Space Charge

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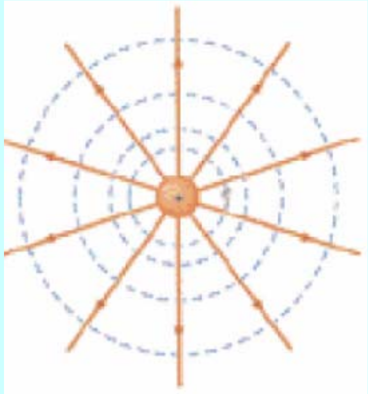


CAS, 25 May 2005

EQUATION OF MOTION

The motion of charged particles is governed by the Lorentz force :

$$\frac{d(m\gamma\,\boldsymbol{v})}{dt} = \boldsymbol{F}^{ext}_{e.m.} = e(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$$

Where ***m*** is the rest mass, γ the relativistic factor and ***v*** the particle velocity

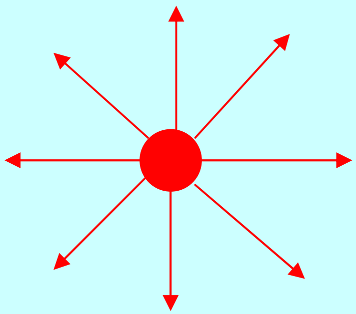
Charged particles are accelerated, guided and confined by external electromagnetic fields.

Acceleration is provided by the electric field of the RF cavity

Magnetic fields are produced in the bending magnets for guiding the charges on the reference trajectory (orbit), in the quadrupoles for the transverse confinement, in the sextupoles for the chromaticity correction.

SELF FIELDS AND WAKE FIELDS

There is another important source of e.m. fields : **the beam itself**



Direct self fields

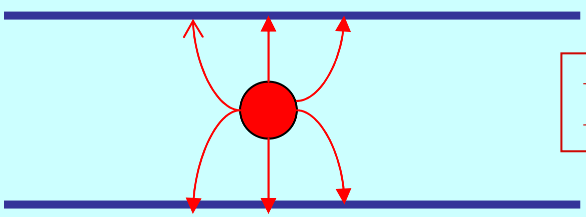
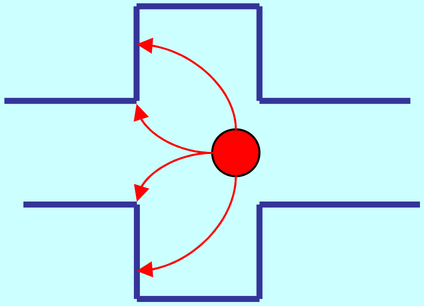


Image self fields

Space Charge



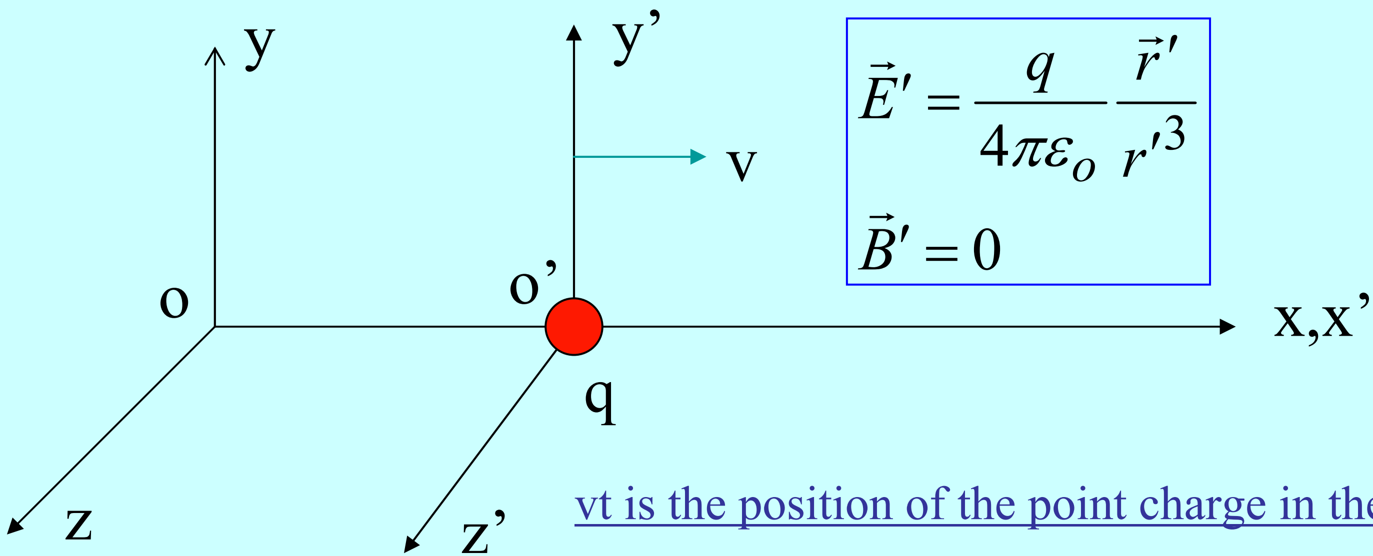
Wake fields

These fields depend on the current and on the charges velocity.

They are responsible of many phenomena of beam dynamics:

- energy loss *(wake-fields)*
- energy spread and emittance degradation
- shift of the synchronous phase and frequency (tune)
- shift of the betatron frequencies (tunes)
- instabilities.

Fields of a point charge with uniform motion



$$\vec{E}' = \frac{q}{4\pi\varepsilon_o}\frac{\vec{r}'}{r'^3}$$

$$\vec{B}' = 0$$

vt is the position of the point charge in the lab. frame O.

- In the moving frame O' the charge is at rest
- The electric field is radial with spherical symmetry
- The magnetic field is zero

$$E'_x = \frac{q}{4\pi\varepsilon_o}\frac{x'}{r'^3} \qquad E'_y = \frac{q}{4\pi\varepsilon_o}\frac{y'}{r'^3} \qquad E'_z = \frac{q}{4\pi\varepsilon_o}\frac{z'}{r'^3}$$

Relativistic transforms of the fields from O' to O

$$E_x = E'_x \qquad B_x = B'_x$$

$$E_y = \gamma(E'_y + vB'_z) \qquad B_y = \gamma(B'_y - vE'_z / c^2)$$

$$E_z = \gamma(E'_z - vB'_y) \qquad B_z = \gamma(B'_z + vE'_y / c^2)$$

$$r' = \left(x'^2 + y'^2 + z'^2\right)^{1/2}$$

$$\begin{cases} x' = \gamma(x - vt) \\ y' = y \\ z' = z \end{cases}$$

$$r' = \left[\gamma^2 (x - vt)^2 + y^2 + z^2\right]^{1/2}$$

$$E_x = E'_x = \frac{q}{4\pi\varepsilon_o}\frac{x'}{r'^3} = \frac{q}{4\pi\varepsilon_o}\frac{\gamma(x-vt)}{\left[\gamma^2(x-vt)^2 + y^2 + z^2\right]^{3/2}}$$

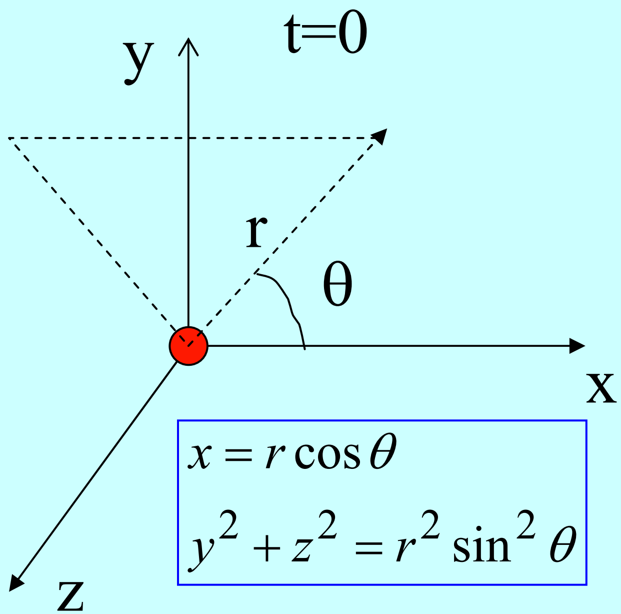
$$E_y = \gamma E'_y = \frac{q}{4\pi\varepsilon_o}\frac{y'}{r'^3} = \frac{q}{4\pi\varepsilon_o}\frac{\gamma y}{\left[\gamma^2(x-vt)^2 + y^2 + z^2\right]^{3/2}}$$

$$E_z = \gamma E'_z = \frac{q}{4\pi\varepsilon_o}\frac{z'}{r'^3} = \frac{q}{4\pi\varepsilon_o}\frac{\gamma z}{\left[\gamma^2(x-vt)^2 + y^2 + z^2\right]^{3/2}}$$

The field pattern is moving with the charge and it can be observed at t=0.

$$\vec{E} = \frac{q}{4\pi\varepsilon_o}\frac{\gamma\,\vec{r}}{\left[\gamma^2 x^2 + y^2 + z^2\right]^{3/2}}$$

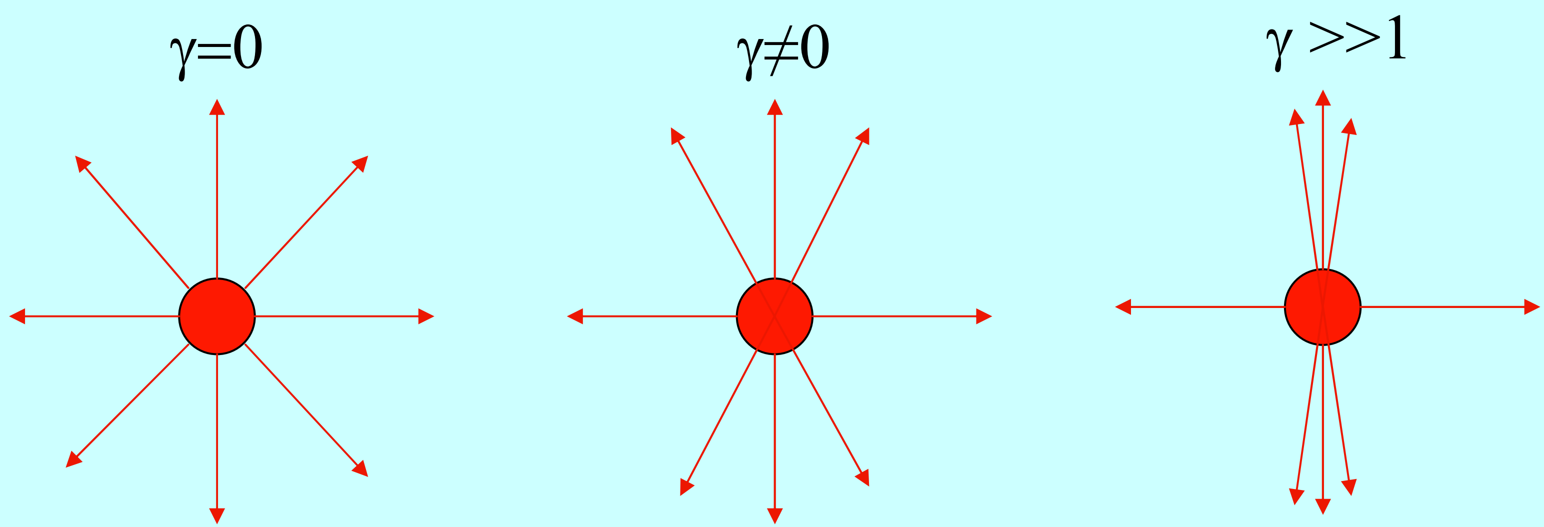
The fields have lost the spherical symmetry but still keep a symmetry with respect to the x-axis.



$$x = r\cos\theta$$

$$y^2 + z^2 = r^2 \sin^2\theta$$

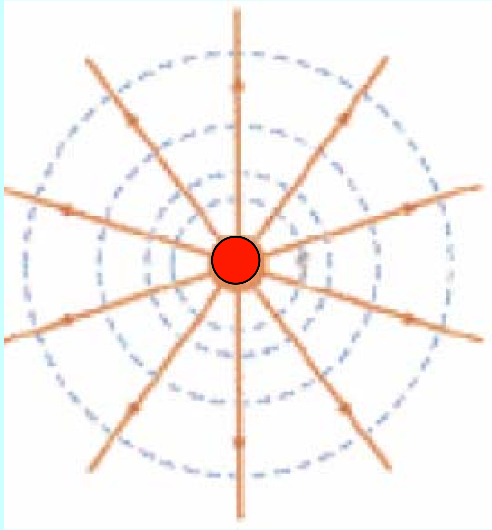
$$\vec{E} = \frac{q}{4\pi\varepsilon_o} \frac{\gamma^2}{\left(1-\beta^2 \sin^2\theta\right)^{3/2}} \frac{\vec{r}}{r^3}$$



B is transverse to the motion direction

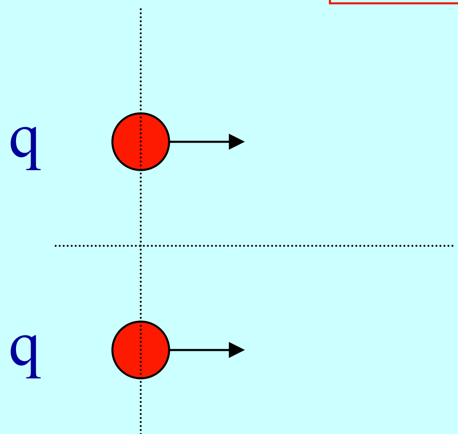
$$B_x = 0$$
$$B_y = -vE_z / c^2$$
$$B_z = vE_y / c^2$$

$$\vec{B}_\perp = \frac{\vec{v} \times \vec{E}}{c^2}$$



$\gamma \longrightarrow \infty$

Two charges in the rest frame O'



$$F'_r = \frac{1}{4\pi\varepsilon_o}\frac{qq}{r^2}$$

Two charges in the laboratory frame O

Relativistic transform

$$F_r = \frac{1}{\gamma}F'_r = \frac{1}{4\pi\varepsilon_o}\frac{qq}{\gamma r^2}$$

Lorentz force

$$F_r = q(E_r - vB_\varphi) = q(E_r - \beta^2 E_r) = \frac{q}{\gamma^2}E_r = \frac{1}{4\pi\varepsilon_o}\frac{qq}{\gamma r^2}$$

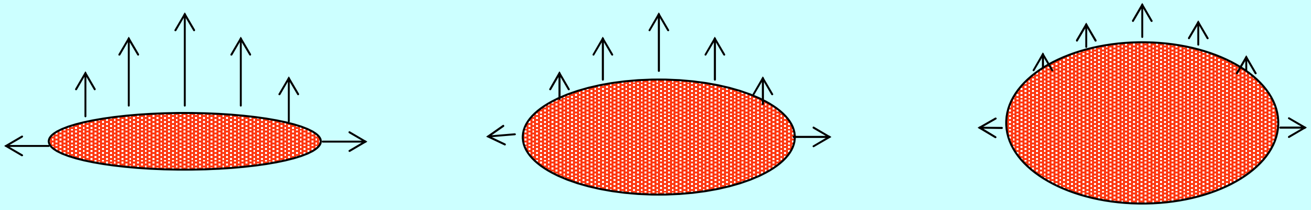
Direct Space Charge Forces

What do we mean with "space charge"?

It is the net effect of the **Coulomb** interactions in a multi-particle system.

Space Charge Regime dominated by the **self field** produced by the particle distribution.

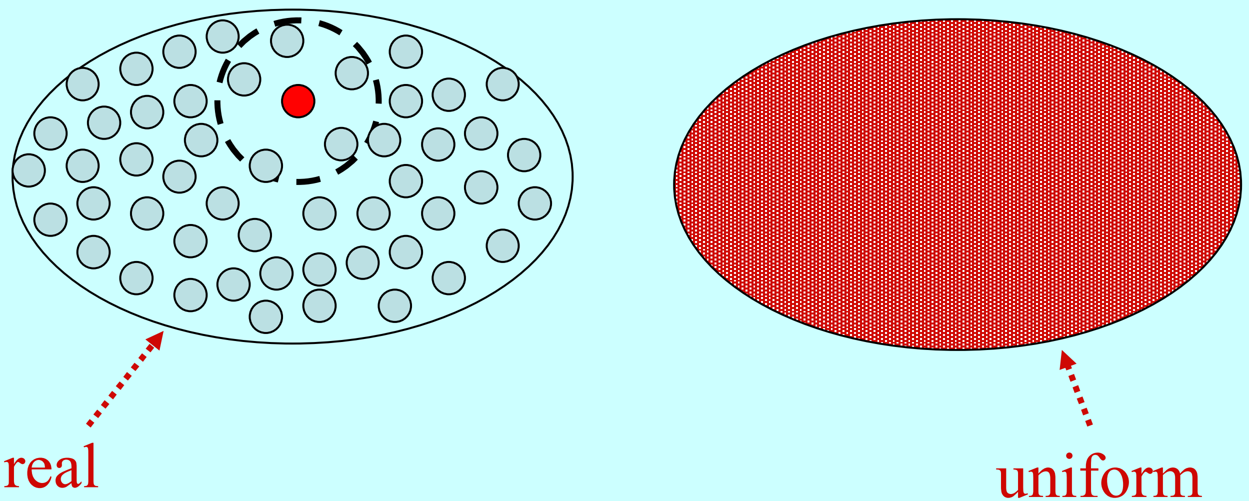
Collective Effects



Debye Length λ_D

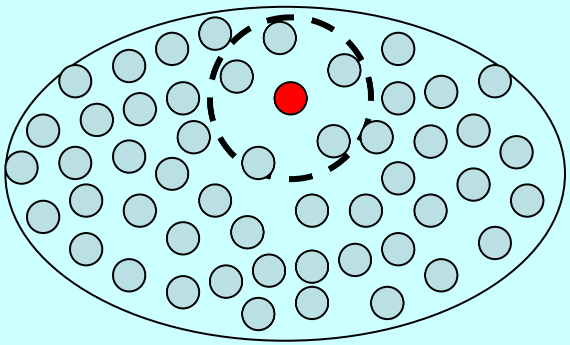
$$\Phi(\vec{r}) = \frac{C}{r}$$

$$C = \frac{e}{4\pi\varepsilon_o}$$



The particle distribution around a test particle will deviate from the continuous distribution.

The effective potential of a test charge can be defined as the sum of the potential of the uniform distribution and a "perturbed" term.



$$\boxed{\Phi_p(\vec{r}) = \frac{C}{r} e^{-r/\lambda_D}}$$

$$\boxed{\lambda_D = \sqrt{\frac{\varepsilon_o k_B T}{e^2 n}}}$$

k_B= Boltzman constant

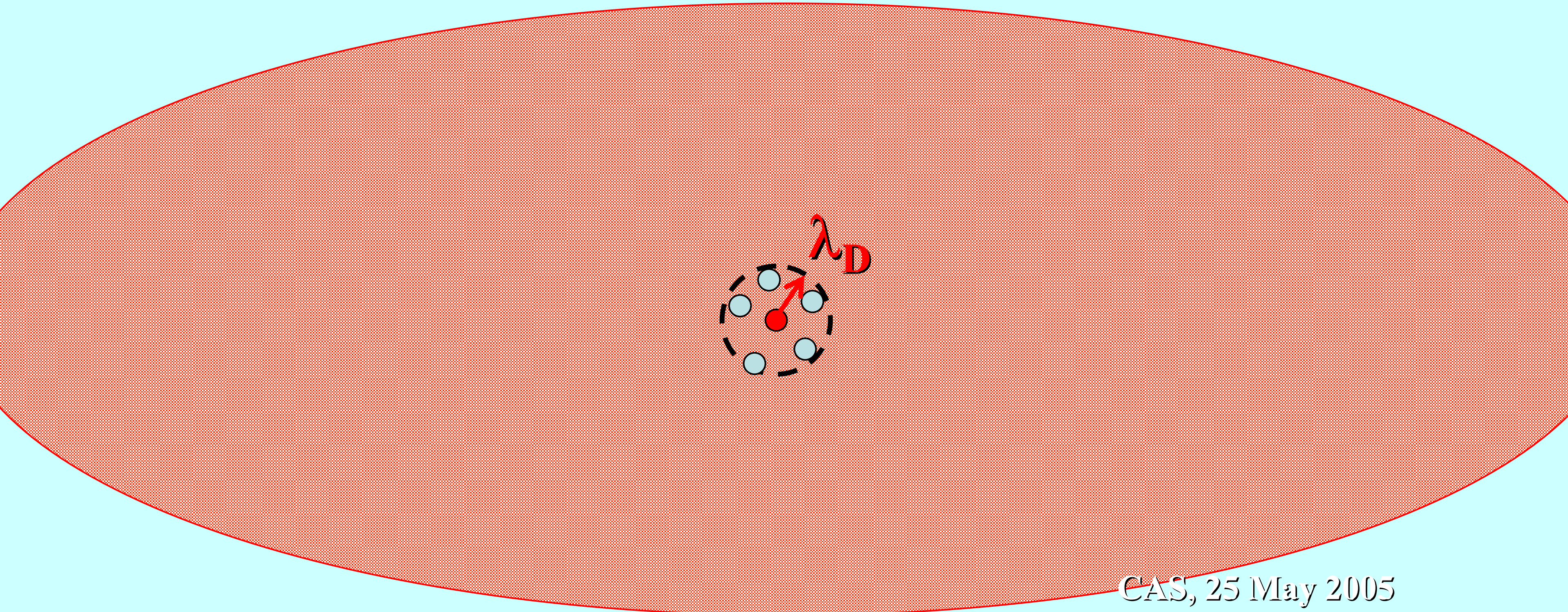
T = Temperature

k_B T = average kinetic energy of the particles

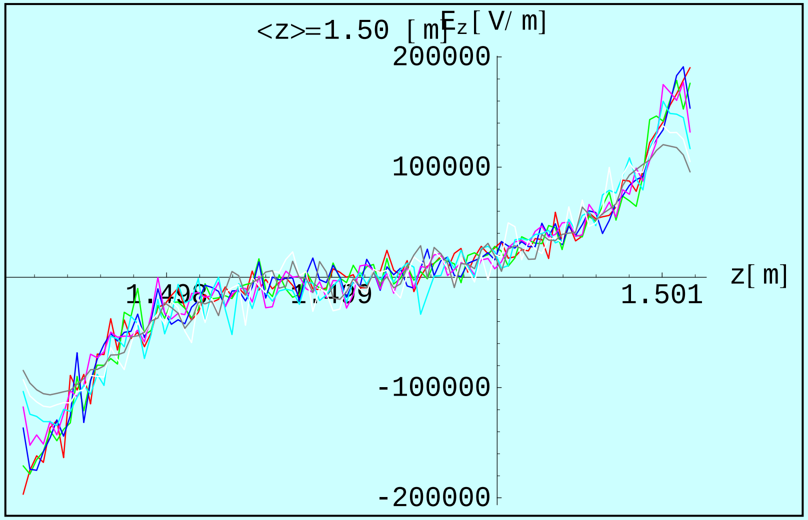
n = particle density (N/V)

The effective interaction range of the test charge is limited to the **Debye length**

Smooth functions for the charge and field distributions can be used as long as the Debye length remains small compared to the particle bunch size

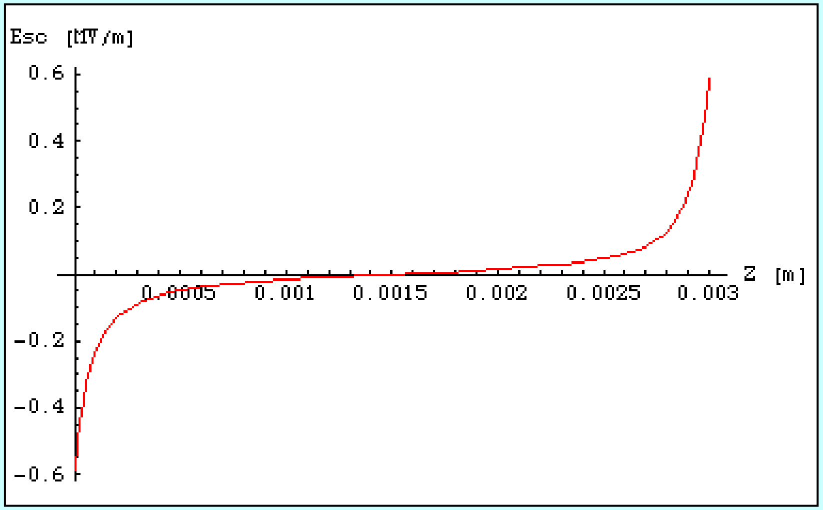


Longitudinal Electric field of a uniform charged cylinder



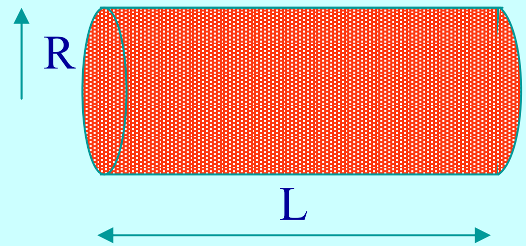
As computed by a multi-particle tracking code

Analytical expression

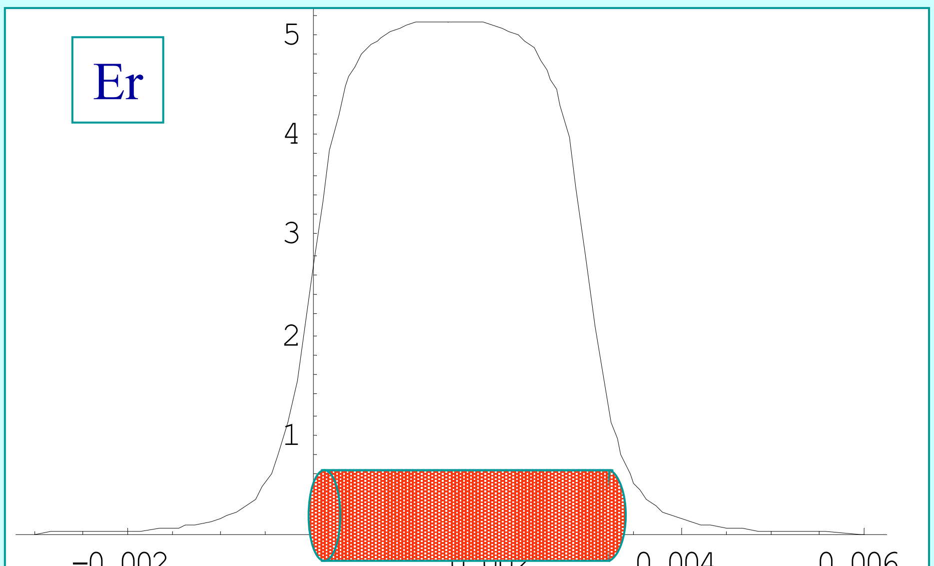


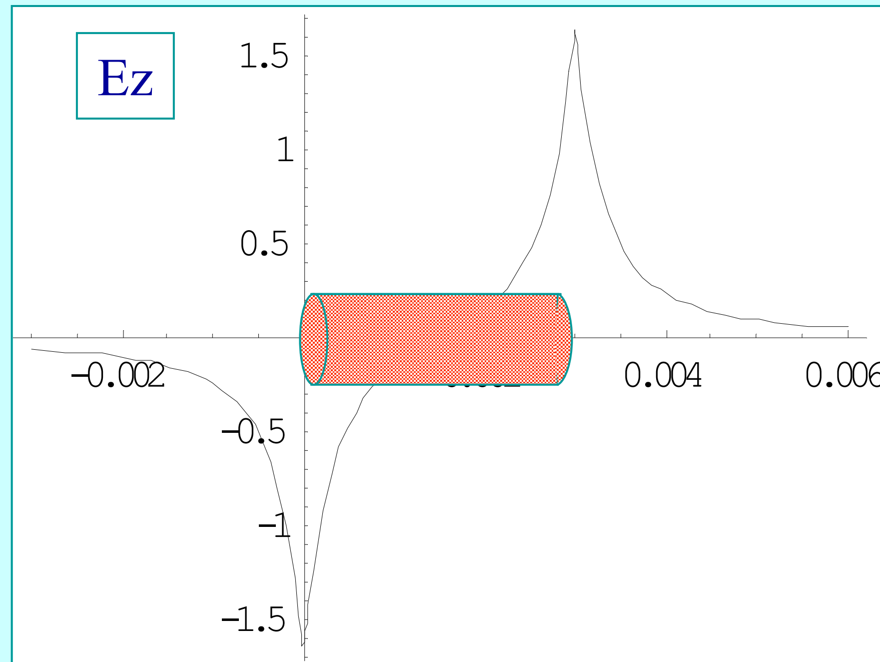
Space charge of a relativistic cylindrical distribution

Cylindrical finite bunch, uniformly charged,
With **circular** cross section



Transverse electric field at r = a

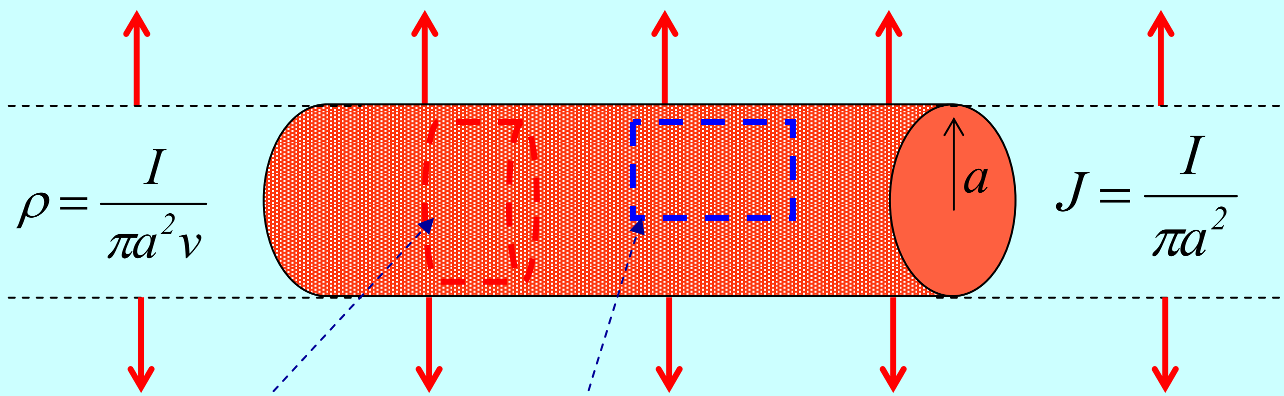




Longitudinal electric field at r = 0

Relativistic Uniform Cylinder

$$\rho = \frac{I}{\pi a^2 v}$$



$$J = \frac{I}{\pi a^2}$$

Gauss's law

$$\int \varepsilon_o E \cdot dS = \int \rho dV$$

Linear with r

$$2\pi r l \varepsilon_o E_r = \rho \pi r^2 l$$

$$\boxed{E_r = \frac{\rho r}{2\varepsilon_o} = \frac{Ir}{2\pi\varepsilon_o a^2 v} \quad \text{for} \quad r \leq a}$$

$$\boxed{B_\varphi = \frac{\beta}{c} E_r}$$

Ampere's law

$$\int B \cdot dl = \mu_o \int J \cdot dS$$

$$2lB_\varphi = \mu_o J l r$$

$$\boxed{B_\varphi = \frac{\mu_o J r}{2} = \mu_o \frac{Ir}{2\pi a^2} \quad \text{for} \quad r \leq a}$$

$$\lambda_o = \rho\pi a^2 (C/m)$$

$$\lambda(r) = \lambda_o (r/a)^2$$

$$J = \beta c \rho (A/m^2)$$
$$I = J\pi a^2 = \beta c \lambda_0 (A)$$

$for\ r < a$

$$E_r(r) = \frac{\lambda_o\, r}{2\pi\varepsilon_o a^2}$$

$$\mathrm{B}_\theta(r) = \frac{\beta}{c} E_r(r) = \frac{\lambda_o \beta}{2\pi\varepsilon_o c}\frac{r}{a^2}$$

The Lorentz Force

$$F_r = e\left(E_r - \beta c B_\vartheta\right) = e\left(1-\beta^2\right)E_r = \frac{eE_r}{\gamma^2} = \frac{e\lambda_o\, r}{2\pi\varepsilon_o \gamma^2 a^2}$$

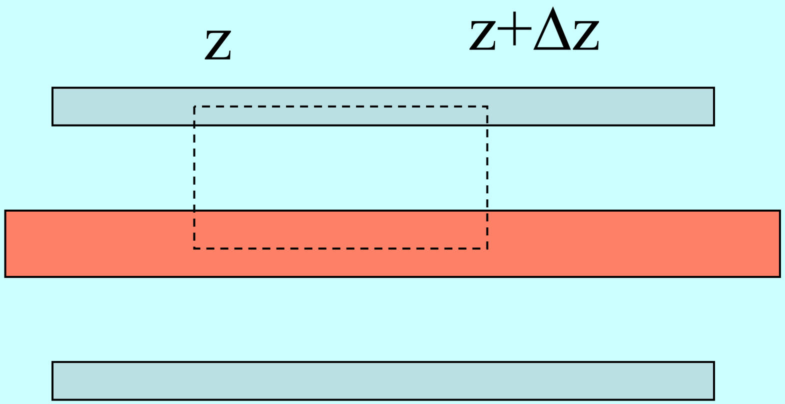
- has only **radial** component
- is a **linear** function of the transverse coordinate

The attractive magnetic force, which becomes significant at high energy, tends to compensate the repulsive electric force.

Longitudinal Space Charge Forces

In order do derive the relationship between the longitudinal and transverse forces inside a beam, let us consider the case of cylindrical symmetry and ultra-relativistic bunches. We know that a varying magnetic field produces a rotational electric field:

$$\oint \mathbf{E} \cdot \mathbf{dl} = -\frac{\partial}{\partial t} \int_S \mathbf{B} \cdot \mathbf{n} dS$$



We choose as path a rectangle going through the beam pipe and the beam, parallel to the axis.

$$E_z(r,z)\Delta z + \int_r^b E_r(r,z+\Delta z)dr - E_z(b,z)\Delta z - \int_r^b E_r(r,z)dr =$$

$$= -\Delta z \frac{\partial}{\partial t}\int_r^b B_\theta(r)dr$$

$$E_z(r,z) = E_z(b,z) + \int_r^b \left[\frac{\partial E_r(r,z)}{\partial z} - \frac{\partial B_\theta(r,z)}{\partial t}\right]dr$$

$$E_z(r,z) = E_z(b,z) + \frac{\partial}{\partial z}\int_r^b \left[E_r(r,z) - vB_\theta(r,z)\right]dr$$

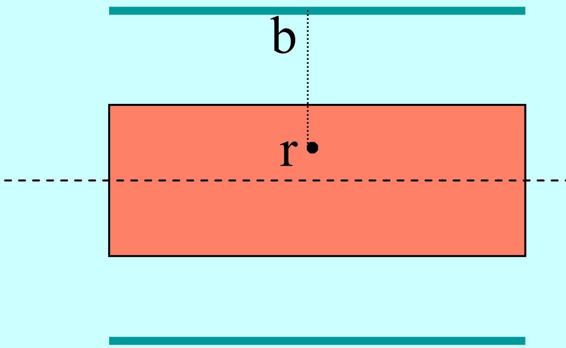
$$E_z(r,z) = E_z(b,z) + \frac{\partial}{\partial z}\int_r^b \left[E_r(r,z) - vB_\theta(r,z)\right]dr$$

$$E_z(r,z) = E_z(b,z) + (1-\beta^2)\frac{\partial}{\partial z}\int_r^b E_r(r,z)dr$$

where $(1-\beta^2)=1/\gamma^2$. For perfectly conducting walls $E_z=0$.

$$E_z(r,z) = \frac{1}{\gamma^2}\frac{\partial}{\partial z}\int_r^b E_r(r,z)dr$$

Uniform beam in a circular p.c. pipe.



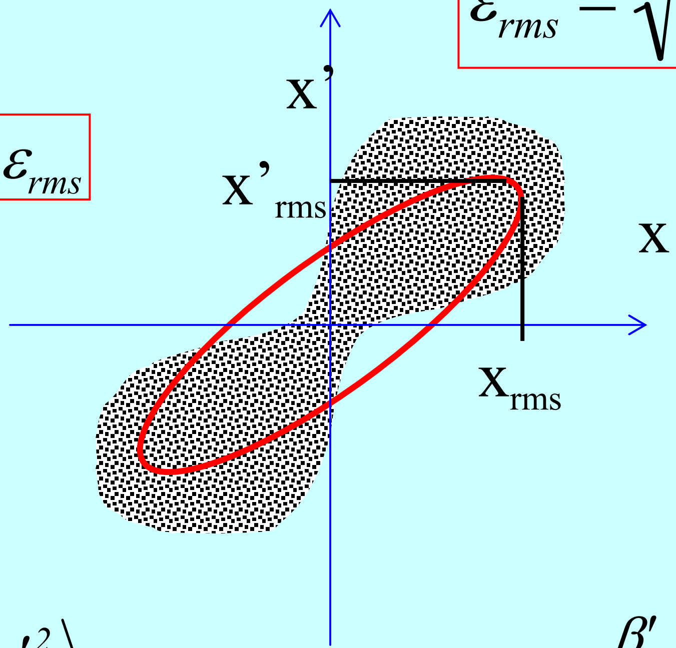
$$F_z(r,z) = \frac{-e}{4\pi\varepsilon_0\gamma^2}\left(1-\frac{r^2}{a^2}+2\ln\frac{b}{a}\right)\frac{\partial\lambda(z)}{\partial z}$$

Beam motion in a linear channel

r.m.s. emittance

$$\varepsilon_{rms} = \sqrt{\left\langle x^2 \right\rangle \left\langle x'^2 \right\rangle - \left\langle xx' \right\rangle^2}$$

$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = \varepsilon_{rms}$$



$$\left\langle x^2 \right\rangle = \beta \varepsilon_{rms} \quad \textbf{\textit{and}} \quad \left\langle x'^2 \right\rangle = \gamma \varepsilon_{rms}$$

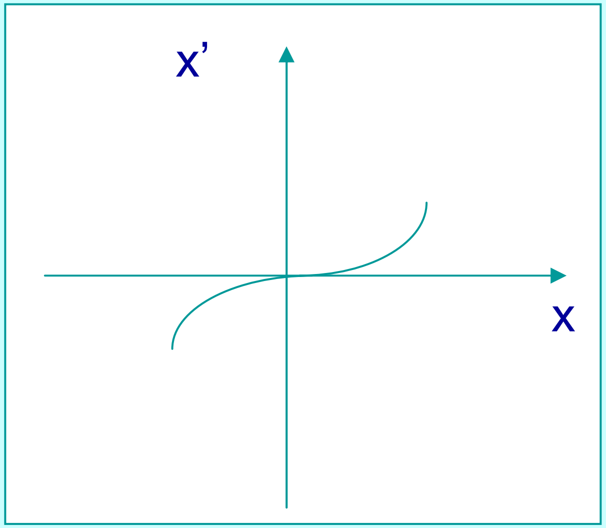
$$\alpha = -\frac{\beta'}{2} = -\frac{1}{2\varepsilon_{rms}} \frac{d}{dz} \left\langle x^2 \right\rangle = -\frac{\left\langle xx' \right\rangle}{\varepsilon_{rms}}$$

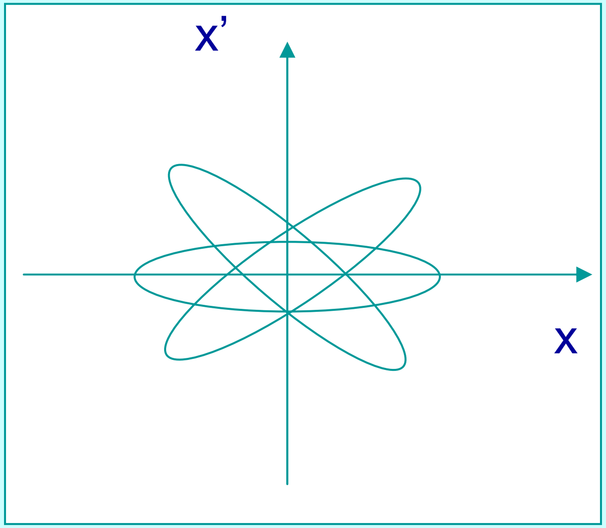
$$\gamma\beta - \alpha^2 = 1$$

Emittance degradation

non linear e.m. fields

Longitudinal correlation along the bunch induced by e.m





RF fields, solenoidal fields, space charge, wake fields

Equation of motion in a drift space:

$$\gamma m \frac{d^2 r}{dt^2} = \frac{eE_r}{\gamma^2} = \frac{eI}{2\pi\gamma^2 \varepsilon_o a^2 v} r$$

$$\frac{d^2 r}{dt^2} = \beta^2 c^2 \frac{d^2 r}{dz^2}$$

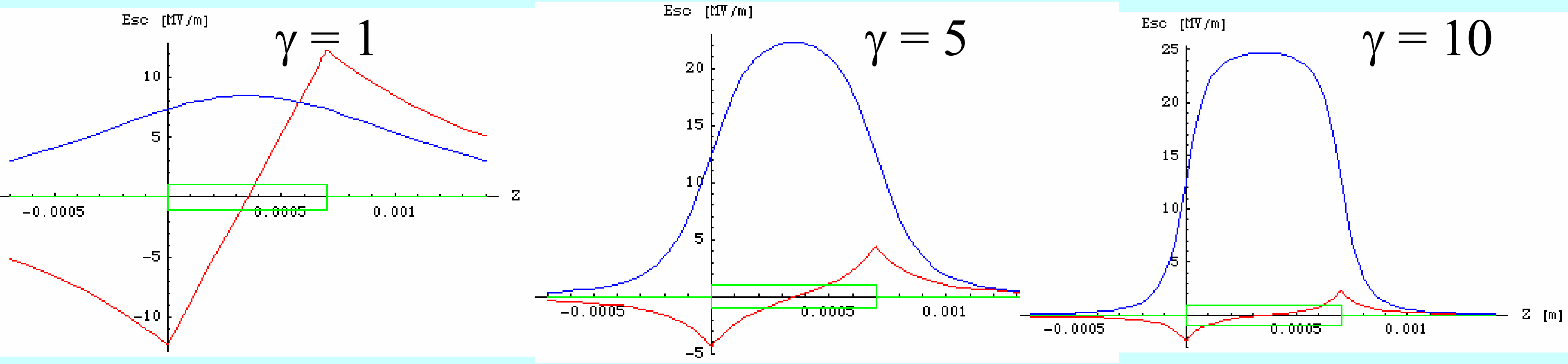
$$\boxed{\frac{d^2 r}{dz^2} = \frac{eI}{2\pi m \gamma^3 \varepsilon_o a^2 v^3} r = \frac{K}{a^2} r}$$

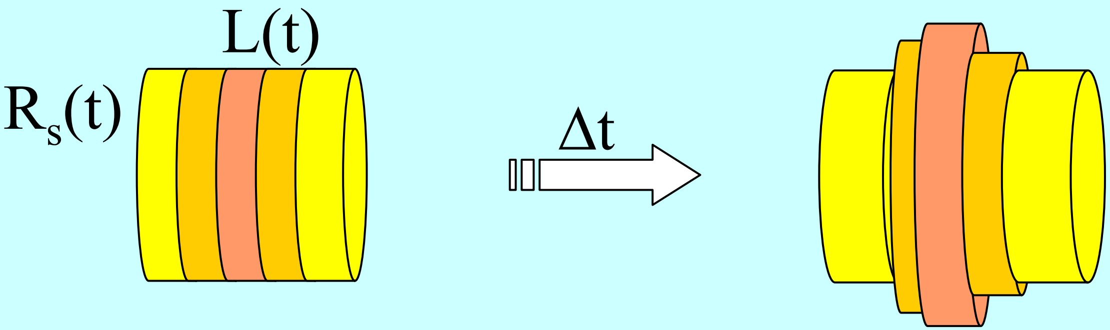
$$K = \frac{eI}{2\pi m \gamma^3 \varepsilon_o v^3} = \frac{2I}{I_o \beta^3 \gamma^3}$$

Generalized perveance

$$I_o = \frac{4\pi\varepsilon_o m c^3}{e}$$

Alfven current

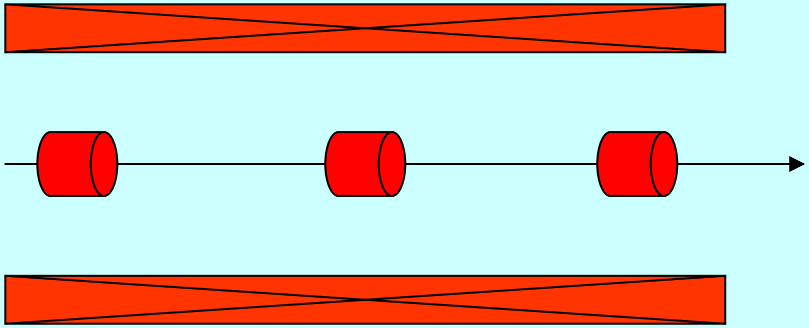

Esc [MV/m]
$\gamma = 1$
10
5
-5
-10
-0.0005
0.0005
0.001
z
Esc [MV/m]
$\gamma = 5$
20
15
10
5
-5
-0.0005
0.0005
0.001
Esc [MV/m]
$\gamma = 10$
25
20
15
10
5
-0.0005
0.0005
0.001
z [m]

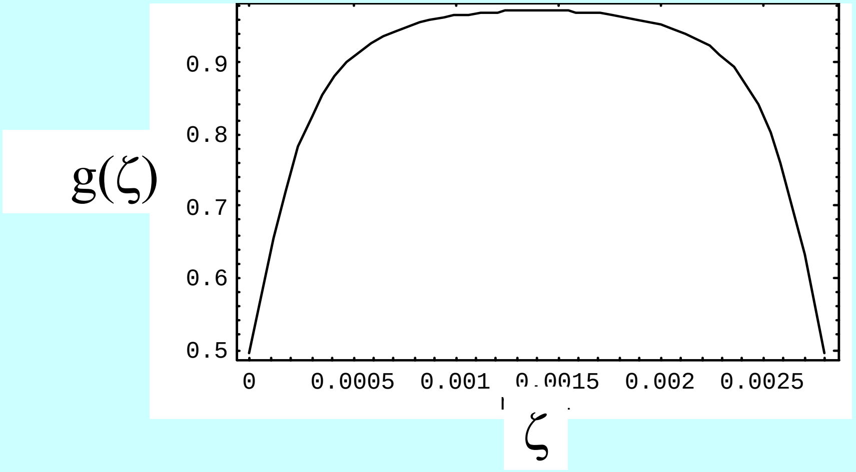

L(t)
$R_s(t)$
Δt

Transport in a Long Solenoid

$$k_s = \frac{qB}{2mc\beta\gamma}$$

$$K(\zeta) = \frac{2Ig(\zeta)}{I_o(\beta\gamma)^3}$$

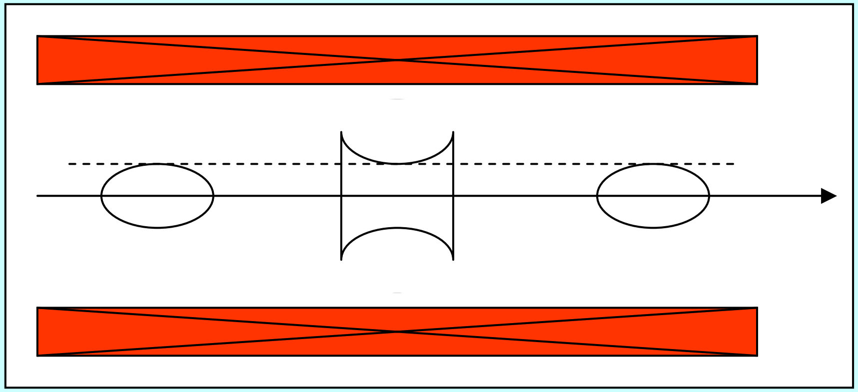




$$\boxed{R'' + k_s^2 R = \frac{K(\zeta)}{R}}$$

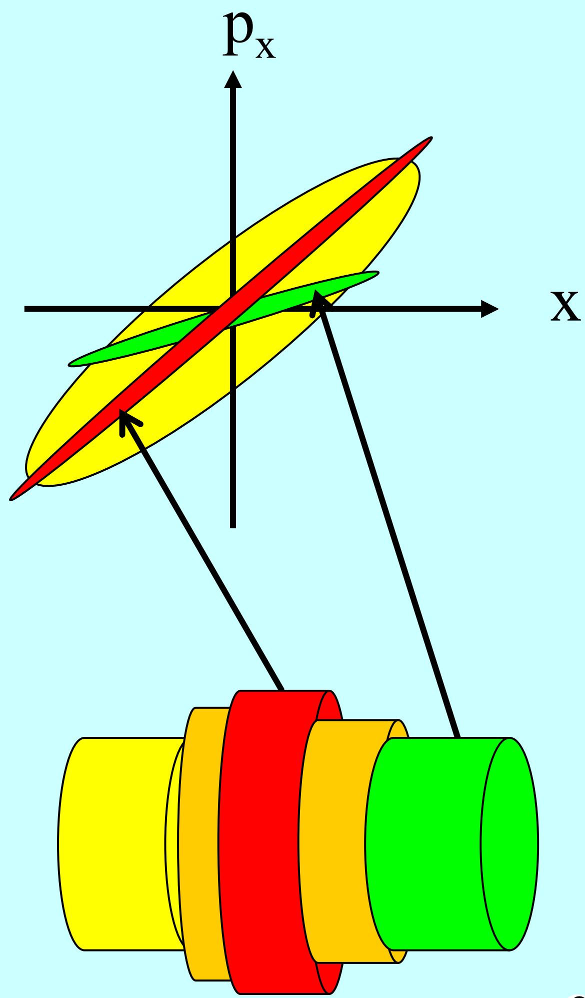
$R'' = 0$ ==> Equilibrium solution ==> $R_{eq}(\zeta) = \dfrac{\sqrt{K(\zeta)}}{k_s}$

Small perturbations around the equilibrium solution



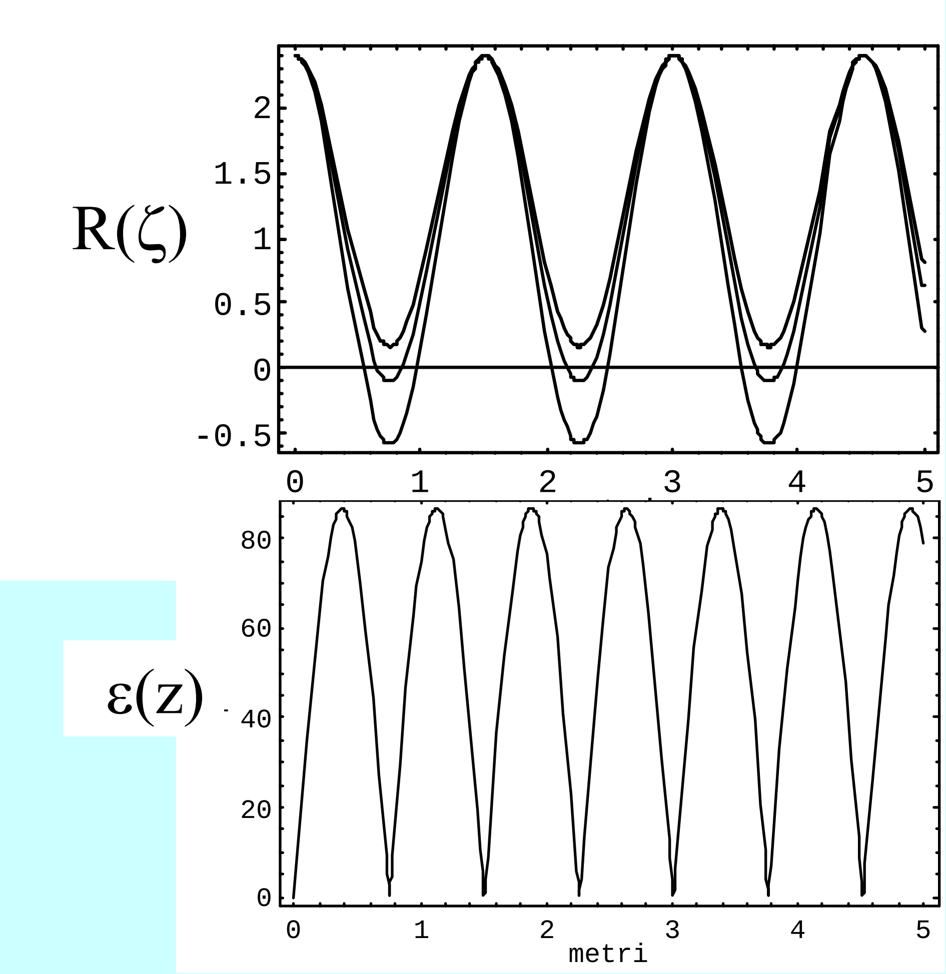
Emittance Oscillations are driven by space charge differential defocusing in core and tails of the beam

Projected Phase Space



Slice Phase Space

Envelope oscillations drive Emittance oscillations

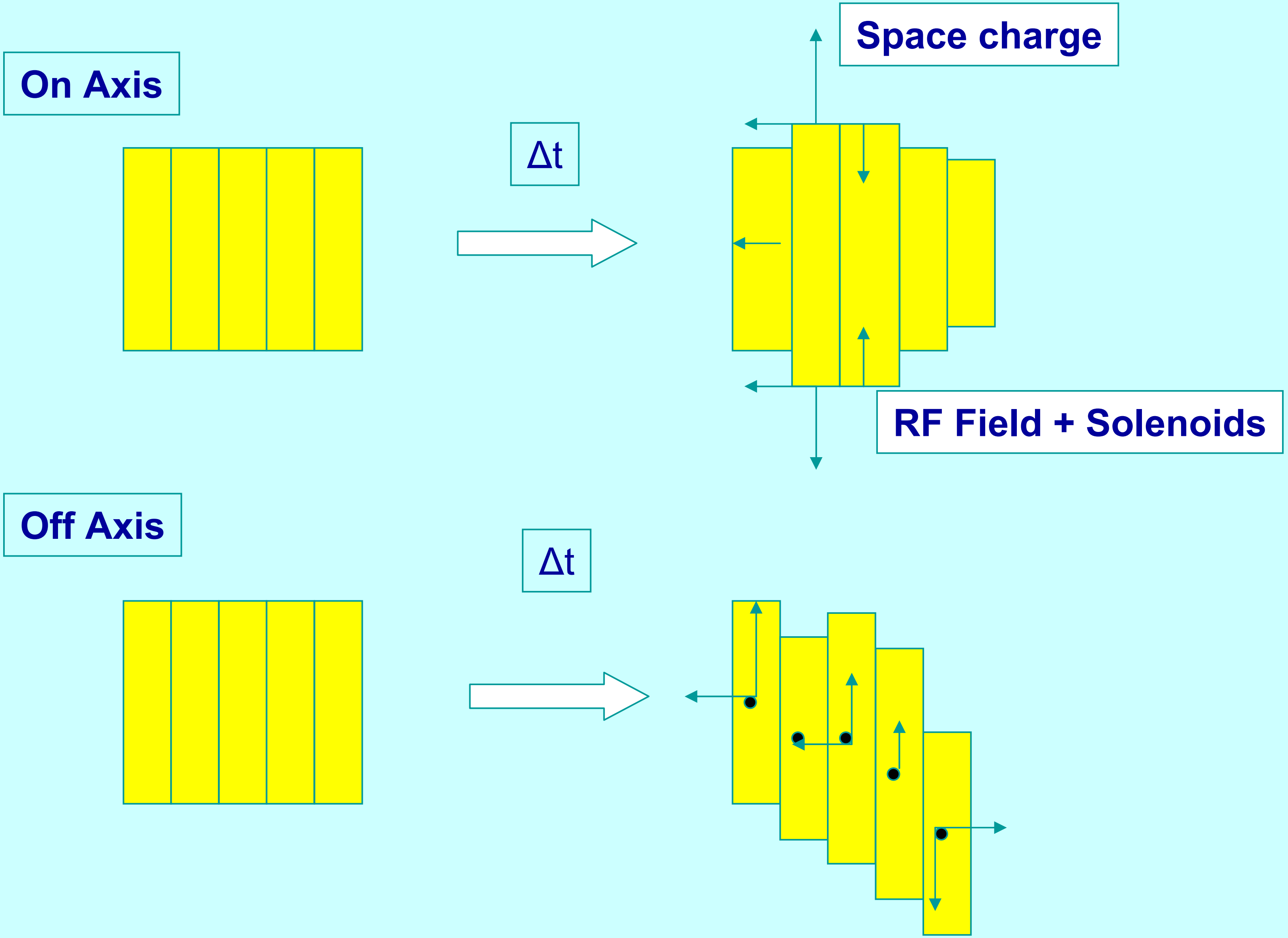


$$\frac{\delta\gamma}{\gamma} = 0$$

$$k_s = \frac{qB}{2mc\beta\gamma}$$

$$\varepsilon(z) = \sqrt{\langle R^2\rangle\langle R'^2\rangle - \langle RR'\rangle^2} \div \left|\sin\left(\sqrt{2}k_s z\right)\right|$$

BEAM DYNAMICS MODELING



On axis → Envelope equation (linear fields)

Space charge

Solenoid field

RF field

$$\ddot{x} + \beta\gamma^2\dot{\beta}x + (k^{rf} + k^{sol})^2 x = \frac{e}{\gamma^3 m} E_x^{sc}(\xi_s, A_{xs}, x) + \left(\frac{4\varepsilon_n^{th}}{\gamma}\right)^2 \frac{1}{x^3}$$

$$\ddot{y} + \beta\gamma^2\dot{\beta}y + (k^{rf} + k^{sol})^2 y = \frac{e}{\gamma^3 m} E_x^{sc}(\xi_s, A_{xs}, y) + \left(\frac{4\varepsilon_n^{th}}{\gamma}\right)^2 \frac{1}{y^3}$$

CODES used for simulations of Space Charge Effects

PARMELA, ASTRA

Multi-particle tracking code, includes space charge but not wake fields

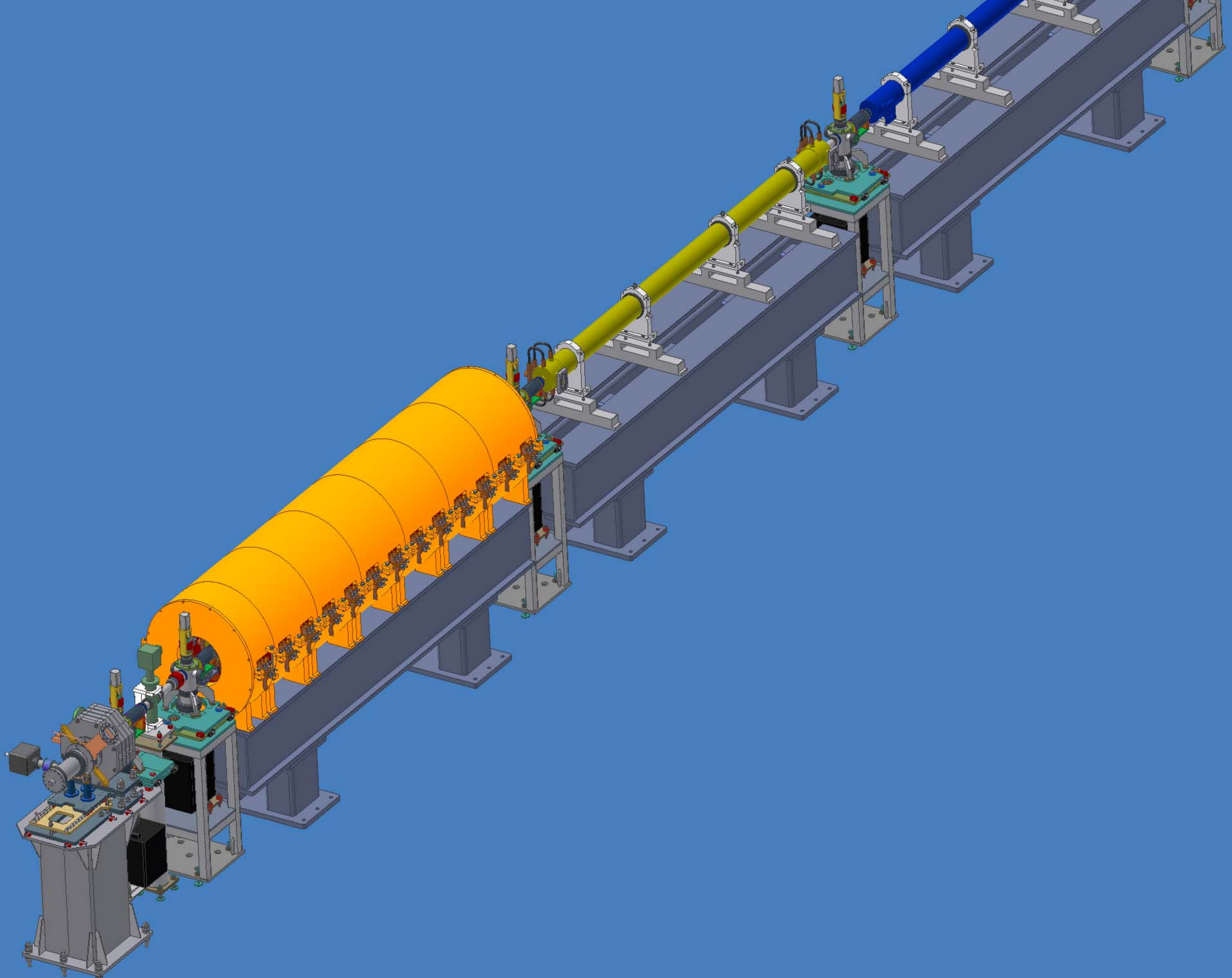
HOMDYN

Relies on a multi-envelope model based on the time dependent evolution of a uniform bunch

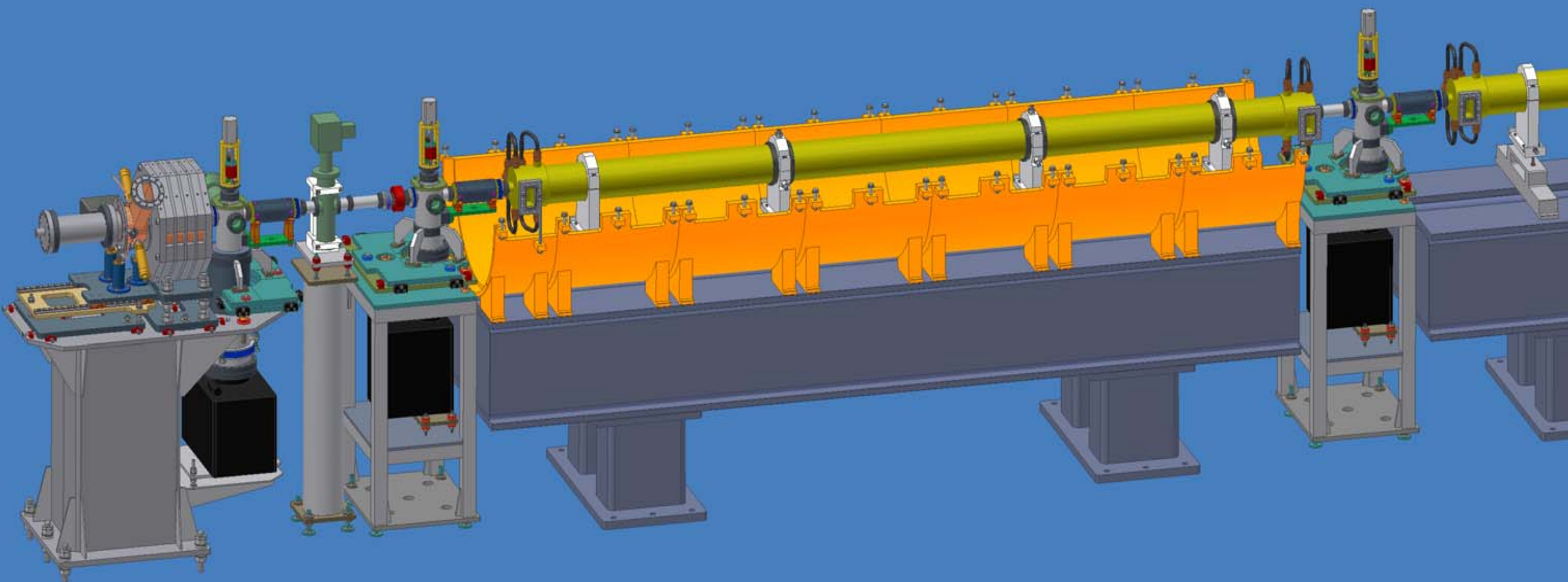
INFN
Istituto Nazionale
di Fisica Nucleare

SPARC

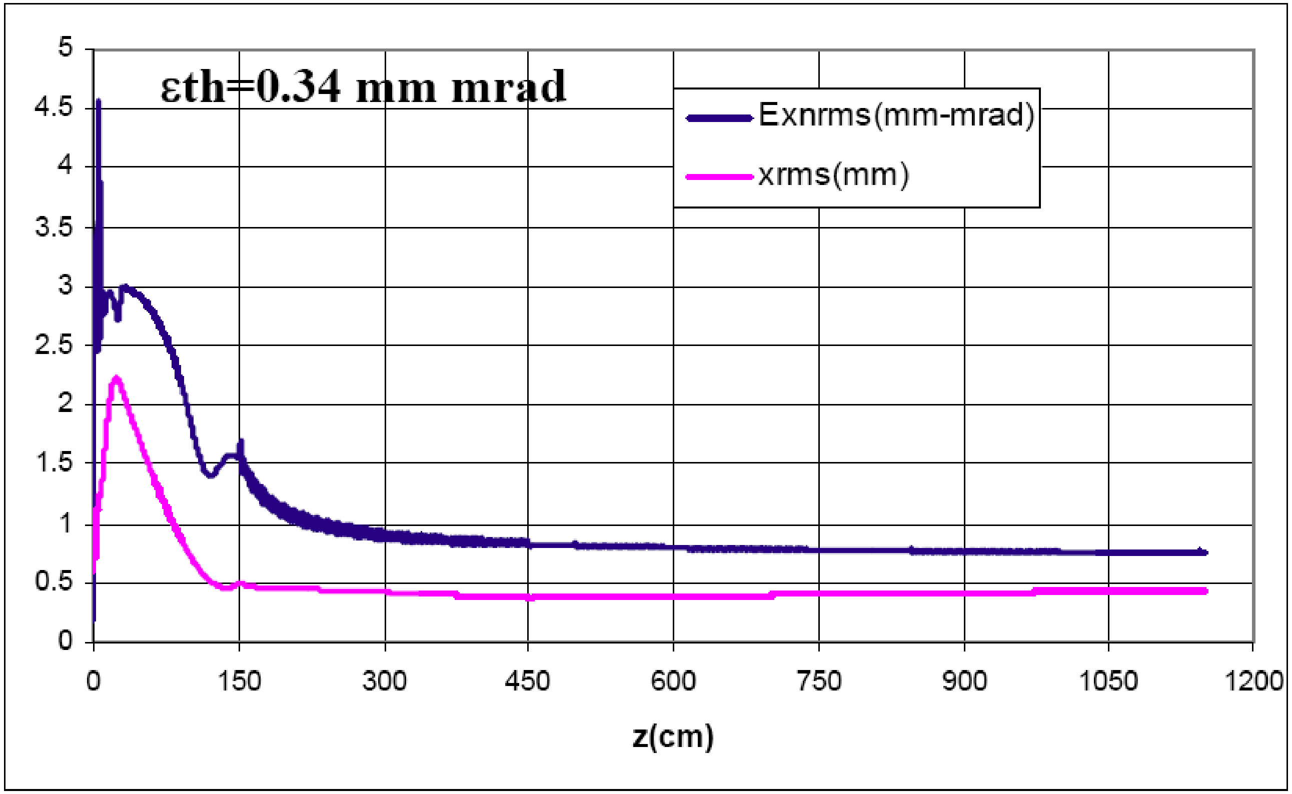
CAS, 25 May 2005



CAS, 25 May 2005



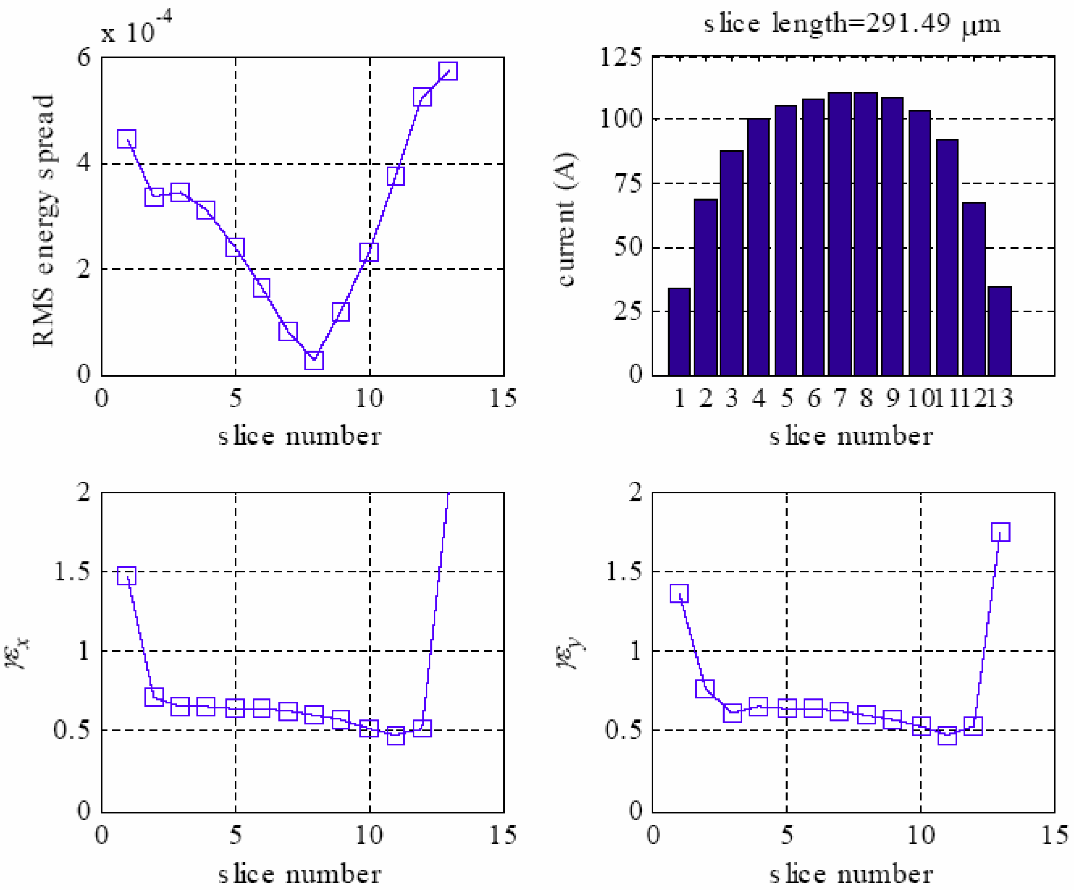
Q=1.1 nC, pulse length (FWHM) = 10 psec, rise time=1 psec



ϕgun=33°, rcathode=1.13 mm, Bgun=2.73 Kgauss,B(TW1)=750 gauss

Slice analysis through the bunch

Q=1.1 nC, pulse length (FWHM) = 10 psec, rise time=1 psec



Max. slice current=110 A

Current ≥ 100 A in 54% beam

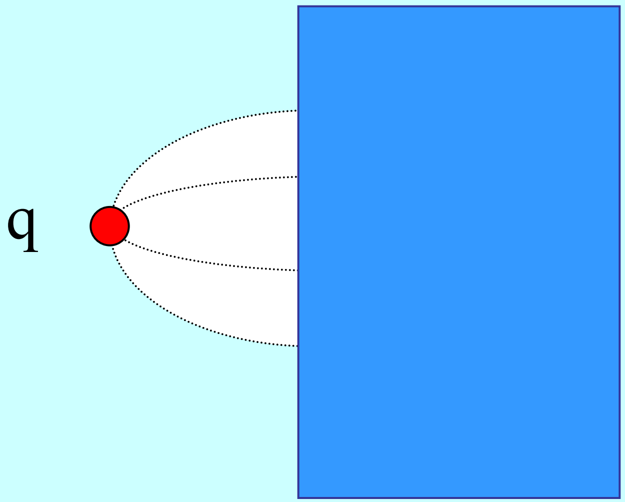
Projected emittance= 0.75 mm mrad

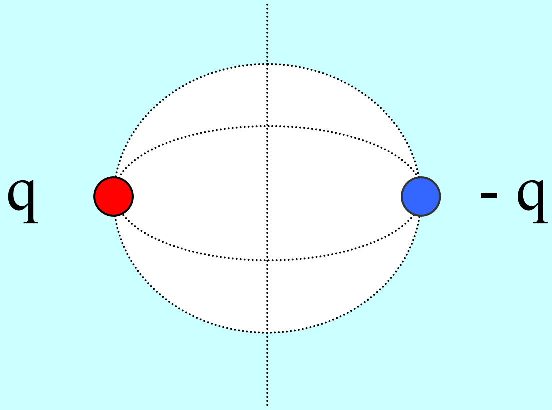
Space charge with image currents

Effects of conducting or magnetic screens

Let us consider a point charge q close to a conducting screen.

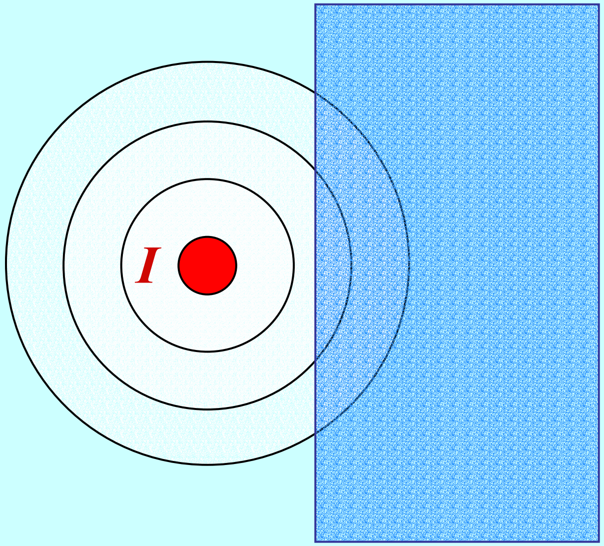
The electrostatic field can be derived through the "image method". Since the metallic screen is an equi-potential plane, it can be removed provided that a "virtual" charge is introduced such that the potential is constant on the screen



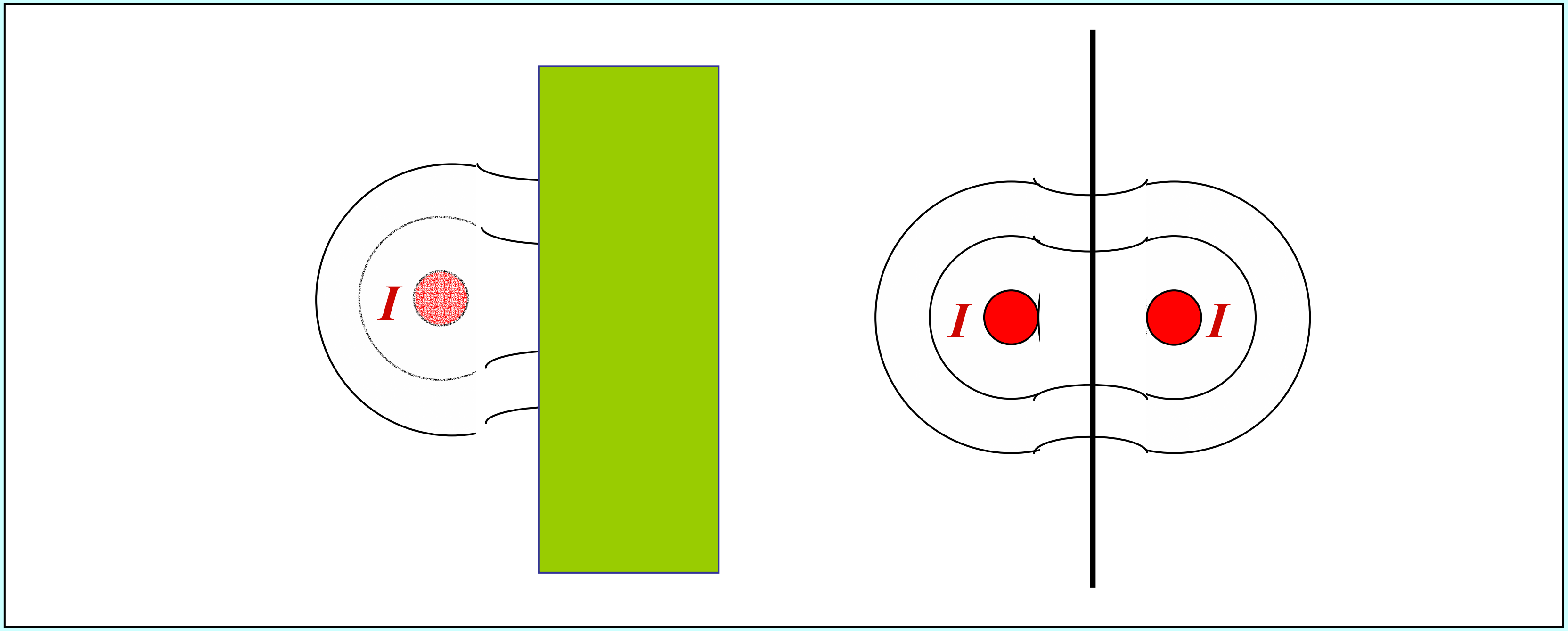


A constant current in the free space produces circular magnetic field.

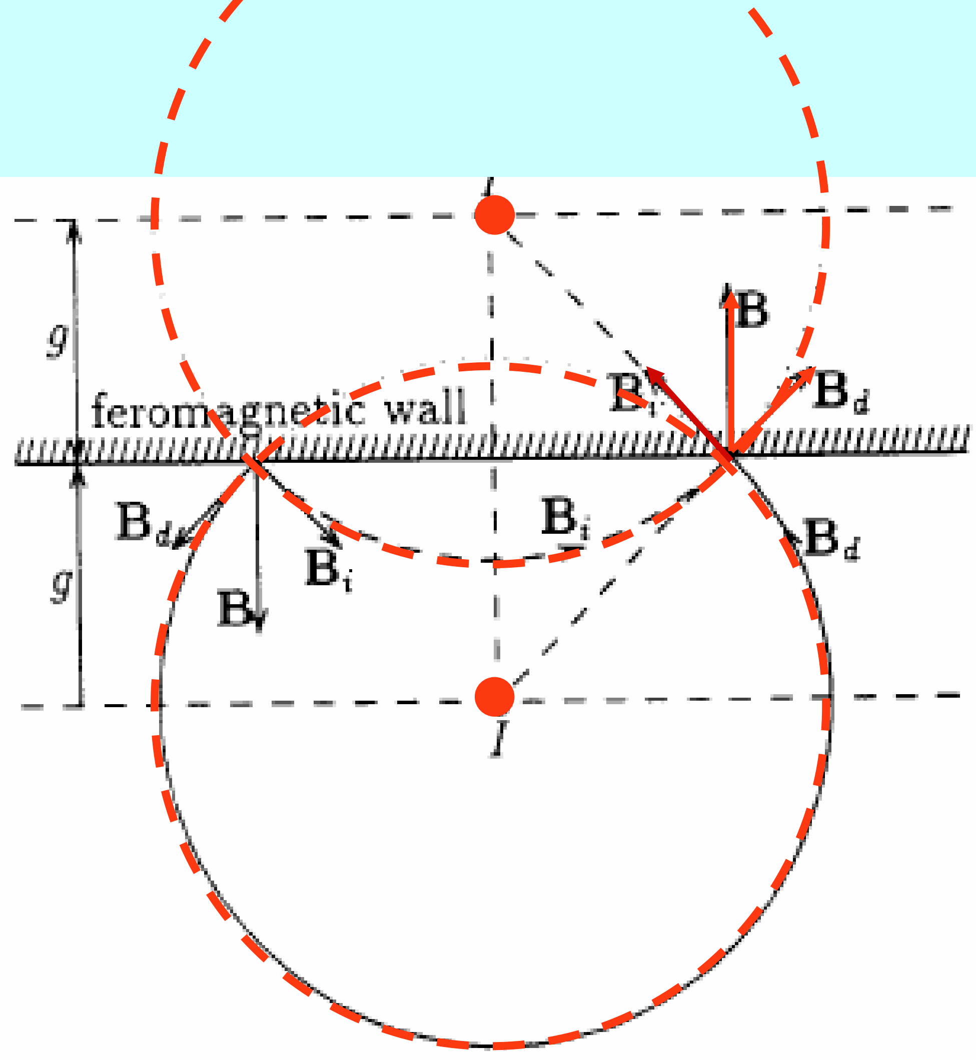
If $\mu_r \approx 1$, the material, even in the case of a good conductor, does not affect the field lines.



For **ferromagnetic type**, with $\mu_r \gg 1$, the very high magnetic permeability makes the tangential magnetic field zero at the boundary so that the magnetic field is perpendicular to the surface, just like the electric field lines close to a conductor.



In analogy with the image method we get the magnetic field, in the region outside the material, as superposition of the fields due to two symmetric equal currents flowing in the same direction.



Satisfying a magnetic boundary condition by an image current

A. Hofmann

Time-varying fields

Static electric fields vanish inside a conductor for any finite conductivity, while magnetic fields pass through unless of an high permeability.

This is no longer true for time changing fields, which can penetrate inside the material only in a region δ_w called skin depth. Inside the conducting material we write the following Maxwell equations:

$$\begin{cases} \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial \mathbf{t}} \\ \nabla \times \mathbf{H} = \mathbf{J} + \dfrac{\partial \mathbf{D}}{\partial \mathbf{t}} \end{cases} \qquad \begin{cases} \mathbf{B} = \mu \mathbf{H} \\ \mathbf{D} = \varepsilon \mathbf{E} \\ \mathbf{J} = \sigma \mathbf{E} \end{cases}$$

Copper $\sigma = 5.8\ 10^7\ (\Omega m)^{-1}$
Aluminium $\sigma = 3.5\ 10^7\ (\Omega m)^{-1}$
Stainless steel $\sigma = 1.4\ 10^6\ (\Omega m)$-1 .

Consider a plane wave (H_z,E_y) propagating in the material

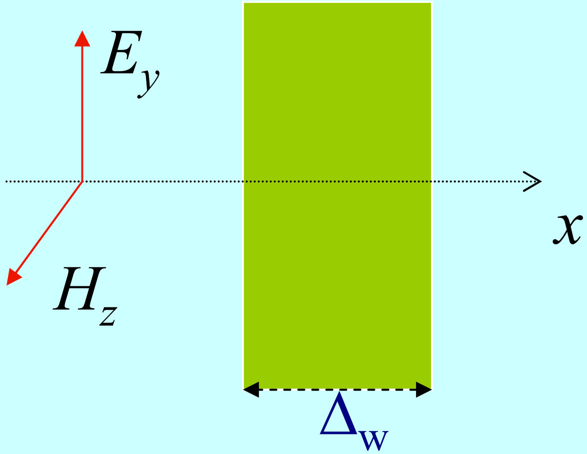
$$\frac{\partial^2 E_y}{\partial x^2} - \varepsilon\mu \frac{\partial^2 E_y}{\partial t^2} - \sigma\mu \frac{\partial E_y}{\partial t} = 0$$

(the same equation holds for H_z). Assuming that fields propagate in the x-direction with the law:

$$H_z = \tilde{H}_o e^{i\omega t - \gamma x}$$

$$E_y = \tilde{E}_o e^{i\omega t - \gamma x}$$

$$(\gamma^2 + \varepsilon\mu\omega^2 - i\omega\mu\sigma)\tilde{E}_o e^{i\omega t - \gamma x} = 0$$



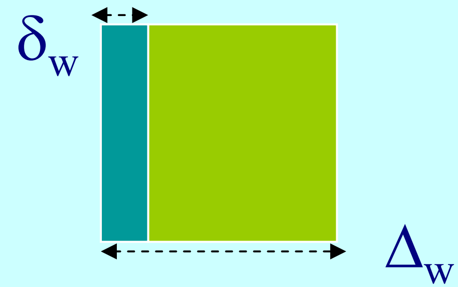
We say that the material behaves like a conductor if σ>>ωε thus:

$$\gamma \cong (1+i)\sqrt{\frac{\sigma\mu\varepsilon}{2}}$$

Fields propagating along "x" are attenuated.

The attenuation constant measured in meters is called skin depth δ_w:

$$\delta_w \cong \frac{1}{\Re(\gamma)} = \sqrt{\frac{2}{\omega\sigma\mu}}$$



The skin depth depends on the material properties and the frequency. Fields pass through the conductor wall if the skin depth is larger than the wall thickness Δ_w. This happens at relatively low frequency.

At higher frequency, for a good conductor $\delta_w \ll \Delta_w$ and both electric and magnetic fields vanish inside the wall.

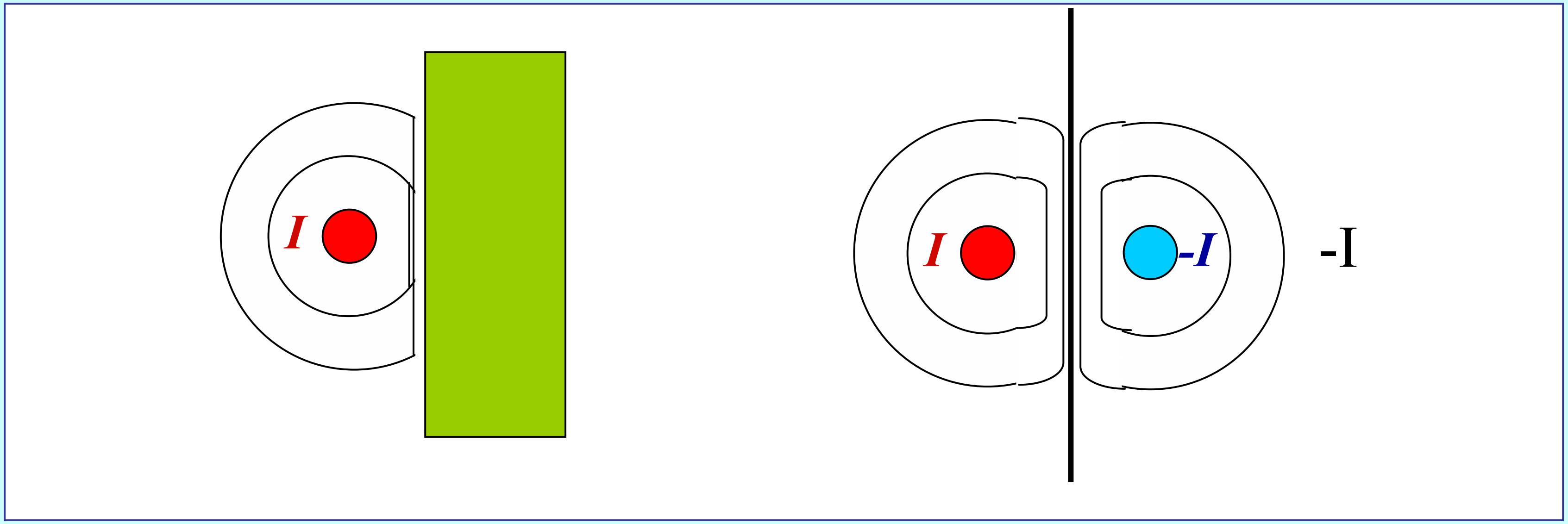
For the copper
$$\delta_w \cong \frac{6.66}{\sqrt{f}}(cm)$$

For a pipe 2mm thick, the fields pass through the wall up to 1 kHz.
(Skin depth of Aluminium is larger by a factor 1.27)

• compare the wall thickness and the skin depth (region of penetration of the e.m. fields) in the conductor.

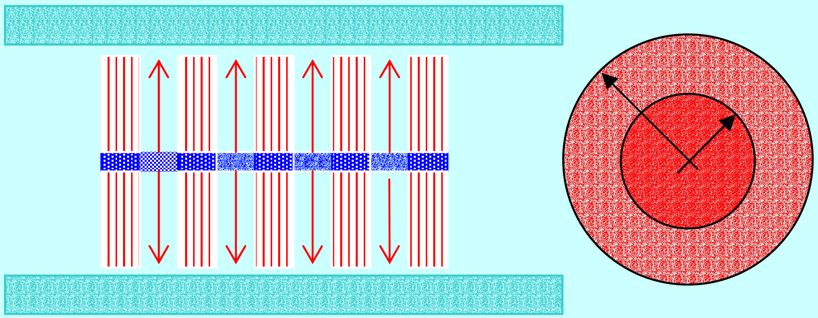
• If the fields penetrate and pass through the material, they can interact with bodies in the outer region.

• If the skin depth is very small, fields do not penetrate, the electric filed lines are perpendicular to the wall, as in the static case, while the magnetic field line are tangent to the surface.



Circular Perfectly Conducting Pipe

(Beam at Center)



In the case of charge distribution, and γ→∞, the electric field lines are perpendicular to the direction of motion. The transverse fields intensity can be computed like in the static case, applying the Gauss and Ampere laws.

$$\lambda(r) = \lambda_o\left(\frac{r}{a}\right)^2 ; \int_S E_r(2\pi r)\Delta z = \frac{\lambda(r)\Delta z}{\varepsilon_o}$$

$$E_r = \frac{\lambda(r)\Delta z}{2\pi\varepsilon_o r}; \quad \boldsymbol{B}_\theta = \frac{\beta}{c}E_r$$

$$E_r(\boldsymbol{r}) = \frac{\lambda_o}{2\pi\,\varepsilon_o}\frac{r}{a^2}\boldsymbol{;} \quad B_\theta(\boldsymbol{r}) = \frac{\lambda_o\beta}{2\pi\varepsilon_o c}\frac{r}{a^2}$$

$$F_\perp(r) = e(E_r - \beta c\,B_\theta) = \frac{e}{\gamma^2}E_r$$

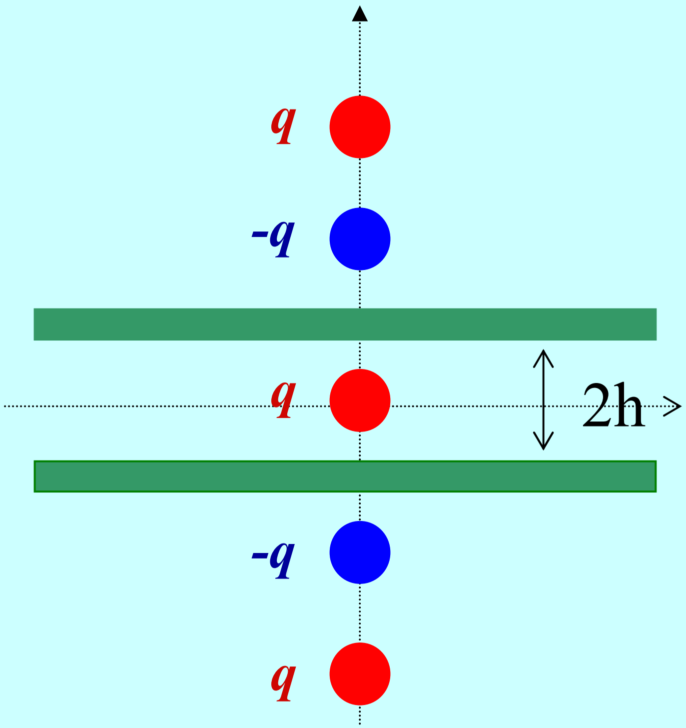
- Due to the symmetry, the transverse fields produced by an ultra-relativistic charge inside the pipe are the same as in the free space.

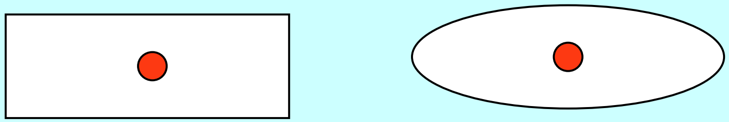
- For a distribution with cylindrical symmetry, in the ultra-relativistic regime, there is a cancellation of the electric and magnetic forces.

- The uniform beam produces exactly the same forces as in the free space.

- This result does not depend on the longitudinal distribution of the beam. In general one has to consider the local charge density $\lambda(z)$

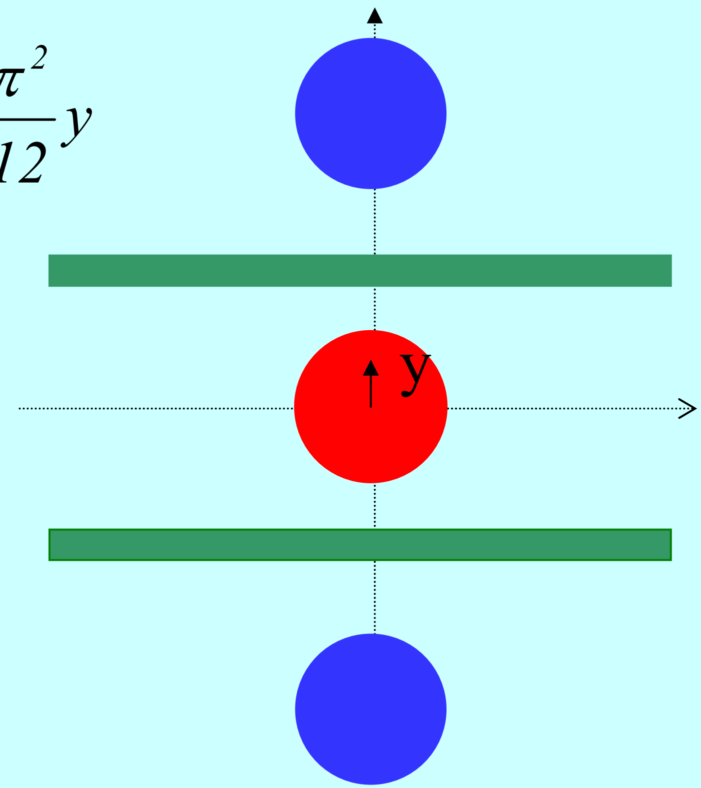
Parallel Plates (Beam at Center)





In some cases, the beam pipe cross section is such that we can consider only the surfaces closer to the beam, which behave like two parallel plates. In this case, we use the image method to a charge distribution of radius a between two conducting plates $2h$ apart. By applying the superposition principle we get the total image field at a position y inside the beam.

$$E_y^{im}(z,y) = \frac{\lambda(z)}{2\pi\varepsilon_o}\sum_{n=1}^{\infty}(-1)^n \frac{-2y}{(2nh)^2 - y^2} \cong \frac{\lambda(z)}{4\pi\varepsilon_o h^2}\frac{\pi^2}{12}y$$



$$E_y^{im}(z,y) = \frac{\lambda(z)}{2\pi\varepsilon_o}\sum_{n=1}^{\infty}(-1)^n\left[\frac{1}{2nh+y} - \frac{1}{2nh-y}\right]$$

Where we have assumed $h >> a > y$.

For d.c. or slowly varying currents, the boundary condition imposed by the conducting plates does not affect the magnetic field.

From the divergence equation we derive also the other transverse component:

$$\frac{\partial}{\partial x}E_x^{im} = -\frac{\partial}{\partial y}E_y^{im} \Rightarrow E_x^{im}(z,x) = \frac{-\lambda(z)}{4\pi\varepsilon_o h^2}\frac{\pi^2}{12}x$$

Including also the direct space charge force, we get:

$$F_x(z,x) = \frac{e\lambda(z)x}{\pi\varepsilon_o}\left(\frac{1}{2a^2\gamma^2} - \frac{\pi^2}{48h^2}\right)$$

$$F_y(z,x) = \frac{e\lambda(z)y}{\pi\varepsilon_o}\left(\frac{1}{2a^2\gamma^2} + \frac{\pi^2}{48h^2}\right)$$

There is no cancellation of the electric and magnetic forces due to the "image" charges.

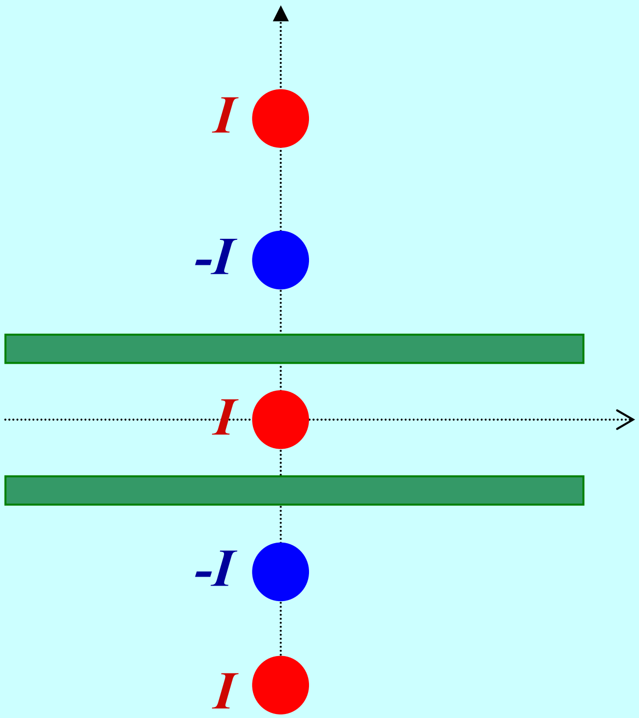
Parallel Plates (Beam at Center) a.c. currents

Usually, the frequency beam spectrum is quite rich of harmonics, especially for bunched beams.

It is convenient to decompose the current into a d.c. component, I, for which $\delta w >> \Delta w$, and an a.c. component, Î, for which $\delta_w << \Delta_w$.

While the d.c. component of the magnetic does not perceives the presence of the material, its a.c. component is obliged to be tangent at the wall. For a charge density λ we have $I=\lambda v$.

We can see that this current produces a magnetic field able to cancel the effect of the electrostatic force.



$$\tilde{E}_y(z,x) = \frac{\tilde{\lambda}(z)y}{\pi\varepsilon_o}\frac{\pi^2}{48h^2}; \qquad \tilde{B}_x(z,x) = \frac{\beta}{c}\tilde{E}_y(z,x)$$

$$\tilde{F}_y(z,x) = \frac{e\tilde{\lambda}(z)y}{\pi\varepsilon_o\gamma^2}\frac{\pi^2}{48h^2}$$

$$\tilde{F}_x(z,x) = \frac{e\tilde{\lambda}(z)x}{2\pi\varepsilon_o\gamma^2}\left(\frac{1}{a^2} - \frac{\pi^2}{24h^2}\right)$$

$$\tilde{F}_y(z,x) = \frac{e\tilde{\lambda}(z)y}{2\pi\varepsilon_o\gamma^2}\left(\frac{1}{a^2} + \frac{\pi^2}{24h^2}\right)$$

There is cancellation of the electric and magnetic forces !!

Parallel Plates - General expression of the force

Taking into account all the boundary conditions for d.c. and a.c. currents, we can write the expression of the force as:

$$F_u = \frac{e}{2\pi\,\varepsilon_o}\left[\frac{1}{\gamma^2}\left(\frac{1}{a^2} \mp \frac{\pi^2}{24h^2}\right)\lambda \mp \beta^2\left(\frac{\pi^2}{24h^2} + \frac{\pi^2}{12g^2}\right)\overline{\lambda}\right]u$$

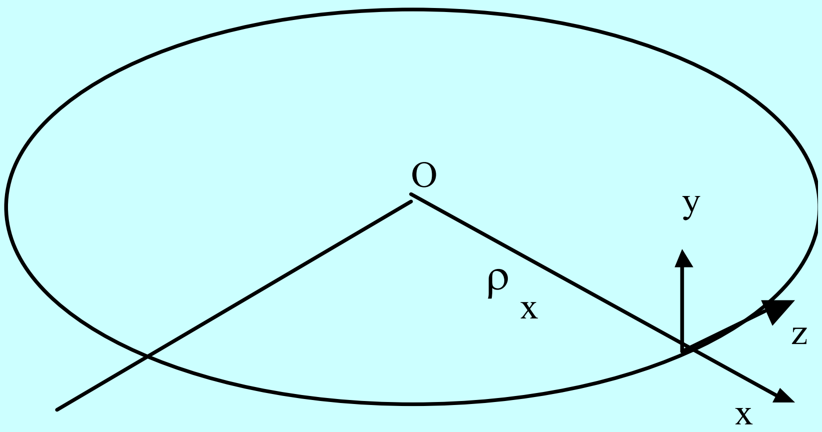
$$u = x, y$$

where λ is the total current, and $\overline{\lambda}$ its d.c. part. We take the sign (+) if u=y, and the sign (–) if u=x.

Space charge effects in storage rings

Self Fields and betatron motion

Consider a perfectly circular accelerator with radius ρ_x. The beam circulates inside the beam pipe. The transverse single particle motion in the linear regime, is derived from the equation of motion. Including the self field forces in the motion equation, we have



$$\frac{d(m\gamma \boldsymbol{v})}{dt} = \boldsymbol{F}^{ext}(\vec{r}) + \boldsymbol{F}^{self}(\vec{r})$$

$$\frac{d\boldsymbol{v}}{dt} = \frac{\boldsymbol{F}^{ext}(\vec{r}) + \boldsymbol{F}^{self}(\vec{r})}{m\gamma}$$

For the single particle "transverse dynamics" we write:

$$\vec{r} = (\rho_x + x)\hat{e}_x + y\hat{e}_y$$
$$\vec{v} = \dot{x}\hat{e}_x + \dot{y}\hat{e}_y + \omega_o(\rho_x + x)\hat{e}_z$$
$$\vec{a} = \left[\ddot{x} - \omega_o^2(\rho_x + x)\right]\hat{e}_x + \ddot{y}\hat{e}_y + \left[\dot{\omega}_o(\rho_x + x) + 2\omega_o\dot{x}\right]\hat{e}_z$$

For the motion along x:

$$\ddot{x} - \omega_o^2(\rho_x + x) = \frac{1}{m\gamma}\left(F_x^{ext} + F_x^{self}\right)$$

Which, with respect to the azimuthal position $s=v_z t$ becomes:

$$\ddot{x} = v_z^2 x'' = \omega_o^2(\rho_x + x)^2 x''$$
$$x'' - \frac{1}{\rho_x + x} = \frac{1}{m v_z^2 \gamma}\left(F_x^{ext} + F_x^{self}\right)$$

We assume small transverse displacements x with respect to the closed orbit, and only dipoles for bending and quadrupole to keep the beam around the closed orbit:

$$F_x^{ext} \approx F_o^{ext} + \left(\frac{\partial F_x^{ext}}{\partial x} \right)_{x=0} x \qquad\qquad x << \rho_x$$

Around the closed orbit, putting $v_z = \beta c$, we get

$$x'' + \left[\frac{1}{\rho_x^2} - \frac{1}{\beta^2 E_o} \left(\frac{\partial F_x^{ext}}{\partial x} \right)_{x=0} \right] x = \frac{1}{\beta^2 E_o} F_x^{self}(x)$$

where E_o is the particle energy. This equation expressed as function of "s" reads:

$$x''(s) + \left[\frac{1}{\rho_x^2(s)} + K_x(s) \right] x(s) = \frac{1}{\beta^2 E_o} F_x^{self}(x,s)$$

•In the analysis of the motion of the particles in presence of the self field, we will adopt a simplified model where particles execute simple harmonic oscillations around the reference orbit.

•This is the case where the focussing term is constant. Although this condition in never fulfilled in a real accelerator, it provides a reliable model for the description of the beam instabilities

$$x''(s)+K_x x(s)=\frac{1}{\beta^2 E_o}F_x^{self}(x)$$

$$x(s)=A_x\cos\left[\sqrt{K_x}s-\varphi_x\right]$$

$$a_x\sqrt{\beta_x}=A_x \Rightarrow \beta_x \ \ const.$$

$$K_x(s)\beta_x^2=1$$

$$\beta_x=\frac{1}{\mu_x'}=\frac{1}{\sqrt{K_x}}$$

$$\mu_x(s)=\sqrt{K_x}s$$

$$Q_x=\frac{\omega_x}{\omega_o}=\frac{1}{2\pi}\int_o^L\frac{ds'}{\beta(s')}=\rho_x\sqrt{K_x}\Rightarrow K_x=\left(\frac{Q_x}{\rho_x}\right)^2$$

$$x''(s)+\left(\frac{Q_x}{\rho_x}\right)^2 x(s)=\frac{1}{\beta^2 E_o}F_x^{self}(x,s)$$

Transverse Incoherent Effects

We take the linear term of the transverse force in the betatron equation:

$$F_x^{s.c.}(x,z) \cong \left(\frac{\partial F_x^{s.c.}}{\partial x} \right)_{x=0} x$$

$$x'' + \left(\frac{Q_x}{\rho_x} \right)^2 x = \frac{1}{\beta^2 E_o} \left(\frac{\partial F_x^{s.c.}}{\partial x} \right)_{x=0} x$$

$$(Q_x + \Delta Q_x)^2 \cong Q_x^2 + 2 Q_x \Delta Q_x \Rightarrow \Delta Q_x = - \frac{\rho_x^2}{2 \beta^2 E_o Q_x} \left(\frac{\partial F_x^{s.c.}}{\partial x} \right)$$

The betatron shift is negative since the space charge forces are defocusing on both planes. Notice that the tune shift is in general function of "z", therefore there is a tune spread inside the beam.

Consequences of the space charge tune shifts

In circular accelerators the values of the betatron tunes should not be close to rational numbers in order to avoid the crossing of linear and non-linear resonances where the beam becomes unstable.

The tune spread induced by the space charge force can make hard to satisfy this basic requirement. Typically, in order to avoid major resonances the stability requires

$$\left|\Delta Q_u\right| < 0.3$$

Example: Incoherent betatron tune shift for an uniform electron beam of radius *a*, length l_o, inside circular perfectly conducting Pipe

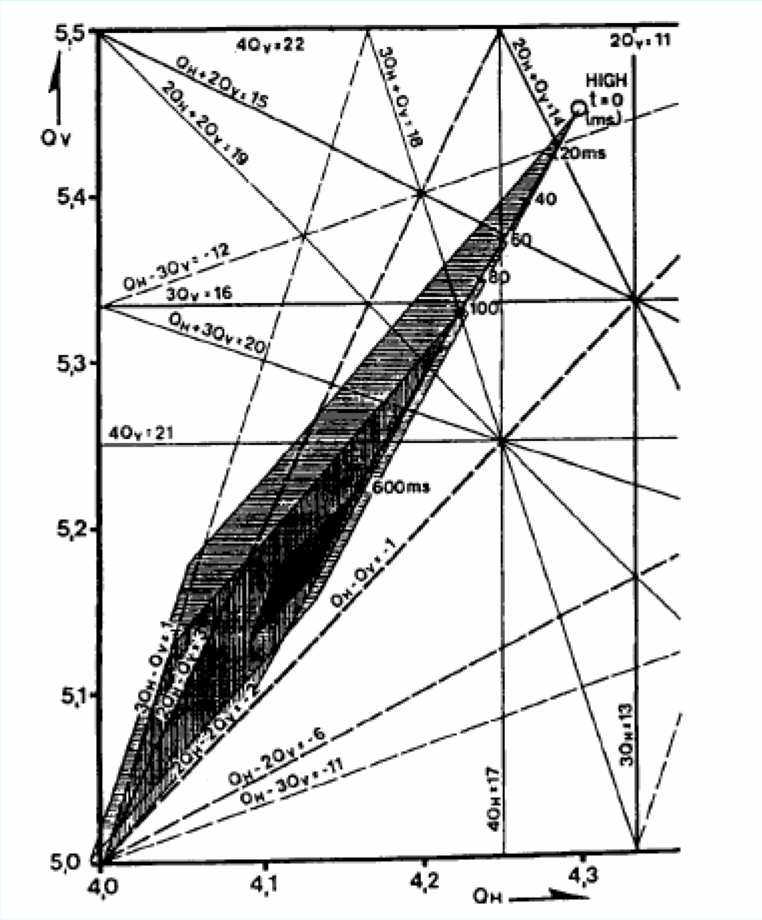
$$\boxed{\left(\frac{\partial F_x^{s.c.}}{\partial x}\right) = \frac{\partial}{\partial x}\frac{e\lambda_o x}{2\pi\varepsilon_o\gamma^2 a^2} = \frac{e\lambda_o}{2\pi\varepsilon_o\gamma^2 a^2}}$$

$$\boxed{\Delta Q_x = -\frac{\rho_x^2 N e^2}{4\pi\varepsilon_o a^2\beta^2\gamma^2 E_o Q_{xo} l_o}}$$

$$r_{e,p} = \frac{e^2}{4\pi\varepsilon_o m_o c^2} \quad (electrons: 2.82\ 10^{-15} m,\ protons: 1.53\ 10^{-18} m)$$

$$\boxed{\Delta Q_x = -\frac{\rho_x^2 N r_{e,p}}{a^2\beta^2\gamma^3 Q_{xo} l_o}}$$

For a real bunched beams the space charge forces, and the tune shift depend on the longitudinal and radial position of the charge.



PS Booster, accelerate proton bunches From 50 to 800 MeV in about 0.6 s. The tunes occupied by the particle are indicated in the diagram by the shaded area. As time goes on, the energy increase and the space charge tune spread gets smaller covering at t=100 ms the tune area shown by the darker area. The point of highest tune correspond to the particles which are least affected by the space charge. This point moves in the Q diagram since the external focusing is adjusted such that the reduced tune spread lies in a region free of harmful resonances.

Finally, the small dark area shows the situation at t=600 ms when the beam has Reached the energy of 800 MeV. The tune spread reduction is lower than expected with the energy increase ($1/\gamma^3$) dependence since the bunch dimensions also decrease during the acceleration.

END