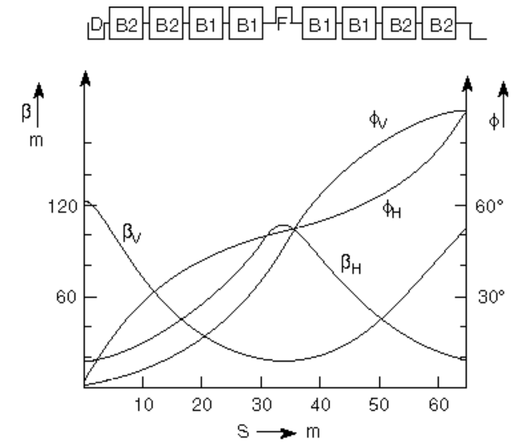
Transverse Dynamics – E. Wilson – CERN – 16th September 2003

- The lattice calculated
- Solution of Hill
- Solution of Hill (conc)
- Meaning of Twiss parameters
- Liouville's Theorem
- Closed orbit of an ideal machine
- Dispersion – from the "sine and cosine" trajectories
- From "three by three" matrices
- Making an orbit bump grow
- Measuring the orbit
- Overlapping beam bumps
- Gradient errors
- Resonance condition
- Multipole field expansion
- Taylor series expansion
- Multipole field shapes
- Correction of Chromaticity
- Luminosity

The lattice



	LENGTH	ANGLE	K(V)	ALPHA(P)	BETA(H)	ALPHA(H)	MUH/2PI	BETA(V)	ALPHA(V)	MUV/2PI	AH/2	AV/2
01	3,085000	0,000000	-,015063	1,388440	104,884885	2,452160	,004571	19,011703	-,520345	,028871	65,715663	9,917560
2	,360000	0,000000	0,000000	1,374653	103,127966	2,426089	,005122	19,395014	-,544408	,029566	66,847813	10,017039
03	6,260000	,008445	0,000000	1,196124	75,348859	2,009521	,016433	28,828710	-,962619	,072198	64,004371	12,212911
4	,400000	0,000000	0,000000	1,186405	73,751941	1,988775	,017287	29,609417	-,989246	,074377	64,781341	12,376828
05	6,260000	,008445	0,000000	1,060742	51,548094	1,564207	,033474	44,610910	-1,407071	,101988	54,174091	15,192432
6	,390000	0,000000	0,000000	1,054559	50,338182	1,538130	,034692	45,718585	-1,433122	,103302	45,428681	15,379447
07	6,260000	,008445	0,000000	,981762	33,701223	1,119863	,058975	66,274961	-1,850527	,121441	44,908056	18,517478
8	,360000	0,000000	0,000000	,978948	32,860011	1,094164	,060793	67,891002	-1,875496	,128344	36,980337	18,713705
09	6,260000	,008445	0,000000	,959017	21,781569	,676566	,098381	93,787678	-2,292753	,134061	30,534921	22,028267
10	2,342700	0,000000	0,000000	,981460	18,983146	,518942	,116750	104,898272	-2,449038	,138621	30,069327	23,295624
11	3,085000	0,000000	,015037	1,034354	18,983088	-,518918	,143368	104,901020	2,447388	,143191	28,349412	23,716825
12	,350000	0,000000	0,000000	1,050730	19,354500	-,542318	,146275	103,198611	2,424067	,143726	28,638028	23,296218
13	6,260000	,008445	0,000000	1,370047	28,764399	-,960879	,189011	75,452122	2,007602	,165027	38,069639	23,100121
14	,380000	0,000000	0,000000	1,391035	29,504322	-,986287	,191088	73,938822	1,982463	,168836	35,546047	19,787412
15	6,260000	,008445	0,000000	1,763219	44,472640	-1,404847	,218731	51,724094	1,565610	,171975	43,780578	19,657880
16	,390000	0,000000	0,000000	1,788053	45,578591	-1,430924	,220109	50,513087	1,539589	,173189	44,298687	16,358398
17	6,260000	,008445	0,000000	2,213103	66,113699	-1,849484	,238298	33,849177	1,122280	,197377	53,470174	16,165762
18	,400000	0,000000	0,000000	2,241952	67,603985	-1,876229	,239281	32,962034	1,098579	,190283	54,079136	13,233307
19	6,260000	,008445	0,000000	2,719886	93,714294	-2,294790	,251780	21,859390	,677943	,230745	63,830251	13,050741
20	2,352700	0,000000	0,000000	2,909420	104,882261	-2,452099	,255555	19,038995	,520847	,288140	67,892709	10,834409
21	3,085000	0,000000	-,015063	2,946010	104,882266	2,452098	,260188	19,038106	-,520546	,281673	68,853088	9,924876
22	,360000	0,000000	0,000000	2,925443	103,126421	2,426027	,260680	19,421551	-,544879	,284683	67,668889	10,023890
23	6,260000	,008445	0,000000	2,594240	75,347037	2,009467	,271992	28,854181	-,962177	,287246	67,105194	12,218305
24	,400000	0,000000	0,000000	2,574785	73,750162	1,982722	,272846	29,634602	-,988874	,289424	57,548939	12,382087
25	6,260000	,008445	0,000000	2,298428	51,546933	1,564162	,289032	44,628208	-1,406186	,386957	56,950187	15,195377
26	,390000	0,000000	0,000000	2,280734	50,337057	1,538085	,290251	45,735180	-1,432204	,358331	47,899867	15,382235
27	6,260000	,008445	0,000000	2,055264	33,700612	1,119526	,314534	66,276862	-1,849098	,376466	47,358928	18,517744
28	,380000	0,000000	0,000000	2,043182	32,889428	1,094117	,316302	67,891805	-1,874435	,377389	39,127022	18,713817
29	6,260000	,008445	0,000000	1,870577	21,781398	,678557	,353941	93,766993	-2,290782	,389888	38,663082	22,026836
30	2,342700	0,000000	0,000000	1,815875	18,983101	,518917	,372318	104,885902	-2,446875	,393646	31,492336	23,292251
31	3,085000	0,000000	,015037	1,873603	18,983178	-,518943	,389928	104,862544	2,447912	,398220	30,027986	23,712598

Solution of Hill

- **Differentiate** $y = \sqrt{\beta(s)\varepsilon}\cos\big(\phi(s) + \phi_o\big)$

substituting $\rightarrow$ $w = \sqrt{\beta}\ , \qquad \phi = \phi(s) + \phi_o$

$$y' = \varepsilon^{1/2}\left\{ w'(s)\cos\phi - \frac{d\phi}{ds} w(s)\sin\phi \right\}$$

- **Necessary condition for solution to be true**

$$\frac{d\phi}{ds} = \frac{1}{\beta(s)} = \frac{1}{w^2(s)}$$

so $$y' = \varepsilon^{1/2}\left\{ w'(s)\cos\phi - \frac{1}{w\ (s)}\sin\phi \right\}$$

- **Differentiate again**

$$y'' = \varepsilon^{1/2}\left\{ w''(s)\cos\phi - \frac{w'(s)}{w^2(s)}\sin\phi + \frac{w'(s)}{w^2(s)}\sin\phi \right\}$$

$$-\frac{1}{w^3(s)}\cos\phi$$

and add to both sides

$$+ky \qquad +kw(s)\cos\phi$$

cancels to 0

must be zero0

Solution of Hill (conc)

$$y'' = \varepsilon^{1/2}\left\{ w''(s)\cos\phi - \frac{w'(s)}{w^2(s)}\sin\phi + \frac{w'(s)}{w^2(s)}\sin\phi \right\}$$

$$-\frac{1}{w^3(s)}\cos\phi$$

cancels to 0

$$+ky \qquad +kw(s)\cos\phi$$

must be zero0

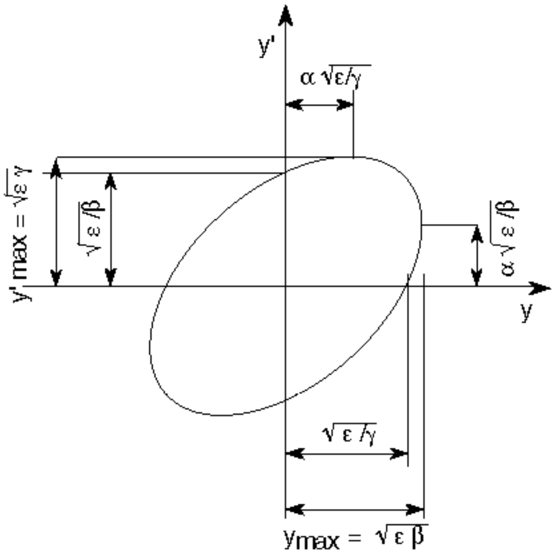
- **The condition that these three coefficients sum to zero is a differential equation for the envelope**

$$\boxed{w''(s) + kw(s) - \frac{1}{w^3(s)} = 0}$$

alternatively

$$\boxed{\frac{1}{2}\beta\beta'' - \frac{1}{4}\beta'^2 + k\beta^2 = 1}$$

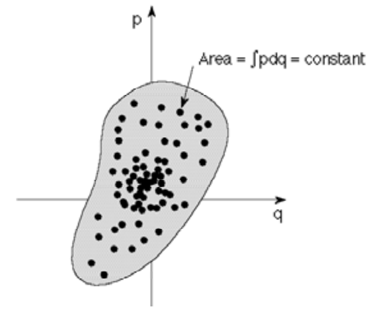
Meaning of Twiss parameters



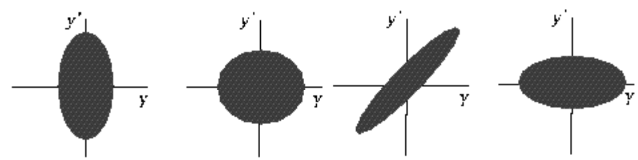
- ε is either :
  - » Emittance of a beam anywhere in the ring
  - » Courant and Snyder invariant for one particle anywhere in the ring

$$\boxed{\gamma(s)^2 + 2\alpha(s)yy' + \beta(s)y'^2 = \varepsilon}$$

Liouville's Theorem

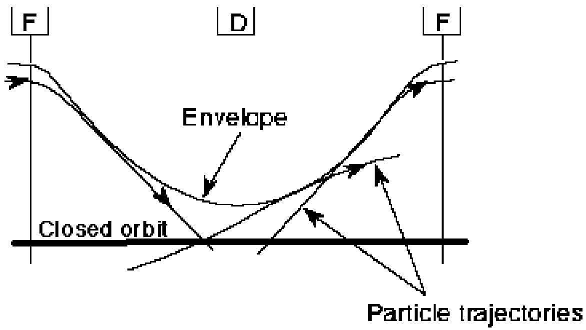


- "*The area of a contour which encloses all the beam in phase space is conserved*"
- This area = πε is the "emittance"
- It is the same all round the ring

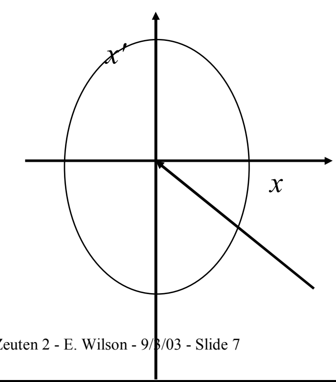


- NOT TRUE:
  - during acceleration
  - in an electron machine where synchrotron emission damps

Closed orbit of an ideal machine

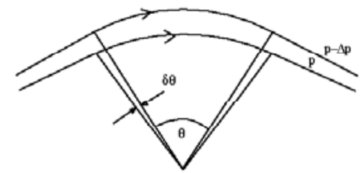


- In general particles executing betatron oscillations have a finite amplitude
- One particle will have zero amplitude and follows an orbit which closes on itself
- In an ideal machine this passes down the axis

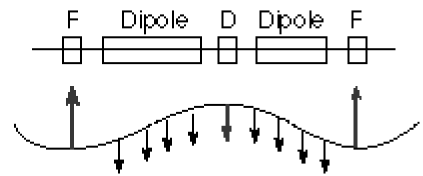


Closed orbit
Zero betatron amplitude

Dispersion- reminder



- Low momentum particle is bent more
- It should spiral inwards but:
- There is a displaced (inwards) closed orbit
- Closer to axis in the D's
- Extra (outward) force balances extra bends



- D(s) is the "dispersion function"

$$x = D(s)\frac{\Delta p}{p}$$

Fig. cas 1.7-7.1C

Dispersion – from the "sine and cosine" trajectories

- **The combination of displacement, divergence and dispersion gives:**

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0} + \frac{\Delta p}{p} \begin{pmatrix} D \\ D' \end{pmatrix}$$

- **Expressed as a matrix**

$$\begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_s = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_{s_0}$$

- **It can be shown that:**

$$\boxed{D(s) = S(s) \int_{s_0}^{s} \frac{1}{\rho(t)} C(t) dt - C(s) \int_{s_0}^{s} \frac{1}{\rho(t)} S(t) dt}$$

- **Fulfils the particular solution of Hill's eqn. when forced :**

$$D''(s) + K(s) D(s) = \frac{1}{\rho(s)}$$

From "three by three" matrices

- **Adding momentum defect to horizontal divergence and displacement vector–**

$$\begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_2 = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_1$$

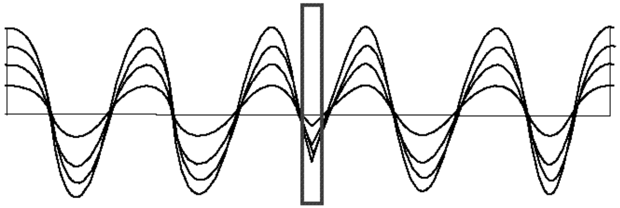
- **Compute the ring as a product of small matrices and then use:**

$$D(s) = \left(\frac{m_{12} m_{13}}{1 - m_{11}} \right) D'(s)$$

$$D'(s) = \frac{m_{13} m_{21} + (1 - m_{11}) m_{23}}{(1 - m_{11})(1 - m_{22}) - m_{21} m_{12}}$$

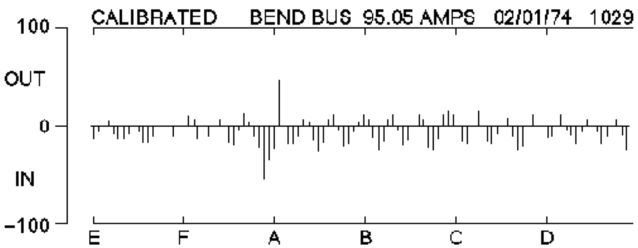
- **To find the dispersion vector at the starting point**
- **Repeat for other points in the ring**

Making an orbit bump grow



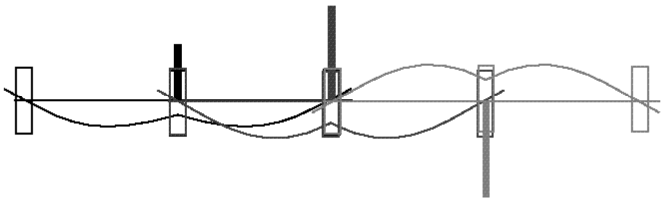
- As we slowly raise the current in a dipole:
- The zero-amplitude betatron particle follows a distorted orbit
- The distorted orbit is CLOSED
- It is still obeying Hill's Equation
- Except at the kink (dipole) it follows a betatron oscillation.
- Other particles with finite amplitudes oscillate about this new closed orbit

FNAL MEASUREMENT



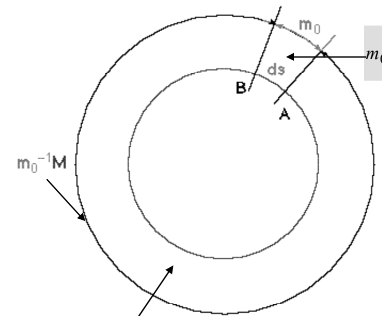
- Historic measurement from FNAL main ring
- Each bar is the position at a quadrupole
- +/- 100 is width of vacuum chamber
- Note mixture of 19th and 20th harmonic
- The Q value was 19.25

Overlapping beam bumps



- **Each colour shows a triad bump centred on a beam position measurement.**
- **A computer calculates the superposition of the currents in the dipoles and corrects the whole orbit simultaneously**

Gradient errors



$$m_0 = \begin{pmatrix} 1 & 0 \\ -k_0(s)ds & 1 \end{pmatrix},$$

$$m = \begin{pmatrix} 1 & 0 \\ -[k_0(s)+\delta k(s)]ds & 1 \end{pmatrix}.$$

$$M_0(s) = \begin{pmatrix} \cos\varphi_0 + \alpha_0 \sin\varphi_0 , & \beta_0 \sin\varphi_0 \\ -\gamma_0 \sin\varphi_0 , & \cos\varphi_0 - \alpha_0 \sin\varphi_0 \end{pmatrix}.$$

$$M(s) = \boxed{m m_0^{-1}} M_0 .$$

$$m m_0^{-1} = \begin{pmatrix} 1 & 0 \\ -\delta k(s_1)ds & 1 \end{pmatrix}.$$

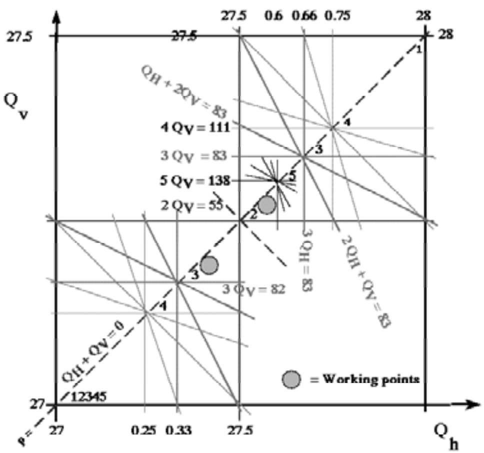
$$M = \begin{pmatrix} \cos\phi_0 + \alpha_0 \sin\phi_0 , & \beta_0 \sin\phi_0 \\ -\delta k(s)ds(\cos\phi_0 + \alpha_0 \sin\phi_0) - \gamma \sin\phi_0 , & -\delta k(s)ds\beta_0 \sin\phi_0 + \cos\phi_0 - \alpha_0 \sin\phi_0 \end{pmatrix}.$$

$$\Delta(\mathrm{Tr\ M})/2 = \Delta(\cos\phi) = -\Delta\phi \sin\phi_0 = \frac{\sin\phi_0}{2}\beta_0(s)\delta k(s)ds$$

$$2\pi\Delta Q = \Delta\phi = \frac{\beta(s)\delta k(s)ds}{2}.$$

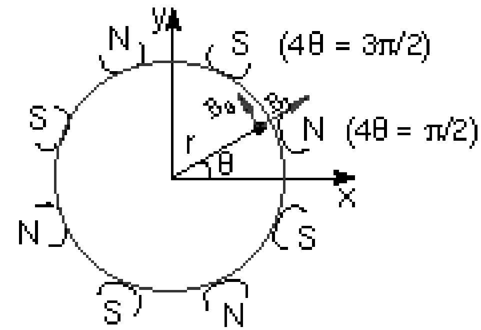
$$\boxed{\Delta Q = \frac{1}{4\pi}\int \beta(s)\delta k(s)\ ds .}$$

Resonance condition $nQ = p$,



$$\ell Q_H + mQ_V = p$$

Multipole field expansion (polar)



Scalar potential $\phi(\mathrm{r}, \theta)$ **obeys Laplace**

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \quad \text{or} \quad \frac{1}{\mathrm{r}^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{1}{\mathrm{r}}\frac{\partial}{\partial \mathrm{r}}\left(\mathrm{r}\frac{\partial \phi}{\partial \mathrm{r}}\right) = 0$$

whose solution is $\phi = \sum_{\mathrm{n}=1}^{\infty} \phi_{\mathrm{n}} \mathrm{r}^{\mathrm{n}} \sin \mathrm{n}\theta$

Example of an octupole whose potential oscillates like sin 4θ around the circle

Taylor series expansion

$$\phi = \sum_{n=1}^{\infty} \phi_n r^n \sin n\theta$$

Field in polar coordinates:

$$B_r = -\frac{\partial \phi}{\partial r}, \quad B_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta}$$

$$B_r = \phi_n n r^{n-1} \sin n\theta, \quad B_\theta = \phi_n n r^{n-1} \cos n\theta$$

To get vertical field

$$\begin{aligned} B_z &= B_r \sin\theta + B_\theta \cos\theta \\ &= -\phi_n n r^{n-1}[\cos\theta \cos n\theta + \sin\theta \sin n\theta] \\ &= \phi_n n r^{n-1} \cos(n-1)\theta = \phi_n n x^{n-1} \quad (\text{when } y = 0) \end{aligned}$$

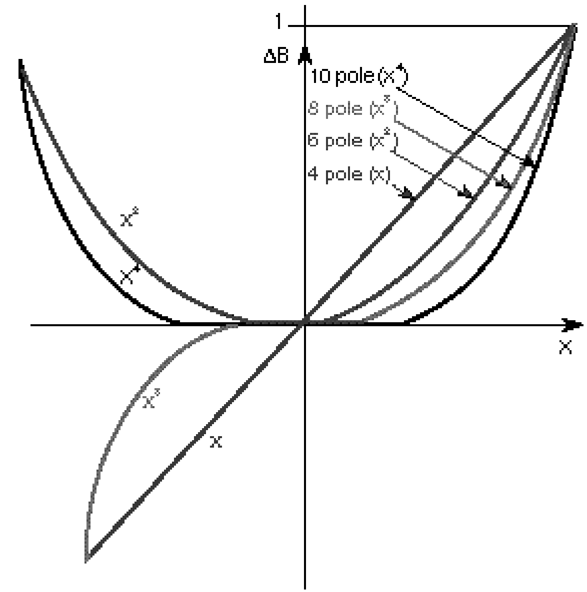
Taylor series of multipoles

$$\begin{aligned} B_z &= \phi_o + \phi_2 \cdot 2x + \phi_3 \cdot 3x^2 + \phi_4 \cdot 4x^3 + \ldots\ldots \\ &= B_o + \frac{1}{1!}\frac{\partial B_z}{\partial x} + \frac{1}{2!}\frac{\partial^2 B_z}{\partial x^2} + \frac{1}{3!}\frac{\partial^3 B_z}{\partial x^3} + \ldots\ldots \end{aligned}$$

Dip. Quad Sext Octupole

Fig. cas 1.2c

Multipole field shapes



Chromaticity- reminder

- The Q is determined by the lattice quadrupoles whose strength is:

$$k = \frac{1}{(B\rho)}\frac{dB_z}{dx} \propto \frac{1}{p}$$

- Differentiating:
- Remember from gradient error analysis

$$\frac{\Delta k}{k} = -\frac{\Delta p}{p} .$$

- Giving by substitution

$$\Delta Q = \frac{1}{4\pi}\int \beta(s)\delta k(s)\ ds .$$

Q' is the chromaticity

- "Natural" chromaticity

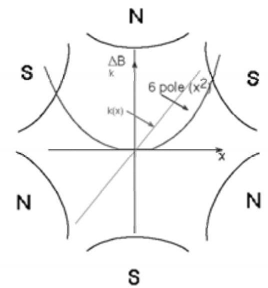
$$\Delta Q = \frac{1}{4\pi}\int \beta(s)\Delta k(s) ds = \left[\boxed{\frac{-1}{4\pi}\int \beta(s)k(s)ds}\right]\frac{\Delta p}{p} .$$

$$\Delta Q = \boxed{Q'}\frac{\Delta p}{p}$$

$$Q' = -\frac{1}{4\pi}\oint \beta(s)k(s)ds \approx -1.3Q$$

N.B. Old books say $\xi = \frac{p}{Q}\frac{dQ}{dp} = \frac{Q'}{Q}$

Correction of Chromaticity



- Parabolic field of a 6 pole is really a gradient which rises linearly with x
- If x is the product of momentum error and dispersion
- The effect of all this extra focusing cancels chromaticity

$$\Delta k = \frac{B''D}{(B\rho)}\frac{\Delta p}{p} .$$

- Because gradient is opposite in v plane we must have two sets of opposite polarity at F and D quads where betas are different

$$\Delta Q = \boxed{\left[\frac{1}{4\pi}\int \frac{B''(s)\beta(s)D(s)ds}{(B\rho)}\right]\frac{dp}{p}} .$$

Luminosity

- Imagine a blue particle colliding with a beam of cross section area - A
- Probability of collision is $\frac{\sigma}{A}\cdot N$
- For N particles in both beams $\frac{\sigma}{A}\cdot N^2$
- Suppose they meet *f* times per second at the revolution frequency
- Event rate $f_{rev} = \frac{\beta c}{2\pi R}$

$$\boxed{\frac{f_{rev}N^2}{A}}\cdot\sigma$$

Make big

e.g. 10^{-25}

Make small

LUMINOSITY $\boxed{\approx 10^{30} \text{ to } 10^{34} \; \left[\mathrm{cm}^{-2}\ \mathrm{s}^{-1}\right]}$

Summary

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- **Solution of Hill**
- **Solution of Hill (conc)**
- **Meaning of Twiss parameters**
- **Liouville's Theorem**
- **Closed orbit of an ideal machine**
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