega\omega_r}{Q} V_0 e^{i\omega t} + \omega_r^2 V_0 e^{i\omega t} = i\frac{\omega_r \omega R}{Q} I_0 e^{i\omega t}$$

$$\Rightarrow \boxed{Z(\omega) = \frac{V_0}{I_0} = R\frac{1}{1 + iQ\dfrac{\omega^2 - \omega_r^2}{\omega\omega_r}}}$$

Impedance of an RLC circuit – also used for longitudinal resonators

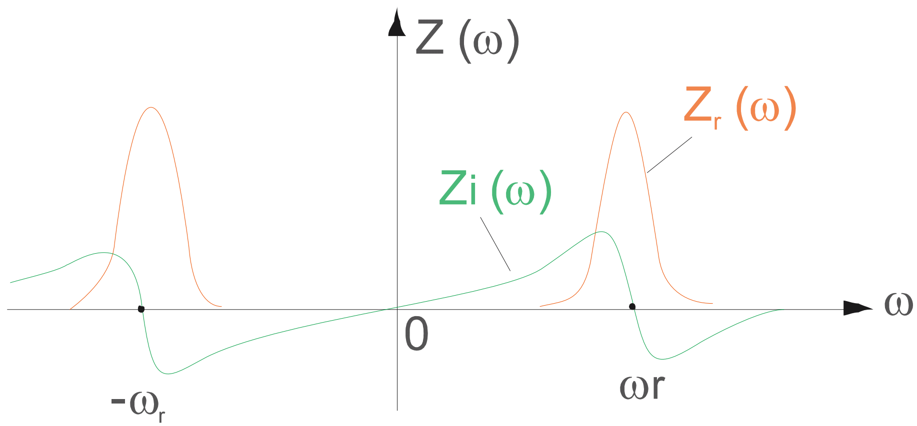
$\mathbf{V_0}$ is complex since in general not in phase with exciting current $\mathbf{I_0}$

Impedance of a Resonator

$$Z(\omega) = Z_r(\omega) + iZ_i(\omega) = R\frac{1 - iQ\dfrac{\omega^2 - \omega_r^2}{\omega\omega_r}}{1 + \left[Q\dfrac{\omega^2 - \omega_r^2}{\omega\omega_r}\right]^2}$$

$Z_r(\omega) = Z_r(-\omega)$ (even)
$Z_i(\omega) = - Z_i(-\omega)$ (odd)

Longitudinal Impedance



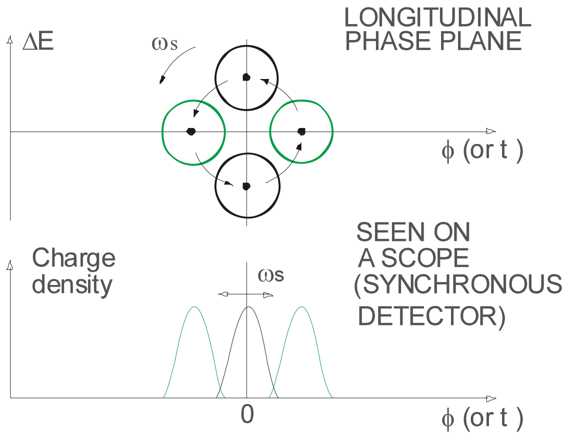
$$Z(\omega) \approx R_s\frac{1 - i2Q\dfrac{\Delta\omega}{\omega_r}}{1 + \left(2Q\dfrac{\Delta\omega}{\omega_r}\right)^2}$$

Impedance of a narrow-band (high-Q) Cavity

with $\Delta\omega = n\omega_0 - \omega_r$, R_S = shunt impedance

The **excitation signal** in such a cavity **decays slowly**: the field induced by the beam is **memorized for many turns**

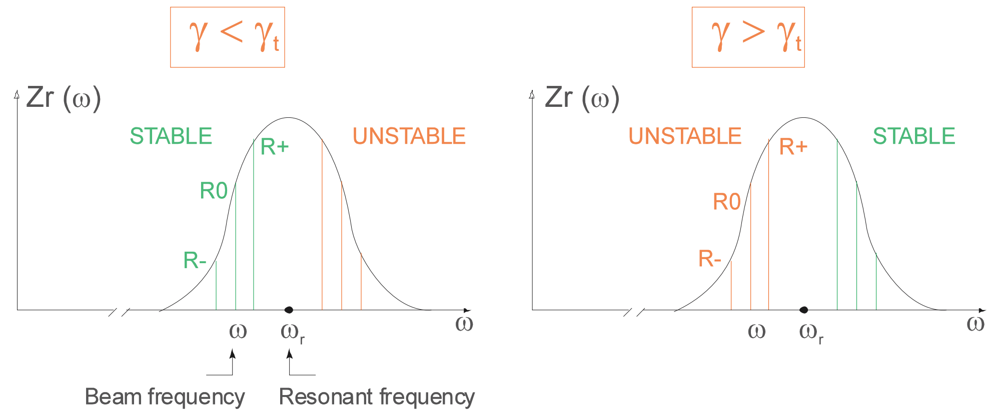
Single Bunch + Narrow-Band Cavity: "Robinson" Instability



"Dipole" mode or
"Rigid Bunch" mode

The single bunch rotates in longitudinal phase plane with ω_s:
synchronous phase ϕ and **energy** also vary with ω_s

Bunch sees resonator at $\omega_r \cong \omega_0$



$\omega < \omega_r$

Whenever ΔE>0:
- ω **increases** (below transition)
- sees **larger** real impedance R_+
- **more** energy taken from beam

➢ **STABILIZATION**

Whenever ΔE>0:
- ω **decreases** (above transition)
- sees **smaller** real impedance R_+
- **less** energy taken from beam

➢ **INSTABILITY**

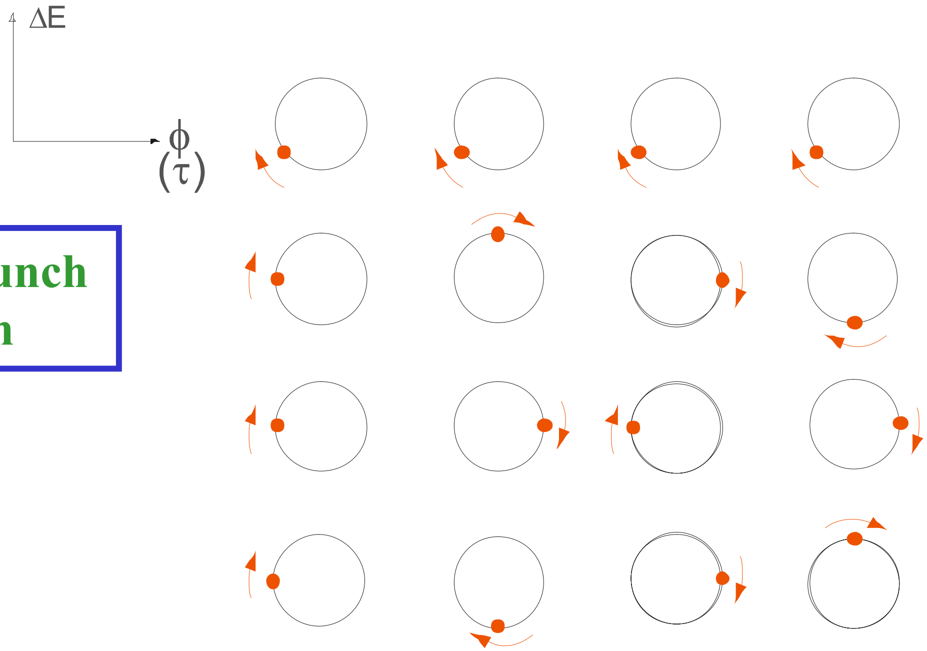
$\omega > \omega_r$

UNSTABLE **STABLE**

Longitudinal Instabilities with Many Bunches

- Fields induced in resonator remain long enough to influence following bunches
- Assume M = 4 bunches performing synchrotron oscillations

Coupled-Bunch Modes n



All bunches in phase
mode $n = 0$

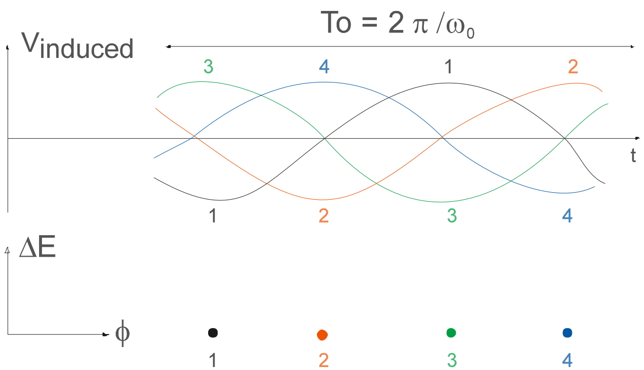
Phase shift between consecutive bunches
$\frac{\pi}{2}$ $n = 1$

π $n = 2$

$\frac{3\pi}{2}$ $n = 3$

- **Four possible phase shifts** between **four bunches**
- **M bunches**, phase shift of Coupled-Bunch mode **n**: $2\pi\frac{n}{M},\ 0 \le n \le M-1 \Rightarrow$ M modes

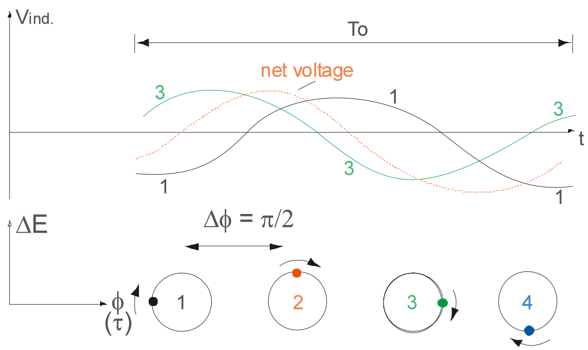
Coupled-Bunch Mode Stability: Qualitative



M = 4 bunches, resonator tuned at ω_0

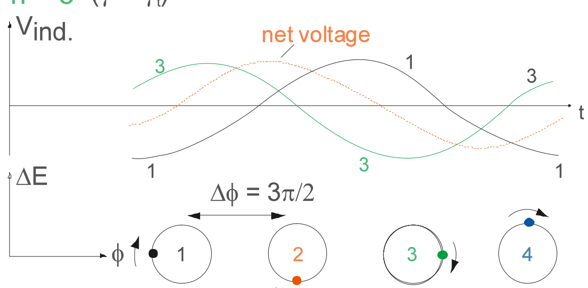
Four stationary buckets (no synchrotron oscillations)
Voltages induced by bunches 2 and 4 **cancel**
Voltages induced by bunches 1 and 3 **cancel**
➔ **NO EFFECT**

n = 1 ($\gamma > \gamma_t$, bunches move clockwise in phase plane)



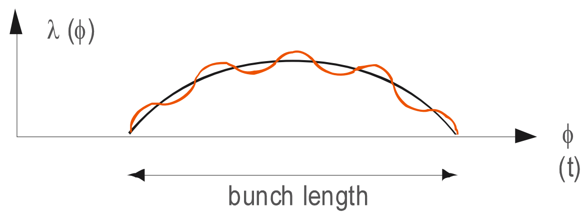
Voltages induced by bunches 2 and 4 cancel, but
bunches 1 and 3 **induce a net voltage**
Bunch 2 **accelerated**, bunch 4 **decelerated**
Synchrotron oscillation amplitude increases
➔ **UNSTABLE**

n = 3 ($\gamma > \gamma_t$)



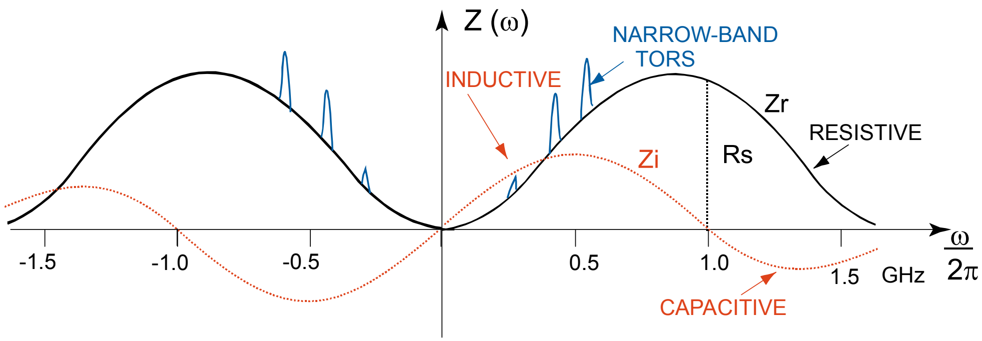
Voltages induced by bunches 2 and 4 cancel, but
bunches 1 and 3 **induce a net voltage**
Bunch 2 **accelerated**, 4 **decelerated**
Same as n=1
Synchrotron oscillation amplitude decreases
➔ **STABLE**

Longitudinal Microwave Instability



- Signature: bunch with **high-frequency** density modulation
- wave length << bunch length (frequencies **100 MHz…1 GHz)**
- **Fast growth rates** – even leptons concerned
- Generated by **BROAD-BAND IMPEDANCE**

All elements of a synchrotron are "lumped" into one low-Q resonator yielding the impedance (p. 12)



$$Z(\omega) = R_s \frac{1 - iQ\dfrac{\omega^2 - \omega_r^2}{\omega\omega_r}}{1 + \left(Q\dfrac{\omega^2 - \omega_r^2}{\omega\omega_r}\right)^2} \qquad Q \approx 1 \qquad \omega_r \approx 1\ \text{GHz}$$

n

For small ω, $Z(\omega) \approx i\frac{R_s\omega}{Q\omega_r} = i\frac{R_s}{Q}\frac{\omega}{\omega_0}\frac{\omega_0}{\omega_r} = i\frac{R_s}{Q}\frac{\omega_0 n}{\omega_r}$ and with (p. 10) $Q = \frac{R_s}{\omega_r L}$

$$\boxed{\left|\frac{Z}{n}\right|_0 = L\omega_0}$$

"Impedance" of a synchrotron in Ω

- This **inductive impedance** is caused mainly by **discontinuities** in the beam pipe
- If value high, the machine is **prone to instabilities**
- Typically **20…50 Ω** for old machines
- **< 1 Ω** for modern synchrotrons

Microwave Instability – Stability Limit

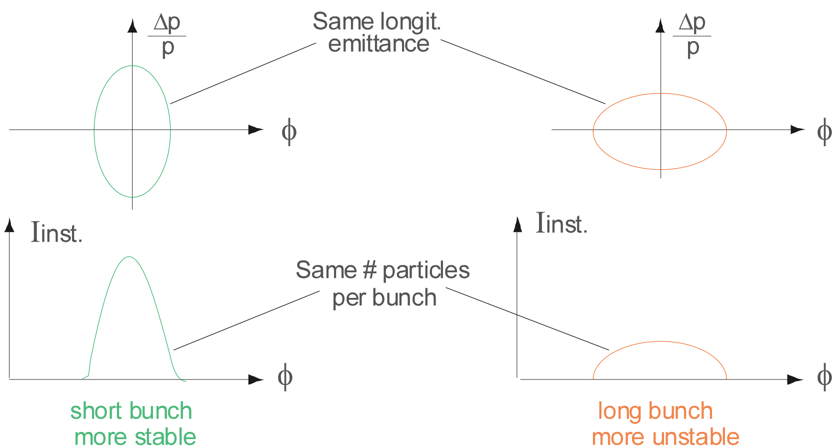
- The **Broad-Band Impedance** with Q=1 has little memory
  - ➢ No coupling between consecutive bunches
  - ➢ Microwave instability is a single bunch effect
- leading to **longitudinal bunch blow-up**
- In lepton machines also called "**turbulent bunch lenthening**"

STABILITY LIMIT: Apply **Keil-Schnell** **criterion for unbunched beams** to **instantaneous current and momentum spread**

$$\boxed{\left|\frac{Z}{n}\right| \leq F\frac{m_0 c^2 \beta^2 \gamma |\eta|}{e}\left[\frac{(\Delta p/p)^2}{I}\right]_{\mathrm{instant.}}}$$

KEIL-SCHNELL-BOUSSARD CRITERION

protons: F ~ 0.65 leptons: F ~ 8

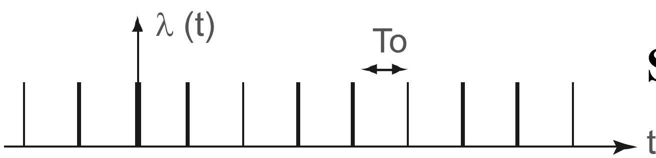


For a equal **bunch population** and **longitudinal emittance**, **short** bunches are **more stable** than **long** ones

Longitudinal Spectrum – Single Particle and Bunch

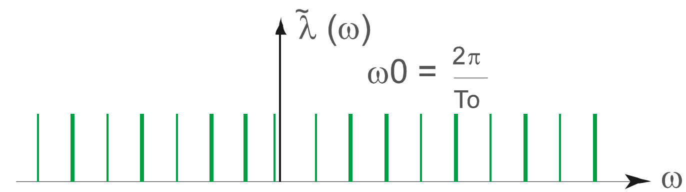
Seen by a current monitor

Spectrum



SINGLE PARTICLE

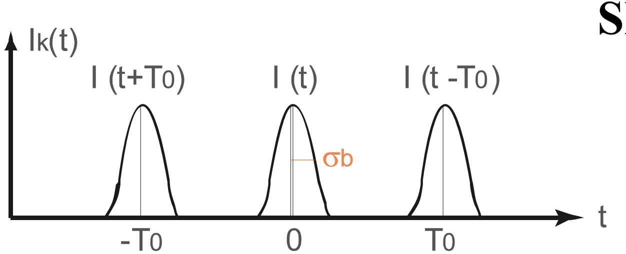
Fourier Series



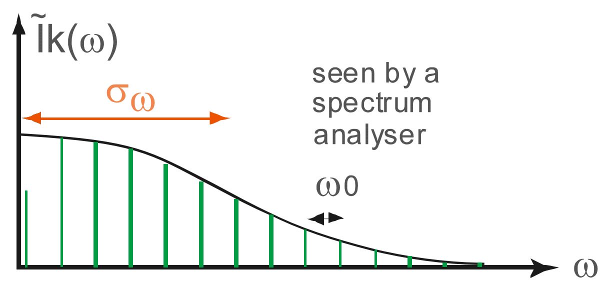
$$\lambda(t) = \frac{e}{\beta c} \sum_{\ell=-\infty}^{+\infty} \delta(t - \ell T_0)$$

$$\lambda(t) = \frac{e\omega_0}{2\pi} \sum_{n=-\infty}^{+\infty} e^{in\omega_0 t}$$

SINGLE BUNCH



Fourier Series



$$I_k(t) = \sum_{k=-\infty}^{+\infty} I(t - kt_0)$$

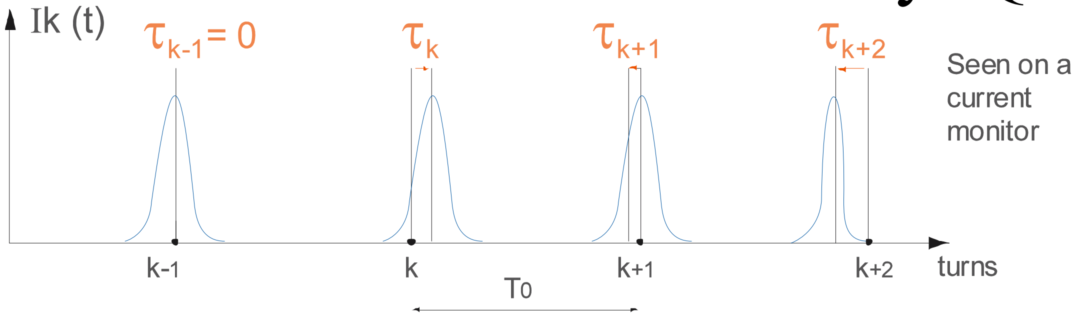
$$I_k(t) = I_0 + \sum_{n=1}^{\infty} I_n \cos(n\omega_0 t)$$

with

$$I_n = \frac{2}{T_0} \int_{-T_0/2}^{T_0/2} I_k(t) \cos(n\omega_0 t)\,dt$$

$\sigma_\omega \sim 2\pi/\sigma_b$: the **shorter** the bunch, the **wider** the spectrum

Robinson Instability: Quantitative



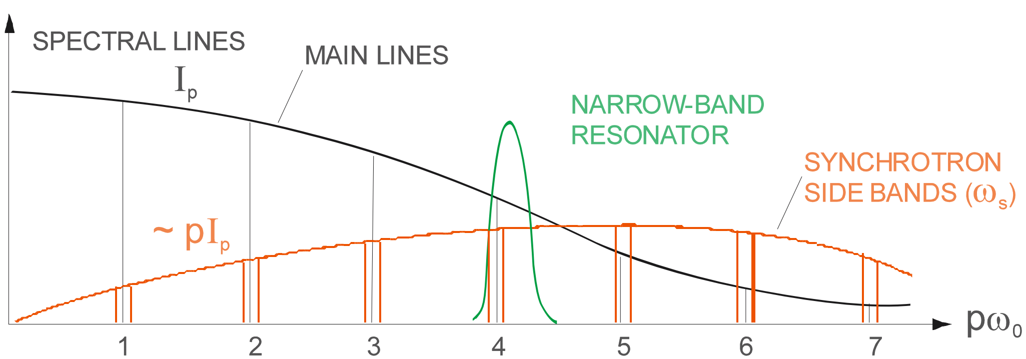
A single bunch performing synchrotron oscillations around a synchronous phase or time

$$\tau_k = \hat{\tau} \cos \omega_s t \quad \text{modulation of bunch passage time}$$

$$I_k(t) = \sum_{k=-\infty}^{+\infty} I(t - kT_0 - \tau_k) = I_0 + \sum_{p=1}^{\infty} I_p \cos(p\omega_0 (t+\tau))$$

Assume $p\omega_0 \hat{\tau} << 1$ (small synchrotron oscillations)

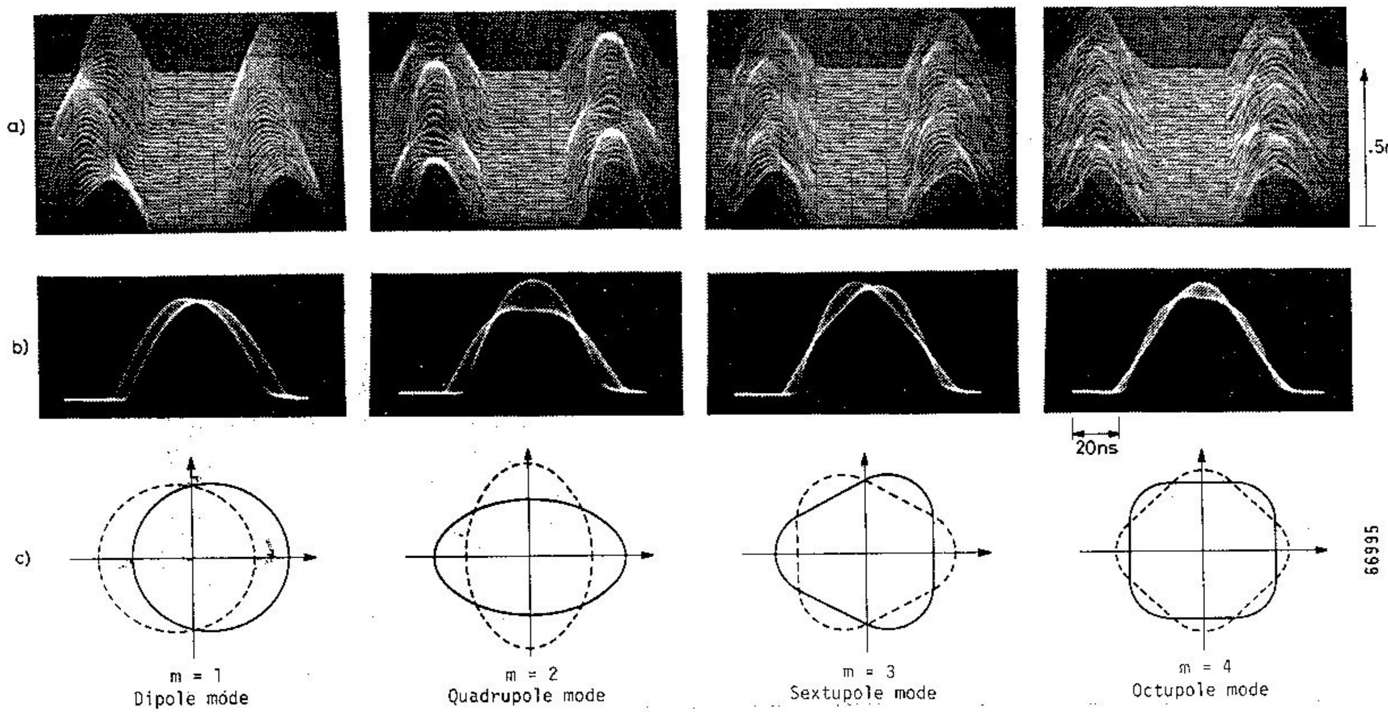
$$\boxed{I_k(t) \approx I_0 + \underbrace{\sum_{p=1}^{\infty} I_p \cos(p\omega_0 t)}_{\text{Main lines}} - \frac{\omega_0 \hat{\tau}}{2} \sum_{p=1}^{\infty} I_p p[\underbrace{\sin((p\omega_0 + \omega_s)t)}_{\text{upper side-bands}} + \underbrace{\sin((p\omega_0 - \omega_s)t)}_{\text{lower side-bands}}]}$$



Spectrum of a single bunch performing small-amplitude synchrotron oscillations

Coupled Bunch Modes, Dipole & Higher Order

Dipole mode | Quadrupole | Sextupole | Octupole



Mountain-range display during one synchrotron oscillation period (0.5 ms)

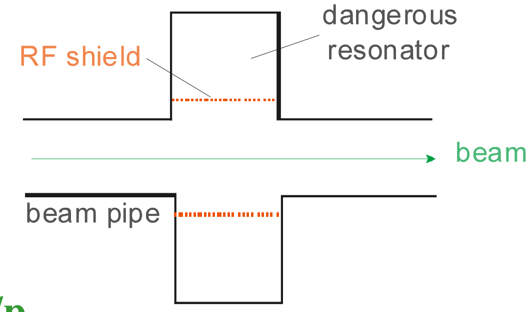
Signals superimposed

Mode pattern in longitudinal phase plane

Dipole (m=1) and higher-order (m=2,3,4) modes in a synchrotron with 5 bunches
Two adjacent bunches shown. *Note phase shifts between adjacent bunches*

Longitudinal Instabilities - Cures

- Robinson Instability, generated by main RF cavities:
  Tune resonance frequency ω_r such that bunch frequency $\boxed{\begin{array}{l} h\omega_0 < \omega_r \text{ for } \gamma < \gamma_t \\ h\omega_0 > \omega_r \text{ for } \gamma > \gamma_t \end{array}}$
- Cavities "Parasitic" Modes are damped by "Higher Order Mode Dampers" (HOM): the unwanted mode is picked up by an antenna and sent to a damping resistor.
- Unwanted Resonators in beam pipe: RF shield protects the beam mimicking a smooth beam pipe
- Microwave Instabilities: Reduce Broad-Band Impedance by smooth changes in beam pipe cross section and shielding cavity-like objects. Large $\Delta p/p$ helpful but costly in RF voltage.

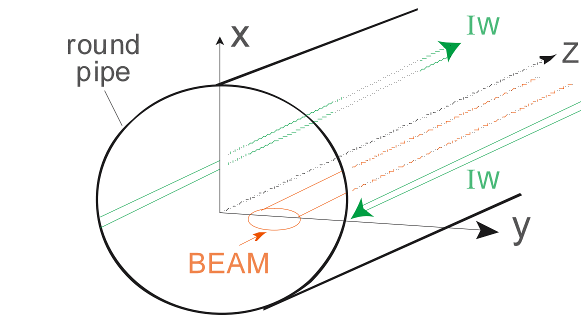


- Coupled-Bunch Mode Instabilities: Run synchrotrons with 1 ore 2 bunches (bunch-to-bunch phase shift of 0 or π are always longitudinally stable) (limited to small synchrotrons)

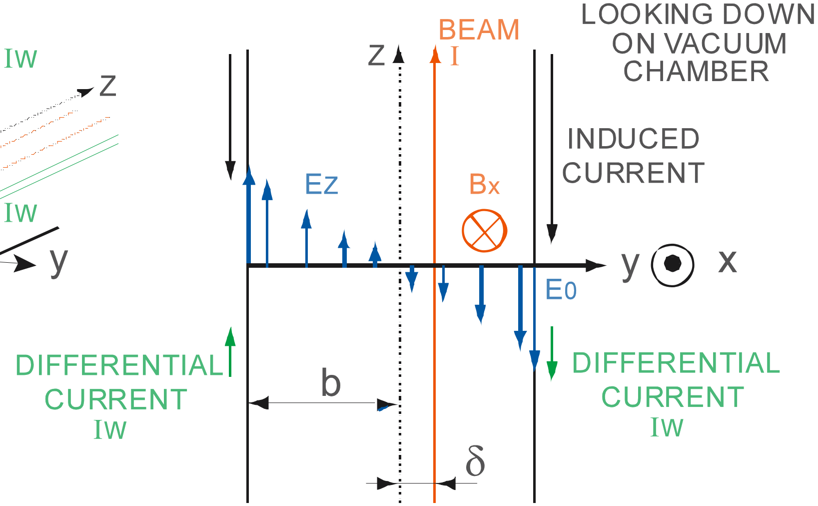
Longitudinal Instabilities – Feedback Systems

- Principle
  The phase (or amplitude) deviation is measured in a synchronous detector and corrected in an accelerating gap which must cover the bandwidth
- In-phase (n=0) dipole (m=1) mode: normally tackled by the phase loop which locks the beam phase to the cavity RF voltage phase
- In-phase (n=0) quadrupole (m=2) mode: These bunch-shape oscillations are treated by feeding back the observed amplitude oscillation to the RF cavity
- Coupled-Bunch instabilities (dipole modes, m=1) are controlled by a feedback system which tackles (i) each bunch (out of M bunches) or (ii) each mode n (n = 0, 1, …, M-1) individually. In both approaches the required bandwidth is ~ $\frac{1}{2} M\omega_0$

Transverse Beam Instabilities – Fields and Forces



δ..displacement of beam in y oscillating with $e^{i\omega t}$

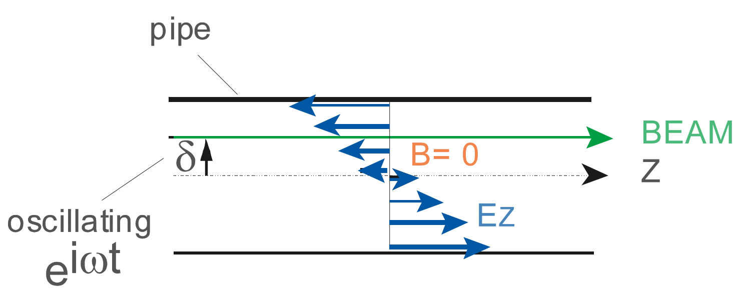


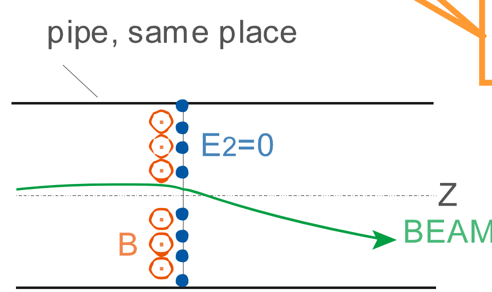
To sustain the **differential wall current** $\mathbf{I_w}$ a longitudinal electric field $\mathbf{E_z}$ **varying across the aperture** is required

➔ $\mathbf{E_z = E_0(y/b)\, e^{i\omega t}}$ in the median plane x = 0

From $\dfrac{\partial \vec{B}}{\partial t} = -\vec{\nabla} \times \vec{E}$ one gets $\dfrac{\partial B_x}{\partial t} = -\dfrac{\partial E_z}{\partial y} = -\dfrac{E_0}{b} e^{i\omega t}, \quad B_x = \dfrac{i}{\omega}\dfrac{E_0}{b} e^{i\omega t}$

Phase-shifted with respect to exciting beam oscillation





t=0, **excitation** by displaced beam

t = (1/4) (2π/ω), **deflection**

Transverse Coupling Impedance

$$\boxed{Z_T(\omega) = i\,\frac{\oint[\vec{E} + \vec{v}\times\vec{B}]_t\,ds}{\beta I \delta}} = \frac{\text{Deflecting field (integrated around the ring)}}{\text{dipole moment of exciting current}} \quad [\Omega/\text{m}]$$

because of phase shift between dipole moment Iδ and deflecting field

Relation between Z_T and Z_L (longitudinal impedance called Z so far), for a **resistive round pipe:**

The wall current **I_w** generates a voltage **V** around the ring:

$$V = 2\pi R E_0 \cong 4 I_w Z_L \Rightarrow E_0 \cong 4\frac{I_w Z_L}{2\pi R}$$

$$I_w = -\frac{1}{2}\frac{\delta}{b} I \quad \left(i.e.\, if\ \delta = b\ \ I_w = -\frac{1}{2} I \right)$$

$$B_x = \frac{i}{\omega}\frac{E_0}{b} e^{i\omega t} = -i\frac{2\delta}{\omega b^2}\frac{Z_L I}{2\pi R} e^{i\omega t}$$

Inserting B_x and putting E = 0 yields

$$\boxed{Z_T(\omega) \cong \frac{2c}{b^2}\frac{Z_L}{\omega}}$$

Handy approximate relation between Z_T and Z_L

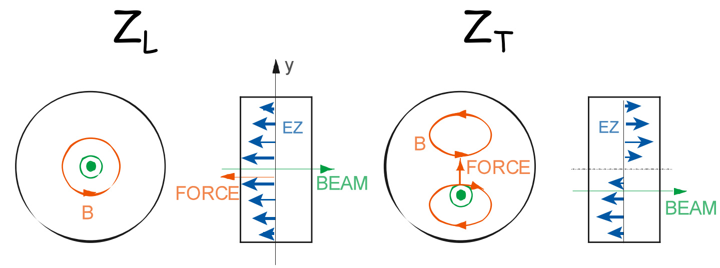
Transverse Impedance Z_T vs. Longitudinal Impedance Z_L

	Z_L	Z_T
Unit	Ω	Ω/m
Symmetry Real Part	$\text{Re}[Z_L(\omega)] = \text{Re}[Z_L(-\omega)]$ even	$\text{Re}[Z_T(\omega)] = -\text{Re}[Z_T(-\omega)]$ odd
Symmetry Imaginary part	$\text{Im}[Z_L(\omega)] = -\text{Im}[Z_L(-\omega)]$ odd	$\text{Im}[Z_T(\omega)] = \text{Im}[Z_T(-\omega)]$ even
Typical values for a synchrotron	$\sim\Omega$	$\sim \text{M}\Omega/\text{m}$

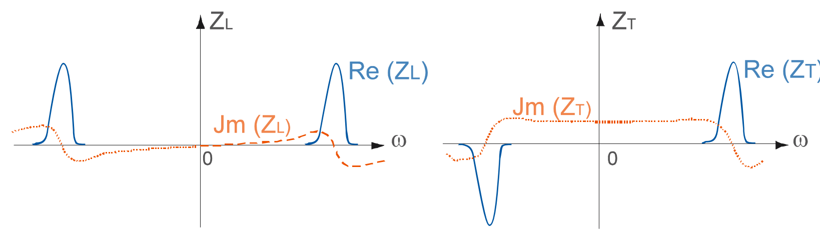
Why negative frequencies?
To make calculations simpler

Transverse and Longitudinal Impedances

Resonator-type object
Fields and Forces



Resonator-type object
Impedance



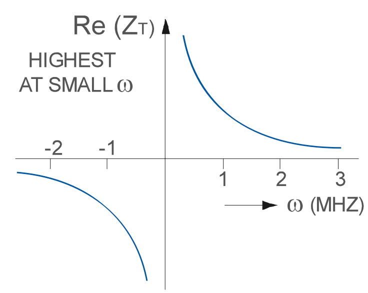
Resistive Wall

R….machine radius
ρ….vacuum chamber resistivity
δ….wall thickness

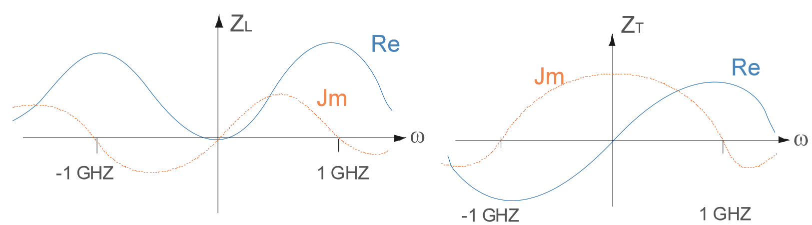
$$\mathrm{Re}(Z_T) = \frac{2cR}{\omega b^3}\frac{\rho}{\delta}\ (\text{low } \omega)$$

$$\boxed{\mathrm{Re}(Z_L) = \frac{R}{b}\frac{\rho}{\delta}}$$

independent of ω



Broad-Band (with Q=1)

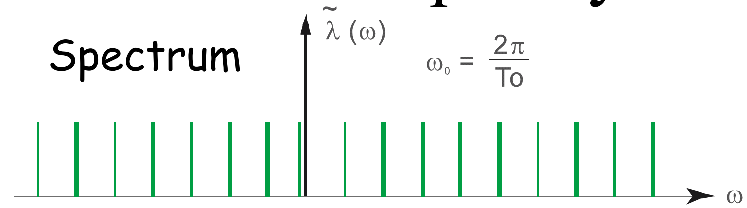


Transverse Beam Signals – Time and Frequency

Single particle on central orbit – longitudinal signal

$$\lambda(t) = \frac{e}{2\pi R} \sum_{n=-\infty}^{+\infty} e^{in\omega_0 t}$$

Spectrum



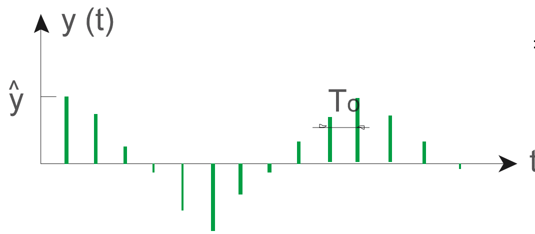
Single particle, oscillating transversally

$$y = \hat{y}\cos(\omega_\beta t + \varphi)$$

$$\omega_\beta = Q\omega_0 = (k + \boxed{q})\omega_0$$

fractional tune

Position monitor signal for q~0.1



Compute spectrum

$$d(t) = \hat{y}\cos(Q\omega_0 t + \phi)\cdot\frac{e}{2\pi R}\sum_{n=-\infty}^{+\infty} e^{in\omega_0 t}$$

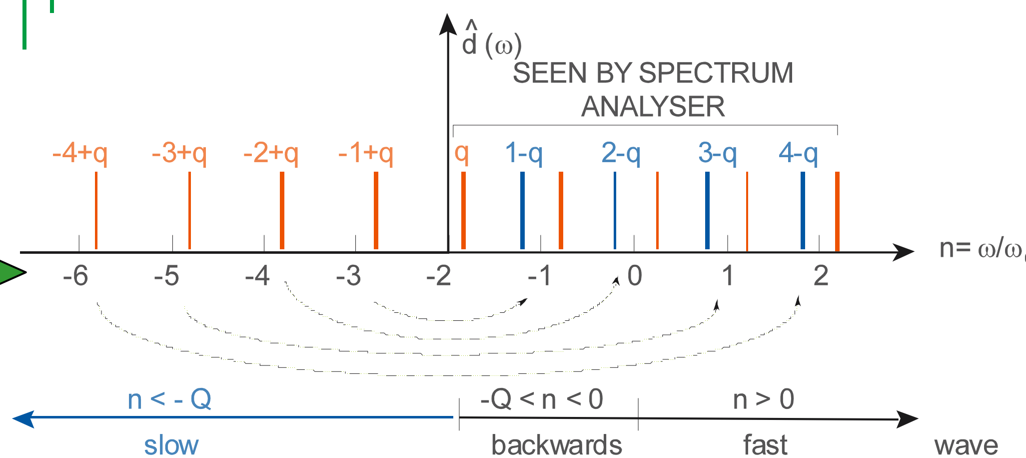
$$= \frac{1}{2}\frac{e\hat{y}}{2\pi R}\left[e^{i(Q\omega_0 t+\phi)} + e^{-i(Q\omega_0 t+\phi)}\right]\sum_{n=-\infty}^{+\infty} e^{in\omega_0 t}$$

$$d(t) = \frac{e\hat{y}}{2\pi R}\sum_{n=-\infty}^{+\infty}\cos[(n+Q)\omega_0 t + \phi]$$

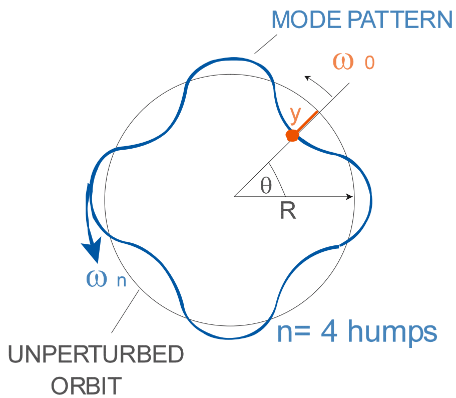
Spectrum $\hat{d}(\omega)$

- constant amplitude
- lines at $(n+Q)\omega_0$, n any integer

Example: Q = 2.25
(q = 0.25)



Transverse Instabilities – Unbunched Beam



MODE: particles are arranged around the synchrotron with a **strict correlation** between **transverse particle positions.**

The mode shown is **n=4**. If one takes a snapshot at t=0 one has

$y(t=0, \theta) = y_4 e^{-4i\theta}$

A single particle always rotates with revolution frequency ω_0 but **the pattern** rotates with $\omega_n \neq \omega_0$; how to compute ω_n?

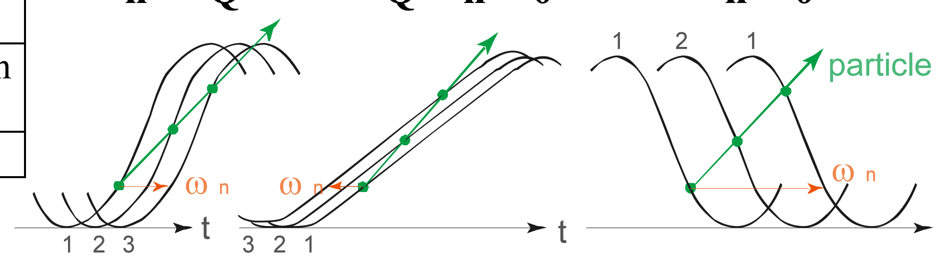
- A particle is at azimuth θ_0 at t=0. Its position evolves as $y_{\theta_0}(t) = y_n e^{i(Q\omega_0 t - n\theta_0)}$
- after time t its azimuth is $\theta = \theta_0 + \omega_0 t$, so $\theta_0 = \theta - \omega_0 t$ and $y(\theta,t) = y_n e^{i[(Q+n)\omega_0 t - n\theta]}$
- condition for $y(t,\theta)$ = const yields $(Q+n)\,\omega_0 t - n\theta = 0$ ➔ $\theta(t) = (1 + Q/n)\,\omega_0 t$ ➔

Rotation frequency of mode pattern

$$\boxed{\omega_n = \dot{\theta} = \left(1 + \frac{Q}{n}\right)\omega_0}$$

	$n < -Q$ $0 < \omega_n < \omega_0$	$-Q < n < 0$ $\omega_n < 0$	$n > 0$ $\omega_n > \omega_0$
pattern moves	slower than particle	backwards	faster than particle
wave	slow	backwards	fast

n < -Q **-Q < n < 0** **n > 0**



Snapshots at t_0 (1), $t_0 + \Delta t$ (2), $t_0 + 2\Delta t$ (3)

Unbunched Beam – Transverse Growth Rate

Only one mode n (one single line) grows, so only Z_T around frequency $(Q+n)\omega_0$ relevant

- Assume $e(\vec{E}+\vec{v}\times\vec{B})_T$ constant around the ring for a given y

$$F = e(\vec{E}+\vec{v}\times\vec{B})_T = -i\frac{e\beta I Z_T}{2\pi R}y(\theta,t) \qquad Z_T = i\frac{\int_0^{2\pi R}(\vec{E}+\vec{v}\times\vec{B})_T ds}{\beta y I}$$

$$F(\theta,t) = -i\frac{e\beta I Z_T}{2\pi R}y_n e^{i[(Q+n)\omega_0 t-n\theta]}$$

- Force on a single particle on azimuth $\theta(t)=\theta_0+\omega_0 t$

$$F(t) = F(\theta_0+\omega_0 t,t) = -i\frac{e\beta I Z_T}{2\pi R}\underbrace{y_n e^{i[Q\omega_0 t-n\theta_0]}}_{y(t)}$$

- This particle's betatron amplitude $y(t)$ satisfies

$$\ddot{y}+Q^2\omega_0^2 y = \frac{\text{Force}}{m_0\gamma} = -i\frac{e\beta I Z_T}{2\pi R m_0\gamma}y$$

$$\ddot{y}+\underbrace{(Q\omega_0+\Delta\Omega)^2}_{\approx Q^2\omega_0^2+2\Delta\Omega Q\omega_0}y=0 \quad\Rightarrow \Delta\Omega = i\frac{e\beta Z_T I}{4Q\pi\omega_0 R\gamma m_0},$$

- With $\omega_0 R=\beta c$ and $\gamma m_0=E/c^2$

$$\boxed{\Delta\Omega = i\frac{cZ_T I}{4\pi QE/e}}$$

Transverse growth rate, unbunched beam, Z_T constant around the ring

- Single particle oscillation changed to

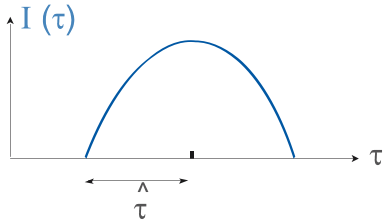
$$y(t) = y_n e^{i[(Q\omega_0+\Delta\Omega)t-n\theta_0]}$$

Unstable if Im(ΔΩ) < 0

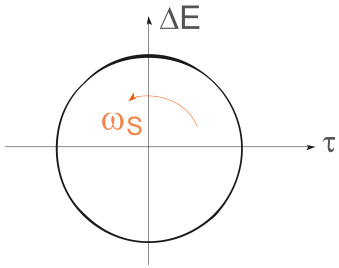
- **Re [$Z_T((Q+n)\omega_0)$] < 0**
- **(Q+n) < 0 slow waves!**

For unbunched beam, only slow wave unstable

Transverse Instabilities – Bunched Beams



Bunch shape observed with current monitor



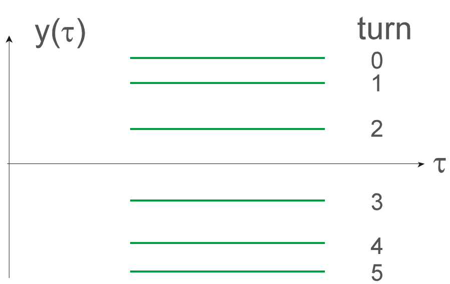
All particles perform synchrotron oscillations – their energy changes with frequency ω_s

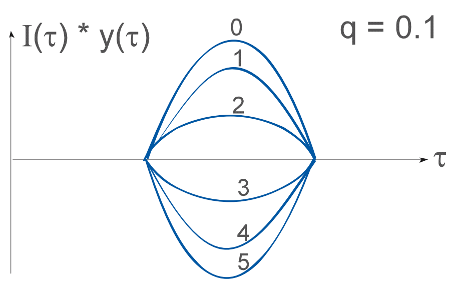
ZERO CHROMATICITY

$$\xi = \frac{dQ}{Q} \Big/ \frac{dp}{p} = 0$$

All particles have same betatron tune Q – even with changing energies

RIGID BUNCH MOTION (m=0) [A. SESSLER ~1960]

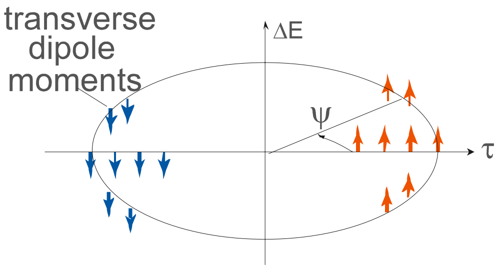




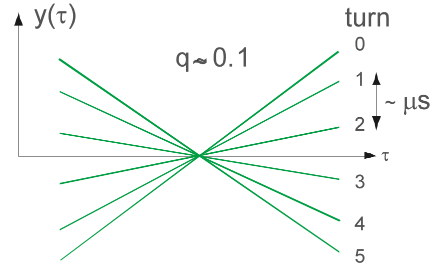
All particles in the bunch start at t=0 with same betatron phase. Although synchrotron motion sweeps them back and forth and changes their energy, they all oscillate in phase

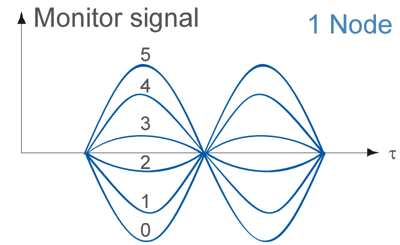
transverse position y(τ)∗current I(τ) = position monitor signal

Transverse Instabilities – Head-Tail Modes



Head-Tail Mode
m=1



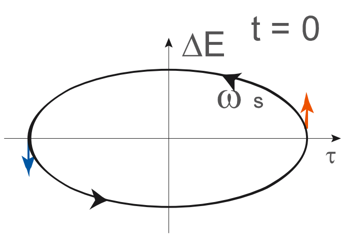


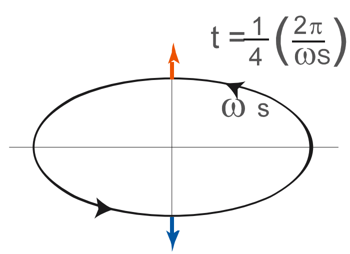
Arrange initial betatron phases so as to have dipole moments **up** near the **head** of the bunch
down near the **tail**

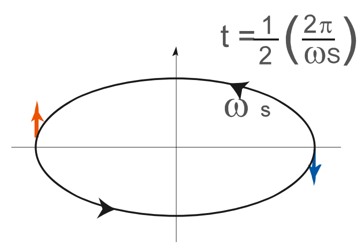
Mode pattern described by $e^{i\psi}$ in longitudinal phase plane

On a slower timescale (~ms): the pattern rotates with ω_s

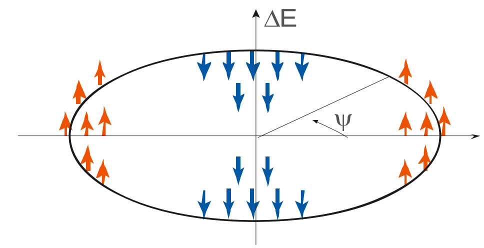
Initial condition (as above)	ups and downs superimposed: signal = 0	ups and downs exchange places

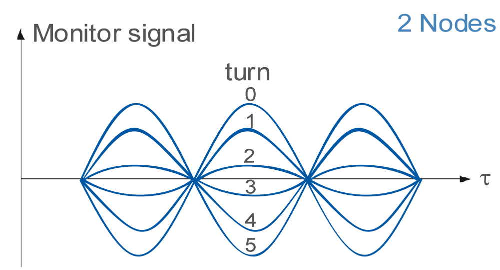






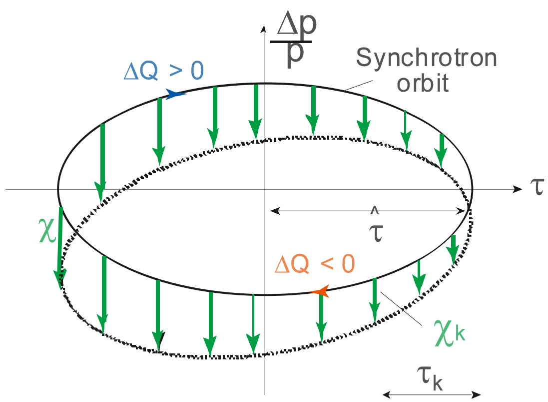
Head-Tail Mode m=2





- **2 nodes**
- pattern described by $e^{i2\psi}$
- pattern rotating with $2\omega_s$

Head-Tail Modes with Non-Zero Chromaticity



$\xi \neq 0$: Q varies along the synchrotron orbits

χ_k….betatron phase slip after k machine turns

χ….. betatron phase slip between head and tail

T_0…..revolution time

$\hat{\tau}$ …..half bunch length

assume

$$\xi = \frac{dQ/Q}{dp/p} > 0, \qquad \gamma < \gamma_t \left[\eta = \frac{1}{\gamma_t^2} - \frac{1}{\gamma^2} < 0 \right]$$

per machine turn

How to calculate χ:

$$\Delta Q = \xi Q \frac{\Delta p}{p}, \quad \frac{\Delta p}{p} = -\frac{1}{\eta}\frac{\Delta f_0}{f_0} = \frac{1}{\eta}\frac{\Delta T_0}{T_0}, \quad \boxed{\Delta Q} = \frac{\xi}{\eta} Q \frac{\Delta T_0}{T_0}$$

Time delay τ_k of a particle relative to the head of the bunch changes per machine turn k:

$$\frac{d\tau_k}{dk} = \Delta T_0$$

Accumulated phase shift χ_k after k machine turns:

$$\chi_k = 2\pi \int_0^k \Delta Q_{per\ turn}\, dk = \underbrace{\frac{2\pi}{T_0}}_{\omega_0} \frac{\xi}{\eta} Q \int_0^k \frac{d\tau_k}{dk} dk = \frac{\xi}{\eta} Q \omega_0 \tau_k$$

Total phase shift between head and tail

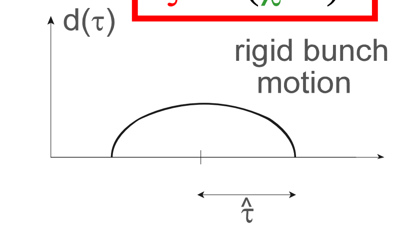
$$\boxed{\chi = \frac{\xi}{\eta} Q \omega_0 \times 2\hat{\tau}}$$

The pattern can be kept stationary if the particles' betatron phases are arranged as in the figure

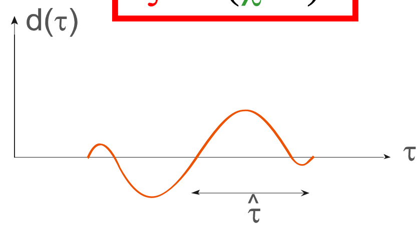
Head-Tail Phase Shift Changes Bunch Spectrum

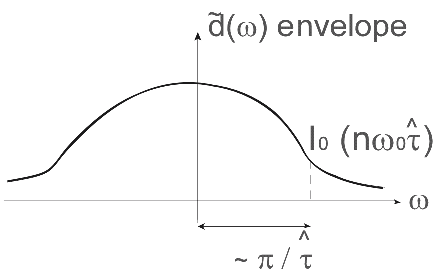
Example: Mode m=0

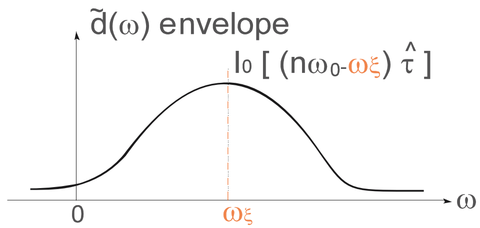
$\xi=0$ $(\chi=0)$



$\xi\neq0$ $(\chi\neq0)$







Head-tail mechanism discovered by C. Pellegrini, M. Sands end 60ies

"Standard model" F.Sacherer mid-70ies

The shorter the bunch length $\hat{\tau}$, the larger the width of the spectrum

The wiggly signal passes through a position monitor which sees

- during bunch passage time $2\,\hat{\tau}$
- a phase shift of χ radians
- the monitor (or an impedance)

"sees" an additional frequency

	η	ξ	ω_ξ
$\gamma < \gamma_t$	< 0	> 0	< 0
		< 0	> 0
$\gamma > \gamma_t$	> 0	> 0	> 0
		< 0	< 0

$\boxed{\omega_\xi = 0}$

Chromaticity Frequency ω_ξ

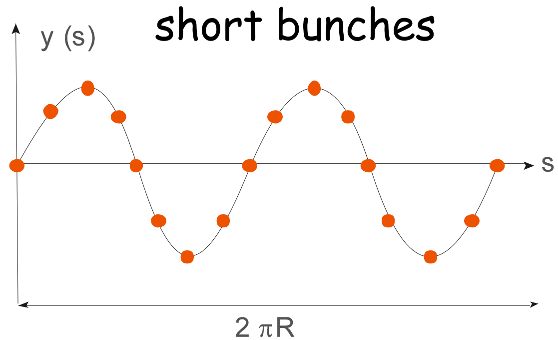
$$\boxed{\omega_\xi = \frac{\xi}{\eta} Q\omega_0}$$

Transverse Instabilities – Many Bunches

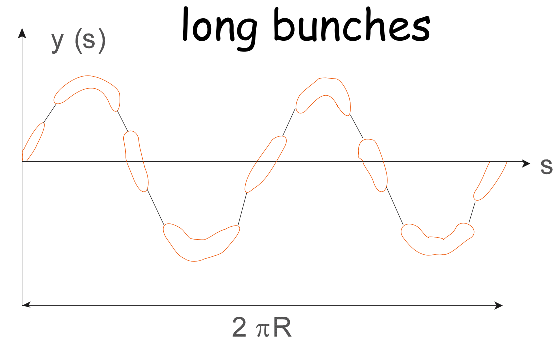
Transverse positions of bunches arranged to form a **pattern of n waves** around the synchrotron

➢ Coupled-bunch mode n

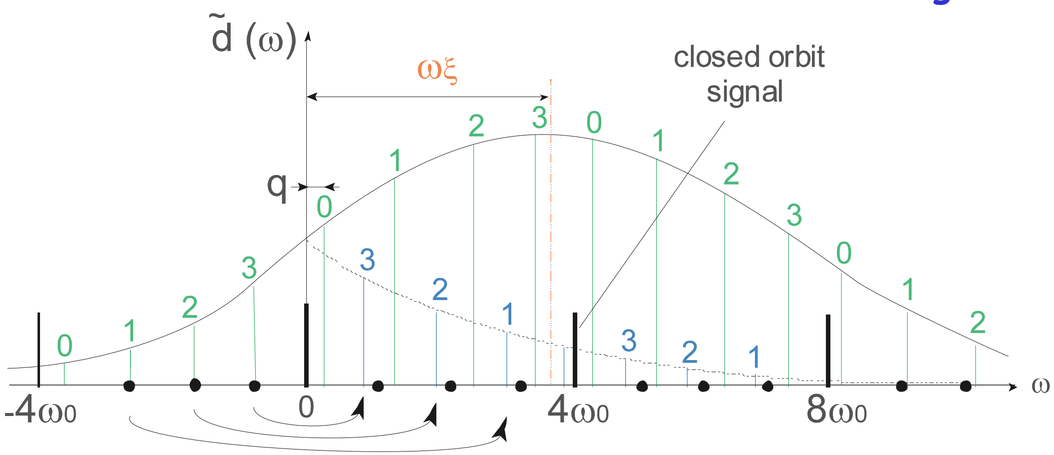
With M bunches, bunch-to-bunch **betatron phase shift $2\pi n/M$**



- n=2 (waves), M=16 (bunches)
- bunch-to-bunch betatron phase shift $\pi/4$
- Head-tail phase shift small
- **behaves like coasting beam**



- n=2, M=8
- bunch-to-bunch betatron phase shift $\pi/2$
- Head-tail phase shift χ large
- can only be sustained with a certain value $\chi \neq 0$



Spectrum for

- M=4 bunches
- m=0 nodes within the bunch
- q = 0.25
- coupled-bunch modes n=0,1,2,3

Bunched Beam – Transverse Growth Rates

Frequency shift, unbunched beam:

Instability if $\mathrm{Im}(\Delta\Omega) < 0 \rightarrow \mathrm{Re}\,(Z_T) < 0$

$$\Delta\Omega = \frac{i}{4\pi Q}\frac{c}{E/e} I Z_T \qquad Z_T \text{ taken at } (n+Q)\omega_0$$

Bunched beam, mode m

- Sum over lines of bunch spectrum $d_m(\omega)$
- Calculate deflecting field $\sim Z_T(\omega)\, d_m(\omega)$ and the force
- Put this force into single-particle equation
- Take sum over $-\infty < p < \infty$

$$\Delta\Omega = \boxed{\frac{1}{1+m}} \frac{i}{4\pi Q}\frac{c}{E/e}\frac{I}{\boxed{B}} \underbrace{\frac{\sum_p Z_T(\omega_p)\tilde{d}_m^2(\omega_p-\omega_\xi)}{\sum_p \tilde{d}_m^2(\omega_p-\omega_\xi)}}_{\text{Impedance summed over all spectral lines}}$$

higher modes more difficult to drive

bunching factor $\frac{\text{bunch length}}{\text{bucket length}}$

Impedance summed over all spectral lines

One bunch: $\omega_p = (p+Q)\omega_0$

M bunches, coupled-bunch **mode n**: $\omega_p = (n + kM + Q)\omega_0, \; -\infty < k < \infty$

a line only every $M\omega_0$

STABILITY?

Unstable if $\mathrm{Im}(\Delta\Omega) < 0$
$\rightarrow \boldsymbol{\sum_p \mathrm{Re}[Z_T(\omega_p)]\, d_m^2(\omega_p - \omega_\xi) < 0}$

as $\mathbf{d_m}^2(\omega_p - \omega_\xi) > 0$

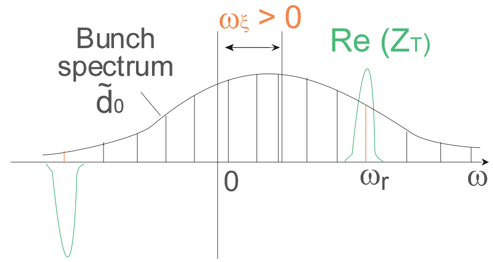
- Unstable if $\mathrm{Re}[Z_T(\omega)] < 0$
- only with negative frequencies
- only slow waves unstable

Bunched Beam Transverse Stability vs. Impedance

Narrow-Band Resonator

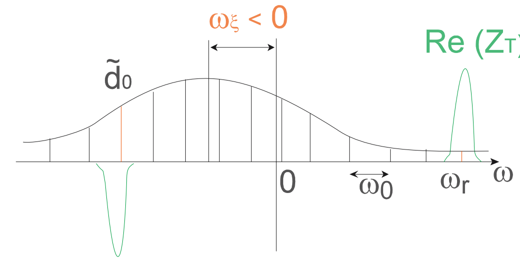
Reminder: $\mathrm{Re}[Z_T(\omega)] = -\,\mathrm{Re}[Z_T(-\omega)]$

- only two spectral lines contribute to the sum
- Fields stored long enough to act on subsequent bunches during several turns



$\omega_\xi > 0$

$\Sigma\, \mathrm{Re}[Z_T]\, d_0^2 > 0 \rightarrow$ stable

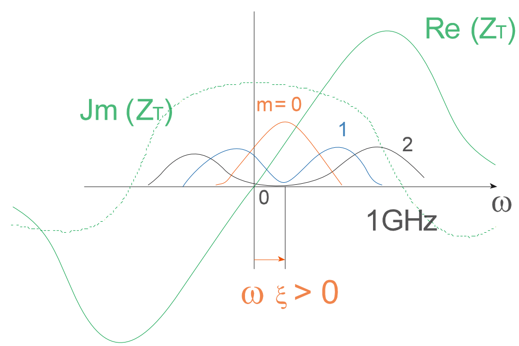


$\omega_\xi < 0$

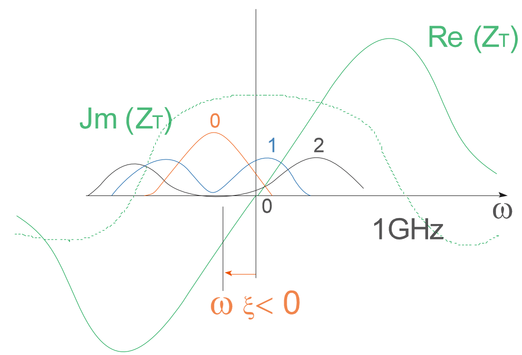
$\Sigma\, \mathrm{Re}[Z_T]\, d_0^2 < 0 \rightarrow$ unstable

Broad-Band Resonator

- extends to ~GHz
- thus spectral lines very dense
- just envelopes I_0, I_1, I_2 shown
- Quality factor Q low → fields not stored long enough to influence subsequent bunches



$\Sigma\, \mathrm{Re}[Z_T]\, d_0^2 > 0 \rightarrow$ stable



$\Sigma\, \mathrm{Re}[Z_T]\, d_0^2 < 0 \rightarrow$ unstable

For any "normal" transverse impedance

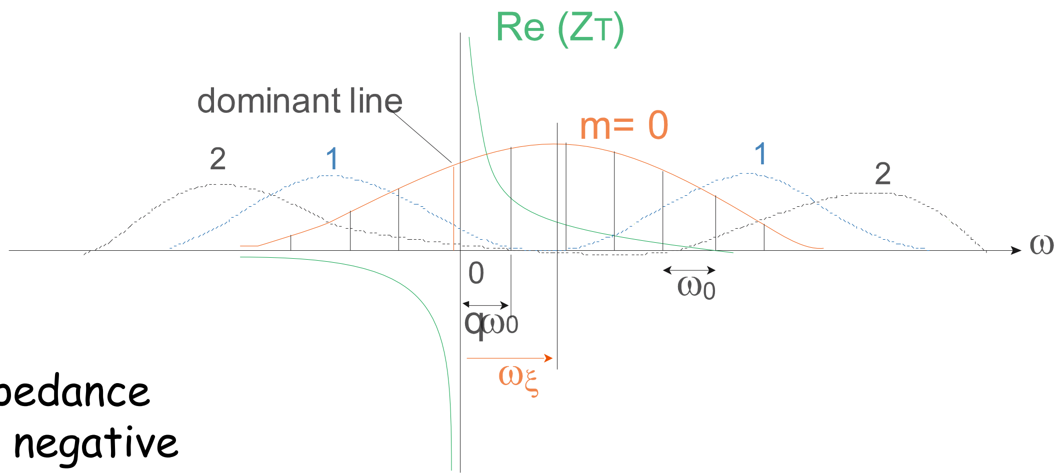
$\gamma < \gamma_t$: set $\xi < \mathbf{0}$ ($\omega_\xi > 0$) to stabilize beam

$\gamma > \gamma_t$: set $\xi > \mathbf{0}$ ($\omega_\xi > 0$) to stabilize beam

Resistive Wall Transverse Instability

$$\mathrm{Re}(Z_T) = \frac{2cR}{\omega b^3}\frac{\rho}{\delta}\ (\mathrm{low}\ \omega)$$

ρ...resistivity of beam pipe
δ…wall thickness (low frequency)
or skin depth (high frequencies)



- not a "normal" transverse impedance
- dominant line at $\mathrm{Re}(Z_T)$ most negative at very low frequency
- dominant mode normally m=0 but cannot be stabilized by setting $\omega_\xi > 0$
- setting Q above an integer (q < 0.5) puts dominant line near the origin but at $\mathrm{Re}(Z_T) > 0$ thus stabilizing the beam

For the resistive wall impedance, fractional tune $q < 0.5$ preferable (A.Sessler 60ies)

Further increasing ω_ξ (by varying ξ with sextupoles) may drive the hump of m=1, 2 etc. onto this dominant line, thus switching from one mode to the next.

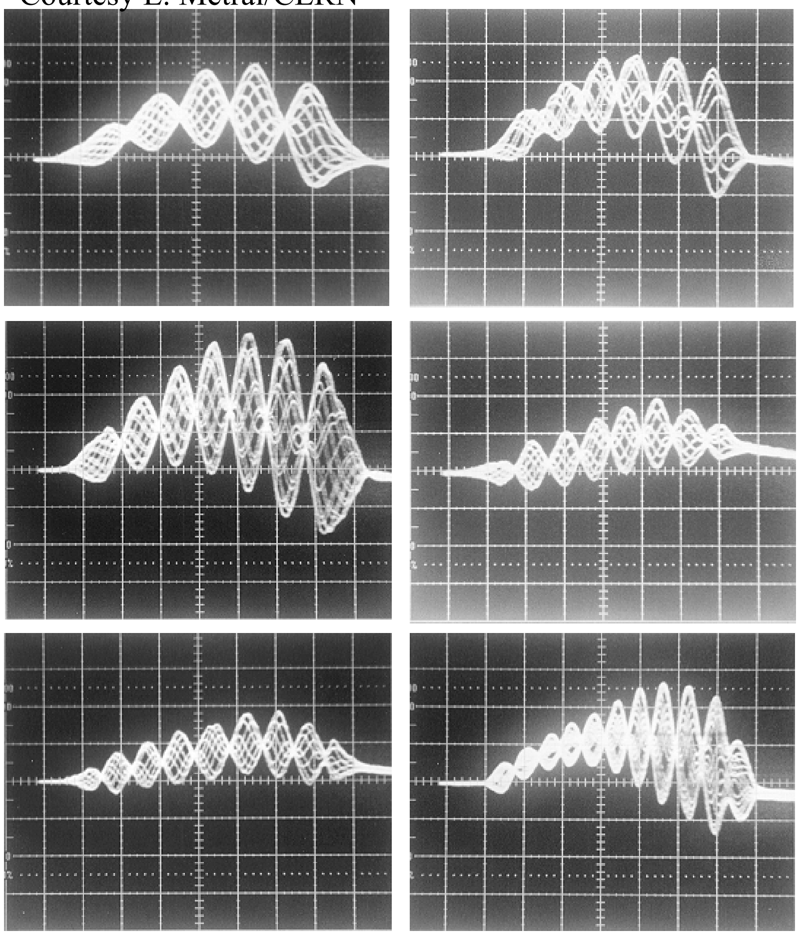
Horizontal Head-Tail Instabilities in CERN PS

Courtesy E. Metral/CERN

m=4, 5

m=6, 7

m=8, 10



20 ns/div

A single bunch with ~10^{12} protons and ~150 ns length on the 1.4 GeV injection plateau in the CERN PS (below transition energy)

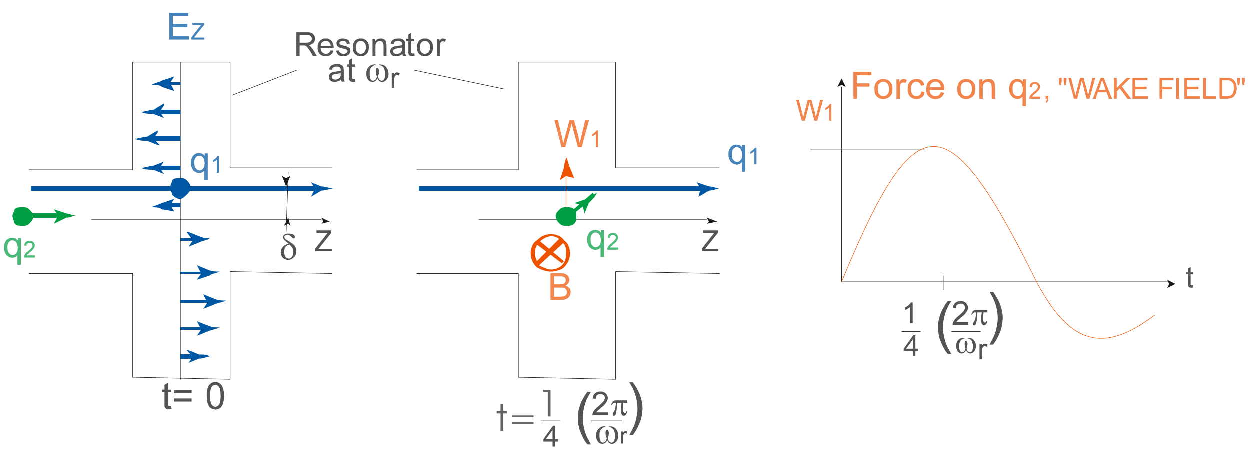
Head-tail mode numbers m=4,…,9 are generated by changing horizontal chromaticity ξ_h from -0.5 (m=4) to -1.3 (m=10). The natural chromaticity, ξ_h= -0.9, yields m=6 (6 nodes). For all pictures, $\omega_\xi > 0$, which normally stabilizes the beam, but not in this case.

→ The impedance responsible for this horizontal instability is the resistive wall impedance

Transverse Wake Fields

Instead of treating instability dynamics in the frequency domain as done so far, one can do it in the time domain by using "Wake Fields"

What is a Wake Field?
Point charge q_1 passes through a resonator with a transverse displacement δ.
The induced Wake field W will act on the subsequent charge q_2.



RLC-circuit (p. 11)

$$W = W_1 e^{-\alpha t} \sin S\omega_r t$$

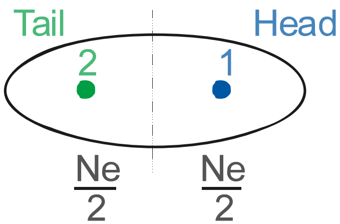
with

$$\alpha = \omega_r/(2Q)$$

$$S = (1 - Q^2/4)^{1/2}$$

The Wake Field concept is very useful for impedances with short memory where the fields do not act on subsequent bunches but only on particles within the same bunch (single-bunch effects). Example: broad-band impedance (low-Q resonator)

Transverse Wake Fields – A Simple Model



Approximate bunch by just two superparticles
"head" (1) and "tail" (2) with Ne/2 charges each

Model by A. Chao

If **head** is displaced by δ, force on particle in **tail** is

$$f = e\frac{Ne}{2}W_1\delta$$

Both **head** **(y_1)** and **tail** **(y_2)** oscillate with same betatron frequency ω_β

same frequency

$$y_1 = \delta\cos\boxed{\omega_\beta}t$$

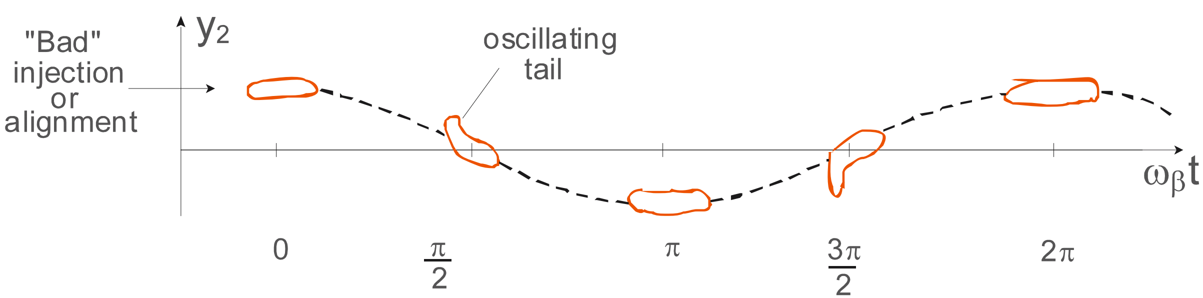
$$\ddot{y}_2 + \boxed{\omega_\beta}^2 y_2 = \frac{f}{m_0\gamma} = \frac{Ne^2W_1}{2m_0\gamma}\boxed{y_1}$$

Excitation on right-hand side **has same frequency**

$$\Rightarrow y_2 = \delta\left[\cos\omega_\beta t + \underbrace{\frac{Ne^2W_1}{4\omega_\beta m_0\gamma}t\sin\omega_\beta t}\right]$$

tail amplitude y_2 **grows linearly** with time

SLAC 50 GeV Electron Linac



Observation: Tail amplitude increasing along the Linac – caused by misalignments

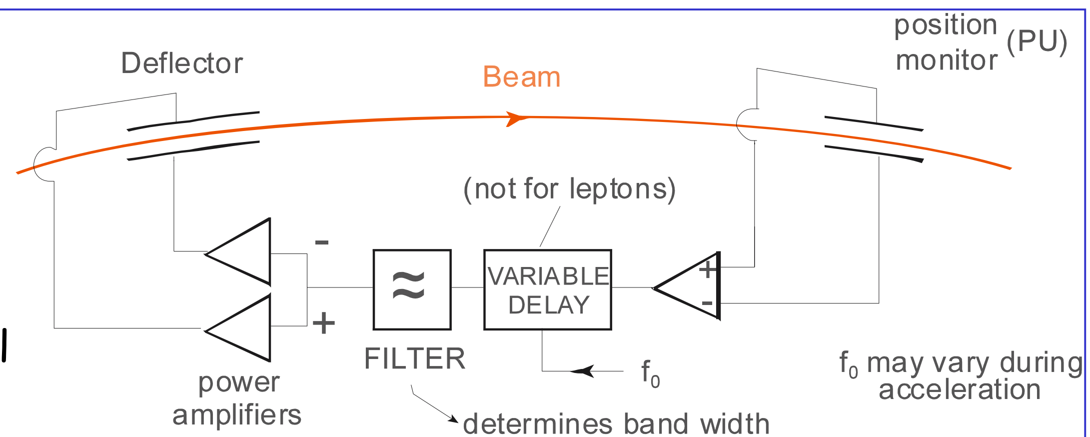
Transverse Instabilities - Cures

- As for longitudinal impedances: damp unwanted HOM's, protect beam by RF shields
- For "normal" transverse impedances, operate with a slightly positive chromaticity frequency ω_ξ → for $\gamma < \gamma_t$ set $\xi < 0$ (by sextupoles)
  → for $\gamma > \gamma_t$ set $\xi > 0$

$$\boxed{\omega_\xi = \frac{\xi}{\eta} Q\omega_0}$$

- For the resistive wall impedance:
  - ➢ operate machine with a betatron tune just above an integer
  - ➢ use highly conductive vacuum pipe material to reduce Re(Z_T) and growth rate
- Landau damping also works in the transverse plane; a betatron frequency spread $\Delta\omega_\beta$ is generated by octupoles (betatron tune depends on oscillation amplitude)

- TRANSVERSE FEEDBACK
  - position error in PU
    → angle error in deflector
  - betatron phase from PU to deflector ~(2n+1)π/2
  - electronic delay ≡ beam travel time from PU to deflector



- Bandwidth: ~ a few 10 kHz to a few MHz if only resistive wall
  ~ up to half the bunch frequency with bunch-by-bunch feedback