LONGITUDINAL BEAM DYNAMICS AND STABILITY

by

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Small Accelerators Course
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summary

- Methods of Acceleration
- Energy Gain
- Small Accelerators (Betatron, Cyclotron, Microtron)
- Synchronous linear accelerator
- Principle of Phase Stability & Consequences
- The capture phenomenon
- RF Gun & RF Quadrupole
- The Synchrotron
- Dispersion Effects in a Synchrotron
- Energy-Phase Equations in a Synchrotron
- Phase Stability in a Synchrotron

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→ And CERN Accelerator Schools (CAS) Proceedings

Main Characteristics of an Accelerator

ACCELERATION is the main job of an accelerator.

- The accelerator provides **kinetic energy** to charged particles, hence increasing their **momentum**.
- In order to do so, it is necessary to have an electric field $\vec{E}$, preferably along the direction of the initial momentum.

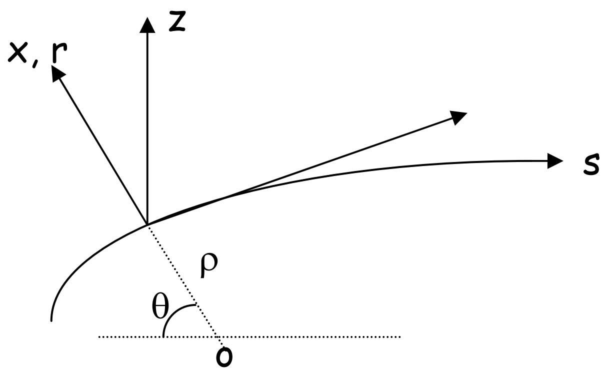
$$\frac{dp}{dt} = eE$$

BENDING is generated by a magnetic field perpendicular to the plane of the particle trajectory. The bending radius ρ obeys to the relation :

$$\frac{p}{e} = B\rho$$

FOCUSING is a second way of using a magnetic field, in which the bending effect is used to bring the particles trajectory closer to the axis, hence to increase the beam density.

Acceleration & Curvature



Within the assumption:

$$\vec{E} \rightarrow E_\theta$$

$$\vec{B} \rightarrow B_z$$

the Newton-Lorentz force:

$$\frac{d\vec{p}}{dt} = e\vec{E} + e\vec{v} \times \vec{B}$$

becomes:

$$\frac{d(mv_\theta)}{dt}\vec{u}_\theta - m\frac{v_\theta^2}{\rho}\vec{u}_r = eE_\theta\vec{u}_\theta - ev_\theta B_z\vec{u}_r$$

leading to:

$$\frac{dp_\theta}{dt} = eE_\theta$$

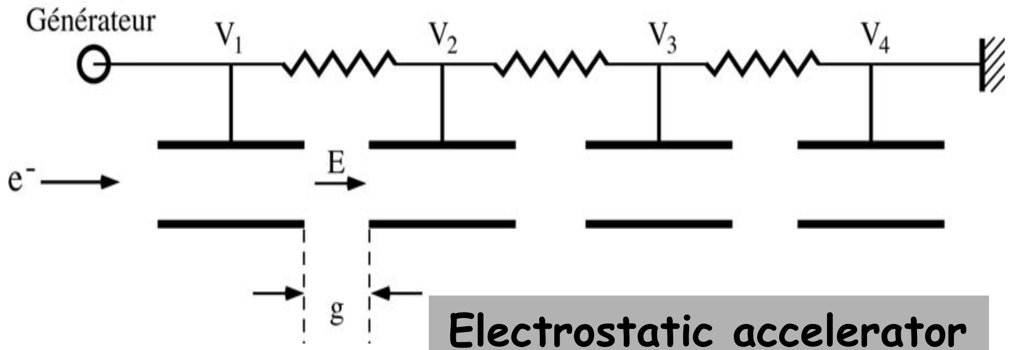
$$\frac{p_\theta}{e} = B_z\rho$$

Methods of Acceleration

1_ Electrostatic Field

Energy gain : $W=n.e(V_2-V_1)$

limitation : $V_{generator} = \Sigma V_i$

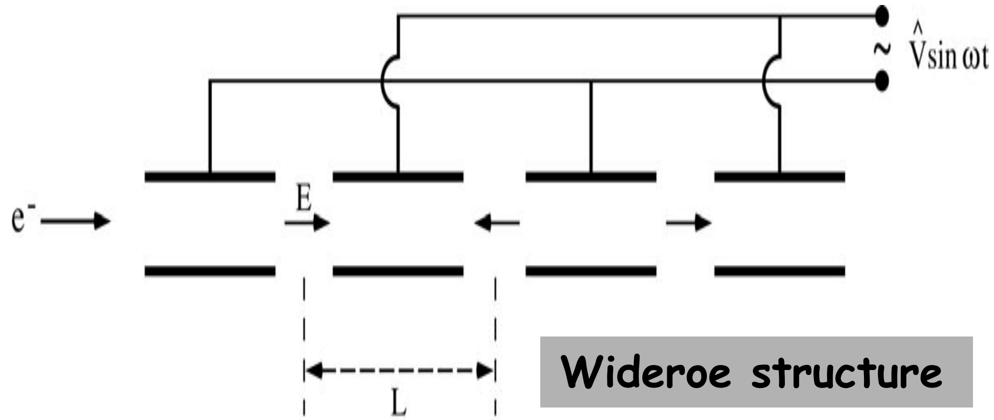


2_ Radio-frequency Field

Synchronism : ➡ L=vT/2

v=particle velocity

T= RF period



Methods of Acceleration (2)

3_ Acceleration by induction

From MAXWELL EQUATIONS :

The electric field is derived from a scalar potential ϕ and a vector potential ***A***
The time variation of the magnetic field **H** generates an electric field **E**

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$$

$$\vec{B} = \mu \vec{H} = \vec{\nabla} \times \vec{A}$$

Energy Gain

In relativistic dynamics, energy and momentum satisfy the relation:

$$E^2 = E_0^2 + p^2c^2 \qquad \left(E = E_0 + W\right)$$

Hence:

$$dE = vdp$$

The rate of energy gain per unit length of acceleration (along z) is then:

$$\frac{dE}{dz} = v\frac{dp}{dz} = \frac{dp}{dt} = eE_z$$

and the kinetic energy gained from the field along the z path is:

$$dW = dE = eE_z dz \qquad \Rightarrow \qquad W = e\int E_z dz = eV$$

where V is just a potential

Energy Gain (2)

RF acceleration

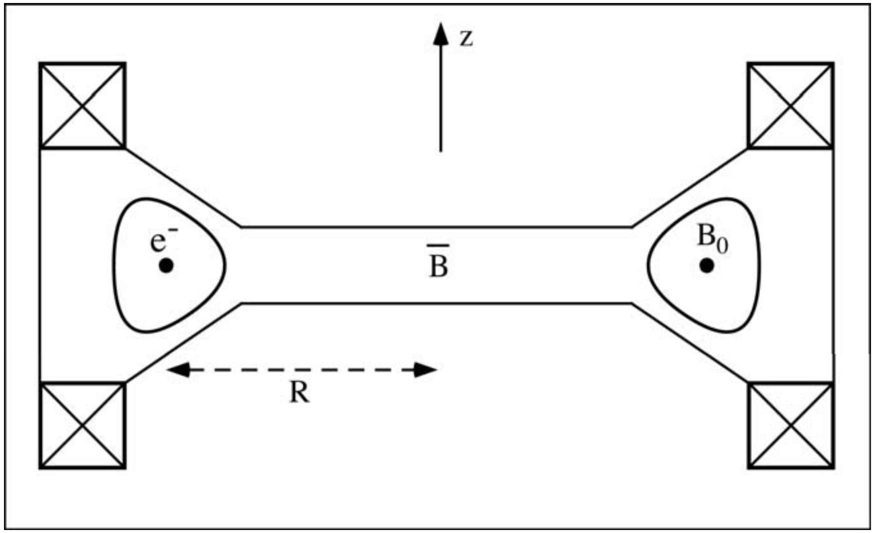
In this case the electric field is oscillating. So it is for the potential. The energy gain will depend on the RF phase experienced by the particle.

$$\int \hat{E}_Z \, dz = \hat{V} \qquad E_Z = \hat{E}_Z \cos \omega_{RF} t \quad = \quad \hat{E}_Z \cos \Phi(t)$$

$$W = e \hat{V} \cos \Phi$$

Neglecting the transit time in the gap.

Betatron



Induction law

$$2\pi R E_\theta = -\frac{d\Phi}{dt} = -\pi R^2 \frac{d\overline{B_z}}{dt}$$

Newton-Lorentz force

$$\frac{dp}{dt} = eE_\vartheta = -\frac{1}{2} e R \frac{d\overline{B_z}}{dt}$$

A constant trajectory also requires :

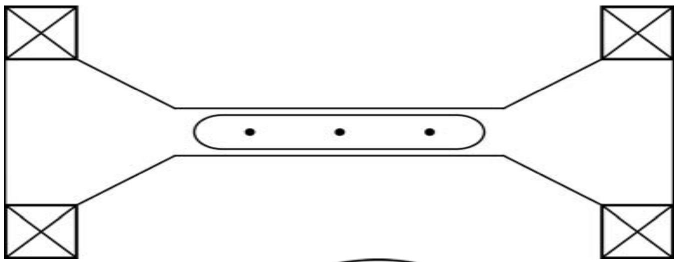
$$p = -e R B_0$$

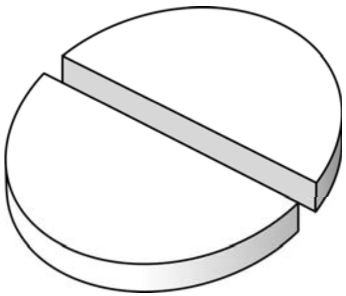
$$\frac{dp}{dt} = -e R \frac{dB_0}{dt}$$

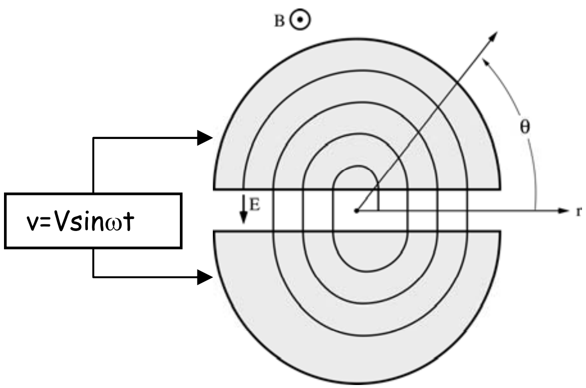
$$\Rightarrow \quad B_o = \frac{1}{2}\overline{B_z}$$

The betatron uses a variable magnetic field with time. The pole shaping gives a magnetic field Bo at the location of the trajectory, smaller than the average magnetic field.

Cyclotron







At each radius r corresponds a velocity v for the accelerated particle. The half circle corresponds to half a revolution period T/2 and B is constant:

$$r = \frac{p}{eB} = \frac{mv}{eB} \Longrightarrow \frac{T}{2} = \frac{\pi\, m}{eB}$$

The corresponding angular frequency is :

$$\omega_r = 2\pi f_r = \frac{2\pi}{T} = \frac{eB}{m}$$

⇒ Synchronism if : $\omega_{RF} = \omega_r$

$m = m_0$ (constant) if $W \ll E_0$

Cyclotron (2)

Energy-phase equation:

Energy gain at each gap transit: $$\Delta E = e\hat{V}\sin\phi$$

Particle RF phase versus time: $$\phi = \omega_{RF}\, t - \theta$$

where θ is the azimuthal angle of trajectory

Differentiating with respect to time gives: $$\dot{\phi} = \omega_{RF} - \omega_r = \omega_{RF} - ec^2\frac{B}{E}$$

Smooth approximation allows: $$\dot{\phi} = \frac{\Delta\phi}{T_r/2} = \frac{\omega_r}{\pi}\Delta\phi$$

Relative phase change at ½ revolution $$\Delta\phi = \frac{\pi}{\omega_r}\dot{\phi} = \pi\left(\frac{\omega_{RF}E}{ec^2B} - 1\right)$$

And smooth approximation again: $$\frac{d\phi}{dE} = \frac{\Delta\phi}{\Delta E} = \frac{\pi}{e\hat{V}\sin\phi}\left(\frac{\omega_{RF}E}{ec^2B} - 1\right)$$

Cyclotron (3)

Separating:

$$d(\cos\phi) = -\frac{\pi}{e\hat{V}}\left(\frac{\omega_{RF}E}{ec^2B} - 1\right)dE$$

Integrating:

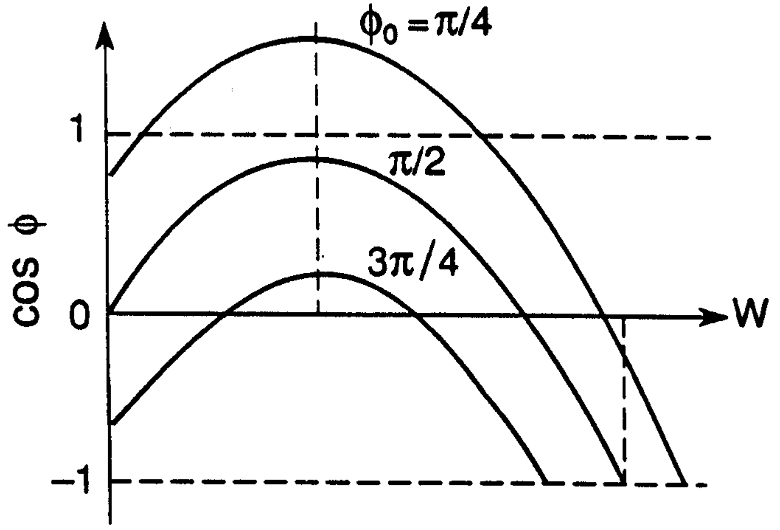
$$\cos\phi = \cos\phi_0 + \frac{\pi}{e\hat{V}}\left(1 - \frac{\omega_{RF}}{\omega_{r0}}\right)(E - E_0) - \frac{\pi}{2e\hat{V}E_0}\frac{\omega_{RF}}{\omega_{r0}}(E - E_0)^2$$

with :

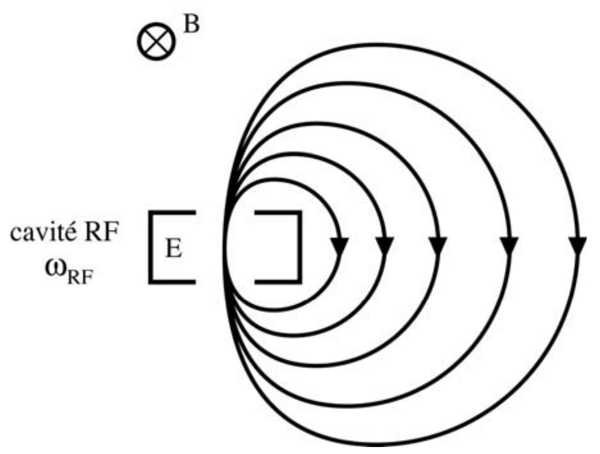
$E_0 =$ Rest energy

$\phi_0 =$ Injection phase

$\omega_{r0} =$ Starting revolution frequency



Microtron (Veksler, 1954)



The expression $\omega_r = \frac{eB}{m}$ shows that if the mass increases, the frequency decreases :

$m \nearrow \Longrightarrow \omega_r \searrow$

Synchronism condition: $T_r \propto m \propto \gamma$

If the first turn is synchronous :

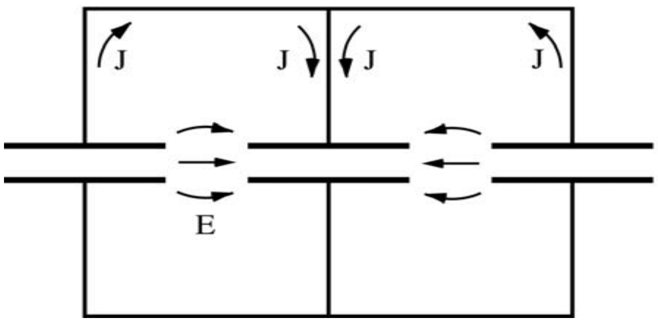
$$\frac{\Delta T_r}{T_{RF}} = \text{integer} \Rightarrow \Delta\gamma_{turn} = \text{integer} \; (\gamma_0 = 1)$$

Energy gain per turn
- electrons → 0.511 MeV
- protons → 0.938 GeV !!!

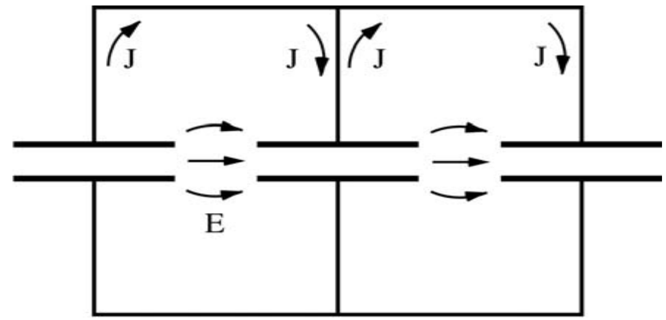
Since required energy gains are large the concept is essentially valid for electrons.

Linear Accelerator

A- Relativistic particles

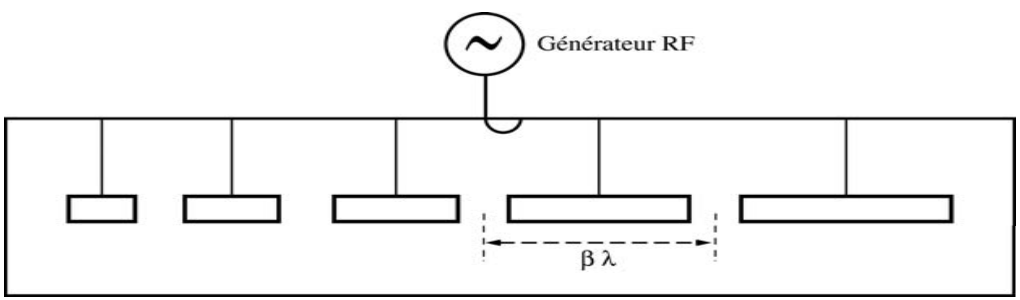


Mode π L= vT/2



Mode 2π L= vT = $\beta\lambda$

In « WIDEROE » the radiated power $\propto \omega$ CV



ALVAREZ structure

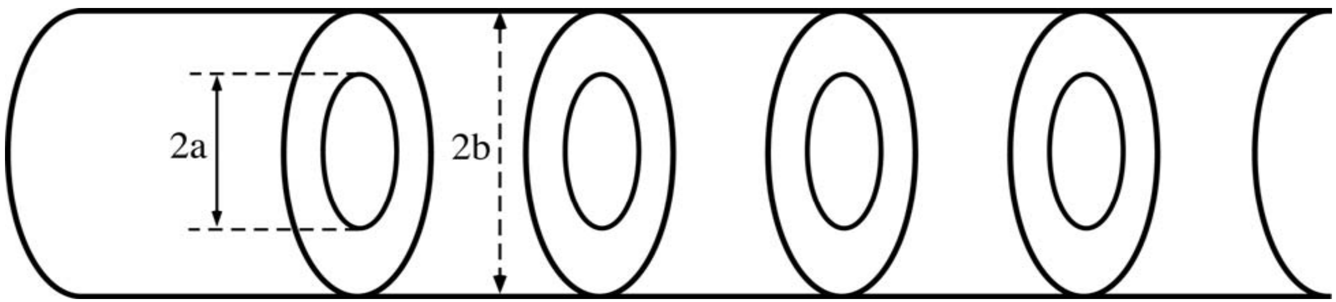
In order to limit the radiated power the gap is enclosed inside a resonant cavity at the operating frequency. A zero circulating current in a wall makes this wall useless (Maxwell).

Linear Accelerator (2)

B- Ultra-relativistic particles v ~ c , β ~ 1

- L increases … unless the frequency ω = 2 π f is increased.
  Following the development of klystrons for radars, it became possible after 1945 to get high RF power at high frequencies, ω ~ 3000 MHz

- Next came the idea of suppressing the drift spaces by using a traveling wave.
  However to benefit from a continuous acceleration the phase velocity of the wave should equal that of the particle (~c).

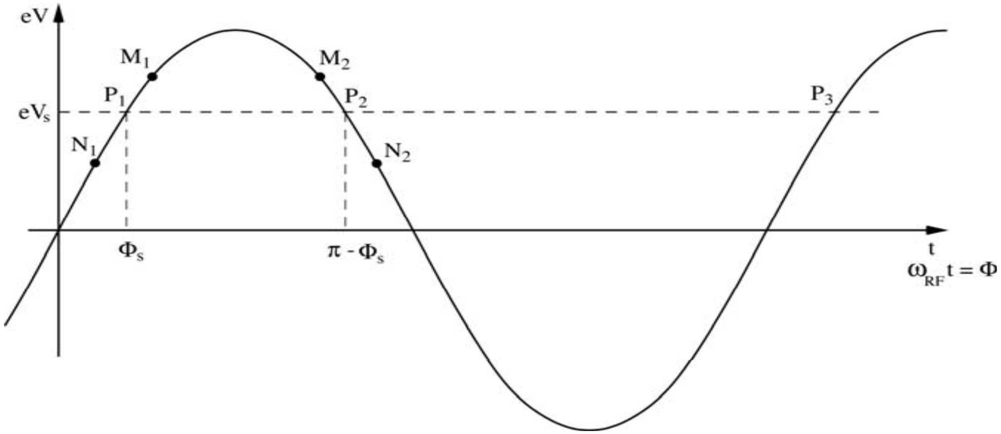


The solution consists of using slow waveguide ⇒ iris loaded waveguide

Principle of Phase Stability

Let's consider a succession of accelerating gaps, operating in the 2π mode, for which the synchronism condition is fulfilled for a phase Φ_s .

For a 2π mode, the electric field is the same in all gaps at any given time.



$$eV_S = e\hat{V}\sin\Phi_S$$

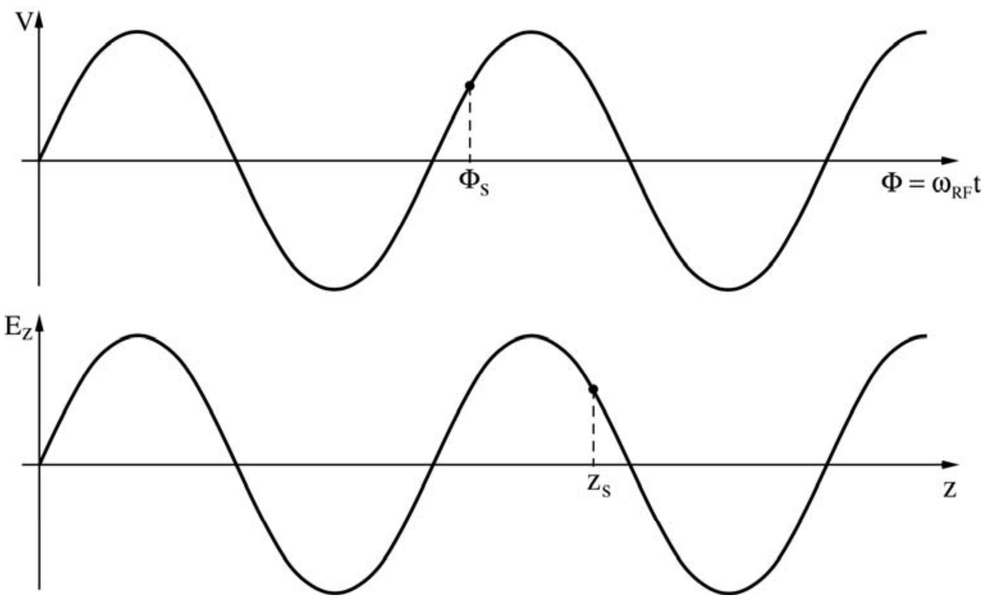
is the energy gain in one gap for the particle to reach the next gap with the same RF phase: P_1 ,P_2, …… are fixed points.

If an increase in energy is transferred into an increase in velocity, M_1 & N_1 will move towards P_1(stable), while M_2 & N_2 will go away from P_2 (unstable).

A Consequence of Phase Stability

Transverse Instability

Longitudinal phase stability means : $\frac{\partial V}{\partial t} > 0 \Rightarrow \frac{\partial E_z}{\partial z} < 0$

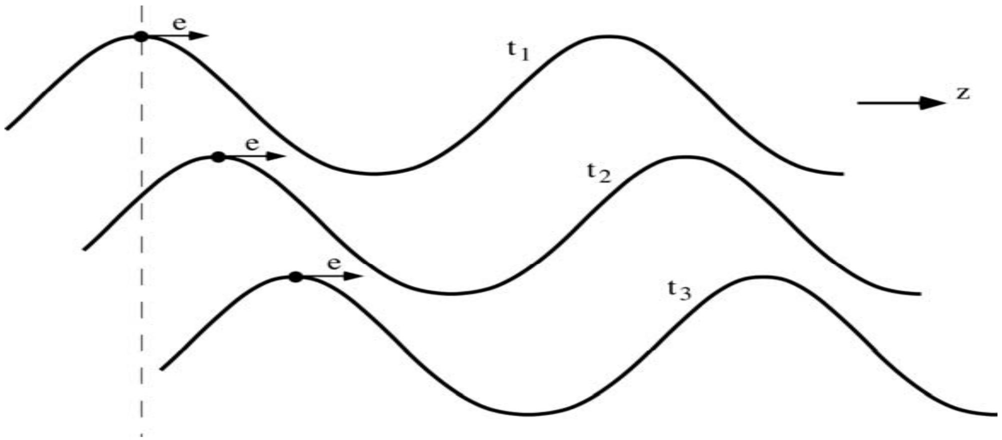


defocusing RF force

The divergence of the field is zero according to Maxwell : $\nabla.\vec{E} = 0 \Rightarrow \frac{\partial E_x}{\partial x} + \frac{\partial E_z}{\partial z} = 0 \Rightarrow \frac{\partial E_x}{\partial x} > 0$

External focusing (solenoid, quadrupole) is then necessary

The Traveling Wave Case



$$E_z = E_0 \cos(\omega_{RF} t - kz)$$

$$k = \frac{\omega_{RF}}{v_\varphi}$$

$$z = v(t - t_0)$$

$$v_\varphi = phase\ velocity$$

$$v = particle\ velocity$$

The particle travels along with the wave, and k represents the wave propagation factor.

$$E_z = E_0 \cos\left(\omega_{RF} t - \omega_{RF} \frac{v}{v_\varphi} t - \phi_0\right)$$

If synchronism satisfied: $v = v_\varphi \quad and \quad E_z = E_0 \cos \phi_0$

where ϕ_0 is the RF phase seen by the particle.

Energy-phase Equations

- Rate of energy gain for the synchronous particle:

$$\frac{dE_s}{dz} = \frac{dp_s}{dt} = eE_0 \sin\phi_s$$

- Rate of energy gain for a non-synchronous particle, expressed in reduced variables, $w = W - W_s = E - E_s$ and $\varphi = \phi - \phi_s$:

$$\frac{dw}{dz} = eE_0\left[\sin\left(\phi_s + \varphi\right) - \sin\phi_s\right] \approx eE_0 \cos\phi_s . \varphi \quad \left(small\ \varphi\right)$$

- Rate of change of the phase with respect to the synchronous one:

$$\frac{d\varphi}{dz} = \omega_{RF}\left(\frac{dt}{dz} - \left(\frac{dt}{dz}\right)_s\right) = \omega_{RF}\left(\frac{1}{v} - \frac{1}{v_s}\right) \cong -\frac{\omega_{RF}}{v_s^2}\left(v - v_s\right)$$

Since:

$$v - v_s = c\left(\beta - \beta_s\right) \cong \frac{c}{2\beta_s}\left(\beta^2 - \beta_s^2\right) \cong \frac{w}{m_0 v_s \gamma_s^3}$$

Energy-phase Oscillations

one gets:

$$\frac{d\varphi}{dz} = -\frac{\omega_{RF}}{m_0 v_s^3 \gamma_s^3} w$$

Combining the two first order equations into a second order one:

$$\frac{d^2\varphi}{dz^2} + \Omega_s^2 \varphi = 0$$

with

$$\Omega_s^2 = \frac{eE_0 \omega_{RF} \cos\phi_s}{m_0 v_s^3 \gamma_s^3}$$

Stable harmonic oscillations imply:

$$\Omega_s^2 > 0 \quad and\ real$$

hence: $\cos\phi_s > 0$

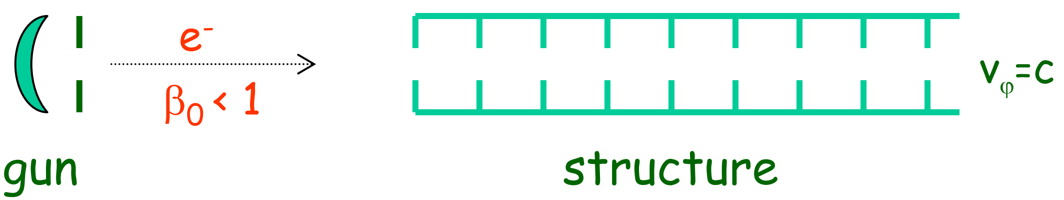
And since acceleration also means: $\sin\phi_s > 0$

One finally gets the results: $0 < \phi_s < \frac{\pi}{2}$

The Capture Problem

- Previous results show that at ultra-relativistic energies ($\gamma \gg 1$) the longitudinal motion is frozen. Since this is rapidly the case for electrons, all traveling wave structures can be made identical (phase velocity=c).

- Hence the question is: can we capture low kinetic electrons energies ($\gamma \approx 1$), as they come out from a gun, using an iris loaded structure matched to c ?



$$E_z = E_0 \sin\phi(t)$$

The electron entering the structure, with velocity v < c, is not synchronous with the wave. The path difference, after a time dt, between the wave and the particle is:

$$dz = (c - v)dt$$

Since: $\phi = \omega_{RF} t - kz \quad \textit{with propagation factor } k = \dfrac{\omega_{RF}}{v_\varphi} = \dfrac{\omega_{RF}}{c}$

one gets: $dz = \dfrac{c}{\omega_{RF}} d\phi = \dfrac{\lambda_g}{2\pi} d\phi$ and $\dfrac{d\phi}{dt} = \dfrac{2\pi}{\lambda_g} c(1-\beta)$

The Capture Problem (2)

From Newton-Lorentz:

$$\frac{d}{dt}(mv) = m_0 c \frac{d}{dt}(\beta\gamma) = m_0 c \frac{d}{dt}\left(\frac{\beta}{\left(1-\beta^2\right)^{\frac{1}{2}}}\right) = eE_0 \sin\phi$$

Introducing a suitable variable:

$$\beta = \cos\alpha$$

the equation becomes:

$$\frac{d\alpha}{dt} = -\frac{eE_0}{m_0 c}\sin\phi \sin^2\alpha$$

Using $\frac{d\phi}{dt} = \frac{d\phi}{d\alpha}\frac{d\alpha}{dt}$ ⟶

$$-\sin\phi\, d\phi = \frac{2\pi m_0 c^2}{\lambda_g e E_0}\frac{1-\cos\alpha}{\sin^2\alpha}\, d\alpha$$

Integrating from t_0 to t (from $\beta=\beta_0$ to $\beta=1$) ⟶

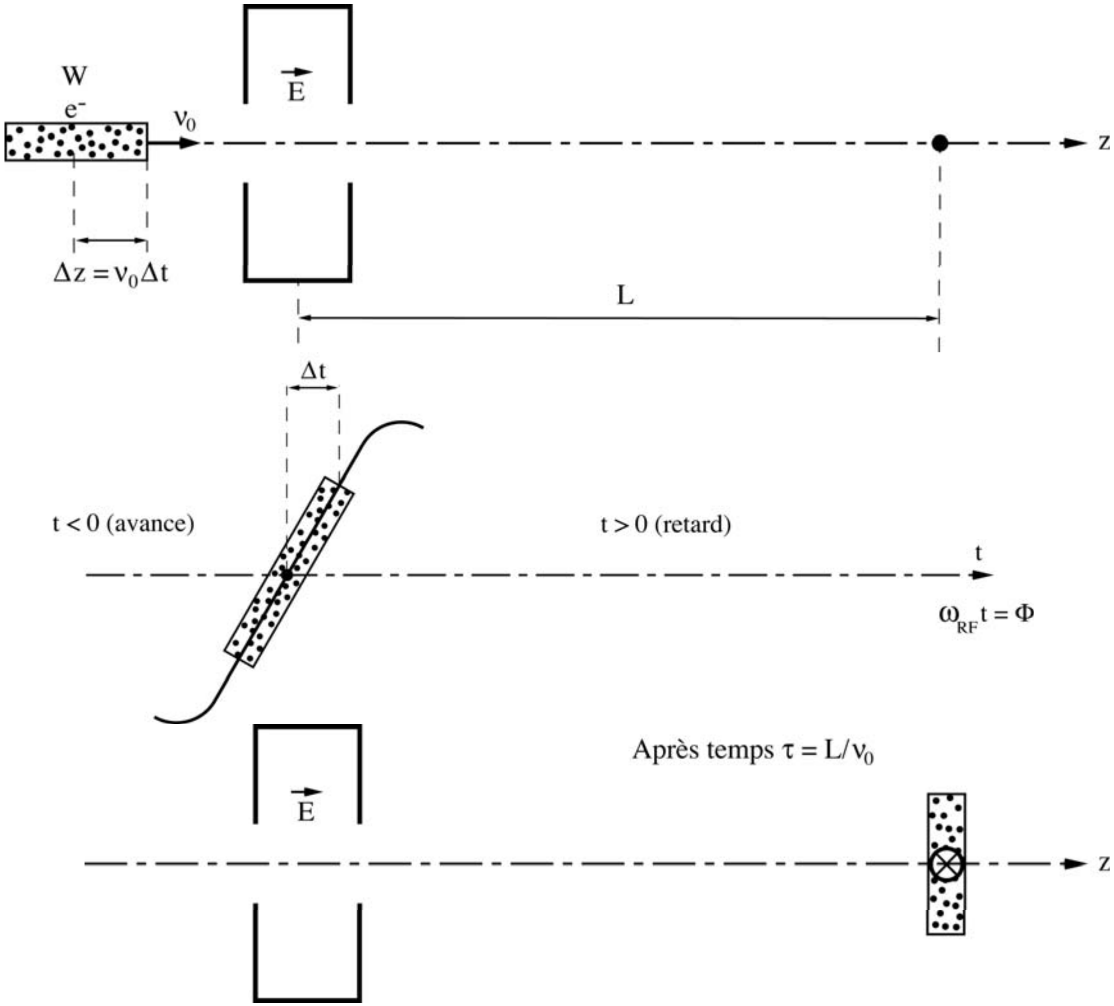
$$\cos\phi_0 - \cos\phi = \frac{2\pi m_0 c^2}{e\lambda_g E_0}\left(\frac{1-\beta_0}{1+\beta_0}\right)^{\frac{1}{2}} \leq 2$$

Capture condition ⟶

$$E_0 \geq \frac{\pi m_0 c^2}{e\lambda_g}\left(\frac{1-\beta_0}{1+\beta_0}\right)^{\frac{1}{2}}$$

Improved Capture With Pre-buncher

A long bunch coming from the gun enters an RF cavity; the reference particle is the one which has no velocity change. The others get accelerated or decelerated. After a distance L bunch gets shorter while energies are spread: bunching effect. This short bunch can now be captured more efficiently by a TW structure (v_φ=c).



Improved Capture With Pre-buncher (2)

The bunching effect is a space modulation that results from a velocity modulation and is similar to the phase stability phenomenon. Let's look at particles in the vicinity of the reference one and use a classical approach.

Energy gain as a function of cavity crossing time:

$$\Delta W = \Delta\left(\frac{1}{2} m_0 v^2\right) = m_0 v_0 \Delta v = eV_0 \sin\phi \approx eV_0 \phi$$

$$\Delta v = \frac{eV_0 \phi}{m_0 v_0}$$

Perfect linear bunching will occur after a time delay τ, corresponding to a distance L, when the path difference is compensated between a particle and the reference one:

$$\Delta v.\tau = \Delta z = v_0 t = v_0 \frac{\phi}{\omega_{RF}}$$

(assuming the reference particle enters the cavity at time t=0)

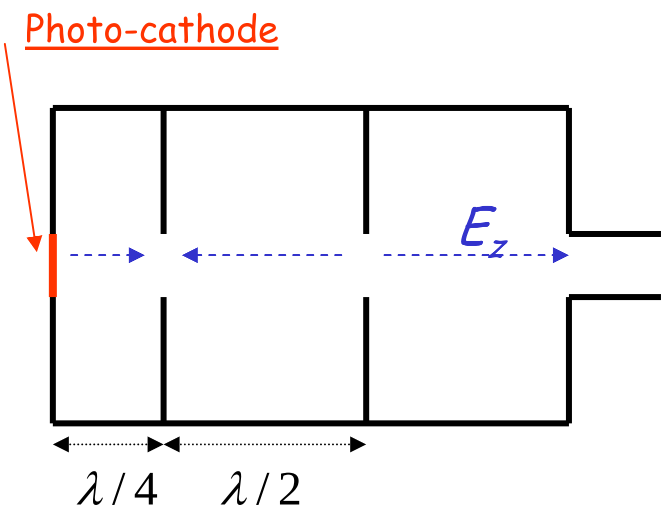
Since L = vτ one gets:

$$L = \frac{2 v_0 W}{eV_0 \omega_{RF}}$$

LABORATOIRE DE L'ACCÉLÉRATEUR LINÉAIRE

IN2P3 Institut National de Physique Nucléaire et de Physique des Particules

Radio-Frequency Gun



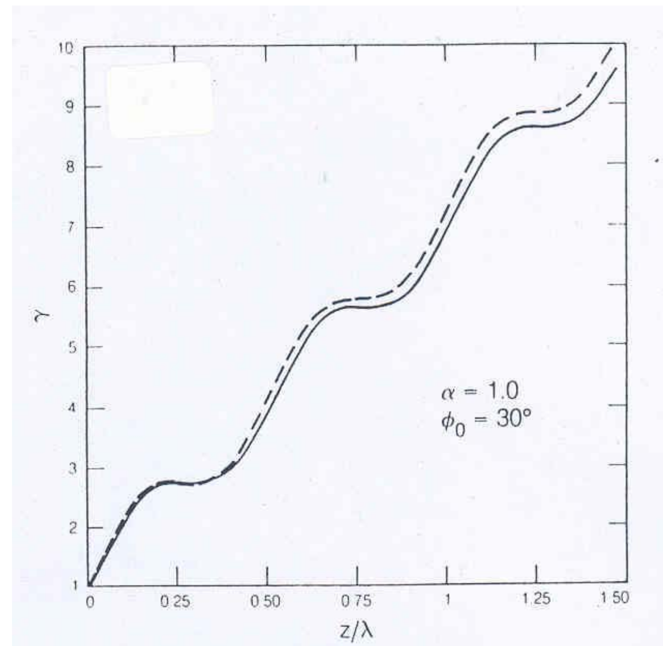
Specifically designed for high intensity, low energy, electron beam; a multi-cells high Q cavity provides a large electric field that rapidly accelerates the beam to ultra-relativistic energy, hence reducing the space charge effect; it also bunches the beam but giving large energy spread.

Generally a short pulse laser hits a photo-cathode to generate short electrons pulses.

$$E_z = E_0 \cos kz \sin(\omega t + \phi_0)$$

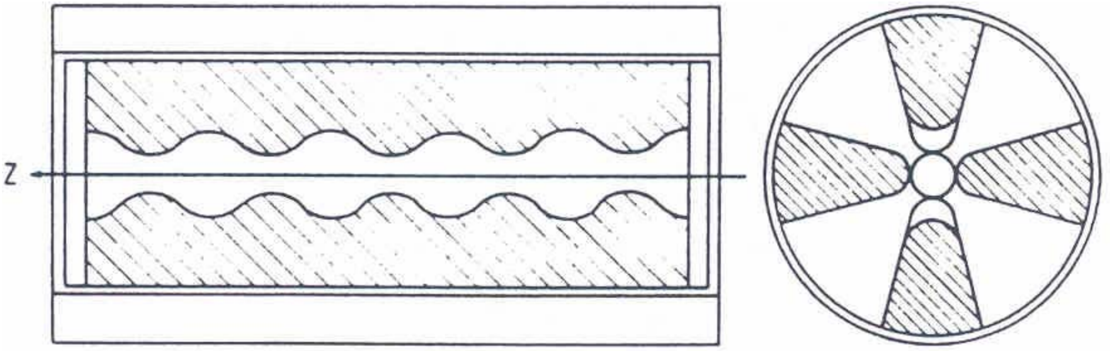
$$k = \frac{2\pi}{\lambda_0} = \frac{\omega}{c}$$

$$\phi_0 = RF \ phase \ of \ the \ particle \ at \ the \ cathode$$



Radio-Frequency Quadrupole

Specifically designed for intense low velocity protons (or ions) beams; it both accelerates and focus to control space charge effects (see A. Lombardi lecture)

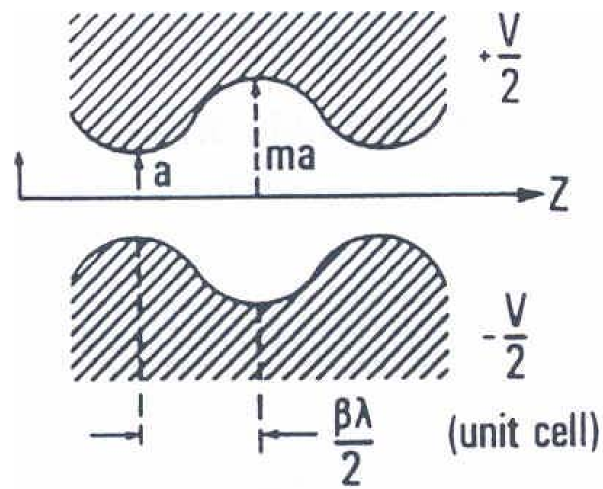


4 vanes resonator that provides a quadrupolar symmetry which gives a transverse E gradient for focusing.

Modulated pole shapes provide a longitudinal E field for acceleration and bunching.

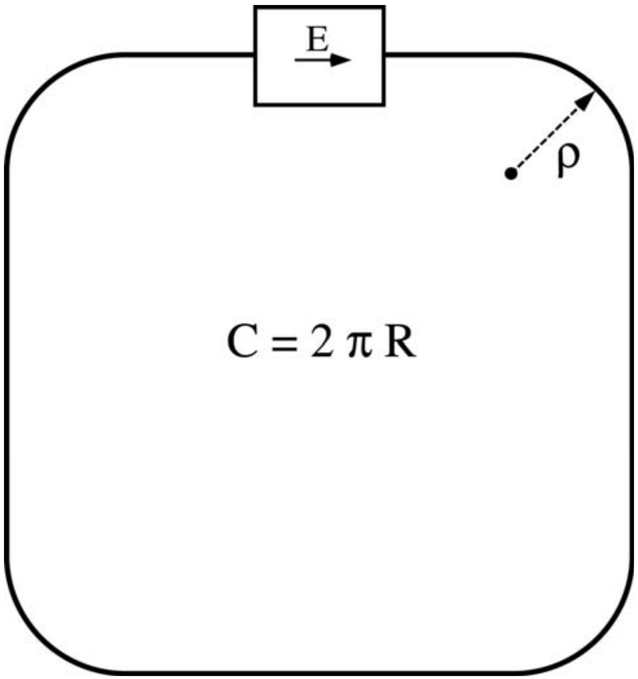
$$U = \frac{V}{2}\left[X\left(\frac{r}{a}\right)^2 \cos 2\psi + A I_0(kr)\cos kz \right]\sin(\omega t+\phi)$$

$$k = \frac{2\pi}{\beta\lambda};\ A = \left(m^2-1\right)/\left(m^2 I_0(ka) + I_0(mka)\right);\ X = 1 - A I_0(ka)$$



The Synchrotron

The synchrotron is a synchronous accelerator since there is a synchronous RF phase for which the energy gain fits the increase of the magnetic field at each turn. That implies the following operating conditions:



$e\hat{V}\sin\Phi \longrightarrow$ Energy gain per turn

$\Phi = \Phi_s = cte \longrightarrow$ Synchronous particle

$\omega_{RF} = h\omega_r \longrightarrow$ RF synchronism

$\rho = cte \quad R = cte \longrightarrow$ Constant orbit

$B\rho = P/e \Rightarrow B \longrightarrow$ Variable magnetic field

If v = c, ω_r hence ω_{RF} remain constant (ultra-relativistic e^-)

The Synchrotron (2)

Energy ramping is simply obtained by varying the B field:

$$p = eB\rho \quad \Rightarrow \quad \frac{dp}{dt} = e\rho\, B' \quad \Rightarrow (\Delta p)_{turn} = e\rho\, B' T_r = \frac{2\pi\, e\rho\, RB'}{v}$$

Since:

$$E^2 = E_0^2 + p^2 c^2 \quad \Rightarrow \quad \Delta E = v\Delta p$$

$$(\Delta E)_{turn} = (\Delta W)_s = 2\pi e\rho RB' = e\hat{V}\sin\phi_s$$

- The number of stable synchronous particles is equal to the harmonic number h. They are equally spaced along the circumference.
- Each synchronous particle satifies the relation p=eBρ. They have the nominal energy and follow the nominal trajectory.

The Synchrotron (3)

During the energy ramping, the RF frequency increases to follow the increase of the revolution frequency :

$$\omega_r = \frac{\omega_{RF}}{h} = \omega(B, R_s)$$

hence :

$$\frac{f_{RF}(t)}{h} = \frac{v(t)}{2\pi R_s} = \frac{1}{2\pi}\frac{e}{m} < B(t) > \quad \Rightarrow \quad \frac{f_{RF}(t)}{h} = \frac{1}{2\pi}\frac{ec^2}{E_s(t)}\frac{r}{R_s}B(t)$$

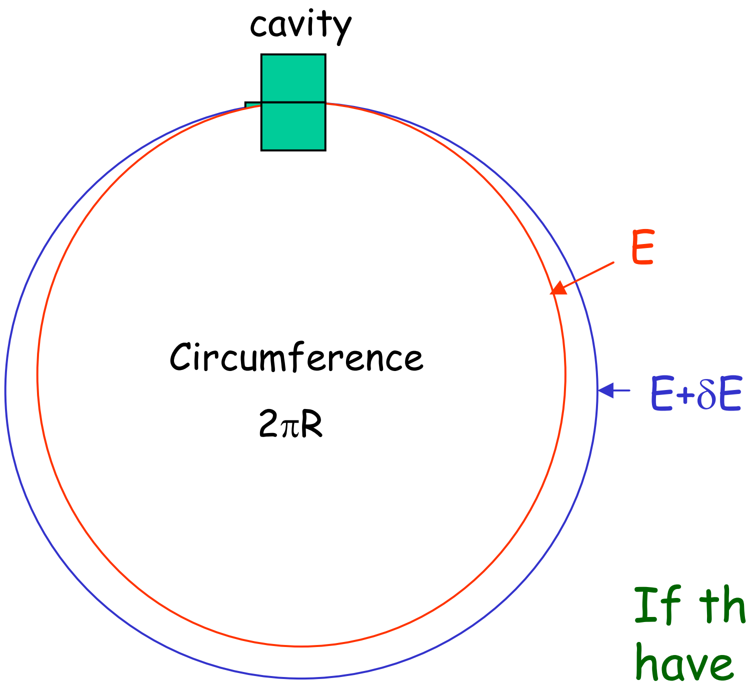
Since $E^2 = m_0^2 c^4 + p^2 c^2$, the RF frequency must follow the variation of the B field with the law :

$$\frac{f_{RF}(t)}{h} = \frac{c}{2\pi R_s}\left\{\frac{B(t)^2}{(m_0 c^2 / ecr)^2 + B(t)^2}\right\}^{1/2}$$

which asymptotically tends towards $f_r \rightarrow \frac{c}{2\pi R}$ when B becomes large compare to $(m_0c^2 / 2\pi r)$ which corresponds to $v \longrightarrow c$ ($pc \gg m_0c^2$). In practice the B field can follow the law:

$$B(t) = \frac{B}{2}(1 - \cos \omega t) = B \sin^2 \frac{\omega}{2} t$$

Dispersion Effects in a Synchrotron



If a particle is slightly shifted in momentum it will have a different orbit:

$$\alpha = \frac{p}{R}\frac{dR}{dp}$$

This is the "momentum compaction" generated by the bending field.

If the particle is shifted in momentum it will have also a different velocity. As a result of both effects the revolution frequency changes:

$$\eta = \frac{p}{f_r}\frac{df_r}{dp}$$

p=particle momentum

R=synchrotron physical radius

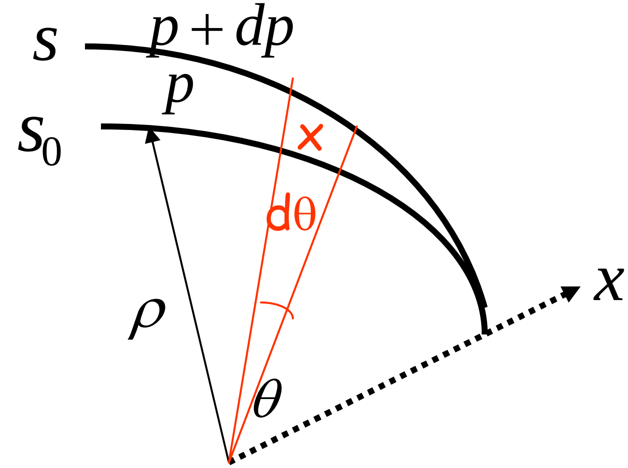
f_r=revolution frequency

Dispersion Effects in a Synchrotron (2)

$$\alpha = \frac{p}{R}\frac{dR}{dp}$$

$$ds_0 = \rho d\theta$$

$$ds = (\rho + x)d\theta$$



The elementary path difference from the two orbits is:

$$\frac{ds - ds_0}{ds_0} = \frac{dl}{ds_0} = \frac{x}{\rho}$$

leading to the total change in the circumference:

$$\int dl = 2\pi dR = \int \frac{x}{\rho} ds_0 = \frac{1}{\rho}\int_m x ds_0 \quad \Rightarrow \quad dR = \langle x \rangle_m$$

< >$_m$ means that the average is considered over the bending magnet only

Since: $$x = D_x \frac{dp}{p}$$ we get: $$\alpha = \frac{\langle D_x \rangle_m}{R}$$

Dispersion Effects in a Synchrotron (3)

$$\eta = \frac{p}{f_r}\frac{df_r}{dp}$$

$$f_r = \frac{\beta c}{2\pi R} \Rightarrow \frac{df_r}{f_r} = \frac{d\beta}{\beta} - \frac{dR}{R}$$

$$p = mv = \beta\gamma\frac{E_0}{c} \Rightarrow \frac{dp}{p} = \frac{d\beta}{\beta} + \frac{d\left(1-\beta^2\right)^{-\frac{1}{2}}}{\left(1-\beta^2\right)^{-\frac{1}{2}}} = \left(1-\beta^2\right)^{-1}\frac{d\beta}{\beta}$$

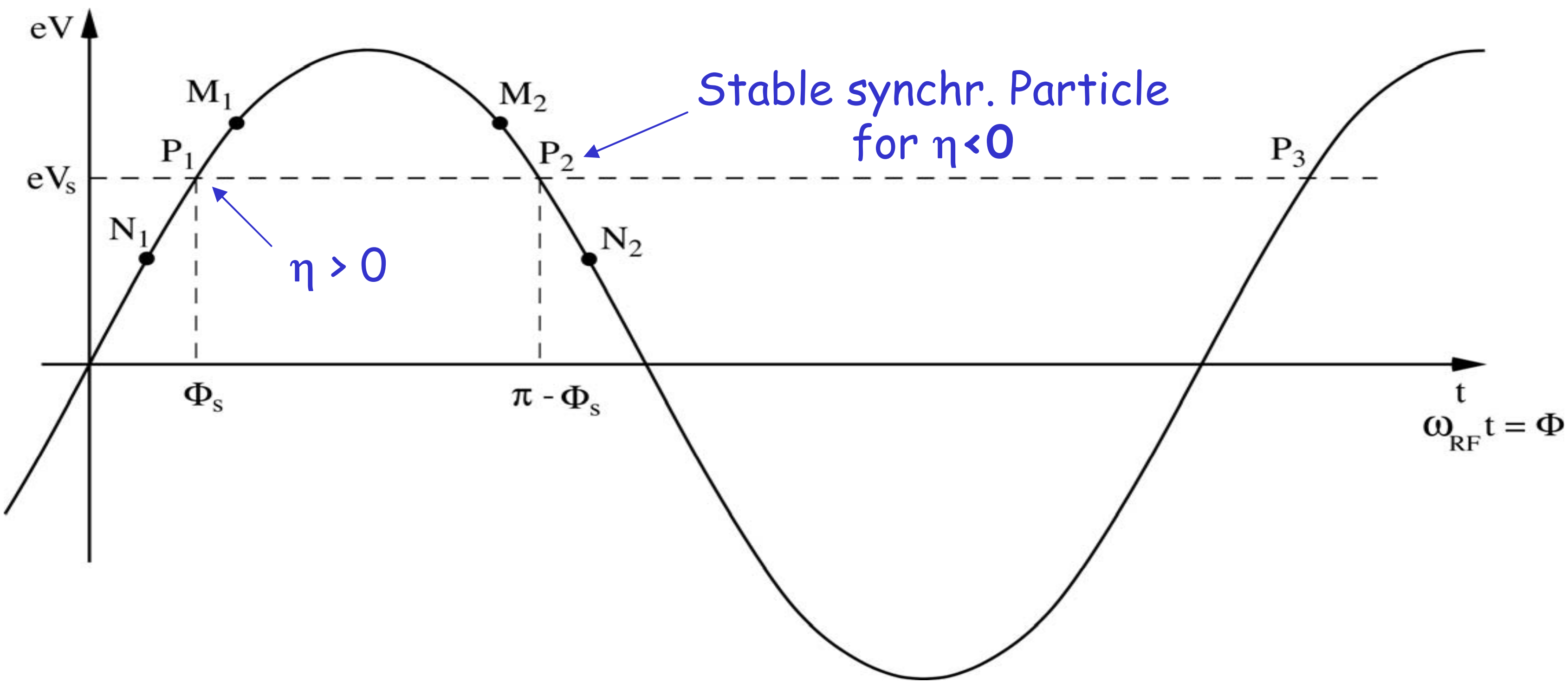
$$\frac{df_r}{f_r} = \left(\frac{1}{\gamma^2} - \alpha\right)\frac{dp}{p} \longrightarrow \eta = \frac{1}{\gamma^2} - \alpha$$

η=0 at the transition energy

$$\gamma_{tr} = \frac{1}{\sqrt{\alpha}}$$

Phase Stability in a Synchrotron

From the definition of η it is clear that below transition an increase in energy is followed by a higher revolution frequency (increase in velocity dominates) while the reverse occurs above transition (v ≈ c and longer path) where the momentum compaction (generally > 0) dominates.



Longitudinal Dynamics in a Synchrotron

It is also often called " synchrotron motion".

The RF acceleration process clearly emphasizes two coupled variables, the energy gained by the particle and the RF phase experienced by the same particle. Since there is a well defined synchronous particle which has always the same phase ϕ_s, and the nominal energy E_s, it is sufficient to follow other particles with respect to that particle. So let's introduce the following reduced variables:

revolution frequency : $\Delta f_r = f_r - f_{rs}$

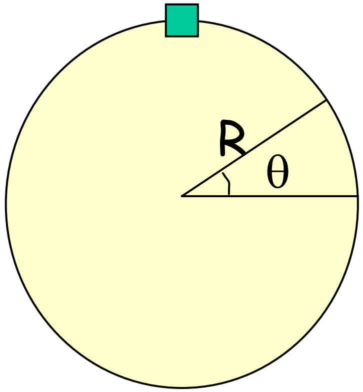
particle RF phase : $\Delta\phi = \phi - \phi_s$

particle momentum : $\Delta p = p - p_s$

particle energy : $\Delta E = E - E_s$

azimuth angle : $\Delta\theta = \theta - \theta_s$

First Energy-Phase Equation



$$f_{RF} = h f_r \quad \Rightarrow \quad \Delta\phi = -h\Delta\theta \quad with \quad \theta = \int \omega_r dt$$

For a given particle with respect to the reference one:

$$\Delta\omega_r = \frac{d}{dt}\left(\Delta\theta\right) = -\frac{1}{h}\frac{d}{dt}\left(\Delta\phi\right) = -\frac{1}{h}\frac{d\phi}{dt}$$

Since: $\eta = \frac{p_s}{\omega_{rs}}\left(\frac{d\omega_r}{dp}\right)_s$ and $E^2 = E_0^2 + p^2 c^2$

$$\Delta E = v_s \Delta p = \omega_{rs} R_s \Delta p$$

one gets:

$$\frac{\Delta E}{\omega_{rs}} = -\frac{p_s R_s}{h\eta\omega_{rs}}\frac{d(\Delta\phi)}{dt} = -\frac{p_s R_s}{h\eta\omega_{rs}}\dot{\phi}$$

Second Energy-Phase Equation

The rate of energy gained by a particle is:

$$\frac{dE}{dt} = e\hat{V}\sin\phi \, \frac{\omega_r}{2\pi}$$

The rate of relative energy gain with respect to the reference particle is then:

$$2\pi \Delta\left(\frac{\dot{E}}{\omega_r}\right) = e\hat{V}(\sin\phi - \sin\phi_s)$$

Expanding the left hand side to first order:

$$\Delta\left(\dot{E}T_r\right) \cong \dot{E}\,\Delta T_r + T_{rs}\,\Delta\dot{E} = \Delta E\,\dot{T}_r + T_{rs}\,\Delta\dot{E} = \frac{d}{dt}\left(T_{rs}\,\Delta E\right)$$

leads to the second energy-phase equation:

$$2\pi \frac{d}{dt}\left(\frac{\Delta E}{\omega_{rs}}\right) = e\hat{V}\left(\sin\phi - \sin\phi_s\right)$$

Equations of Longitudinal Motion

$$\frac{\Delta E}{\omega_{rs}} = -\frac{p_s R_s}{h\eta\omega_{rs}}\frac{d(\Delta\phi)}{dt} = -\frac{p_s R_s}{h\eta\omega_{rs}}\dot{\phi}$$

$$2\pi\frac{d}{dt}\left(\frac{\Delta E}{\omega_{rs}}\right) = e\hat{V}(\sin\phi - \sin\phi_s)$$

deriving and combining

$$\frac{d}{dt}\left[\frac{R_s p_s}{h\eta\omega_{rs}}\frac{d\phi}{dt}\right] + \frac{e\hat{V}}{2\pi}(\sin\phi - \sin\phi_s) = 0$$

This second order equation is non linear. Moreover the parameters within the bracket are in general slowly varying with time…………………

Small Amplitude Oscillations

Let's assume constant parameters R_s, p_s, ω_s and η:

$$\ddot{\phi}+\frac{\Omega_s^2}{\cos\phi_s}(\sin\phi-\sin\phi_s)=0$$

with

$$\Omega_s^2=\frac{h\eta\omega_{rs}e\hat{V}\cos\phi_s}{2\pi R_s p_s}$$

Consider now small phase deviations from the reference particle:

$$\sin\phi-\sin\phi_s=\sin(\phi_s+\Delta\phi)-\sin\phi_s\cong\cos\phi_s\,\Delta\phi$$

(for small $\Delta\phi$)

and the corresponding linearized motion reduces to a harmonic oscillation:

$$\ddot{\phi}+\Omega_s^2\Delta\phi=0$$

stable for $\Omega_s^2>0$ and Ω_s real

$\gamma < \gamma_{tr}$	$\eta > 0$	$0 < \phi_s < \pi/2$	$\sin\phi_s > 0$
$\gamma > \gamma_{tr}$	$\eta < 0$	$\pi/2 < \phi_s < \pi$	$\sin\phi_s > 0$

Large Amplitude Oscillations

For larger phase (or energy) deviations from the reference the second order differential equation is non-linear:

$$\ddot{\phi} + \frac{\Omega_s^2}{\cos\phi_s}\left(\sin\phi - \sin\phi_s\right) = 0$$

(Ω_s as previously defined)

Multiplying by $\dot{\phi}$ and integrating gives an invariant of the motion:

$$\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos\phi_s}\left(\cos\phi + \phi\sin\phi_s\right) = I$$

which for small amplitudes reduces to:

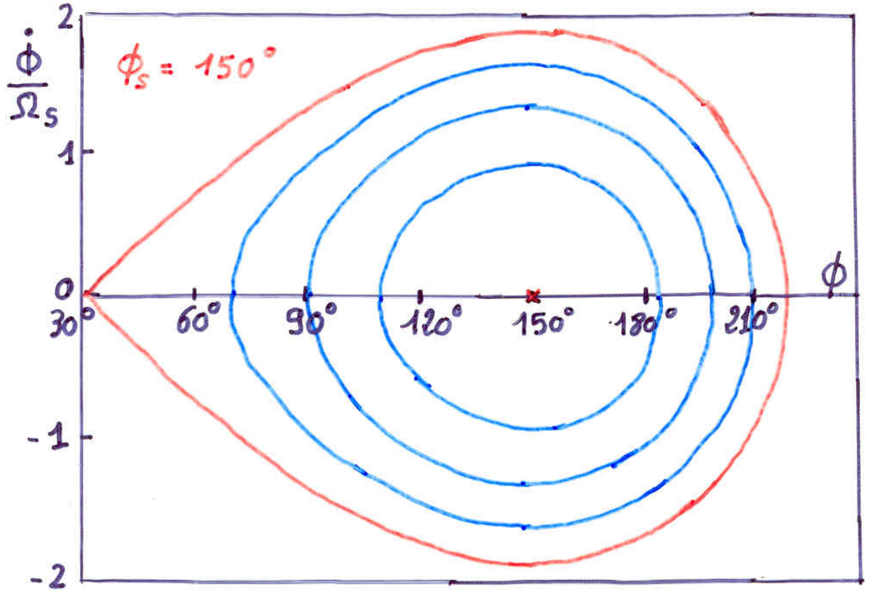
$$\frac{\dot{\phi}^2}{2} + \Omega_s^2\frac{(\Delta\phi)^2}{2} = I$$

(the variable is $\Delta\phi$ and ϕ_s is constant)

Similar equations exist for the second variable : $\Delta E \propto d\phi/dt$

Large Amplitude Oscillations (2)

When ϕ reaches $\pi-\phi_s$ the force goes to zero and beyond it becomes non restoring. Hence $\pi-\phi_s$ is an extreme amplitude for a stable motion which in the phase space($\frac{\dot{\phi}}{\Omega_s}, \Delta\phi$) is shown as closed trajectories.



Equation of the separatrix:

$$\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos\phi_s}\left(\cos\phi + \phi\sin\phi_s\right) = -\frac{\Omega_s^2}{\cos\phi_s}\left(\cos(\pi-\phi_s) + (\pi-\phi_s)\sin\phi_s\right)$$

Second value ϕ_m where the separatrix crosses the horizontal axis:

$$\cos\phi_m + \phi_m\sin\phi_s = \cos(\pi-\phi_s) + (\pi-\phi_s)\sin\phi_s$$

Energy Acceptance

From the equation of motion it is seen that $\dot{\phi}$ reaches an extremum when $\ddot{\phi} = 0$, hence corresponding to $\phi = \phi_s$.

Introducing this value into the equation of the separatrix gives:

$$\dot{\phi}^2_{max} = 2\Omega_s^2 \{2 + (2\phi_s - \pi)\tan\phi_s\}$$

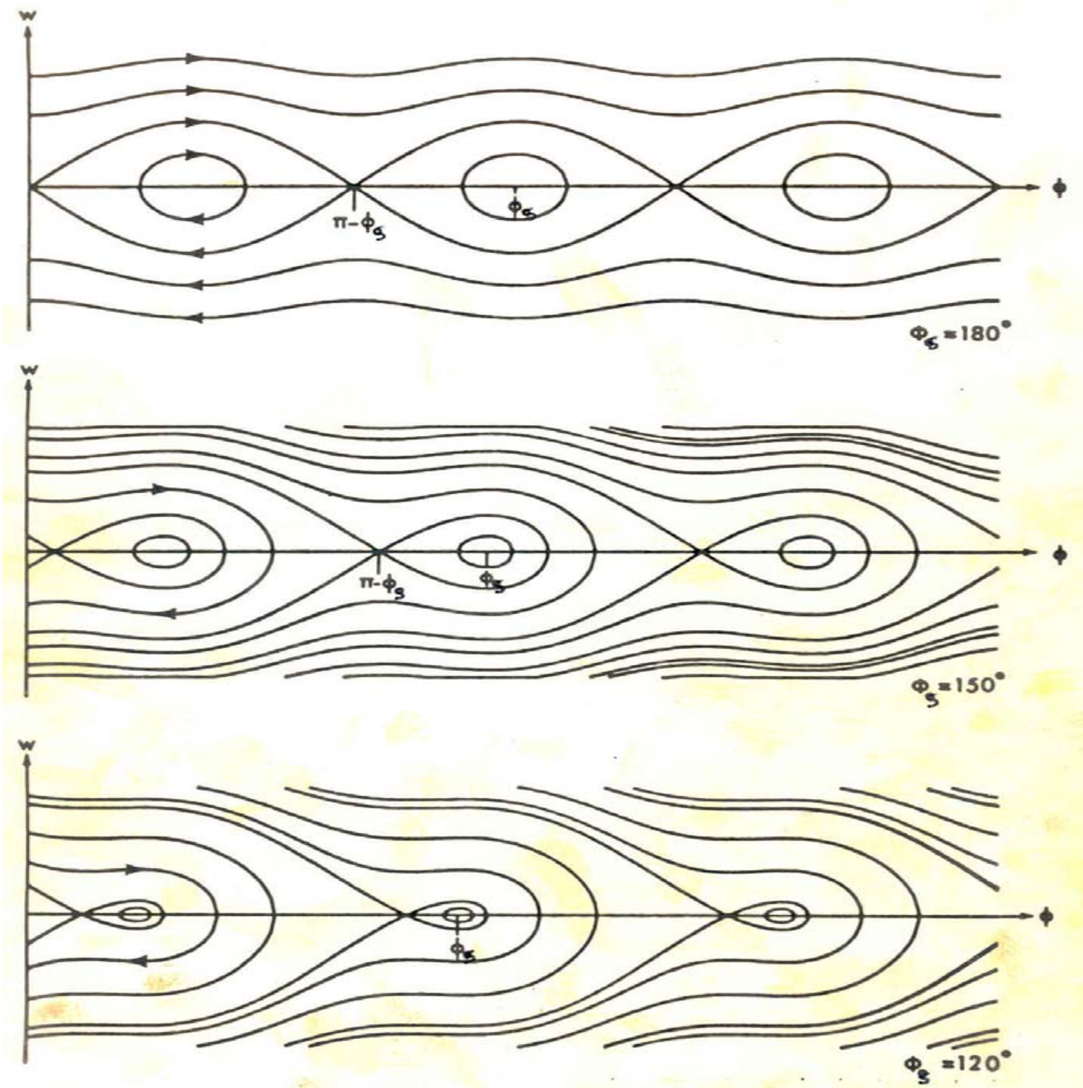
That translates into an acceptance in energy:

$$\left(\frac{\Delta E}{E_s}\right)_{max} = \mp \beta \left\{ -\frac{e\hat{V}}{\pi h \eta E_s} G(\phi_s) \right\}^{\frac{1}{2}}$$

$$G(\phi_s) = [2\cos\phi_s + (2\phi_s - \pi)\sin\phi_s]$$

This “RF acceptance” depends strongly on ϕ_s and plays an important role for the electron capture at injection, and the stored beam lifetime.

RF Acceptance versus Synchronous Phase



As the synchronous phase gets closer to 90° the area of stable motion (closed trajectories) gets smaller. These areas are often called "BUCKET".

The number of circulating buckets is equal to "h".

The phase extension of the bucket is maximum for ϕ_s =180° (or 0°) which correspond to no acceleration . The RF acceptance increases with the RF voltage.

Ions in Circular Accelerators

A =atomic number

Q =charge state

$q = Q\,e$

$W = E - E_r$

$P = q\,B\,r$

$E^2 = p^2c^2 + E_r^2$

$E_r = A\,E_0$

$m = \gamma\,m_r$

$P = m\,v$

$E = \gamma\,E_r$

$$E^2 - E_r^2 = (qcBr)^2$$

$$\frac{W}{A}\left(\frac{W}{A} + 2E_0\right) = \left(\frac{Q}{A}\right)^2 (ecBr)^2$$

Moreover:

$$dW = dE = \frac{E^2 - E_r^2}{E}\left[\frac{dB}{B} + \frac{dr}{r}\right]$$

dr/r = 0 synchrotron   dB/B = 0 cyclotron