SPACE CHARGE EFFECTS

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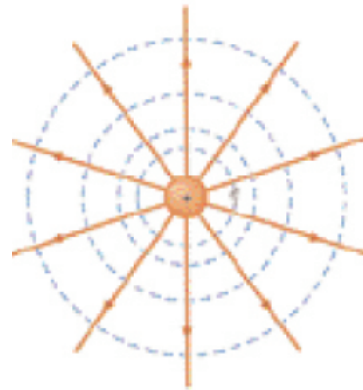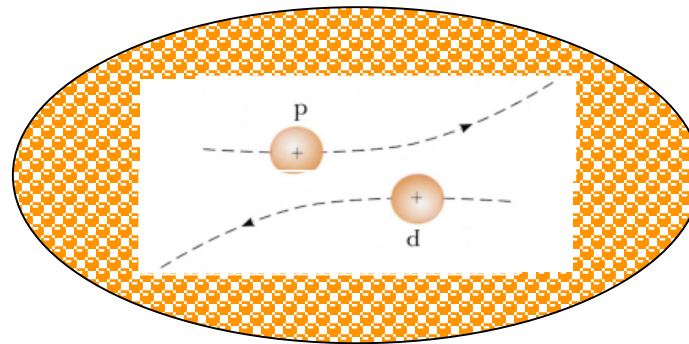
- **Space Charge Dominated Beams**

- **Low Energy Case**

- **High Energy Case**

Space Charge: What does it mean?

The net effect of the **Coulomb** interactions in a multi-particle system can be classified into two regimes:

1) **Collisional Regime** ==> dominated by **binary collisions** caused by close particle encounters ==> **Single Particle Effects**

2) **Space Charge Regime** ==> dominated by the **self field** produced by the particle distribution, which varies appreciably only over large distances compare to the average separation of the particles ==> **Collective Effects**
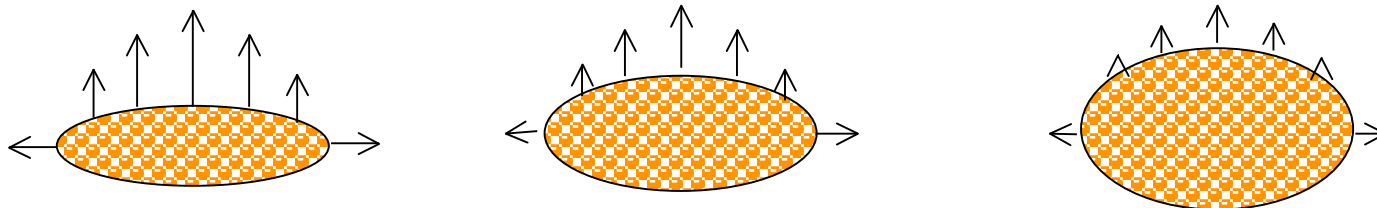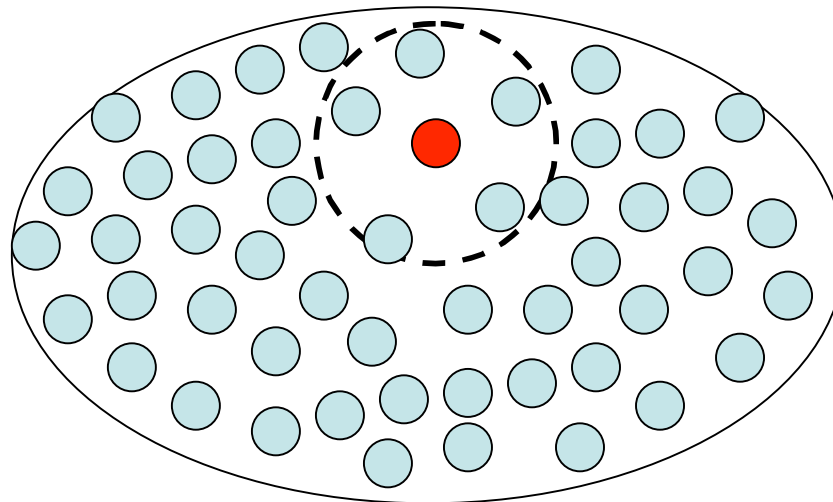
A measure for the relative importance of collisional versus collective effects is the

Debye Length λ_D

Let consider a **non-neutralized** system of **identical charged particles**

We wish to calculate the effective potential of a fixed test charged particle surrounded by other particles that are statistically distributed.

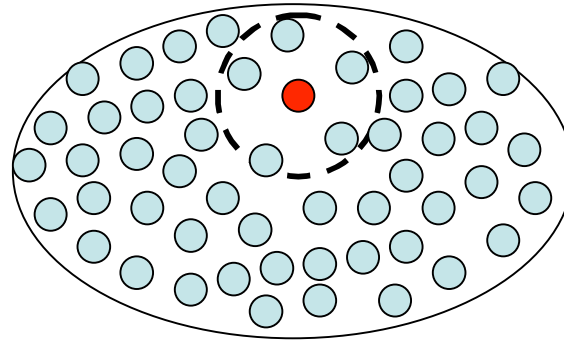$$\Phi(\vec{r}) = \frac{C}{r}$$

$$C = \frac{e}{4\pi\varepsilon_o}$$

$$\Phi_s(\vec{r}) = ?$$

N total number of particles

n average particle density

The particle distribution around the test particle will deviate from a uniform distribution so that the effective potential of the test particle is now defined as the sum of the original and the induced potential:

Poisson Equation

$$\nabla^2 \Phi_s(\vec{r}) = \frac{e}{\varepsilon_o}\delta(\vec{r}) + \frac{e}{\varepsilon_o}\Delta n(\vec{r})$$

Local deviation from n

$$\Delta n(\vec{r}) = n_m(\vec{r}) - n$$

Local microscopic distribution

$$n_m(\vec{r}) = \frac{1}{N}\sum_{i=1}^{N} e\delta(\vec{r} - \vec{r}_i)$$

Presupposing thermodynamic equilibrium, n_m will obey Maxwell-Boltzmann statistic:
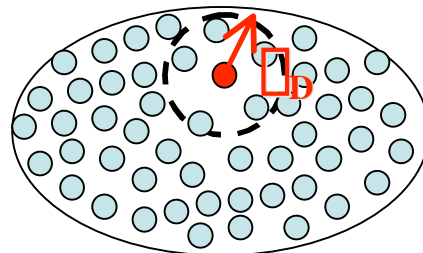
$$n_m(\vec{r}) = n e^{-e\Phi_s(\vec{r})/k_B T}$$

$$\Delta n(\vec{r}) = n_m(\vec{r}) - n = n\left(e^{-e\Phi_s(\vec{r})/k_B T} - 1\right) \approx -\frac{e\Phi_s(\vec{r})}{k_B T}$$

Where the potential energy of the particles is assumed to be much smaller than their kinetic energy

$$\nabla^2\Phi_s(\vec{r}) + \frac{\Phi_s(\vec{r})}{\lambda_D^2} = \frac{e}{\varepsilon_o}\delta(\vec{r})$$

The solution with the boundary condition that Φ_s vanishes at infinity is:

$$\Phi_s(\vec{r}) = \frac{C}{r}e^{-r/\lambda_D}$$

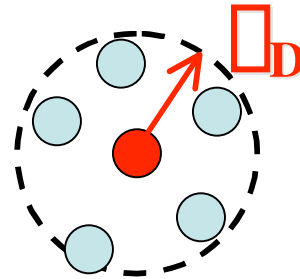$$\lambda_D = \sqrt{\frac{\varepsilon_o k_B T}{e^2 n}}$$

Conclusion: the effective interaction range of the test particle is limited to the **Debye length**

The charges sourrounding the test particles have a screening effect

$$\Phi_s(\vec{r}) = \frac{C}{r}e^{-r/\lambda_D}$$

$$\Phi_s(\vec{r}) \approx \Phi(\vec{r}) \quad for \quad r << \lambda_D$$

$$\Phi_s(\vec{r}) << \Phi(\vec{r}) \quad for \quad r \geq \lambda_D$$

λ_D

<u>Smooth functions for the charge and field distributions can be used as long as the Debye length remains small compared to the particle bunch size</u>

Ex: Longitudinal Electrict field of a uniform charged cylinder
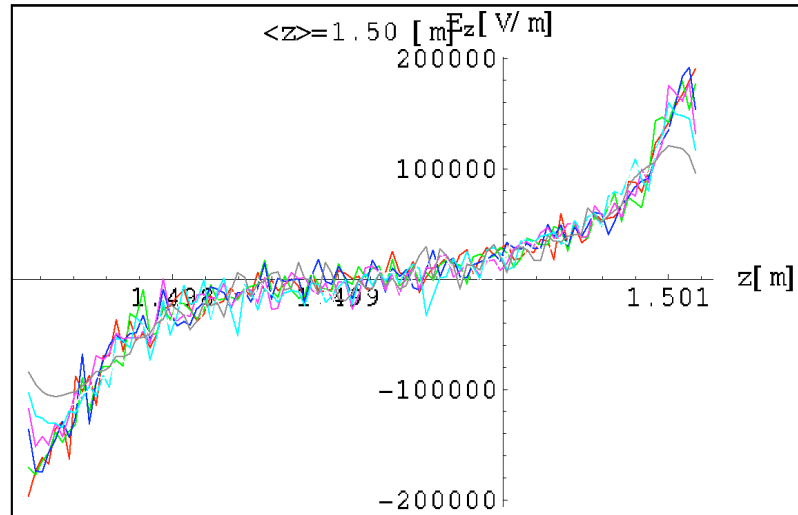
As computed by a multi-particle tracking code

Analyticall expression

Important consequences

If collisions can be neglected the Liouville's theorem holds in the 6-D phase space (**r**,**p**). This is possible because the smoothed space-charge for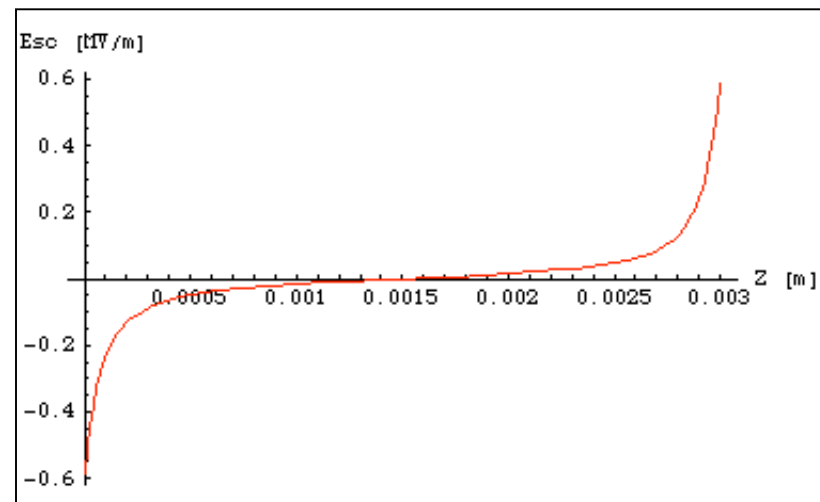ces acting on a particle can be treated like an applied force. Thus the 6-D phase space volume occupied by the particles remains constant during acceleration.

In addition if all forces are linear functions of the particle displacement from the beam center and there is no coupling between degrees of freedom, the normalized emittance associated with each plane (2-D phase space) remains a constant of the motion

Continuous Uniform Cylindrical Beam Model

$$\rho = \frac{I}{\pi a^2 v}$$

$$J = \frac{I}{\pi a^2}$$
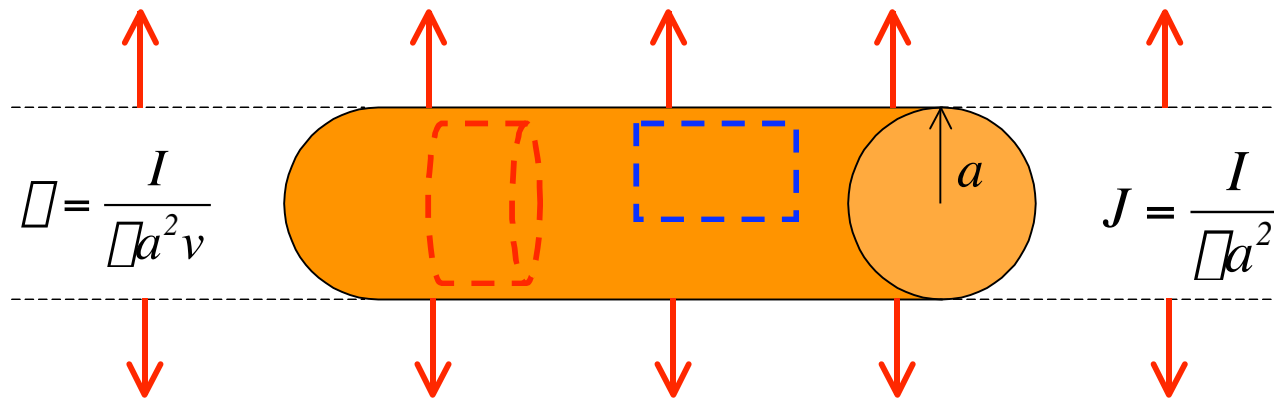
Gauss's law

$$\int \varepsilon_o E \cdot dS = \int \rho \, dV$$

Linear with r

$$2\pi r l \varepsilon_o E_r = \rho 2\pi r^2 l$$

$$\boxed{E_r = \frac{\rho r}{2\varepsilon_o} = \frac{Ir}{2\pi \varepsilon_o a^2 v} \quad \textbf{for} \quad \textbf{\textit{r}} \leq \textbf{\textit{a}}}$$

Ampere's law

$$\int B \cdot dl = \mu_o \int J \cdot dS$$

$$\boxed{B_\vartheta = \frac{\beta}{c} E_r}$$

$$2l B_\vartheta = \mu_o J l r$$

$$\boxed{B_\vartheta = \frac{\mu_o J r}{2} = \mu_o \frac{Ir}{2\pi a^2} \quad \textbf{for} \quad \textbf{\textit{r}} \leq \textbf{\textit{a}}}$$

Lorentz Force

$$F_r = e\left(E_r - \beta c B_\vartheta\right) = e\left(1 - \beta^2\right)E_r = \frac{eE_r}{\gamma^2}$$

has only **radial** component and

is a **linear** function of the transverse coordinate

The attractive magnetic force , which becomes significant at high velocities, tends to compensate for the repulsive electric force. Therefore, space charge defocusing is primarily a non-relativistic

Equation of motion:

$$\gamma m \frac{d^2 r}{dt^2} = \frac{eE_r}{\gamma^2} = \frac{eI}{2\pi\gamma^2 \varepsilon_o a^2 v} r$$

$$\frac{d^2 r}{dt^2} = \beta^2 c^2 \frac{d^2 r}{dz^2}$$

$$\frac{d^2 r}{dz^2} = \frac{eI}{2\pi m \gamma^3 \varepsilon_o a^2 v^3} r = \frac{K}{a^2} r$$

$$K = \frac{eI}{2\pi m \gamma^3 \varepsilon_o v^3} = \frac{2I}{I_o \beta^3 \gamma^3} \qquad \text{Generalized perveance}$$

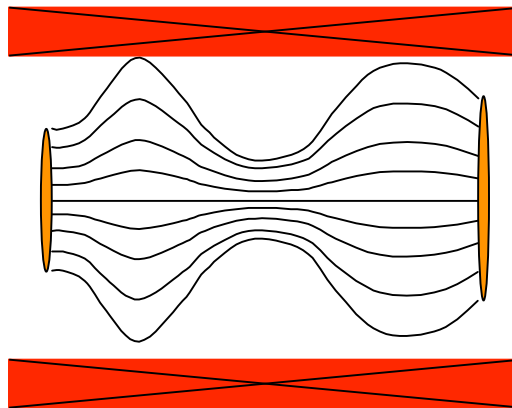$$I_o = \frac{4\pi \varepsilon_o m c^3}{e} \qquad \text{Alfven current}$$

Laminar Beam

If the initial particle velocities depend linearly on the initial coordinates

$$\frac{dr(0)}{dt} = Ar(0)$$

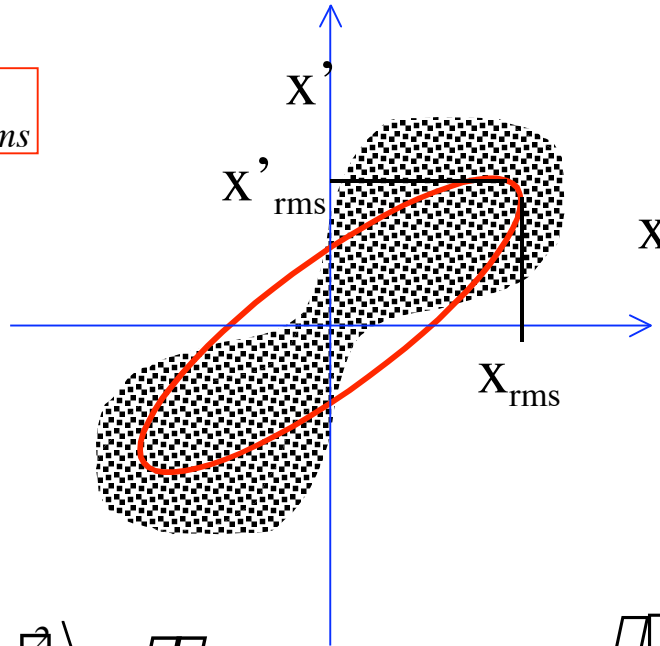then the linear dependence will be conserved during the motion, because of the linearity of the equation of motion.

This means that particle trajectories do not cross ==> **Laminar Beam**

What about RMS Emittance (Lawson)?

$$\gamma x^2 + 2\alpha xx' + \beta x'^2 = \varepsilon_{rms}$$

$$\langle x^2 \rangle = \beta\varepsilon_{rms} \quad \textbf{and} \quad \langle x'^2 \rangle = \gamma\varepsilon_{rms}$$

$$\alpha = -\frac{\beta'}{2} = -\frac{1}{2\varepsilon_{rms}}\frac{d}{dz}\langle x^2 \rangle = -\frac{\langle xx' \rangle}{\varepsilon_{rms}}$$

$$\gamma\beta - \alpha^2 = 1$$

$$\varepsilon_{rms} = \sqrt{\langle x^2 \rangle\langle x'^2 \rangle - \langle xx' \rangle^2}$$

In the phase space (x,x') all particles lie in the interval bounded by the points (a,a').
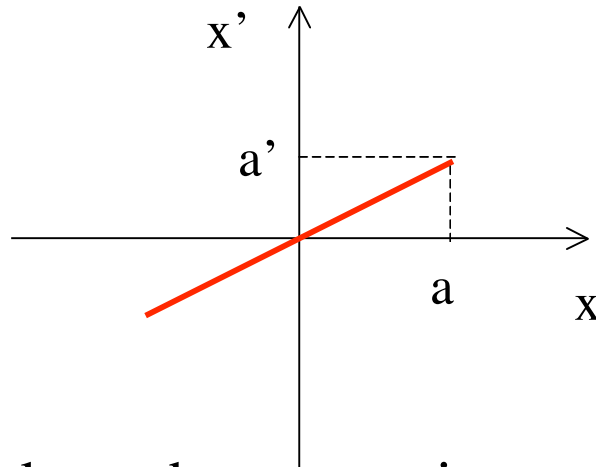
What about the rms emittance?

$$\varepsilon_{rms}^2 = \left\langle x^2 \right\rangle\left\langle x'^2 \right\rangle - \left\langle xx' \right\rangle^2$$

$$x' = Cx^n$$

$$\varepsilon_{rms}^2 = C^2\left(\left\langle x^2 \right\rangle\left\langle x^{2n} \right\rangle - \left\langle x^{n+1} \right\rangle^2\right)$$

When n = 1   ==> $\varepsilon_r = 0$

When n ≠ 1   ==> $\varepsilon_r \neq 0$

The presence of nonlinear space charge forces can distort the
phase space contours and causes emittance growth

$$\rho(r) = \frac{I}{\pi a^2 v}\left(1 - \frac{r^2}{a^2}\right)$$

$$\boxed{E_r = \frac{\rho r}{2\varepsilon_o} = \frac{I}{2\pi\varepsilon_o a^2 v}\left(r - \frac{r^3}{a^2}\right)}$$
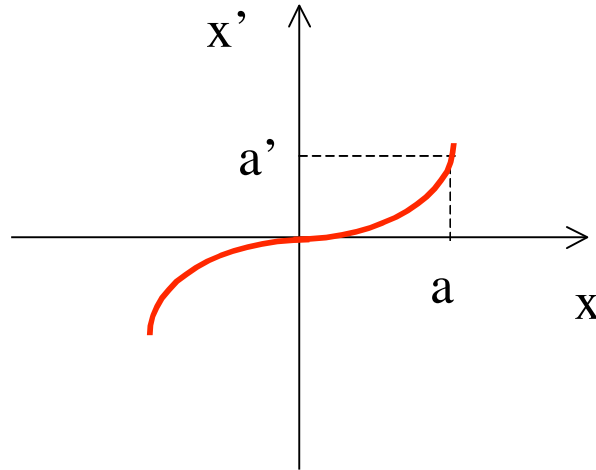
$$\varepsilon_{rms}^2 = C^2\left(\left\langle x^2\right\rangle\left\langle x^{2n}\right\rangle - \left\langle x^{n+1}\right\rangle^2\right) \neq 0$$

Low Energy Case

Space Charge induced emittance oscillations

Bunched Uniform Cylindrical Beam Model

$$v_z = \beta c$$
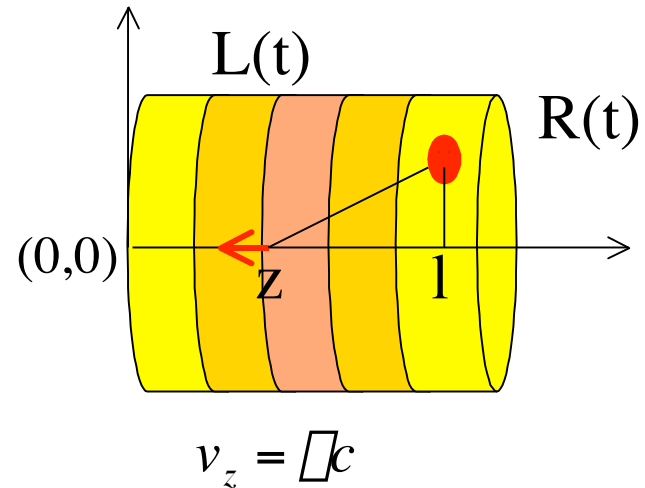
Longitudinal Space Charge field in the bunch moving frame:

$$\tilde{\rho} = \frac{Q}{\pi R^2 \tilde{L}}$$

$$E_z\left(\tilde{z}, r = 0\right) = \frac{\tilde{\rho}}{4\pi\varepsilon_o} \int\limits_0^R \int\limits_0^{2\pi} \int\limits_0^{\tilde{L}} \frac{\left(\tilde{l} - \tilde{z}\right)}{\left[\left(\tilde{l} - \tilde{z}\right)^2 + r^2\right]^{3/2}} \, r \, dr \, d\varphi \, d\tilde{l}$$

$$E_z(\tilde{z}, r = 0) = \frac{\tilde{\rho}}{2\varepsilon_0}\left[\sqrt{R^2 + (\tilde{L} - \tilde{z})^2} - \sqrt{R^2 + \tilde{z}^2} + \left(2\tilde{z} - \tilde{L}\right)\right]$$

Radial Space Charge field in the bunch moving frame by series representation of axisymmetric field:

$$E_r(r,\tilde{z}) \cong \left[\frac{\tilde{\rho}}{\varepsilon_0} - \frac{\partial}{\partial \tilde{z}} E_z(\tilde{z},0) \right] \frac{r}{2} + \left[\cdots \right] \frac{r^3}{16} +$$

$$E_r(r,\tilde{z}) = \frac{\tilde{\rho}}{2\varepsilon_0} \left[\frac{(\tilde{L}-\tilde{z})}{\sqrt{R^2 + (\tilde{L}-\tilde{z})^2}} + \frac{\tilde{z}}{\sqrt{R^2 + \tilde{z}^2}} \right] \frac{r}{2}$$

Lorentz Transformation to the Lab frame

$$L = \frac{\tilde{L}}{\gamma} \quad \Rightarrow \quad \rho = \gamma \tilde{\rho}$$

$$E_r = \gamma \tilde{E}_r$$

$$B_\vartheta = \gamma \frac{\beta}{c} \tilde{E}_r = \frac{\beta}{c} E_r$$

$$F_r = e \frac{E_r}{\gamma^2}$$

$$E_z(z) = \frac{\rho}{\gamma 2\varepsilon_0}\left[\sqrt{R^2 + \gamma^2(L-z)^2} - \sqrt{R^2 + \gamma^2 z^2} + \gamma(2z - L)\right]$$

$$= \frac{\rho}{\gamma 2\varepsilon_0}(\gamma L)\left[\sqrt{\frac{R^2}{\gamma^2 L^2} + (1 - \frac{z}{L})^2} - \sqrt{\frac{R^2}{\gamma^2 L^2} + \frac{z^2}{L^2}} + \left(2\frac{z}{L} - 1\right)\right]$$

$$\zeta = \frac{z}{L} \qquad = \frac{\rho L}{2\varepsilon_0}\left[\sqrt{A + (1-\zeta)^2} - \sqrt{A + \zeta^2} + (2\zeta - 1)\right]$$

$$A = \frac{R}{\gamma L} \qquad \textbf{\textcolor{red}{Beam Aspect Ratio}}$$

$$E_r(r,z) = \frac{\rho}{2\varepsilon_0}\left[\frac{(1-\zeta)}{\sqrt{A^2 + (1-\zeta)^2}} + \frac{\zeta}{\sqrt{A^2 + \zeta^2}}\right]\frac{r}{2}$$

$$= \frac{Ir}{4\pi\varepsilon_0 R^2 v}g(\zeta) \qquad\qquad \rho = \frac{Q}{\pi R^2 L} = \frac{I}{\pi R^2 v}$$

It is still a linear field with r but with a longitudinal correlation ζ
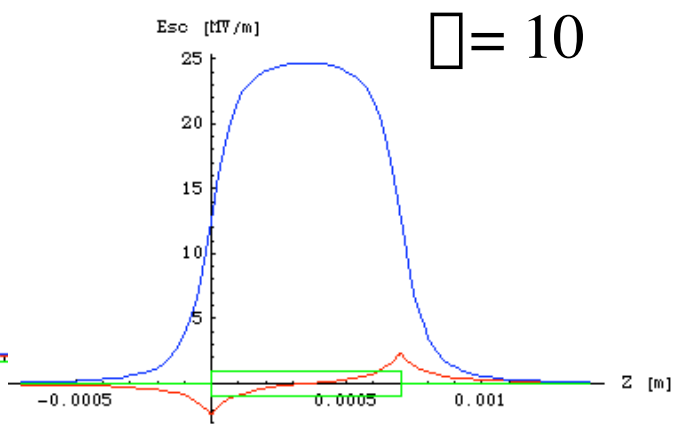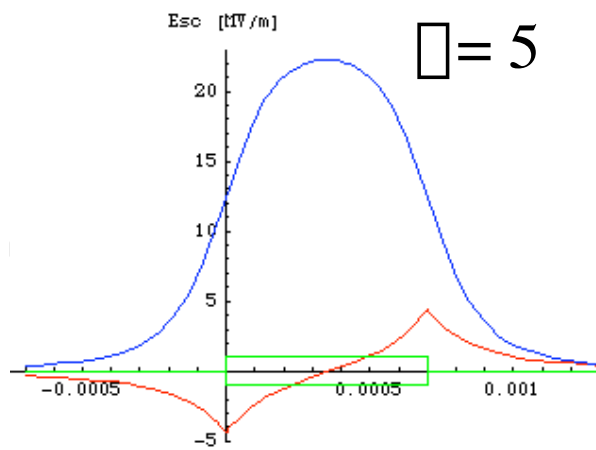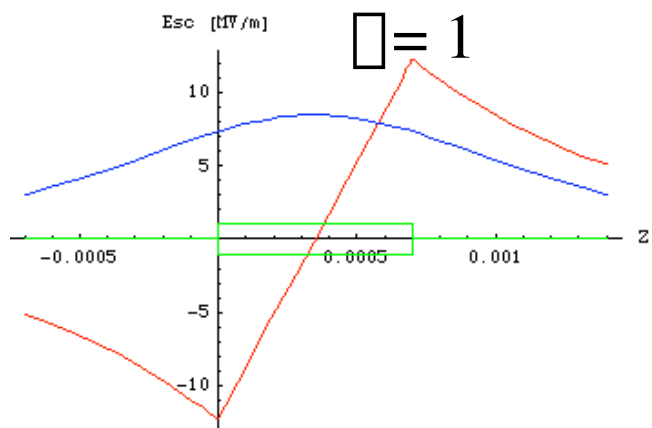
$$A_{r,s} \equiv R_s / (\gamma_s L)$$
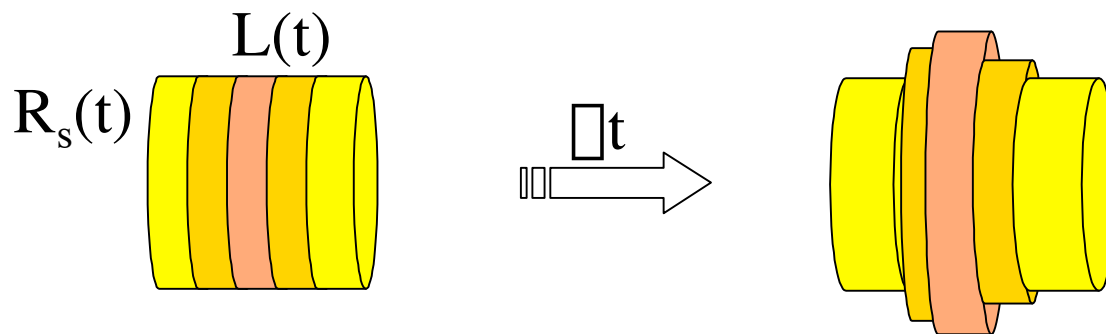
Simple Case: Transport in a Long Solenoid

$$k_s = \frac{qB}{2mc\beta\gamma}$$

$$K(\zeta) = \frac{2Ig(\zeta)}{I_o(\beta\gamma)^3}$$

g(ζ) plot with vertical axis labeled g(ζ) showing values 0.5, 0.6, 0.7, 0.8, 0.9 and horizontal axis ζ with values 0, 0.0005, 0.001, 0.0015, 0.002, 0.0025
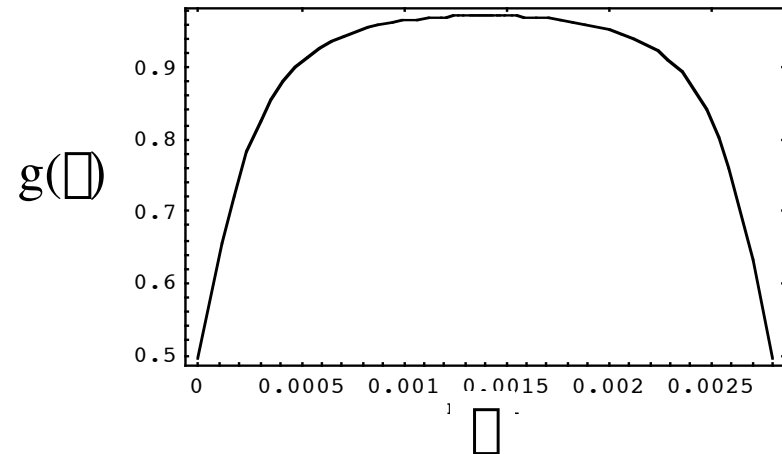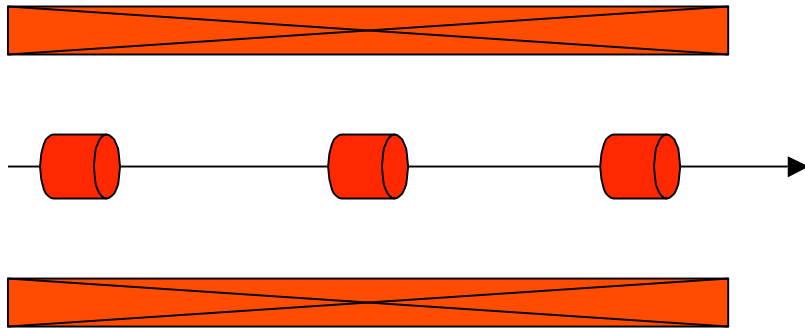
$$R'' + k_s^2 R = \frac{K(\zeta)}{R}$$

$$R'' = 0 \quad \Longrightarrow \text{ Equilibrium solution ? } \Longrightarrow \quad R_{eq}(\zeta) = \frac{\sqrt{K(\zeta)}}{k_s}$$

Small perturbations around the equilibrium solution

$$R'' + k_s^2 R = \frac{K(\zeta)}{R}$$

$$R(\zeta) = R_{eq}(\zeta) + \delta r(\zeta)$$

$$\delta r'' + k_s^2 \left(R_{eq} + \delta r \right) = \frac{K(\zeta)}{\left(R_{eq} + \delta r \right)}$$

$$\delta r'' + k_s^2 \left(R_{eq} + \delta r \right) = \frac{K(\zeta)}{R_{eq}\left(1 + \dfrac{\delta r}{R_{eq}} \right)} = \frac{K(\zeta)}{R_{eq}}\left(1 - \frac{\delta r}{R_{eq}} \right)$$

$$\delta r''(\zeta) + 2k_s^2 \delta r(\zeta) = 0$$

$$\delta r''(\zeta) + 2k_s^2 \delta r(\zeta) = 0$$

$$R(\zeta) = R_{eq}(\zeta) + \delta r(\zeta) \cos\left(\sqrt{2}\,k_s z\right)$$

$$R'(\zeta) = -\delta r(\zeta) \sin\left(\sqrt{2}\,k_s z\right)$$
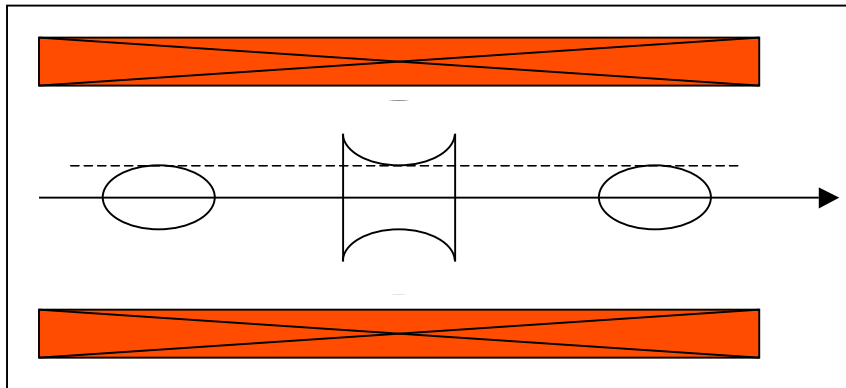
Plasma frequency

$$k_p = \sqrt{2}\,k_s$$

Emittance Oscillations are driven by space charge differential defocusing in core and tails of the beam
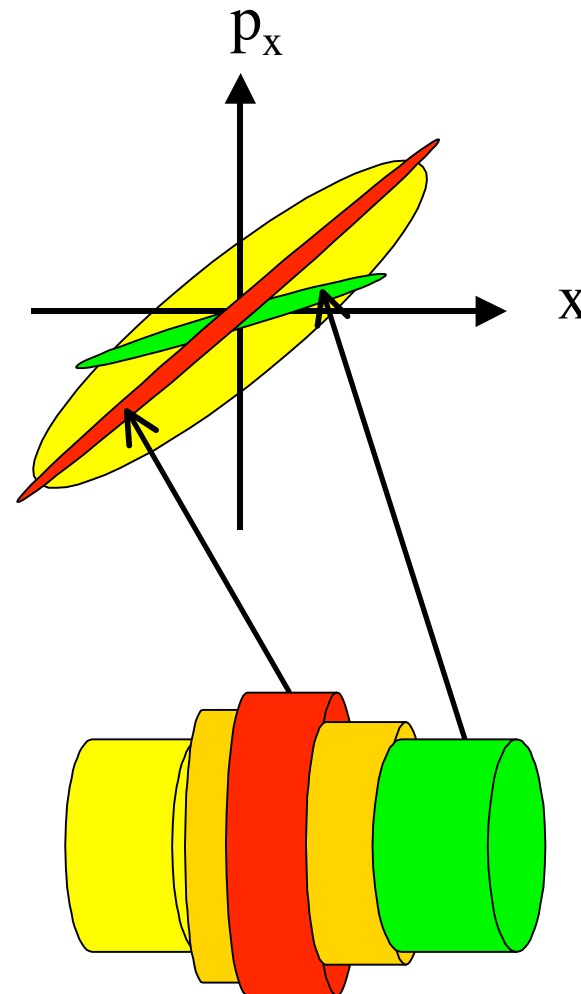
Envelope oscillations drive Emittance oscillations
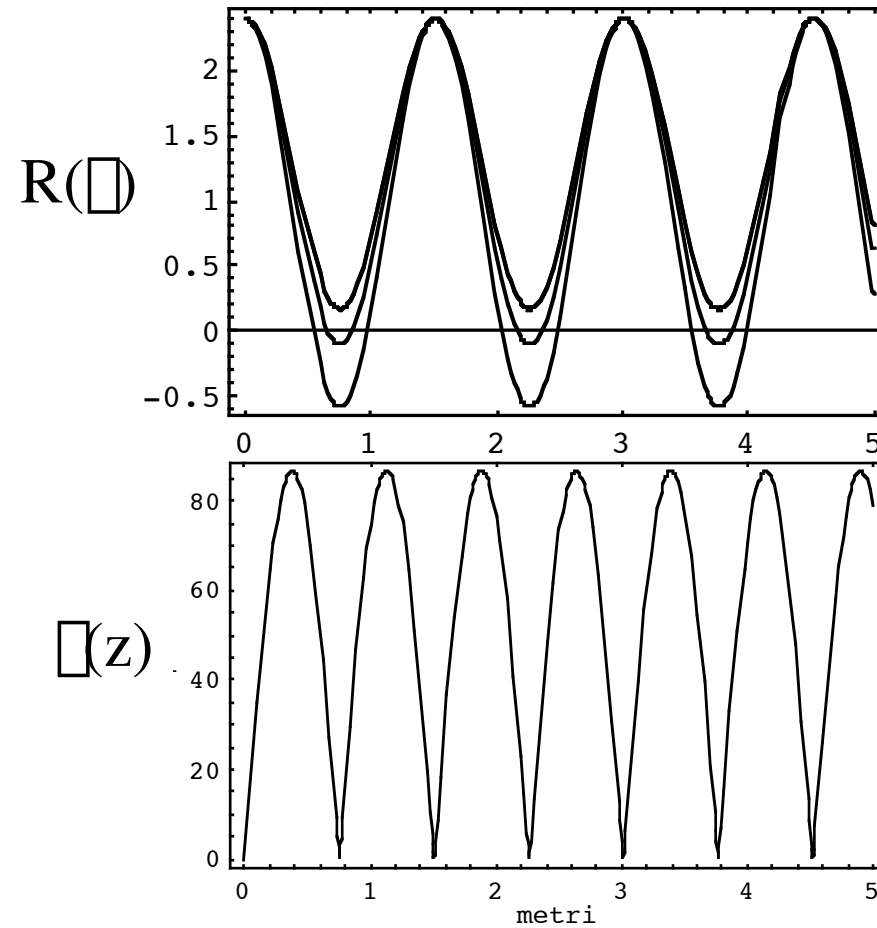
$$\frac{\delta\gamma}{\gamma} = 0$$

$$k_s = \frac{qB}{2mc\beta\gamma}$$

$$\varepsilon(z) = \sqrt{\left\langle R^2\right\rangle\left\langle R'^2\right\rangle - \left\langle RR'\right\rangle^2}\ \left|sin\left(\sqrt{2}k_s z\right)\right|$$

Perturbed trajectories oscillate around the equilibrium with the same frequency but with different amplitudes
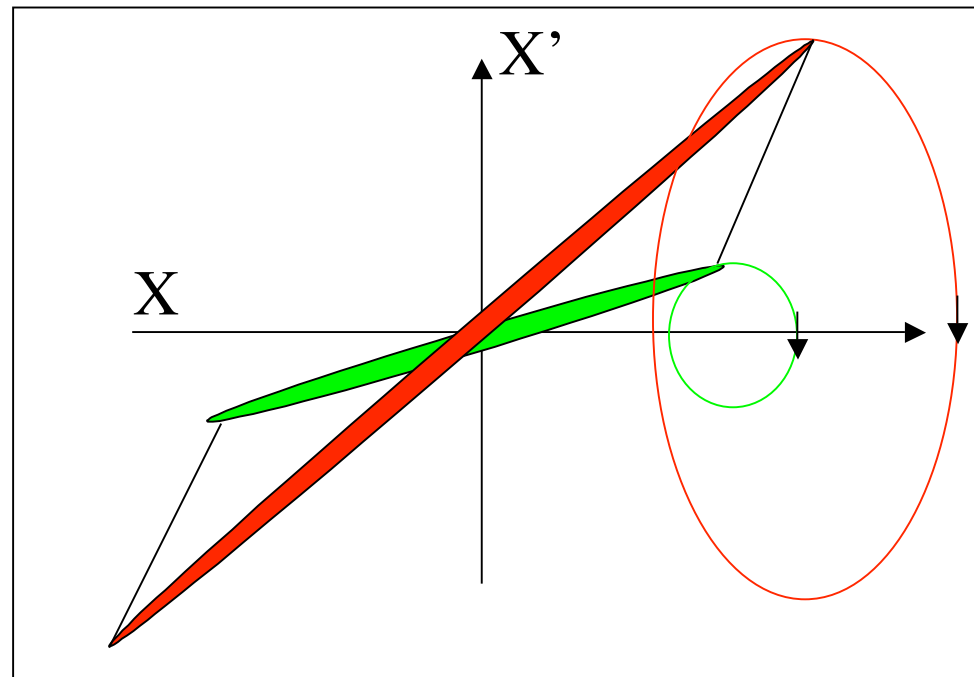
HOMDYN Simulation of a L-band photo-injector

$Q = 1$ nC
$R = 1.5$ mm
$L = 20$ ps
$\varepsilon_{th} = 0.45$ mm-mrad
$E_{peak} = 60$ MV/m (Gun)
$E_{acc} = 13$ MV/m (Cryo1)
$B = 1.9$ kG (Solenoid)

$I = 50$ A
$E = 120$ MeV
$\varepsilon_n = 0.6$ mm-mrad

sigma_x_[mm]
enx_[um]

ε_n
[mm-mrad]

6 MeV

3.5 m

Z [m]
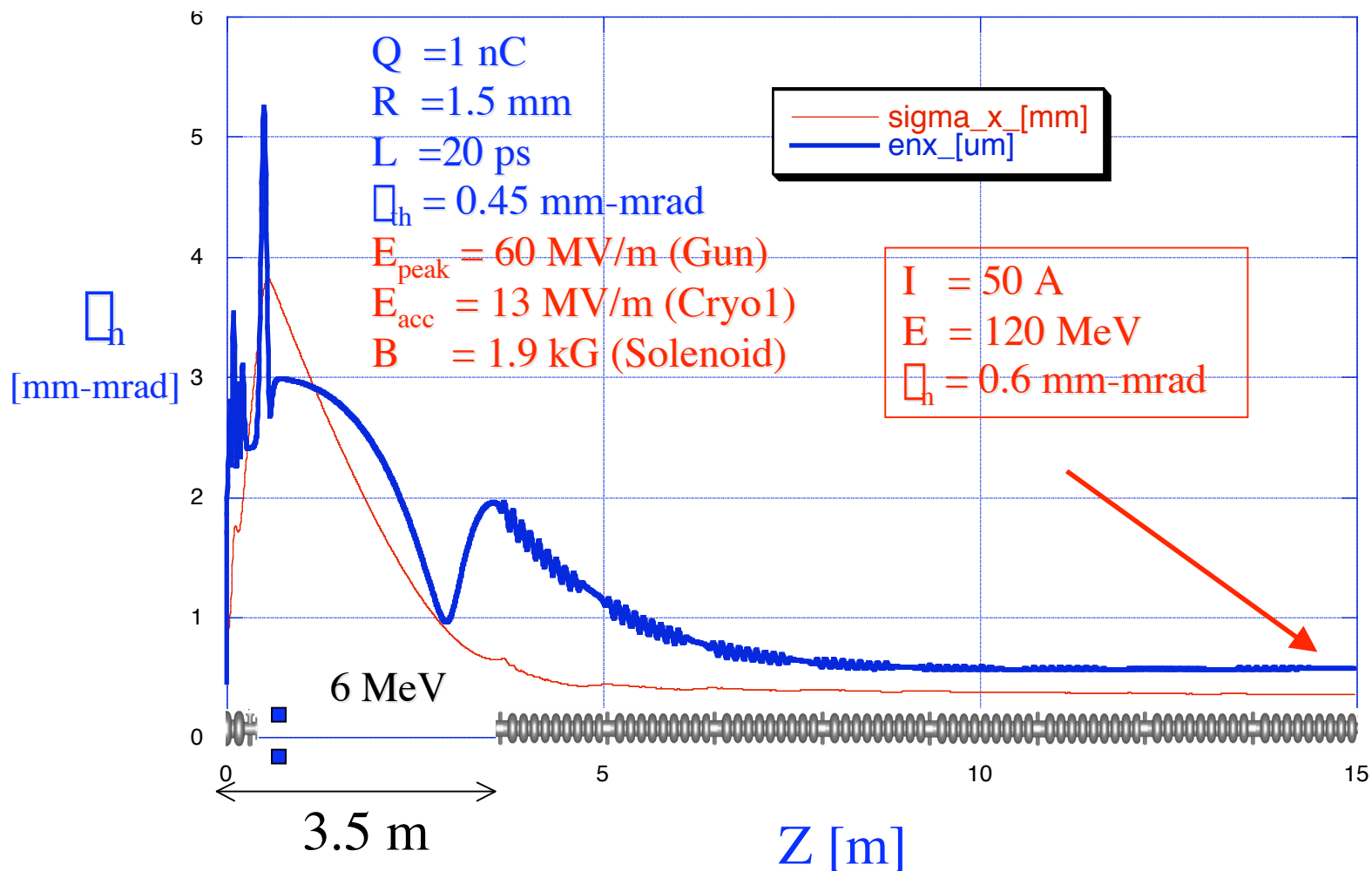
High Energy Case

Effects of conducting or magnetic screens

Let us consider a point charge close to a conducting screen.

The electrostatic field can be derived through the "image method". Since the metallic screen is an equi-potential plane, it can be removed provided that a "virtual" charge is introduced such that the potential is constant at the position of the screen

A constant current in the free space produces circular magnetic field.

If $\mu_r \approx 1$, the material, even in the case of a good conductor, does not affect the field lines.
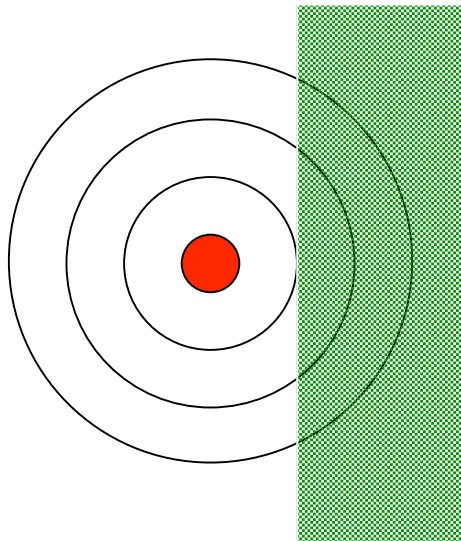
However, if the material is of **ferromagnetic type**, with $\mu_r \gg 1$, due to its magnetisation, the magnetic field lines are strongly affected, inside and outside the material. In particular a very high magnetic permeability makes the tangential magnetic field zero at the boundary so that the magnetic field is perpendicular to the surface, just like the electric field lines close to a conductor.
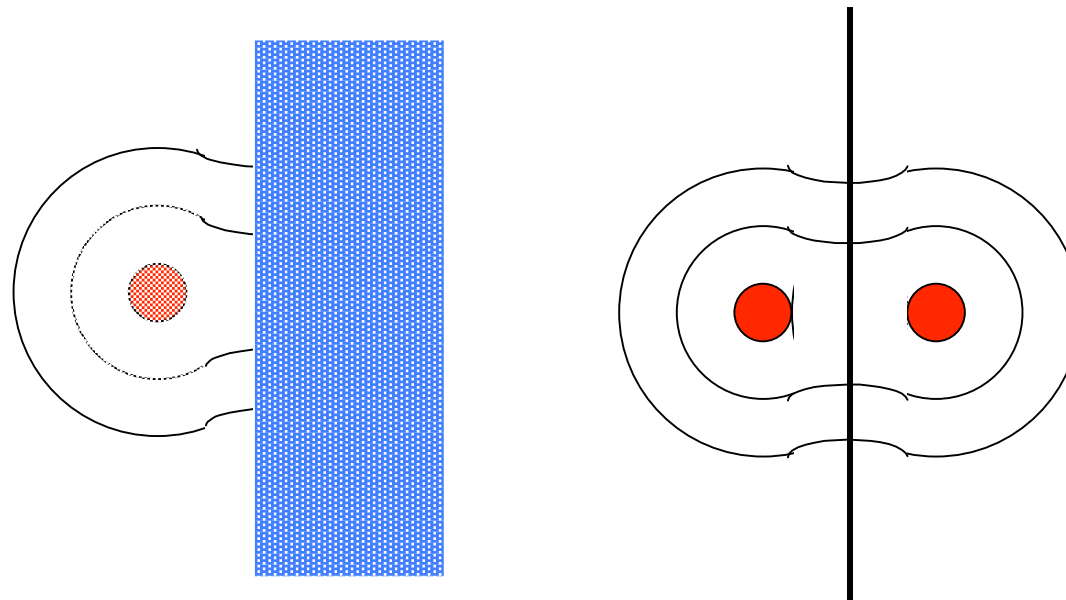
In analogy with the image method for charges close to conducting screens, we get the magnetic field, in the region outside the material, as superposition of the fields due to two symmetric equal currents flowing in the same direction.

The **scenario changes** when we deal with **time-varying fields** for which it is necessary to compare the **wall thickness** and the **skin depth** (region of penetration of the e.m. fields) in the conductor.

If the fields penetrate and pass through the material, we are practically in the static boundary conditions case. Conversely, if the skin depth is very small, fields do not penetrate, the electric filed lines are perpendicular to the wall, as in the static case, while the magnetic field line are tangent to the surface.

In this case, the magnetic field lines can be obtained by considering two linear charge distributions with opposite sign, flowing in the same direction (opposite charges, opposite currents).
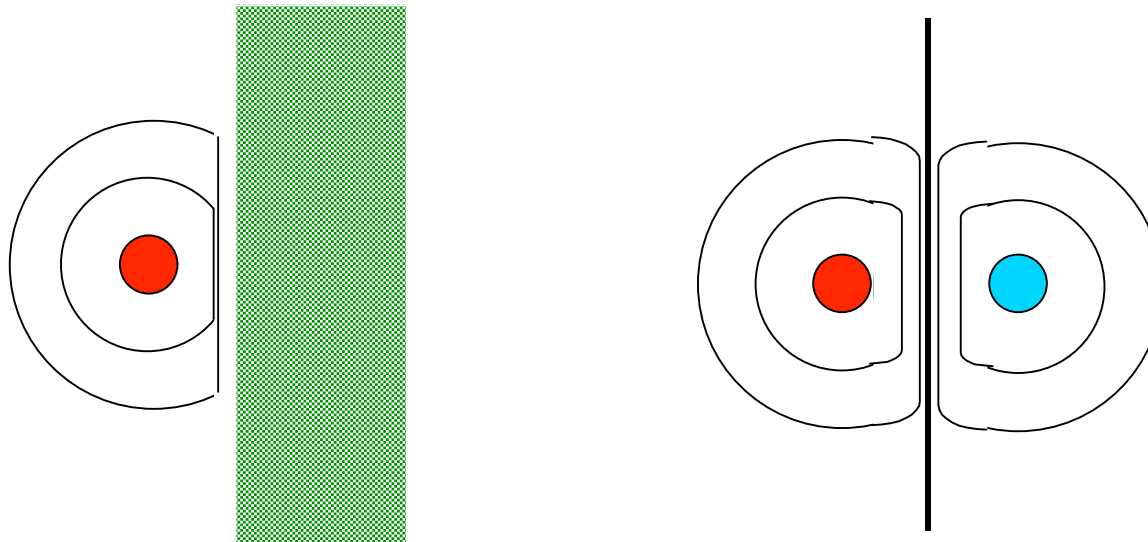
Circular Perfectly Conducting Pipe (Beam at Center)

In the case of charge distribution, and $\gamma \to \infty$, the electric field lines are perpendicular to the direction of motion. The transverse fields intensity can be computed like in the static case, applying the Gauss and Ampere laws. Due to the symmetry, the transverse fields produced by an ultra-relativistic charge inside the pipe are the same as in the free space. This implies that for a distribution with cylindrical symmetry, in the ultra-relativistic regime, there is a cancellation of the electric and magnetic forces. Therefore, the uniform beam produces exactly the same forces as in the free space. It is interesting to note that this result does not depend on the longitudinal distribution of the beam.

$$\lambda(r) = \lambda_o \left(\frac{r}{a}\right)^2 \; ; \int_S E_r(2\pi r)\Delta z = \frac{\lambda(r)\Delta z}{\varepsilon_o}$$

$$E_r = \frac{\lambda(r)\Delta z}{2\pi\varepsilon_o r}; \quad B_\theta = \frac{\beta}{c} E_r$$

$$E_r(\boldsymbol{r}) = \frac{\lambda_o}{2\pi\,\varepsilon_o}\frac{r}{a^2}; \quad B_\theta(\boldsymbol{r}) = \frac{\lambda_o\beta}{2\pi\varepsilon_o c}\frac{r}{a^2}$$
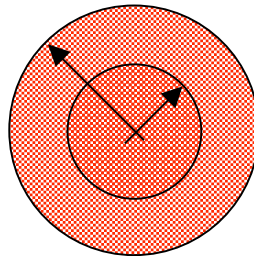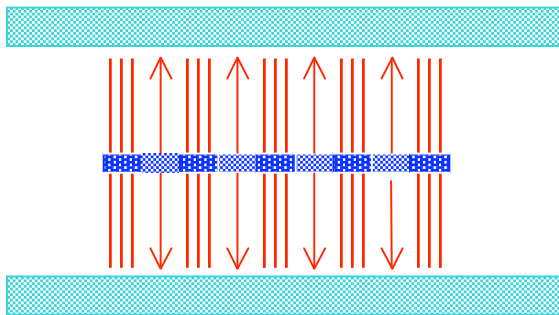
$$F_\perp(r) = e(E_r - \beta c\, B_\theta) = \frac{e}{\gamma^2} E_r$$

Parallel Plates (Beam at Center)

In some cases, the beam pipe cross section is such that we can consider only the surfaces closer to the beam, which behave like two parallel plates. In this case, we use the image method to a charge distribution of radius a between two conducting plates $2h$ apart. By applying the superposition principle we get the total image field at a position y inside the beam.

$$E_y^{im}(z,y) = \frac{\lambda(z)}{2\pi\,\varepsilon_o} \sum_{n=1}^{\infty} (-1)^n \left[\frac{1}{2nh+y} - \frac{1}{2nh-y} \right]$$

$$E_y^{im}(z,y) = \frac{\lambda(z)}{2\pi\,\varepsilon_o} \sum_{n=1}^{\infty} (-1)^n \frac{-2y}{(2nh)^2 - y^2} \cong \frac{\lambda(z)}{4\pi\,\varepsilon_o h^2} \frac{\pi^2}{12} y$$

Where we have assumed $h \gg a > y$.
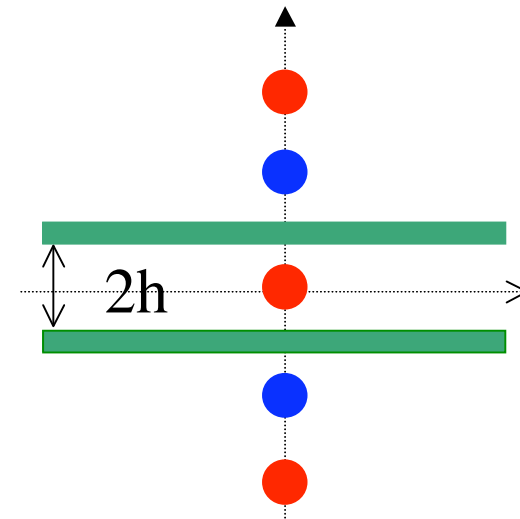
For d.c. or slowly varying currents, the boundary condition imposed by the conducting plates does not affect the magnetic field. As a consequence there is no cancellation effect for the fields produced by the "image" charges.

From the divergence equation we derive also the other transverse component:

$$\frac{\partial}{\partial x}E_x^{im} = -\frac{\partial}{\partial y}E_y^{im} \Rightarrow E_x^{im}(z,x) = \frac{-\lambda(z)}{4\pi\,\varepsilon_o h^2}\frac{\pi^2}{12}x$$

Including also the direct space charge force, we get:

$$F_x(z,x) = \frac{e\lambda(z)x}{\pi\,\varepsilon_o}\left(\frac{1}{2a^2\gamma^2} - \frac{\pi^2}{48h^2}\right)$$

$$F_y(z,x) = \frac{e\lambda(z)y}{\pi\,\varepsilon_o}\left(\frac{1}{2a^2\gamma^2} + \frac{\pi^2}{48h^2}\right)$$

Therefore, for γ>>1, and for d.c. or slowly varying currents the cancellation effect applies only for the direct space charge forces. There is no cancellation of the electric and magnetic forces due to the "image" charges.

Parallel Plates - General expression of the force

Taking into account all the boundary conditions for d.c. and a.c. currents, we can write the expression of the force as:

$$F_u = \frac{e}{2\pi\,\varepsilon_o}\left[\frac{1}{\gamma^2}\left(\frac{1}{a^2} \mp \frac{\pi^2}{24h^2}\right)\lambda \mp \beta^2\left(\frac{\pi^2}{24h^2} + \frac{\pi^2}{12g^2}\right)\overline{\lambda}\right]u$$

where λ is the total current, and $\overline{\lambda}$ its d.c. part. We take the sign (+) if u=y, and the sign (−) if u=x.

The betatron motion

We consider a perfectly circular accelerator with radius ρ_x. The beam circulates inside the beam pipe. The transverse single particle motion in the linear regime, is derived from the equation of motion. Including the self field forces in the motion equation, we have
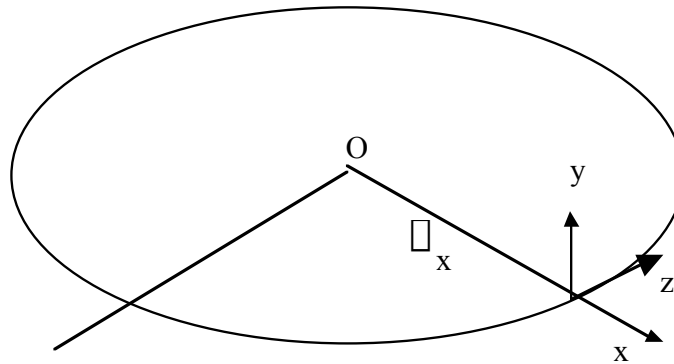
$$\frac{d(m\gamma\,\boldsymbol{v})}{dt} = \boldsymbol{F}^{ext}(\vec{r}) + \boldsymbol{F}^{self}(\vec{r})$$

$$\frac{d\boldsymbol{v}}{dt} = \frac{\boldsymbol{F}^{ext}(\vec{r}) + \boldsymbol{F}^{self}(\vec{r})}{m\gamma}$$

Following the same steps already seen in the "transverse dynamics" lectures, we write:

$$\vec{r} = \left(\rho_x + x\right)\hat{e}_x + y\hat{e}_y$$

$$\vec{v} = \dot{x}\hat{e}_x + \dot{y}\hat{e}_y + \omega_o\left(\rho_x + x\right)\hat{e}_z$$

$$\vec{a} = \left[\ddot{x} - \omega_o^2\left(\rho_x + x\right)\right]\hat{e}_x + \ddot{y}\hat{e}_y + \left[\dot{\omega}_o\left(\rho_x + x\right) + 2\omega_o\dot{x}\right]\hat{e}_z$$

For the motion along x:

$$\ddot{x} - \omega_o^2\left(\rho_x + x\right) = \frac{1}{m\gamma}\left(F_x^{ext} + F_x^{self}\right)$$

which is rewritten with respect to the azimuthal position s=v_zt:

$$\ddot{x} = v_z^2 x'' = \omega_o^2\left(\rho_x + x\right)^2 x''$$

$$x'' - \frac{1}{\rho_x + x} = \frac{1}{m v_z^2 \gamma}\left(F_x^{ext} + F_x^{self}\right)$$

We assume small transverse displacements x, and only transverse quadrupole forces which keep the beam around the closed orbit:

$$F_x^{ext} \approx \left(\frac{\partial F_x^{ext}}{\partial x} \right)_{x=0} x \qquad\qquad x << \rho_x$$

Putting $v_z = \beta c$, we get

$$x'' + \left[\frac{1}{\rho_x^2} - \frac{1}{\beta^2 E_o} \left(\frac{\partial F_x^{ext}}{\partial x} \right)_{x=0} \right] x = \frac{1}{\beta^2 E_o} F_x^{self}(x)$$

where E_o is the particle energy. This equation expressed as function of "s" reads:

$$x''(s) + \left[\frac{1}{\rho_x^2(s)} + K_x(s) \right] x(s) = \frac{1}{\beta^2 E_o} F_x^{self}(x,s)$$

In the analysis of the motion of the particles in presence of the self field, we will adopt a simplified model where particles execute simple harmonic oscillations around the reference orbit.

This is the case where the focussing term is constant. Although this condition in never fulfilled in a real accelerator, it provides a reliable model for the description of the beam instabilities

$$x''(s) + K_x x(s) = \frac{1}{\beta^2 E_o} F_x^{self}(x)$$

$$x''(s) + \left(\frac{Q_x}{\rho_x}\right)^2 x(s) = \frac{1}{\beta^2 E_o} F_x^{self}(x,s)$$

$$x(s) = A_x \cos\left[\sqrt{K_x}\, s - \varphi_x\right]$$

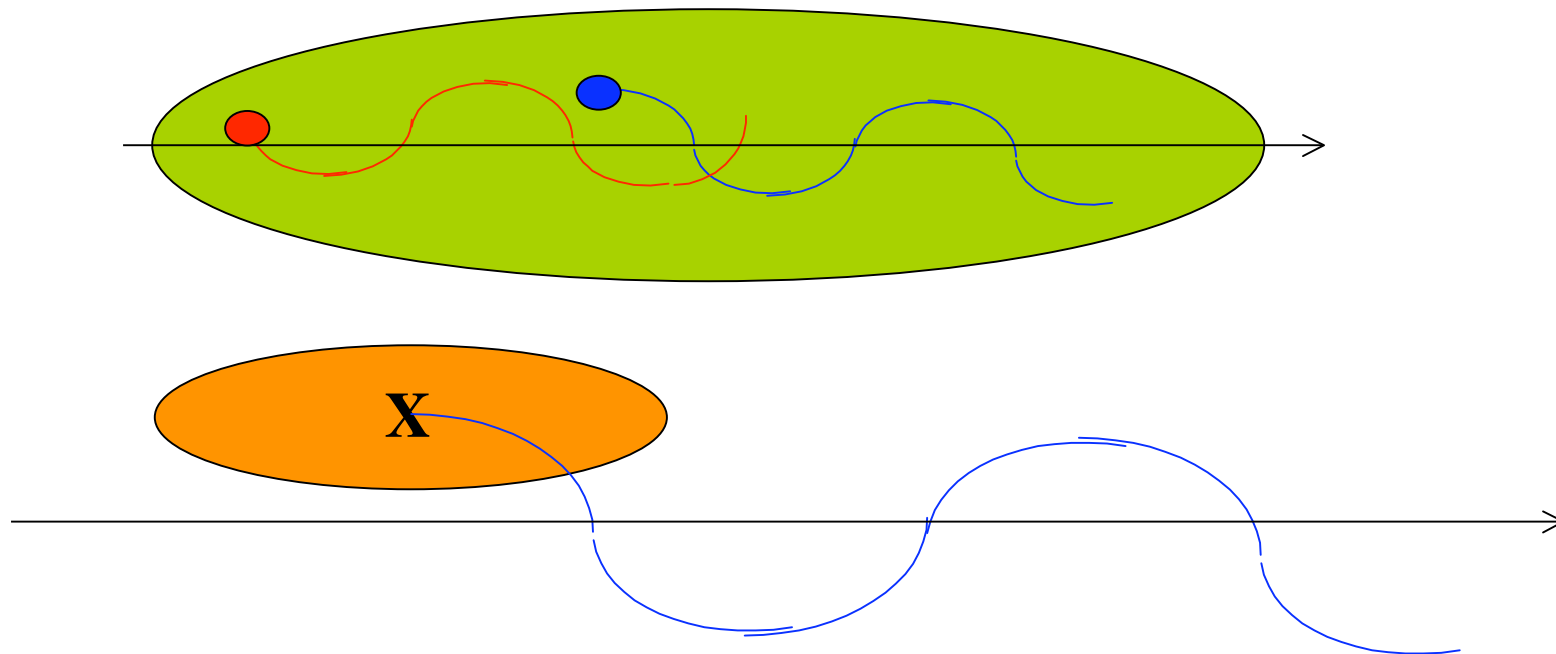$$a_x \sqrt{\beta_x} = A_x \Rightarrow \beta_x \;\; const.$$

$$K_x(s)\beta_x^2 = 1$$

$$\beta_x = \frac{1}{\mu_x'} = \frac{1}{\sqrt{K_x}}$$

$$\mu_x(s) = \sqrt{K_x}\, s$$

$$Q_x = \frac{\omega_x}{\omega_o} = \frac{1}{2\pi}\int_o^L \frac{ds'}{\beta(s')} = \rho_x\sqrt{K_x} \Rightarrow K_x = \left(\frac{Q_x}{\rho_x}\right)^2$$

Shift and Spread of the Incoherent Tunes

When the beam is located at the centre of symmetry of the pipe, the e.m. forces due to space charge and images cannot affect the motion of the centre of mass **(coherent),** but change the trajectory of individual charges in the beam **(incoherent).**

X

These force may have a complicate dependence on the charge position. A simple analysis is done considering only the linear expansion of the self-fields forces around the equilibrium trajectory.

Transverse Incoherent Effects

We take the linear term of the transverse force in the betatron equation:

$$F_u^{s.c.}(u,z) \cong \left(\frac{\partial F_u^{s.c.}}{\partial u}\right)_{u=0} u$$

$$u'' + \left(\frac{Q_u}{\rho_x}\right)^2 u = \frac{1}{\beta^2 E_o}\left(\frac{\partial F_u^{s.c.}}{\partial u}\right)_{u=0} u$$

$$\left(Q_u + \Delta Q_u\right)^2 \cong Q_u + 2Q_u\Delta Q_u \Rightarrow \Delta Q_u = -\frac{\rho_x^2}{2\beta^2 E_o Q_u}\left(\frac{\partial F_u^{s.c.}}{\partial u}\right)$$

The betatron shift is negative since the space charge forces are defocusing on both planes. Notice that the tune shift is in general function of "z", therefore there is a tune spread inside the beam.

Consequences of the space charge tune shifts

In circular accelerators the values of the betatron tunes should not be close to rational numbers in order to avoid the crossing of linear and non-linear resonances where the beam becomes unstable. The tune spread induced by the space charge force can make hard to satisfy this basic requirement. Typically, in order to avoid major resonances the stability requires

$$\left|\Delta Q_u\right| < 0.5$$

In a LINAC or a beam transport line, the space charge forces cause an energy spread and perturb the equilibrium beam size. It is required that the defocusing space charge forces must not be larger than the external forces.