LANDAU DAMPING

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CAS 2003, DESY Zeuthen; Albert Hofmann

Introduction
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Introduction

Mechanism of Landau damping

A single oscillator with resonant frequency ω_r reacts to a pulse excitation with an free oscillation lasting for a long time or being slightly damped. A lasting harmonic excitation with frequency ω results in a forced oscillation at the same frequency ω but a phase which depends on the difference $\omega - \omega_r$. For $\omega = \omega_r$ the oscillation amplitude grows linear with time.
We consider a set of oscillators having different resonant frequencies ω_{ri} with distribution $f(\omega_{ri})$. A pulse excitation results in an oscillation of each oscillator with the same initial velocity $\dot{x}(0)$ followed by a free oscillation with individual frequencies ω_{rj}. For instabilities or beam observation the **center-of-mass motion** of the particles is relevant. Due to their different ω_{ri} the freely oscillating particles change their phase with respect to each other and the center-of-mass motion is slowly reduced. In case of a harmonic excitation the phases of the individual particle oscillations are different and the center-of-mass motion has a smaller amplitude than the individual particles.
This represents a kind of damping where the **coherent** center-of-mass motion is reduced compared to the **incoherent** motion of the particles. This damping is usually not exponential and differs in many respect from other damping mechanism. It depends on the form of the resonant frequency distribution $f(\omega_{rj})$ but mainly on its width, i.e. the frequency spread. For a frequency distribution given by an external parameter the damping is proportional to $f(\omega)$ itself. If this distribution is determined by the amplitude dependence of ω_{rj} it is affected by the excitation giving a damping proportional to the derivative $df(\omega_r)/d\omega_r$.

Treatment of Landau damping

Landau damping can be understood from different points of view and presented in different ways. We treat it here in a manner close to beam observation and experiment.

We calculate the center-of-mass response of a beam with resonant frequency distribution $f(\omega_r)$ to a pulse or harmonic excitation and compare it with experiments.

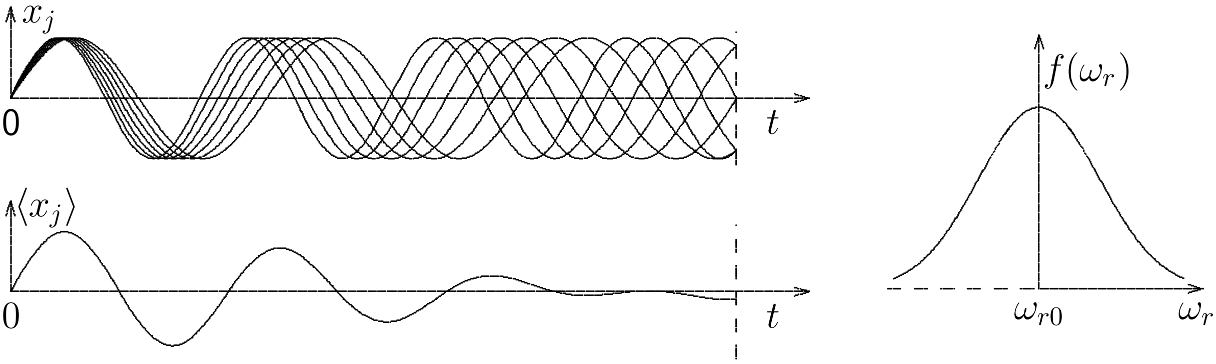
This center-of-mass motion induces fields in the beam surroundings which act back on the beam and can enhance the excitation. The electrical properties of the components surrounding the beam, relevant for this effect, can be expressed by an impedance. The fields induced in this impedance can be sufficiently large to keep this process going leading to a self excitation. This leads to an instability having a threshold determined by beam response and impedance. Below this threshold the frequency spread eliminates any coherent motion at **infinitesimal small amplitudes** before it can grow, we have stability. Above, the voltage induced in the resistive part of the impedance leads to an increase of initial coherent motion and we have an instability.

The amount of Landau damping is proportional to the frequency distribution $f(\omega)$ or its derivative at the frequency ω at which the instability occurs. It can happen that the coherent (center-of-mass) motion has a different frequency than the incoherent individual particle frequencies. In this case Landau damping might become ineffective and we can get an instability for a very small resistive impedance.

We will calculate the beam response and Landau damping for transverse and longitudinal oscillation of a coasting (un-bunched) beam. From this we can determine the maximum transverse and longitudinal impedance which still does not create an instability and represent this in the so-called stability diagram.

Response of an oscillator set to excitation

Response of an oscillator set to a pulse excitation



A set of oscillators j of different resonant frequencies ω_{rj} receive at $t = 0$ a kick giving each the same velocity $\dot{x}_j(0^+) = \dot{x}_0$. Each performs a harmonic oscillation of the form

$$\dot{x}_j(t) = \dot{x}_0 \cos(\omega_{rj} t)\ ,\ x_j = \frac{\dot{x}_0}{\omega_{rj}} \sin(\omega_{rj} t)$$

The observer measures only the center-of-mass motion $\langle x_j(t) \rangle = \frac{1}{N} \Sigma\, x_j(t)$ of particles which is 'damped' but each particles still has its original amplitude $\hat{x}_j(t) = \dot{x}_0/\omega_{rj}$. Many particles with normalized, narrow distribution $f(\omega_r)$, around ω_{r0} with $\int f(\omega_r) d\omega_r = 1$, $\Delta\omega_r = \omega_r - \omega_{r0} \ll \omega_{r0}$ have a center-of- mass motion

$$\begin{aligned} \langle \dot{x}(t) \rangle &= \dot{x}_0 \int f(\omega_r) \cos(\omega_r t) d\omega_r \\ \langle x(t) \rangle &= \dot{x}_0 \int \frac{f(\omega_r)}{\omega_r} \sin(\omega_r t) d\omega_r \approx \frac{\dot{x}_0}{\omega_{r0}} \int f(\omega_r) \sin(\omega_r t) d\omega_r \end{aligned}$$

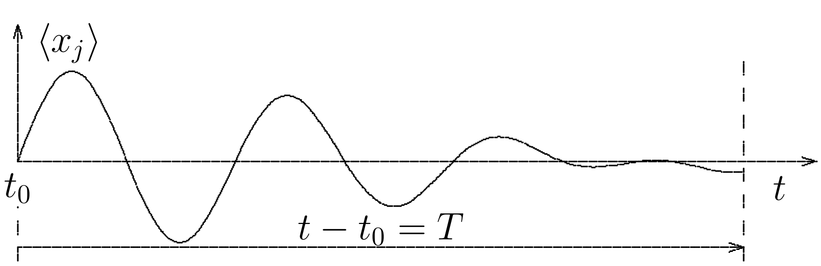
Expressing the velocity response by the difference frequency $\Delta\omega_r$

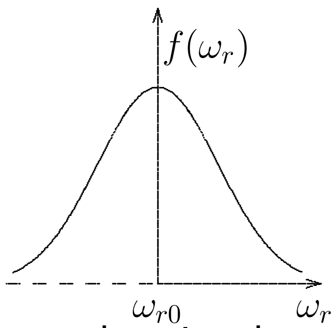
$$\begin{aligned} g(t) = \frac{\langle \dot{x}(t) \rangle}{\dot{x}_0} &= \cos(\omega_{r0} t) \int f(\Delta\omega_r) \cos(\Delta\omega_r t) d\omega_r \\ &\quad - \sin(\omega_{r0} t) \int f(\Delta\omega_r)) \sin(\Delta\omega_r t) d\omega_r \end{aligned}$$

$$\begin{aligned} g(t) &= \mathcal{F}^{-1}_{\cos}(f(\Delta\omega_r)) \cos(\omega_{r0} t) - \mathcal{F}^{-1}_{\sin}(f(\Delta\omega_r)) \sin(\omega_{r0} t) \\ &= \mathcal{F}^{-1}(f(\Delta\omega_r)) \mathrm{e}^{i\omega_{r0} t} \end{aligned}$$

The center-of-mass velocity response $g(t)$ of an oscillator set with resonance frequency distribution $f(\Delta\omega_r)$ to a pulse excitation (Green function) is the inverse Fourier transform of this distribution times an oscillation with the central frequency ω_{r0}.

Response of an oscillator set to a harmonic excitation





We assume oscillators having a very small damping and excite them at the time $t = t_0$. The resulting velocity response of a single particle is

$$\dot{x}_j(t) = \mathrm{e}^{-a(t-t_0)} \cos(\omega_{rj}(t - t_0))$$

A harmonic excitation with frequency ω starting at $t_0 = -\infty$, lasting until observation time t is treated as infinitesimal kicks with harmonic modulation

$$d\dot{x}_0 = \frac{d\hat{\dot{x}}}{dt_0} \cos(\omega t_0) dt_0 = G dt_0 = \hat{G} \cos(\omega t_0) dt_0$$

where G is an acceleration. The velocity obtained at the time t is

$$\dot{x}(t) = \hat{G} \int_{-\infty}^{t} \cos(\omega t_0) \mathrm{e}^{-a(t-t_0)} \cos(\omega_r(t - t_0)) dt_0$$

substituting $T = t - t_0$ and developing $\cos(\omega t_0) = \cos(\omega(t - T))$

$$\begin{aligned} \dot{x}(t) &= \hat{G} \int_0^{\infty} (\cos(\omega t)\cos(\omega T) + \sin(\omega t)\sin(\omega T))\, \mathrm{e}^{-aT} \cos(\omega_r T) dT \\ &= \hat{G}\,(I_1 \cos(\omega t) + I_2 \sin(\omega t)) \end{aligned}$$

with the two integrals

$$I_1 = \int_0^{\infty} \mathrm{e}^{-aT} \cos(\omega T)\cos(\omega_r T) dT = \left(\frac{a/2}{a^2 + (\omega - \omega_r)^2} + \frac{a/2}{a^2 + (\omega + \omega_r)^2} \right)$$

$$I_2 = \int_0^{\infty} \mathrm{e}^{-aT} \sin(\omega T)\cos(\omega_r T) dT = \left(\frac{(\omega - \omega_r)/2}{a^2 + (\omega - \omega_r)^2} + \frac{(\omega + \omega_r)/2}{a^2 + (\omega + \omega_r)^2} \right)$$

We assume $\omega \approx \omega_{r0}$ and weak damping $a \ll \omega$.

$$I_1 \approx \frac{a}{2(a^2 + (\omega - \omega_r)^2)}\,,\ I_2 \approx \frac{\omega - \omega_r}{2(a^2 + (\omega - \omega_r)^2)}$$

$$\text{with} \quad \int_{-\infty}^{\infty} I_1 d\omega_r = \frac{1}{2} \arctan\left(\frac{\omega_r - \omega}{a}\right)\Bigg|_{-\infty}^{\infty} = \frac{\pi}{2}$$

$$I_1 \approx \frac{a}{2(a^2+(\omega-\omega_r)^2)} \ , \ I_2 \approx \frac{\omega-\omega_r}{2(a^2+(\omega-\omega_r)^2)}$$
$$\text{with} \ \int_{-\infty}^{\infty} I_1 d\omega_r = \frac{\pi}{2}0$$

For $a \to 0$ the first expression vanishes except for the point $\omega - \omega_r \to 0$ where it becomes infinite. We approximate it with the Dirac δ-function. The second integral can be simplified, for vanishing a it jumps from $-\infty$ to $+\infty$ when ω goes through ω_r

$$I_1 = \frac{1}{2}\pi\delta(\omega - \omega_r) \text{ for } a \to 0 \ , \ I_2 = \frac{1}{2(\omega-\omega_r)} \text{ for } a \to 0 \ , \ \omega - \omega_r \neq 0$$

We get for the single particle response to a harmonic excitation

$$\dot{x} = \frac{\hat{G}}{2}\left(\pi\delta(\omega-\omega_r)\cos(\omega t) + \frac{1}{(\omega-\omega_r)}\sin(\omega t)\right)$$

and for the set of oscillators with resonant frequency distribution $f(\Delta\omega_r)$

$$\begin{aligned} \langle \dot{x} \rangle &= \int_{-\infty}^{\infty} \dot{x} f(\Delta\omega_r) d\omega_r \\ &= \frac{\hat{G}}{2}\left(\pi f(\omega)\cos(\omega t) + PV\int_{-\infty}^{\infty} \frac{f(\Delta\omega_r)d\omega_r}{\omega-\omega_r}\sin(\omega t)\right) \end{aligned}$$

This response to a harmonic excitation is often called **transfer function**.

Short derivation using complex notation

Taking oscillators with symmetric resonant frequency distribution, no damping and complex notation with positive and negative frequencies

$$f(\omega_r) = f(-\omega_r)\ ,\ \ddot{x} + \omega_r^2 x = \frac{\hat{G}}{2}\mathrm{e}^{j\omega t}\ ,\ (-\omega^2 + \omega_r^2)x = \frac{\hat{G}}{2}\mathrm{e}^{j\omega t}$$

we have a displacement response

$$x = \frac{-\hat{G}\mathrm{e}^{j\omega t}}{2(\omega^2 - \omega_r^2)} = \frac{-\hat{G}\mathrm{e}^{j\omega t}}{2(\omega - \omega_r)(\omega + \omega_r)} = \frac{-\hat{G}\mathrm{e}^{j\omega t}}{4\omega}\left(\frac{1}{\omega - \omega_r} + \frac{1}{\omega + \omega_r}\right)$$

$$\langle x \rangle = \frac{-\hat{G}\mathrm{e}^{j\omega t}}{4\omega}\int_{-\infty}^{\infty}\left(\frac{f(\Delta\omega_r)}{\omega - \omega_r} + \frac{f(\Delta\omega_r)}{\omega + \omega_r}\right)d\omega_r = \frac{-\hat{G}\mathrm{e}^{j\omega t}}{2\omega}\int_{-\infty}^{\infty}\frac{f(\Delta\omega_r)}{\omega - \omega_r}d\omega_r$$

If the exciting frequency ω is within the distribution $f(\omega_r)$ the integral contains a pole leading to a residue and principle value integral

$$\langle x \rangle = \frac{-\hat{G}\mathrm{e}^{j\omega t}}{2\omega}\left(\pm j\pi f(\omega) + PV\int\frac{f(\Delta\omega_r)}{\omega - \omega_r}d\omega_r\right)$$

with

$$PV\int\frac{f(\Delta\omega_r)}{\omega - \omega_r}d\omega_r = \lim_{\epsilon \to 0}\left[\int_{-\infty}^{\omega-\epsilon}\frac{f(\Delta\omega_r)}{\omega - \omega_r} + \int_{\omega+\epsilon}^{\infty}\frac{f(\Delta\omega_r)}{\omega - \omega_r}\right]d\omega_r$$

The velocity response is $\langle \dot{x} \rangle = j\omega\langle x \rangle$

$$\langle \dot{x} \rangle = \frac{\hat{G}}{2}\mathrm{e}^{j\omega t}\left(\pm\pi f(\omega) - jPV\int\frac{f(\Delta\omega_r)}{\omega - \omega_r}d\omega_r\right)$$

The real (resistive) part of the velocity response is in phase with the excitation and can absorb energy. The imaginary (reactive) part is out of phase with the excitation and does not absorb energy. There is an ambiguity in the sign of the resistive part because the initial condition is not specified. If there is no oscillation to begin with, the exciter gives energy to the oscillators ($+$ sign), however, there could already exist a coherent oscillation at an early time giving now energy to the exciter (- sign).

Caution: Sometimes for the oscillation in complex notation $\exp(-i\omega t)$ is used instead of $\exp(j\omega t)$. The corresponding results can be converted by exchanging $j = -i$.

Relation between pulse and harmonic excitation

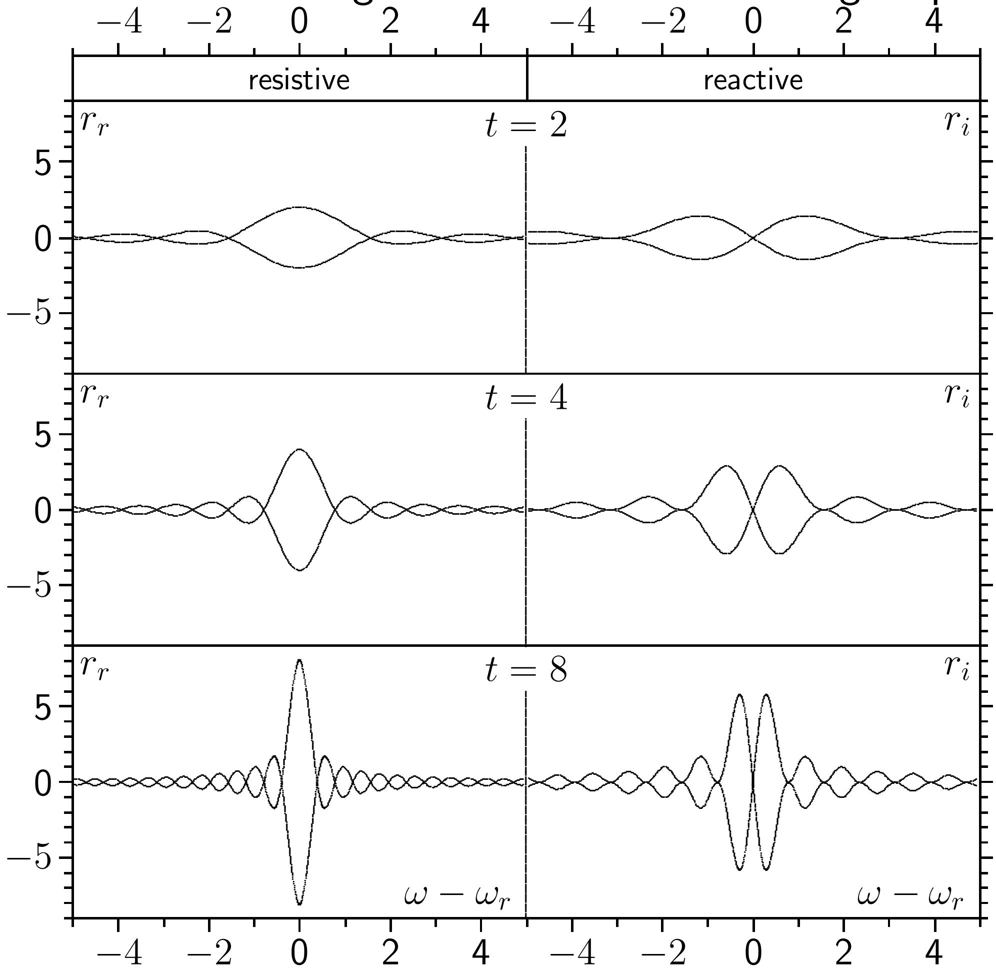
The center of mass velocity response of an oscillator set with resonant frequency distribution $f(\omega_r)$ to a pulse excitation is

$$g(t) = \frac{\langle \dot{x}(t) \rangle}{\dot{x}_0} = \cos(\omega_{r0} t) \int f(\Delta\omega_r) \cos(\Delta\omega_r t) d\omega_r - \sin(\omega_{r0} t) \int f(\Delta\omega_r) \sin(\Delta\omega_r t) d\omega_r$$

The response to a harmonic excitation $C\cos(\omega t)$ has a resistive and reactive term and is related to the pulse response by a Fourier transform

$$\frac{\langle \dot{x} \rangle}{G} = \frac{1}{2}(r_r(\omega) + j r_i(\omega)) = \frac{1}{2}\left(\pi f(\omega)\cos(\omega t) - j\mathrm{PV}\int \frac{f(\omega_r)}{\omega - \omega_r} d\omega_r \sin(\omega t)\right)$$

This represents actually the response after a harmonic excitation for a sufficiently long time. This can be illustrated by exposing a flat frequency distribution to a harmonic of excitation at a central frequency ω starting at $t = 0$ and observing the development. As time goes on the band of responding oscillators are concentrated around ω with decreasing band width and increasing amplitudes.



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Response of oscillators with a Gaussian distribution

$$f(\Delta\omega_r) = \frac{1}{\sqrt{2\pi}\sigma_\omega}\mathrm{e}^{-(\Delta\omega_r/\sigma_\omega)^2/2} \quad \text{with} \quad \int_{-\infty}^{\infty} f(\Delta\omega_r)d\omega_r = 1$$

Pulse response $g(t)$ and its envelope $E(t)$ are

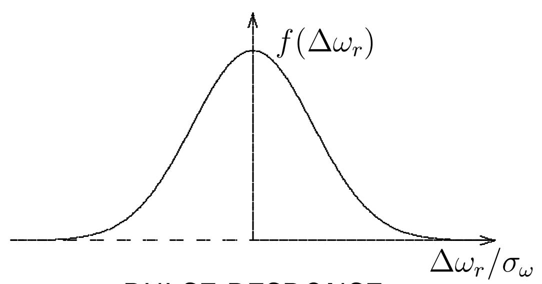
$$\begin{aligned} g(t) &= \frac{1}{\sqrt{2\pi}\sigma_\omega}\int_{-\infty}^{\infty} \mathrm{e}^{-\Delta\omega_r^2/2\sigma_\omega^2}\cos(\Delta\omega_r t)d\omega_r \cos(\omega_{r0}) \\ &= \mathrm{e}^{-\sigma_\omega^2 t^2/2}\cos(\omega_{r0}t) \\ E(t) &= \mathrm{e}^{-\sigma_\omega^2 t^2/2}. \end{aligned}$$

The transfer function is obtained by a Fourier transform of $E(t)$

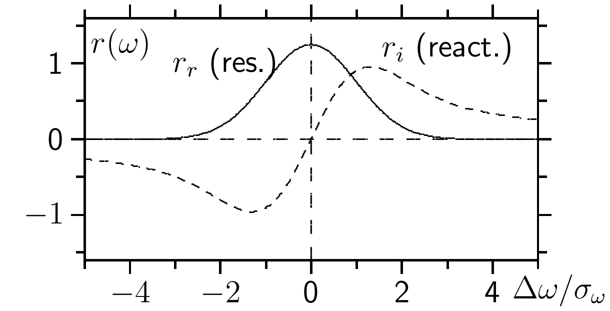
$$\begin{aligned} r_r(\omega) &= \int_0^{\infty} \mathrm{e}^{-\sigma_\omega^2 t^2/2}\cos(\Delta\omega t)dt = \frac{\pi}{\sqrt{2\pi}\sigma_\omega}\mathrm{e}^{-\Delta\omega^2/2\sigma_\omega^2} \\ r_i(\omega) &= \int_0^{\infty} \mathrm{e}^{-\sigma_\omega^2 t^2/2}\sin(\Delta\omega t)dt = \frac{\sqrt{2}}{\sigma_\omega}\mathrm{e}^{-(\Delta\omega/\sigma_\omega)^2/2}\int_0^{\Delta\omega/(\sqrt{2}\sigma_\omega)} \mathrm{e}^{t'^2}dt' \end{aligned}$$

The integral on the right is called Dawson integral.

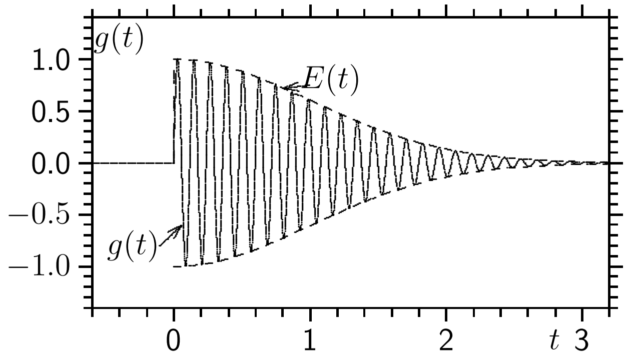
FREQUENCY DISTRIBUTION

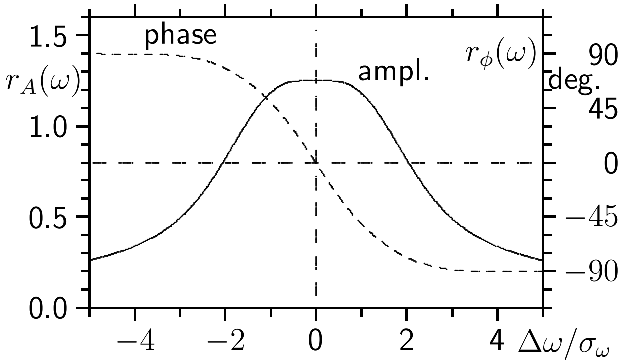


TRANSFER FUNCTION

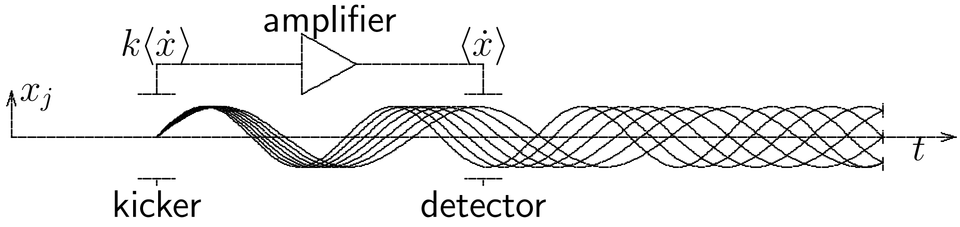


PULSE RESPONSE





Landau damping of oscillator set



Based on the center of mass response of a set of oscillators we illustrate how the frequency spread leads to Landau damping of coherent oscillation which would otherwise grow.
The velocity of center of mass motion of a set of oscillators is measured by a detector, the signal is amplified and fed to a kicker to produce an acceleration G_s in phase with the velocity which should lead to a growing oscillation, i.e. a negative feed-back system. The center of mass velocity response to an acceleration $G = \hat{G}\exp(j\omega t)$ is in general and for a Gaussian distribution

$$\langle\dot{x}\rangle = G[r_r + jr_i] = G\frac{1}{2}\left(\pi f(\omega) - jPV\int\frac{f(\Delta\omega_r)}{\omega-\omega_r}d\omega_r\right)$$
$$= G\left(\frac{\pi}{\sqrt{2\pi}\sigma_\omega}\mathrm{e}^{-\Delta\omega^2/2\sigma_\omega^2} + j\frac{\sqrt{2}}{\sigma_\omega}\mathrm{e}^{-(\Delta\omega/\sigma_\omega)^2/2}\int_0^{\Delta\omega/(\sqrt{2}\sigma_\omega)}\mathrm{e}^{t'^2}dt'\right).$$

We assume now that the excitation happens at the central frequency for the Gaussian distribution $\Delta\omega = 0$ for which $r_i = 0$

$$\langle\dot{x}\rangle = G\frac{\pi}{\sqrt{2\pi}\sigma_\omega}$$

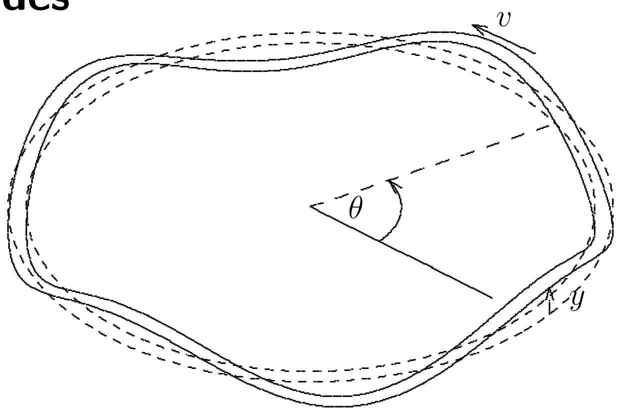
We replace the external excitation G by the one of the feed-back $G_s = k\langle\dot{x}\rangle$ and assume a gain k just sufficient to keep the oscillation going. This is the limit of stability since a slightly larger gain would increase the oscillation leading to an exponential growth

$$\langle\dot{x}\rangle = k\langle\dot{x}\rangle\frac{\pi}{\sqrt{2\pi}\sigma_\omega} \;\rightarrow\; k \le \sqrt{\frac{2}{\pi}}\sigma_\omega.$$

This maximum gain k still giving stability is proportional to the frequency spread. Landau damping works by making an accidental coherent oscillation incoherent at infinitesimal levels without having first a growth reaching finite amplitudes. It does not lead to a growth of the incoherent oscillations.

Transverse coasting beam instability

Oscillation modes



A uniform coasting beam of N particles circulates with revolution frequency ω_0, current $I = eN\omega_0/(2\pi)$ in a ring of uniform focusing. Each particle executes a betatron oscillation of frequency $Q\omega_0$

$$\theta_i = \theta_{0i} + \omega_0 t \ \ , \ \ y_i(t) = \hat{y}\cos(Q\omega_0(t - t_i)).$$

Depending on the phases $Q\omega_0 t_i$ between adjacent particle we have different modes. We choose a set of modes having a form as seen at a fixed location θ

$$y(t) = \hat{y}\cos(n\theta - \omega t)\ ,\ y(0) = \hat{y}\cos(n\theta).$$

Frozen in time $t = 0$ we have a closed wave with n periods. Following a particle $\theta_s(t) = \theta_0 + \omega_0 t$ give us the betatron oscillation with frequency $Q\omega_0$.

$$y_s = \hat{y}\cos(n\theta_0 - (\omega - n\omega_0)t) = \hat{y}\cos(n\theta_s - Q\omega_0 t)$$

giving for the frequency ω seen by a stationary observer

$$\omega = (n + Q)\omega_0 = \omega_\beta \ \text{ with } \ -\infty < n < \infty.$$

We divide modes into fast and slow waves according to the sign of the phase difference between adjacent particle

$$\begin{aligned} \omega_{\beta f} &= (n_f + Q)\omega_0\ ,\ n_f > -Q \\ \omega_{\beta s} &= (n_s - Q)\omega_0\ ,\ n_s > Q. \end{aligned}$$

Effect of momentum spread

The betatron frequencies of a beam with nominal momentum are:

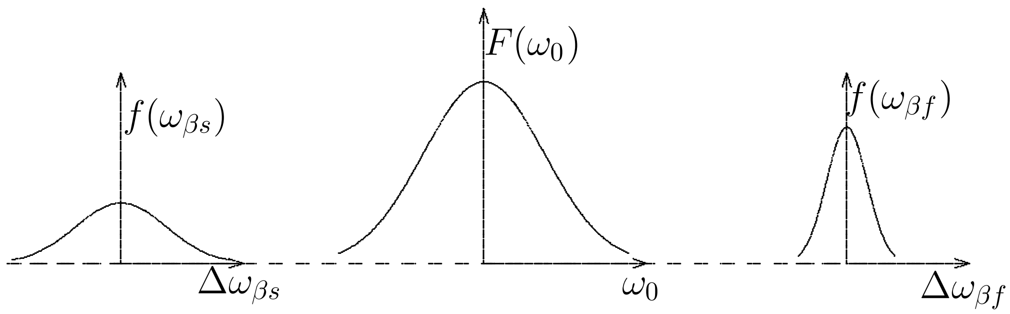
$$\omega_{\beta f} = (n_f + Q)\omega_0 \ \ , \ \ \omega_{\beta s} = (n_s - Q)\omega_0.$$

Through $\dfrac{\Delta E}{E} = \beta^2 \dfrac{\Delta p}{p} = -\dfrac{\beta^2}{\eta_c}\dfrac{\Delta \omega_0}{\omega_0}$, and $\Delta Q = Q' \dfrac{\Delta p}{p}$.

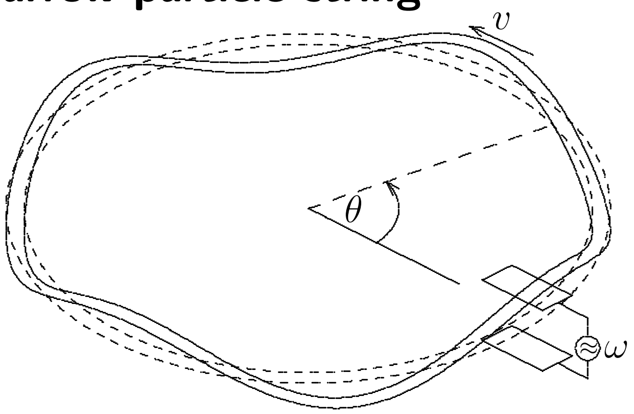
they are affected by a momentum deviation

$$\begin{aligned} \Delta\omega_{\beta f} &= (Q' - \eta_c(n_f + Q))\omega_0 \frac{\Delta p}{p} \\ \Delta\omega_{\beta s} &= (Q' - \eta_c(n_s - Q))\omega_0 \frac{\Delta p}{p}. \end{aligned}$$

resulting in two frequency distributions $f(\omega_{\beta f})$, $f(\omega_{\beta s})$.



Response of narrow particle string



All particle have the same momentum forming uniform ring. We excite them

$$\ddot{y} + \omega_0 Q^2 y = \hat{G}\cos(\omega)$$

and seek a solution $y(t) = \hat{y}\cos(n\theta - \omega)$. To excite such a mode n each particle has to be driven by the proper phase corresponding to its longitudinal position θ. Therefore, we expect to find a excitation frequency which is not $\omega_0 Q$ but close to the fast or slow wave frequencies $\omega_{\beta f} = (n_f + Q)\omega_0$ or $\omega_{\beta s} = (n_s - Q)\omega_0$.
Substituting the desired solution form in the differential equation form gives

$$\left(-(n\omega_0 - \omega)^2 + Q^2\omega_0^2\right)\hat{y}\cos(n\theta - \omega t) = \hat{G}\cos(\omega t).$$

We assume excitation and observation is done at the location $\theta = 0$

$$\begin{aligned}\hat{y} &= \frac{\hat{G}}{\omega_0^2 Q^2 - (n\omega_0 - \omega)^2} = \frac{-\hat{G}}{(\omega - \omega_0(n+Q)(\omega - \omega_0(n-Q)} \\ &= \frac{-\hat{G}}{(\omega - \omega_{\beta f})(\omega - \omega_{\beta s})} = \frac{\hat{G}}{2\omega_0 Q}\left(\frac{1}{\omega - \omega_{\beta s}} - \frac{1}{\omega - \omega_{\beta f}}\right).\end{aligned}$$

to excite the fast wave we use $\omega \approx (n_f + Q)\omega_0$ and the first term is much smaller than the second one. Correspondingly for the slow wave we use $\omega \approx (n_s - Q)\omega_0$ and the second term is much smaller than the first one. We approximate for the two waves

$$\left(\frac{\hat{y}}{\hat{G}}\right)_f \approx -\frac{1}{2\omega_0 Q}\left(\frac{1}{\omega - \omega_{\beta f}}\right), \quad \left(\frac{\hat{y}}{\hat{G}}\right)_s \approx \frac{1}{2\omega_0 Q}\left(\frac{1}{\omega - \omega_{\beta s}}\right).$$

The two responses have opposite sign, this will be discussed later.

Response of the whole beam

The whole beam has frequency distribution $f(\omega_{\beta f})$ and $f(\omega_{\beta s})$ The center of mass responses in displacement and velocity are related $\langle \dot{y} \rangle = j\omega \langle y \rangle$

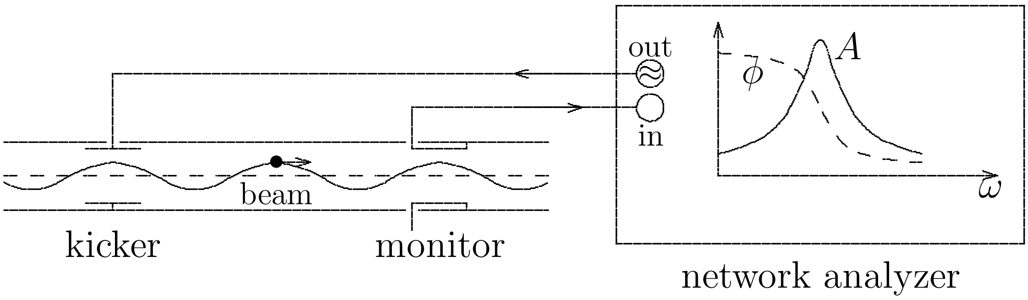
$$\langle \hat{y} \rangle_f = -\frac{\hat{G}\omega}{2Q\omega_0} \int \frac{f(\omega_{\beta f})}{\omega_{\beta f} - \omega} d\omega_{\beta f}) = -\frac{\hat{G}\omega}{2Q\omega_0} \left(\pi f(\omega) - jPV \int \frac{f(\omega_{\beta f})}{\omega - \omega_{\beta f}} \right) d\omega_{\beta f}.$$

$$\langle \hat{y} \rangle_s = \frac{\hat{G}\omega}{2Q\omega_0} \int \frac{f(\omega_{\beta s}}{\omega_{\beta s} - \omega} d\omega_{\beta s} = \frac{\hat{G}\omega}{2Q\omega_0} \left(\pi f(\omega) - jPV \int \frac{f(\omega_{\beta s})}{\omega - \omega_{\beta s}} \right) d\omega_{\beta s}.$$

The term $\pi f(\omega)$ is real, exciting acceleration and responding velocity are in phase resulting in an absorption of energy and damping, called Landau damping. It is only present if the excitation frequency ω is within the frequency distribution of the individual particles. The second term is imaginary and gives the out-of-phase response being of less interest.

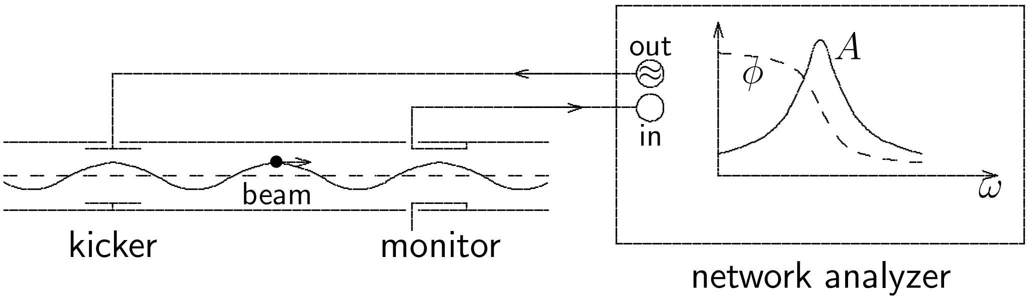
The spread in betatron frequencies is given by the momentum spread and the dependence of revolution frequency ω_0 and betatron tune Q on momentum deviation $\Delta p/p$. It is therefore determined by an **external parameter** which is not affected by the excitation of betatron oscillations.

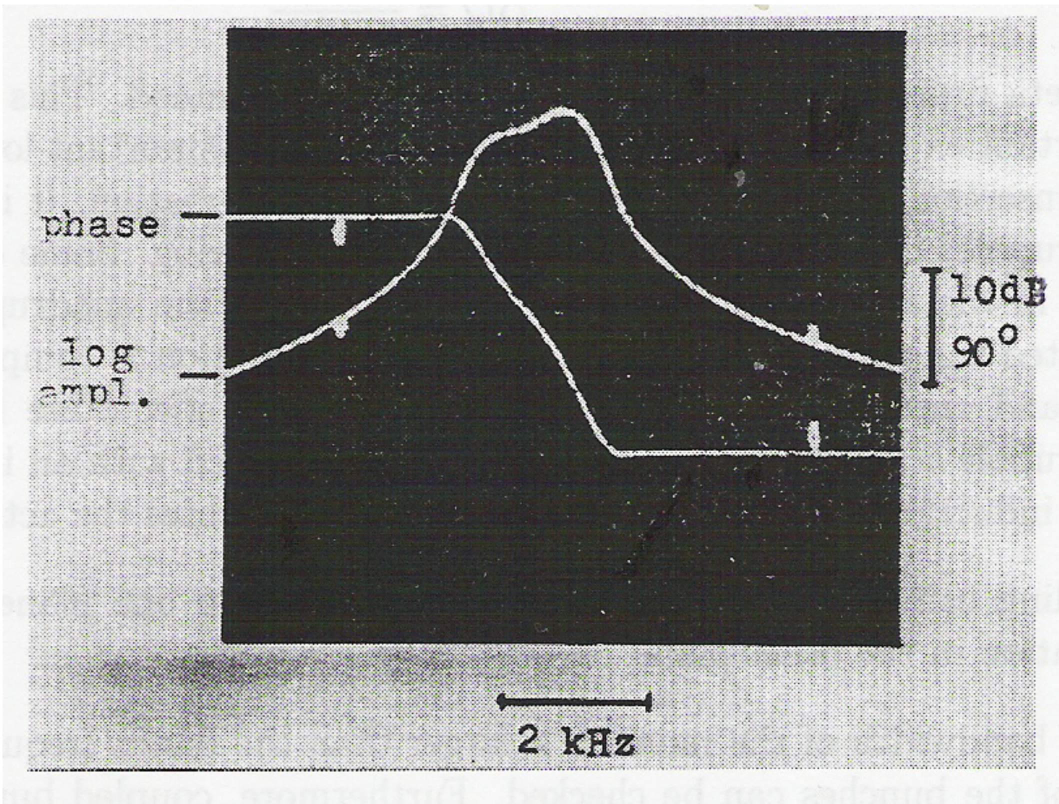
Measuring the beam response



The center of mass displacement response can directly be measured with a network analyzer. Here, we derived the velocity response which is more transparent for understanding the resistive and reactive behavior of the beam. In measurements the displacement is observed and our equation have to be converted to analyze the results. Due to cable delays the real and imaginary part of the response are often mixed. It is easier to measure amplitude and phase response and correct the latter off-line.

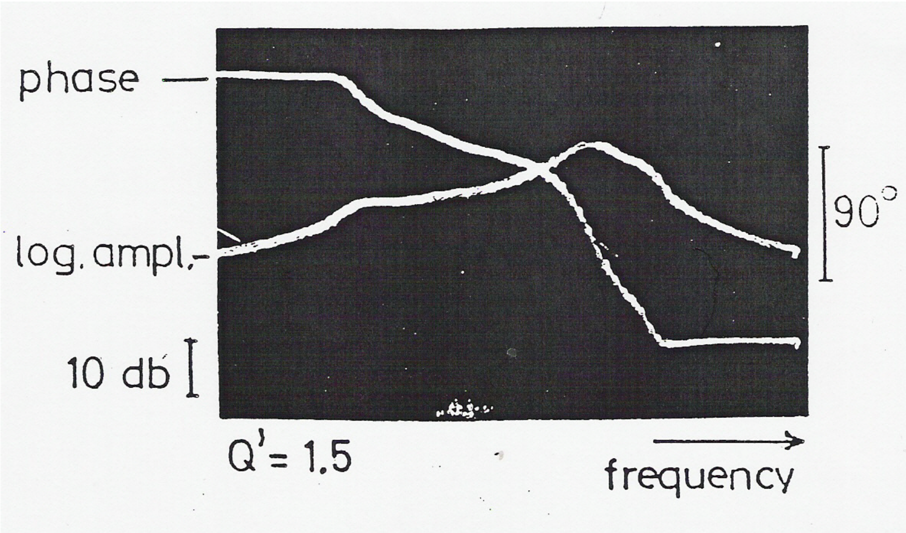
Measurement of the transverse beam response

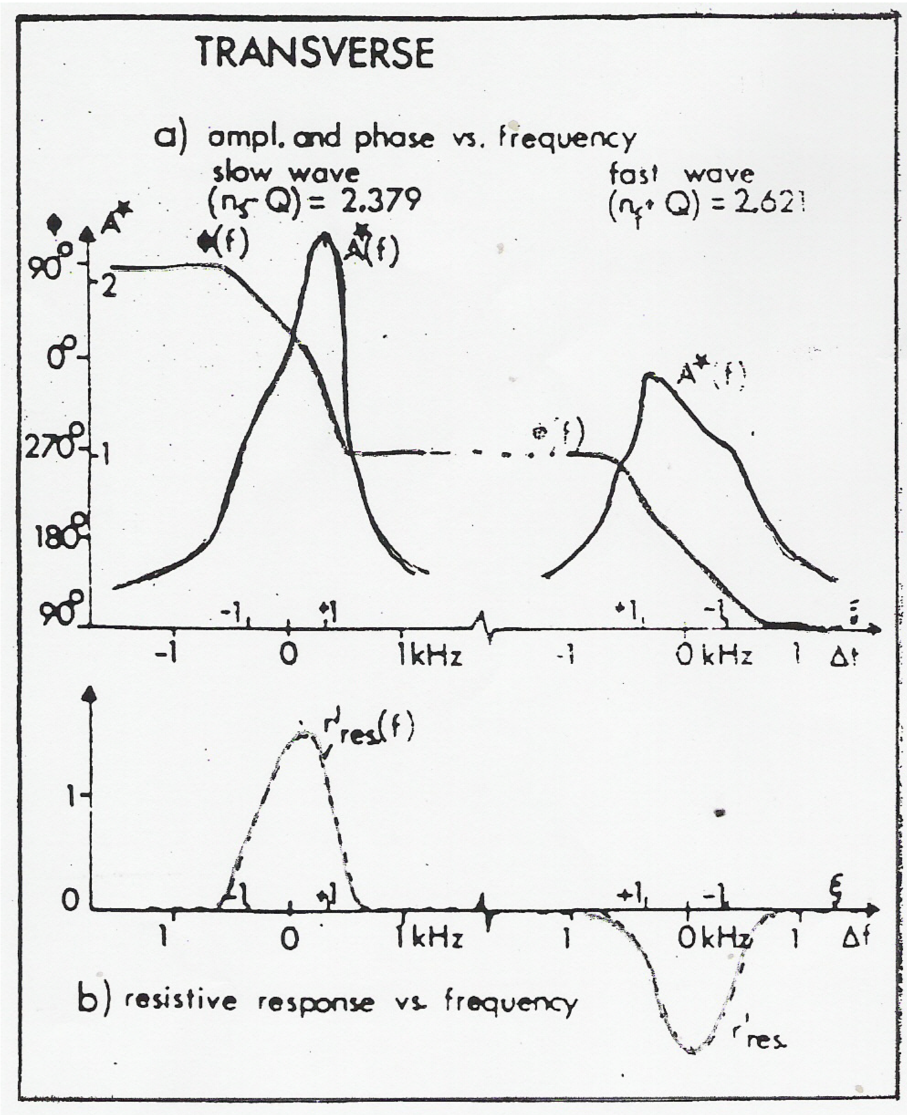




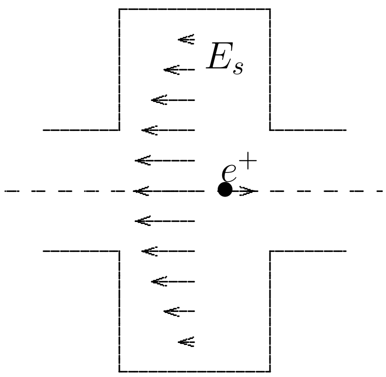
Vertical TF of an unbunched beam in the ISR

Measurement of upper and lower side-band





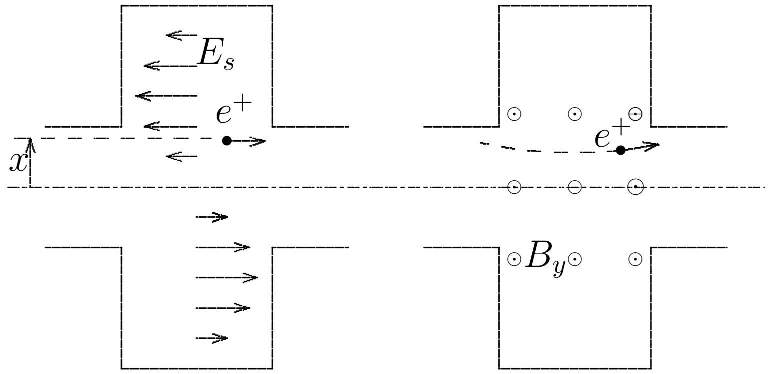
Longitudinal impedance



$$Z(\omega) = -\frac{\int \tilde{E}_s(\omega) ds}{\tilde{I}(\omega)} = \frac{\tilde{V}(\omega)}{\omega}.$$

The longitudinal impedance is defined as the ratio between the integrated longitudinal field to the current which excites it. It has a real (resistive) part for which voltage and current are in phase and an imaginary (reactive) part for which they are out of phase. It is measured in Ohm=V/A.

Transverse impedance



The transverse impedance is defined as the ratio between a longitudinal integral over the transverse deflecting fields and the dipole moment of the current which excites it

$$
\begin{aligned}
Z_T(\omega) &= j\frac{\int \left(\tilde{\mathbf{E}}(\omega + [\mathbf{v} \times \tilde{\mathbf{B}}(\omega)]\right)_T ds}{\widetilde{Ix}(\omega)} \\
&= \frac{-\omega \int \left(\tilde{\mathbf{E}}(\omega + [\mathbf{v} \times \tilde{\mathbf{B}}(\omega)]\right)_T ds}{\widetilde{I\dot{x}}(\omega)}.
\end{aligned}
$$

It is illustrated by a cavity mode having a transverse electric field with a gradient $\partial E_s/\partial x$ which is first induced by the dipole moment of the current. After 1/4 oscillation this is converted into a transverse magnetic field B_y which produces a deflection in the x-direction. The 'j' in front of the first definition indicates that the exciting dipole moment and the deflecting field are out of phase. However, the second definition relates the transverse deflection to the transverse velocity is real indicating the transfer of energy. Like in the longitudinal case the transverse impedance has a real (resistive) and an imaginary (reactive) part, furthermore it has a horizontal and vertical component. It is measured in units of Ohm/m=V/(A m).

Stability limit

The oscillating beam can induce a voltage in a transverse impedance which in turn applies a self acceleration G_s to the beam

$$Z_T(\omega) = -\frac{\omega}{I\dot{y}(\omega)} \oint \left(\vec{E}(\omega) + [\vec{\beta} \times \vec{B}(\omega)]\right)_T ds \quad , \quad \hat{G}_s = -\frac{eZ_T I \langle \dot{y} \rangle}{\gamma m_0 2\pi R \omega}$$

If $\hat{G}_s = \hat{G}$ we can have a steady self sustained oscillation without external excitation, i.e. a threshold of an instability. Introducing this into the response we get for this threshold

$$1 = -\frac{jecIZ_T(\omega)}{4\pi QE} \int \frac{f(\omega_{\beta f})}{\omega - \omega_{\beta f}} d\omega_{\beta f} = -\frac{ecIZ_T(\omega)}{4\pi QE} \left(\pi f(\omega) - jPV \int \frac{f(\omega_{\beta f})}{\omega - \omega_{\beta f}} d\omega_{\beta f}\right).$$

$$1 = \frac{jecIZ_T(\omega)}{4\pi QE} \int \frac{f(\omega_{\beta s})}{\omega_{\beta s} - \omega} d\omega_{\beta s} = \frac{ecIZ_T(\omega)}{4\pi QE} \left(\pi f(\omega) + jPV \int \frac{f(\omega_{\beta s})}{\omega - \omega_{\beta s}} d\omega_{\beta s}\right).$$

These equations represent relations between the complex impedance and the complex beam response to an excitation. We plot this as a stability diagram shown for a Gaussian distribution. If the impedances lies inside the central curve we have stability, outside an instability. The curve itself represents the threshold. Its shape is determined by the frequency distribution of the particles.

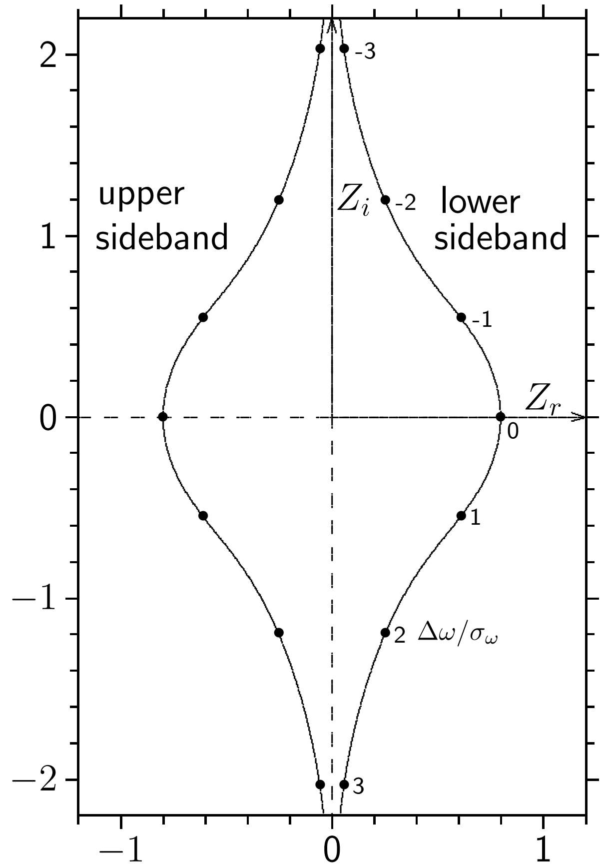
The stability diagram is the inverse response of the beam plotted the inverse amplitude against the negative phase, i.e. it is an inverse Nyquist diagram.

lower sideband

$$\frac{jecIZ_T(\omega)}{4\pi QE} \leq \frac{1}{\int \frac{f(\omega_{\beta s})}{\omega - \omega_{\beta s}} d\omega_{\beta s}}$$

upper sideband

$$\frac{jecIZ_T(\omega)}{4\pi QE} \leq -\frac{1}{\int \frac{f(\omega_{\beta f})}{\omega - \omega_{\beta f}} d\omega_{\beta f}}$$



Response in the presence of an impedance

The response of a beam to an external acceleration is for the lower side band

$$\langle\hat{y}\rangle_s = \frac{\hat{G}\omega}{2Q\omega_0}\int\frac{f(\omega_{\beta s})}{\omega_{\beta s}-\omega}d\omega_{\beta s}$$

The oscillating beam can induce a voltage in a transverse impedance which in turn applies a self acceleration G_Z to the beam

$$Z_T(\omega) = -\frac{\omega}{I\dot{y}(\omega)}\oint\left(\vec{E}(\omega)+[\vec{\beta}\times\vec{B}(\omega)]\right)_T ds \quad , \quad \hat{G}_Z = -\frac{eZ_TI\langle\dot{y}\rangle}{\gamma m_0 2\pi R\omega}$$

This self excitation has to be added to the external one. We take the inverse response (stability diagram) due to both

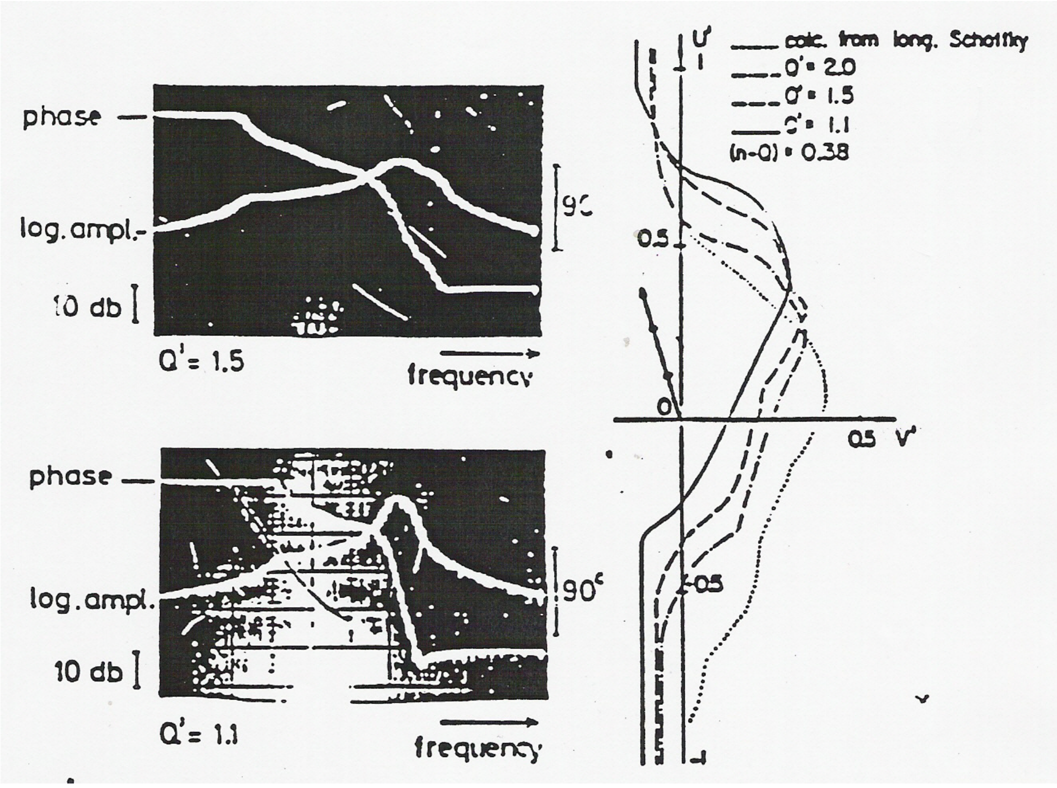
$$\frac{(\hat{G}+\hat{G}_Z)}{\langle\hat{y}\rangle_s} = \frac{\omega}{2Q\omega_0\int\frac{f(\omega_{\beta s})}{\omega_{\beta s}-\omega}d\omega_{\beta s}}$$

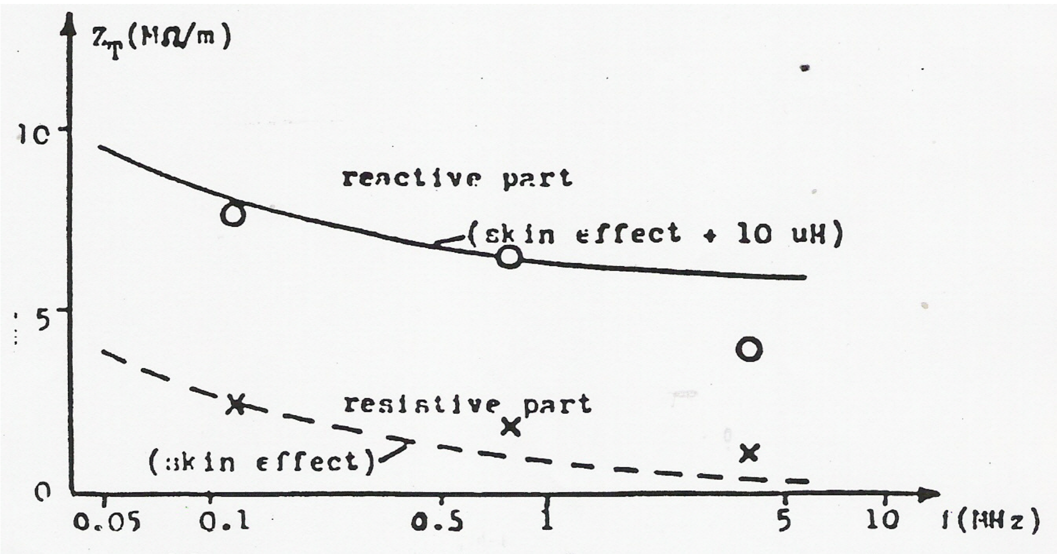
However, we know only the external excitation and would like the relation of the response to it. The inverse of this response is

$$\frac{\hat{G}}{\langle\hat{y}\rangle_s} = \frac{\omega}{2Q\omega_0\int\frac{f(\omega_{\beta s})}{\omega_{\beta s}-\omega}d\omega_{\beta s}} - \frac{\hat{G}_Z}{\langle\hat{y}\rangle_s} = \frac{\omega}{2Q\omega_0\int\frac{f(\omega_{\beta s})}{\omega_{\beta s}-\omega}d\omega_{\beta s}} - \frac{eZ_TI}{\gamma m_0 2\pi R\omega}$$

The presence of an impedance shifts the stability diagram by a vector which is proportional to the negative complex impedance.

Measurement of transverse beam response in the presence of impedance





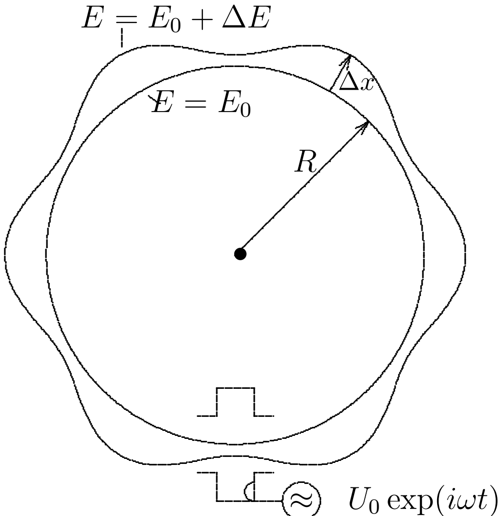
Longitudinal coasting beam instability

The longitudinal dynamics of a coasting beam is governed by the relation between the deviations in momentum and revolution frequency

$$\frac{\Delta E}{E} = \beta^2 \frac{\Delta p}{p} = -\frac{\beta^2}{\eta_c}\frac{\Delta\omega_0}{\omega_0}\,,\text{with }\ \eta_c = \alpha_c - \frac{1}{\gamma^2}.$$

The beam has an equilibrium energy distribution which translates into a distribution in revolution frequency

$$f_0(\Delta E) = \frac{1}{N}\frac{d^2N}{d\theta dE}\ \rightarrow F_0(\Delta\omega_0) = \frac{1}{N}\frac{d^2N}{d\theta d\omega_0}$$



A stable beam has a continuous current I_0, however, exciting it with $U_0 \exp(j\omega t)$ close to $n\omega_0$ give a current perturbation

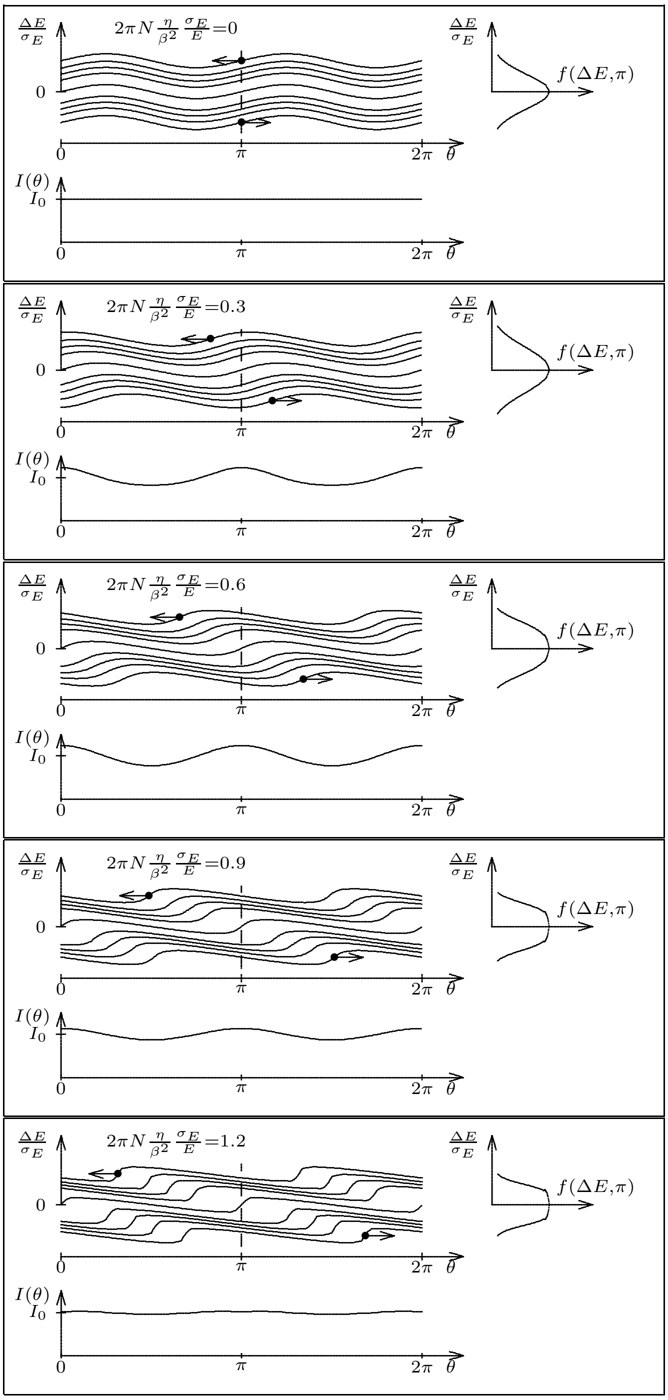
$$I_1(t) = \frac{-jNe^2\omega_0^3 U_0}{2\pi\beta^2 E)}\int \frac{dF_0(\omega_0)/dt}{\omega - n\omega_0}d\omega_0 = \frac{Ne^2\omega_0^3 U_0}{2\pi\beta^2 E)}\left(\pi\frac{dF_0}{d\omega_0}(\omega) - jPV\int\right).$$

This current I_1 can induce a voltage in an impedance Z. If it is as large or larger than U_0 it can replace the external excitation and keep the current modulation going or increase it. We get for this stability limit

$$1 = \frac{N\mathrm{e}^2\omega_0^3\eta Z(\omega)}{2\pi\beta^2 E}\left(\frac{\pi dF_0}{d\omega_0}(\omega) - jPV\int\frac{dF_0(\omega_0)/dt}{\omega - n\omega_0}d\omega_0\right).$$

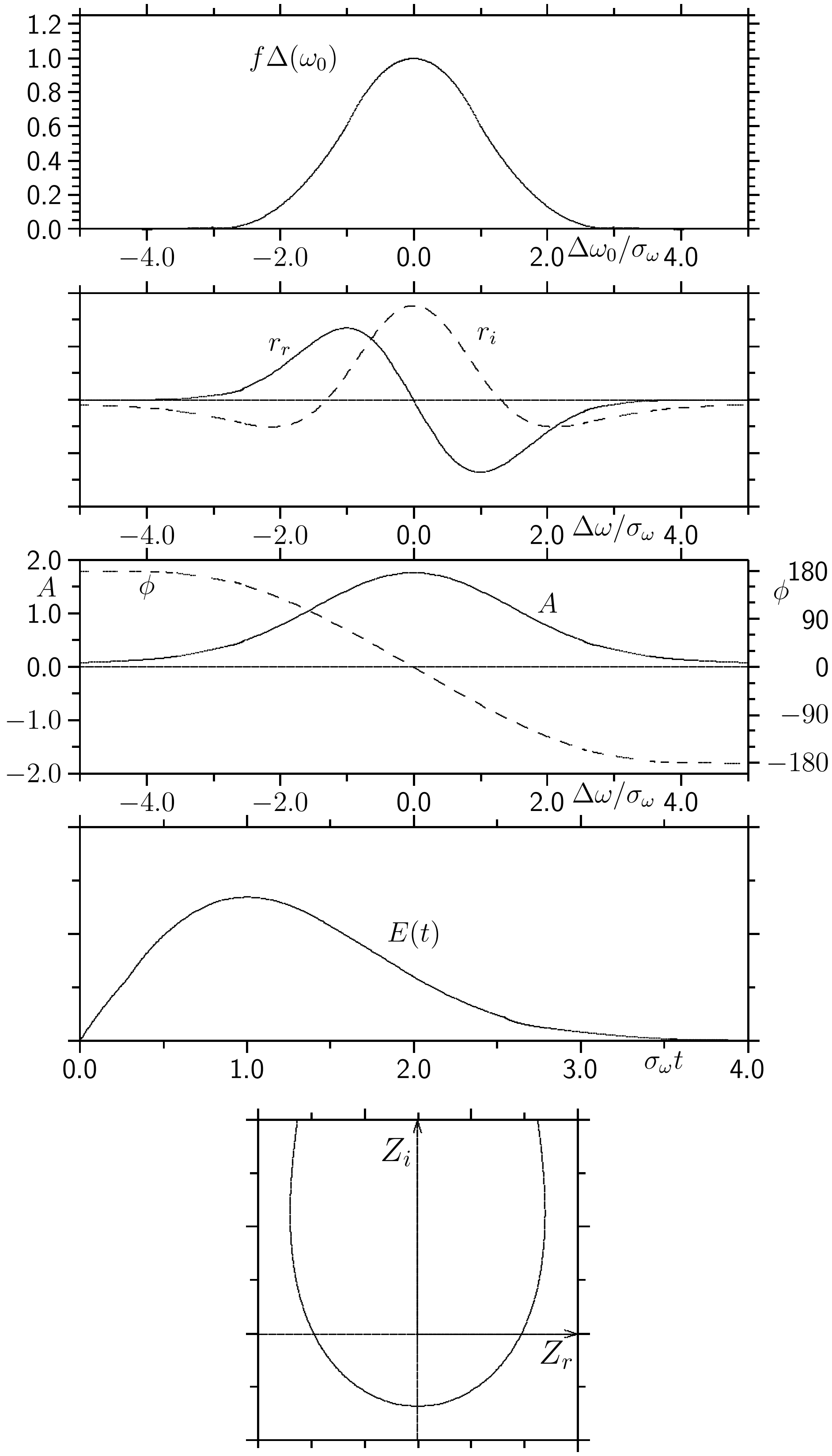
This equation is a complex mapping which can be represented in form of a stability diagram which depends on the energy or revolution frequency distribution of the particles

Development of an initial energy modulation.



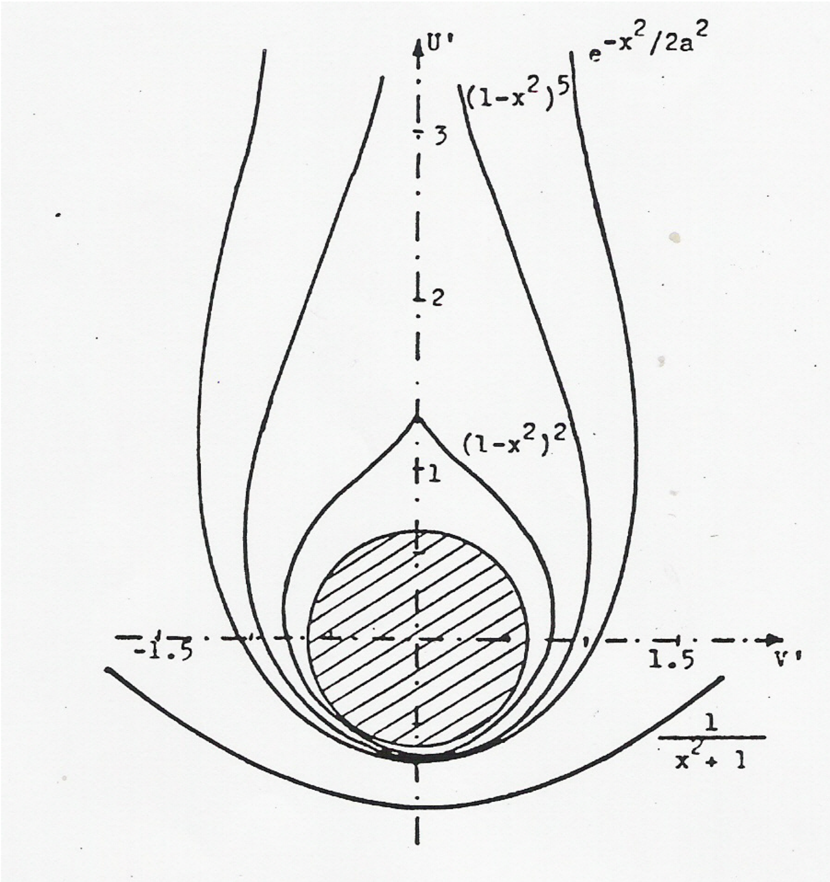
Development of an initial energy perturbation $\Delta E = \Delta E_0 \cos(2\theta)$. This modulation of energy is converted into one of revolution frequency ω_0. Due to the dependence of ω_0 on ΔE the modulation smears out resulting after some time again in a stationary beam but with increased momentum spread.

Longitudinal response of a coasting beam



Gaussian distribution, longitudinal response in real and imaginary part and in amplitude and phase to a harmonic and a pulse excitation of a coasting beam and stability diagram.

Stability diagrams for different distributions, (A. Ruggiero, V. Vaccaro)



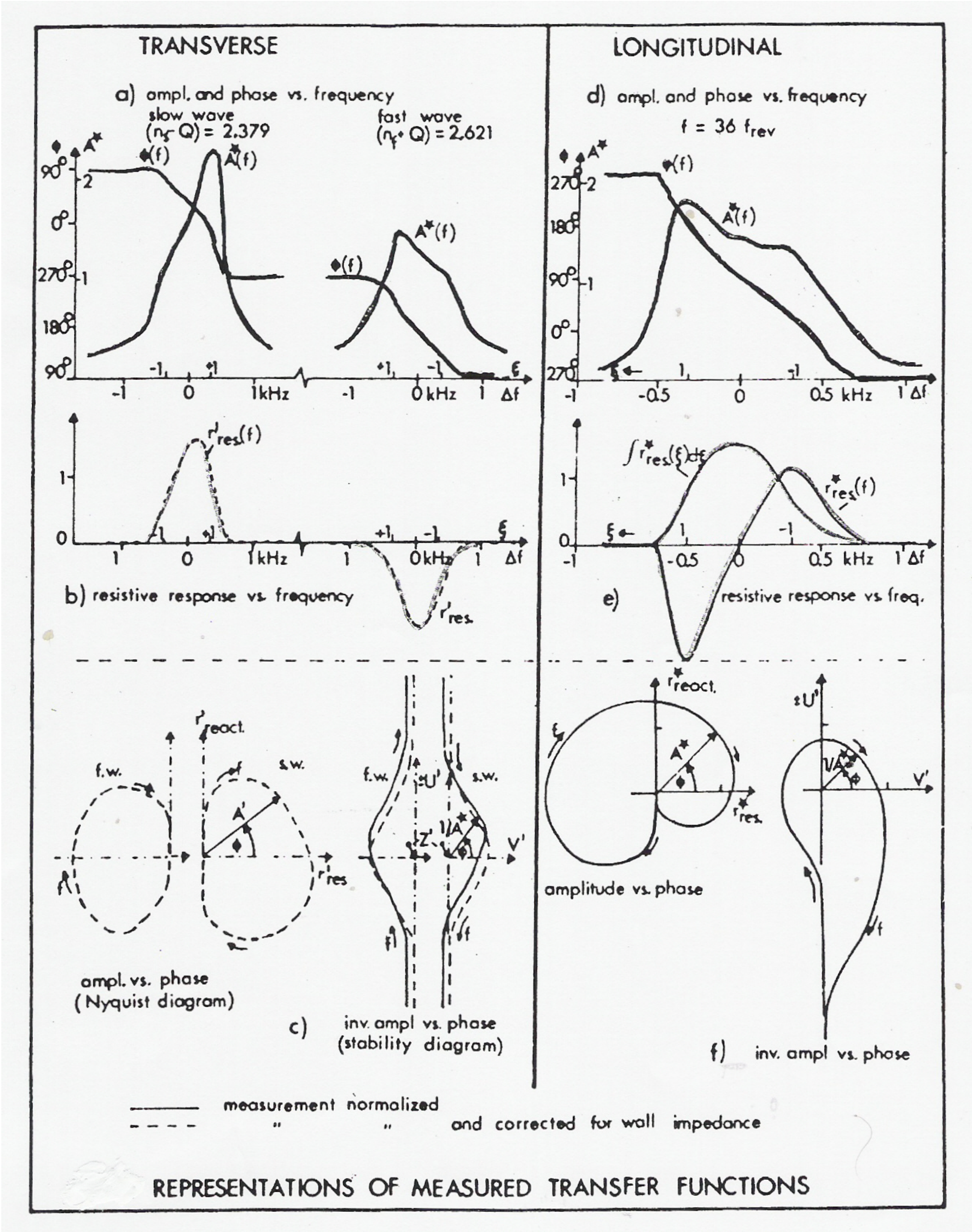
$$1 = \frac{Ne^2\omega_0^3\eta Z(\omega)}{2\pi\beta^2 E}\left(\frac{\pi dF_0}{d\omega_0}(\omega) - jPV\int\frac{dF_0(\omega_0)/dt}{\omega - n\omega_0}d\omega_0\right).$$

To separate the dependence on the form of the distribution from the one on physical parameters like E, I_0, $\Delta p/p$ and η_c the stability diagram is normalized with the width the momentum spread. Taking many such diagrams and approximating them with a circle gives the (Keil-Schnell) stability criterion

$$\left|\frac{Z}{n}\right| \leq \frac{2\pi\beta^2 E\eta_c(\,\Delta p/p)^2}{eI_0}.$$

Important is the strong dependence on the momentum spread, or the connected frequency spread, which gives rise to Landau damping.

Measurement of longitudinal and transverse beam response in the ISR



Simple demonstration of frequency spread and shift