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Electromagnetism

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Contents

- ❑ Maxwell's equations and Lorentz Force Law
- ❑ Motion of a charged particle under constant Electromagnetic fields
- ❑ Relativistic transformations of fields
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- ❑ Waves in a uniform conducting guide
 - Simple example TE_{01} mode
 - Propagation constant, cut-off frequency
 - Group velocity, phase velocity
 - Illustrations

Reading

- ❑ J.D. Jackson: *Classical Electrodynamics*
- ❑ H.D. Young and R.A. Freedman: *University Physics (with Modern Physics)*
- ❑ P.C. Clemmow: *Electromagnetic Theory*
- ❑ *Feynmann Lectures on Physics*
- ❑ W.K.H. Panofsky and M.N. Phillips: *Classical Electricity and Magnetism*
- ❑ G.L. Pollack and D.R. Stump: *Electromagnetism*

What is electromagnetism?

- ❑ The study of Maxwell's equations, devised in 1863 to represent the relationships between electric and magnetic fields in the presence of electric charges and currents, whether steady or rapidly fluctuating, in a vacuum or in matter.
- ❑ The equations represent one of the most elegant and concise way to describe the fundamentals of electricity and magnetism. They pull together in a consistent way earlier results known from the work of Gauss, Faraday, Ampère, Biot, Savart and others.
- ❑ Remarkably, Maxwell's equations are perfectly consistent with the transformations of special relativity.

Maxwell's Equations



- Relate Electric and Magnetic fields generated by charge and current distributions.

\mathbf{E} = electric field

\mathbf{D} = electric displacement

\mathbf{H} = magnetic field

\mathbf{B} = magnetic flux density

ρ = charge density

\mathbf{j} = current density

μ_0 (permeability of free space) = $4\pi \cdot 10^{-7}$

ϵ_0 (permittivity of free space) = $8.854 \cdot 10^{-12}$

c (speed of light) = $2.99792458 \cdot 10^8$ m/s

$$\nabla \cdot \vec{D} = \rho$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \wedge \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \wedge \vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t}$$

In vacuum $\vec{D} = \epsilon_0 \vec{E}$, $\vec{B} = \mu_0 \vec{H}$, $\epsilon_0 \mu_0 c^2 = 1$

Maxwell's 1st Equation

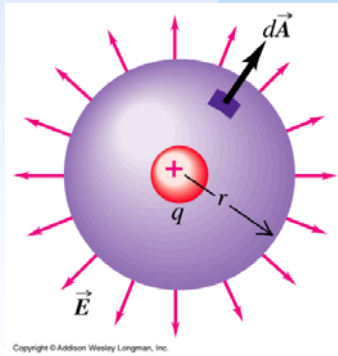
$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

Equivalent to Gauss' Flux Theorem:

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \Leftrightarrow \iiint_V \nabla \cdot \vec{E} dV = \oiint_S \vec{E} \cdot d\vec{S} = \frac{1}{\epsilon_0} \iiint_V \rho dV = \frac{Q}{\epsilon_0}$$

The flux of electric field out of a closed region is proportional to the total electric charge Q enclosed within the surface.

A point charge q generates an electric field



$$\vec{E} = \frac{q}{4\pi\epsilon_0 r^3} \vec{r}$$

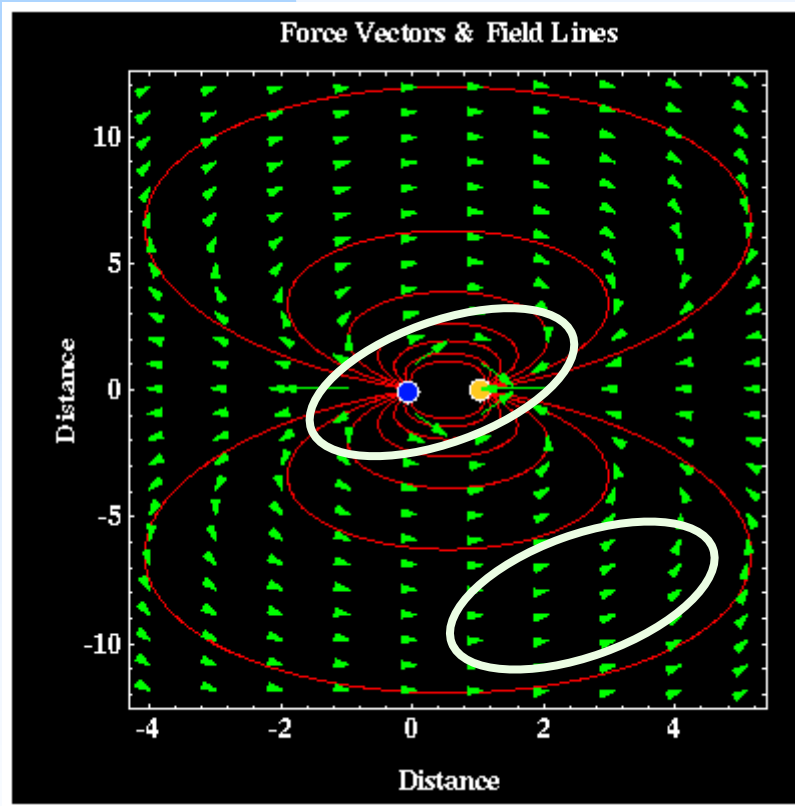
$$\iint_{\text{sphere}} \vec{E} \cdot d\vec{S} = \frac{q}{4\pi\epsilon_0} \iint_{\text{sphere}} \frac{dS}{r^2} = \frac{q}{\epsilon_0}$$



Area integral gives a measure of the net charge enclosed; divergence of the electric field gives the density of the sources.

Maxwell's 2nd Equation

$$\nabla \cdot \vec{B} = 0$$



$$\nabla \cdot \vec{B} = 0 \quad \Leftrightarrow \quad \oiint \vec{B} \cdot d\vec{S} = 0$$

Gauss' law for magnetism:

The net magnetic flux out of any closed surface is zero. Surround a magnetic dipole with a closed surface. The magnetic flux directed inward towards the south pole will equal the flux outward from the north pole.

If there were a magnetic monopole source, this would give a non-zero integral.

Gauss' law for magnetism is then a statement that

There are no magnetic monopoles

Maxwell's 3rd Equation

$$\nabla \wedge \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

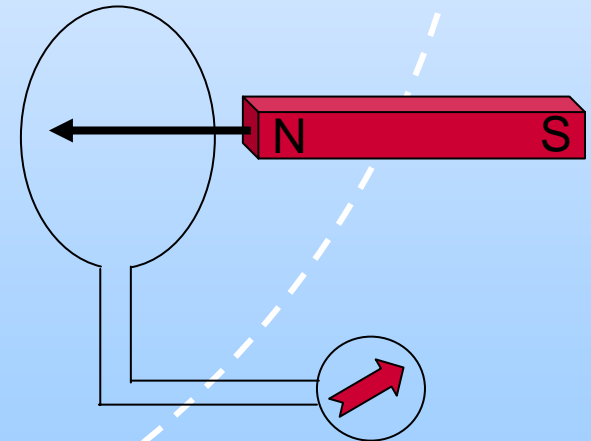
Equivalent to Faraday's Law of Induction:

$$\iint_S \nabla \wedge \vec{E} \cdot d\vec{S} = -\iint_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S}$$

(for a fixed circuit C) $\Leftrightarrow \oint_C \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \iint_S \vec{B} \cdot d\vec{S} = -\frac{d\Phi}{dt}$



The electromotive force round a circuit $\varepsilon = \oint \vec{E} \cdot d\vec{l}$ is proportional to the rate of change of flux of magnetic field, $\Phi = \iint \vec{B} \cdot d\vec{S}$ through the circuit.



Faraday's Law is the basis for electric generators. It also forms the basis for inductors and transformers.

Maxwell's 4th Equation

$$\nabla \wedge \vec{B} = \mu_0 \vec{j} + \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t}$$



Ampère

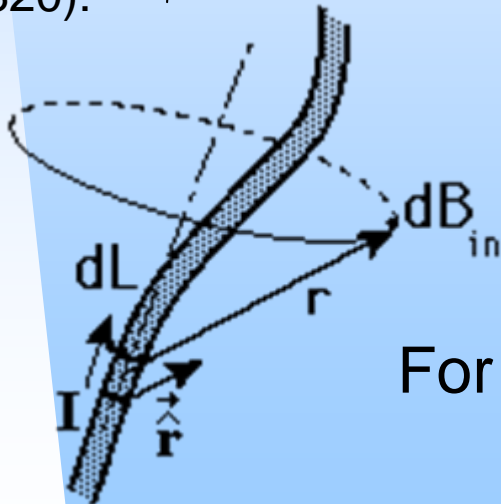
Originates from Ampère's (Circuital) Law : $\nabla \wedge \vec{B} = \mu_0 \vec{j}$

$$\oint_C \vec{B} \cdot d\vec{l} = \iint_S \nabla \wedge \vec{B} \cdot d\vec{S} = \mu_0 \iint_S \vec{j} \cdot d\vec{S} = \mu_0 I$$

Satisfied by the field for a steady line current (Biot-Savart Law, 1820):



Biot

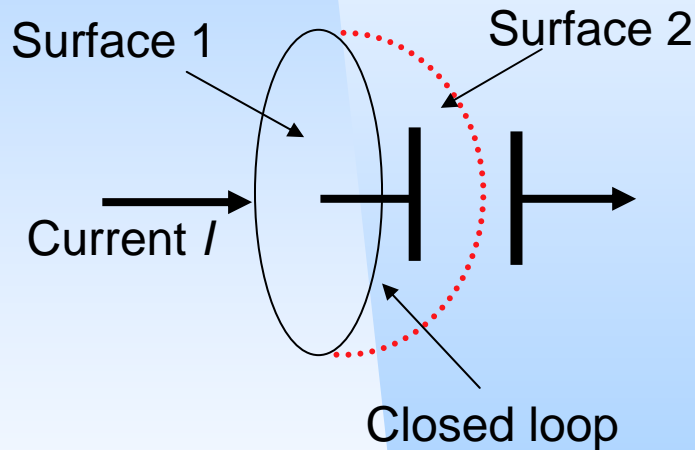


$$\vec{B} = \frac{\mu_0 I}{4\pi} \oint \frac{d\vec{l} \wedge \vec{r}}{r^3}$$

For a straight line current $B_\theta = \frac{\mu_0 I}{2\pi r}$

Need for displacement current

- ❑ Faraday: vary B-field, generate E-field
- ❑ Maxwell: varying E-field should then produce a B-field, but not covered by Ampère's Law.



- ❑ Apply Ampère to surface 1 (flat disk): line integral of $\mathbf{B} = \mu_0 I$
- ❑ Applied to surface 2, line integral is zero since no current penetrates the deformed surface.

- ❑ In capacitor, $E = \frac{Q}{\epsilon_0 A}$, so $I = \frac{dQ}{dt} = \epsilon_0 A \frac{dE}{dt}$

- ❑ Displacement current density is $\vec{j}_d = \epsilon_0 \frac{\partial \vec{E}}{\partial t}$

$$\nabla \wedge \vec{B} = \mu_0 (\vec{j} + \vec{j}_d) = \mu_0 \vec{j} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Consistency with charge conservation

- Charge conservation: Total current flowing out of a region equals the rate of decrease of charge within the volume.

$$\oiint \vec{j} \cdot d\vec{S} = -\frac{d}{dt} \iiint \rho dV$$

$$\Leftrightarrow \iiint \nabla \cdot \vec{j} dV = -\iiint \frac{\partial \rho}{\partial t} dV$$

$$\Leftrightarrow \nabla \cdot \vec{j} + \frac{\partial \rho}{\partial t} = 0$$

- From Maxwell's equations:
Take divergence of (modified)
Ampère's equation

$$\nabla \cdot \nabla \wedge \vec{B} = \mu_0 \nabla \cdot \vec{j} + \frac{1}{c^2} \frac{\partial}{\partial t} (\nabla \cdot \vec{E})$$

$$\Rightarrow 0 = \mu_0 \nabla \cdot \vec{j} + \varepsilon_0 \mu_0 \frac{\partial}{\partial t} \left(\frac{\rho}{\varepsilon_0} \right)$$

$$\Rightarrow 0 = \nabla \cdot \vec{j} + \frac{\partial \rho}{\partial t}$$

Maxwell's Equations in Vacuum

- In vacuum

$$\vec{D} = \epsilon_0 \vec{E}, \quad \vec{B} = \mu_0 \vec{H}, \quad \epsilon_0 \mu_0 = \frac{1}{c^2}$$

- Source-free equations:

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \wedge \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

- Source equations

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla \wedge \vec{B} - \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{j}$$

- Equivalent integral forms (sometimes useful for simple geometries)

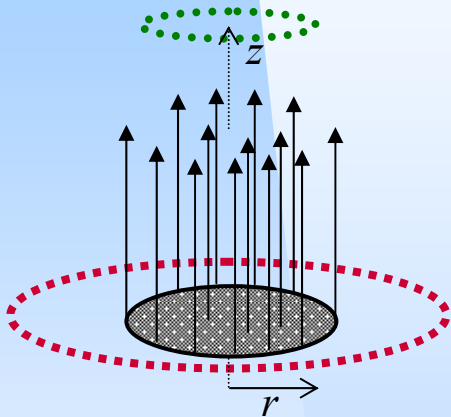
$$\oiint \vec{E} \cdot d\vec{S} = \frac{1}{\epsilon_0} \iiint \rho dV$$

$$\oiint \vec{B} \cdot d\vec{S} = 0$$

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \iint \vec{B} \cdot d\vec{S} = -\frac{d\Phi}{dt}$$

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 \iint \vec{j} \cdot d\vec{S} + \frac{1}{c^2} \frac{d}{dt} \iint \vec{E} \cdot d\vec{S}$$

Example: Calculate E from B



$$B_z = \begin{cases} B_0 \sin \omega t & r < r_0 \\ 0 & r > r_0 \end{cases}$$

Also from $\nabla \wedge \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \iint \vec{B} \cdot d\vec{S}$$

$$r < r_0 \quad 2\pi r E_\theta = -\pi r^2 B_0 \omega \cos \omega t$$

$$\Rightarrow \quad E_\theta = -\frac{B_0 \omega r}{2} \cos \omega t$$

$$r > r_0 \quad 2\pi r E_\theta = -\pi r_0^2 B_0 \omega \cos \omega t$$

$$\Rightarrow \quad E_\theta = -\frac{\omega r_0^2 B_0}{2r} \cos \omega t$$

$$\nabla \wedge \vec{B} = \mu_0 \vec{j} + \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} \quad \text{then gives current density necessary to sustain the fields}$$

Lorentz force law

- Supplement to Maxwell's equations, gives force on a charged particle moving in an electromagnetic field:

$$\vec{f} = q(\vec{E} + \vec{v} \wedge \vec{B})$$

- For continuous distributions, have a force density $\vec{f}_d = \rho\vec{E} + \vec{j} \wedge \vec{B}$
- Relativistic equation of motion

- 4-vector form: $F = \frac{dP}{d\tau} \Rightarrow \gamma \left(\frac{\vec{v} \cdot \vec{f}}{c}, \vec{f} \right) = \gamma \left(\frac{1}{c} \frac{dE}{dt}, \frac{d\vec{p}}{dt} \right)$

- 3-vector component:

$$\frac{d}{dt}(m_0\gamma\vec{v}) = \vec{f} = q(\vec{E} + \vec{v} \wedge \vec{B})$$

Motion of charged particles in constant electromagnetic fields

$$\frac{d}{dt}(m_0\gamma\vec{v}) = \vec{f} = q(\vec{E} + \vec{v} \wedge \vec{B})$$

- Constant E-field gives uniform acceleration in straight line

- Solution of $\frac{d}{dt}(\gamma\vec{v}) = \frac{q}{m_0}E$

$$x = \frac{m_0c^2}{qE} \left[\sqrt{1 + \left(\frac{qEt}{m_0c}\right)^2} - 1 \right]$$

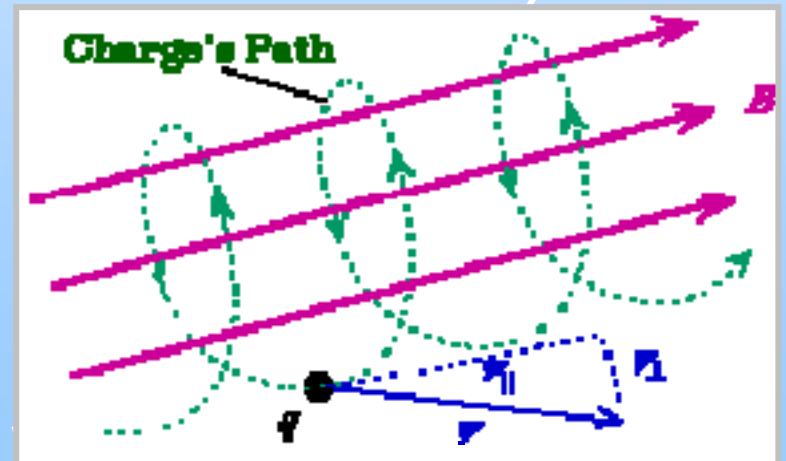
$$\approx \frac{1}{2} \frac{qE}{m_0} t^2 \quad \text{for } qE \ll m_0c$$

- Energy gain = qEx

- Constant magnetic field gives uniform spiral about B with constant energy.

$$\frac{d\vec{v}}{dt} = \frac{q}{m_0\gamma} \vec{v} \wedge \vec{B} \quad \vec{v}_{//} = \text{constant}$$

$$|\vec{x}_{\perp}| = \text{constant}$$



Relativistic Transformations of E and B

- According to observer O in frame F, particle has velocity \vec{v} , fields are \vec{E} and \vec{B} and Lorentz force is

$$\vec{f} = q(\vec{E} + \vec{v} \wedge \vec{B})$$

- In Frame F', particle is at rest and force is $\vec{f}' = q'\vec{E}'$

- Assume measurements give same charge and force, so

$$q = q' \quad \text{and} \quad \vec{E}' = \vec{E} + \vec{v} \wedge \vec{B}$$

- Point charge q at rest in F: $\vec{E} = \frac{q}{4\pi\epsilon_0} \frac{\vec{r}}{r^3}$, $\vec{B} = 0$

- See a current in F', giving a field $\vec{B} = -\frac{\mu_0 q}{4\pi} \frac{\vec{v} \wedge \vec{r}}{r^3} = -\frac{1}{c^2} \vec{v} \wedge \vec{E}$

- Suggests $\vec{B}' = \vec{B} - \frac{1}{c^2} \vec{v} \wedge \vec{E}$

Rough idea

Exact:

$$\vec{E}'_{\perp} = \gamma \left(\vec{E}_{\perp} + \vec{v} \wedge \vec{B} \right), \quad \vec{E}'_{\parallel} = \vec{E}_{\parallel}$$

$$\vec{B}'_{\perp} = \gamma \left(\vec{B}_{\perp} - \frac{\vec{v} \wedge \vec{E}}{c^2} \right), \quad \vec{B}'_{\parallel} = \vec{B}_{\parallel}$$

Electromagnetic waves

- Maxwell's equations predict the existence of electromagnetic waves, later discovered by Hertz.
- No charges, no currents:

$$\begin{aligned}\nabla \wedge (\nabla \wedge \vec{E}) &= -\nabla \wedge \frac{\partial \vec{B}}{\partial t} \\ &= -\frac{\partial}{\partial t} (\nabla \wedge \vec{B}) \\ &= -\mu \frac{\partial^2 \vec{D}}{\partial t^2} = -\mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2}\end{aligned}$$

$$\begin{aligned}\nabla \wedge \vec{H} &= \frac{\partial \vec{D}}{\partial t} & \nabla \wedge \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\ \nabla \cdot \vec{D} &= 0 & \nabla \cdot \vec{B} &= 0\end{aligned}$$

$$\begin{aligned}\nabla \wedge (\nabla \wedge \vec{E}) &= \nabla (\nabla \cdot \vec{E}) - \nabla^2 \vec{E} \\ &= -\nabla^2 \vec{E}\end{aligned}$$

3D wave equation :

$$\nabla^2 \vec{E} = \frac{\partial^2 \vec{E}}{\partial x^2} + \frac{\partial^2 \vec{E}}{\partial y^2} + \frac{\partial^2 \vec{E}}{\partial z^2} = \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

Nature of electromagnetic waves

- A general plane wave with angular frequency ω travelling in the direction of the wave vector \vec{k} has the form

$$\vec{E} = \vec{E}_0 \exp[i(\omega t - \vec{k} \cdot \vec{x})] \quad \vec{B} = \vec{B}_0 \exp[i(\omega t - \vec{k} \cdot \vec{x})]$$

- Phase $\omega t - \vec{k} \cdot \vec{x} = 2\pi \times \text{number of waves}$ and so is a Lorentz invariant.
- Apply Maxwell's equations

$$\begin{aligned} \nabla &\leftrightarrow -i\vec{k} \\ \frac{\partial}{\partial t} &\leftrightarrow i\omega \end{aligned}$$

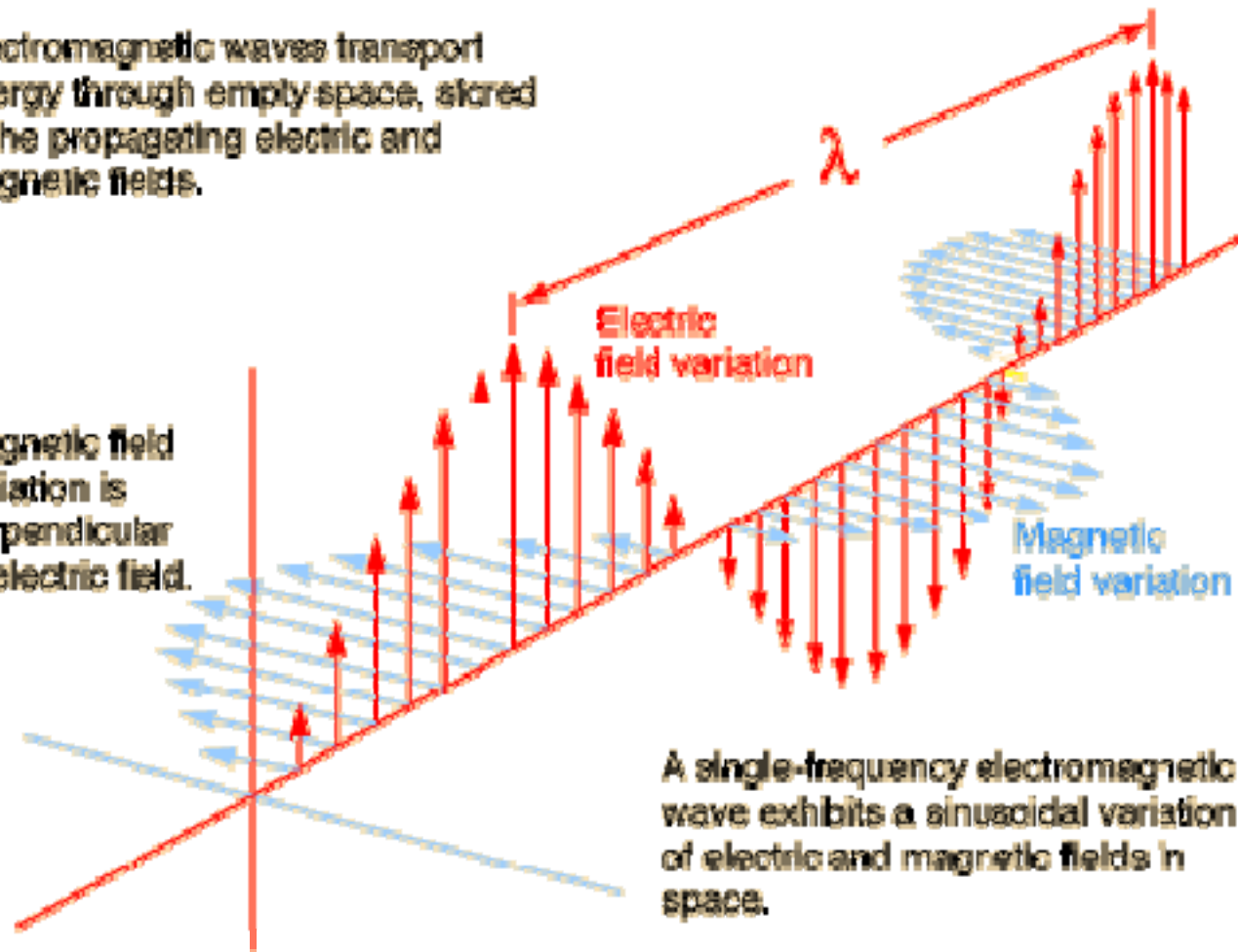
$$\begin{aligned} \nabla \cdot \vec{E} = 0 = \nabla \cdot \vec{B} &\leftrightarrow \vec{k} \cdot \vec{E} = 0 = \vec{k} \cdot \vec{B} \\ \nabla \wedge \vec{E} = -\dot{\vec{B}} &\leftrightarrow \vec{k} \wedge \vec{E} = \omega \vec{B} \end{aligned}$$

Waves are transverse to the direction of propagation, and \vec{E} , \vec{B} and \vec{k} are mutually perpendicular

Plane electromagnetic wave

Electromagnetic waves transport energy through empty space, stored in the propagating electric and magnetic fields.

Magnetic field variation is perpendicular to electric field.



A single-frequency electromagnetic wave exhibits a sinusoidal variation of electric and magnetic fields in space.

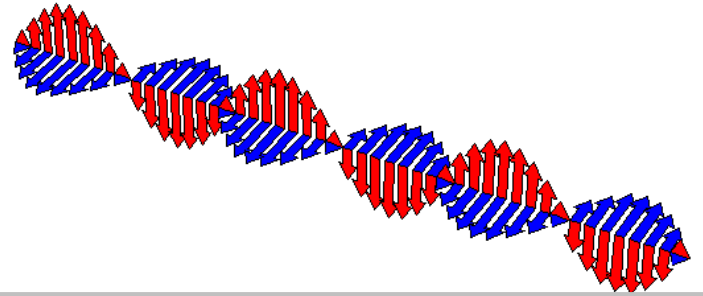
Plane Electromagnetic Waves

$$\nabla \wedge \vec{B} = \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} \quad \leftrightarrow \quad \vec{k} \wedge \vec{B} = -\frac{\omega}{c^2} \vec{E}$$

Combined with $\vec{k} \wedge \vec{E} = \omega \vec{B}$

deduce that $\frac{|\vec{E}|}{|\vec{B}|} = \frac{\omega}{k} = \frac{kc^2}{\omega}$

\Rightarrow speed of wave in vacuum is $\frac{\omega}{|\vec{k}|} = c$



Wavelength $\lambda = \frac{2\pi}{|\vec{k}|}$

Frequency $\nu = \frac{\omega}{2\pi}$

The fact that $\omega t - \vec{k} \cdot \vec{x}$ is an invariant tells us that

$$\Lambda = \left(\frac{\omega}{c}, \vec{k} \right)$$

is a Lorentz 4-vector, the 4-Frequency vector. Deduce frequency transforms as

$$\omega' = \gamma(\omega - \vec{v} \cdot \vec{k}) = \omega \sqrt{\frac{c-v}{c+v}}$$

Waves in a conducting medium

- For a medium of conductivity σ , $\vec{j} = \sigma \vec{E}$
- Modified Maxwell: $\nabla \wedge \vec{H} = \vec{j} + \epsilon \dot{\vec{E}} = \sigma \vec{E} + \epsilon \dot{\vec{E}}$
- Put $\vec{E} = \vec{E}_0 \exp[i(\omega t - \vec{k} \cdot \vec{x})]$ $\vec{B} = \vec{B}_0 \exp[i(\omega t - \vec{k} \cdot \vec{x})]$

$$-i\vec{k} \wedge \vec{H} = \sigma \vec{E} + i\omega\epsilon \vec{E}$$

Dissipation factor

$$D = \frac{\sigma}{\omega\epsilon}$$

conduction
current

displacement
current

$$\text{Copper: } \sigma = 5.8 \times 10^7, \epsilon = \epsilon_0 \Rightarrow D = 10^{12}$$

$$\text{Teflon: } \sigma = 3 \times 10^{-8}, \epsilon = 2.1\epsilon_0 \Rightarrow D = 2.57 \times 10^{-4}$$

Attenuation in a Good Conductor

$$-i\vec{k} \wedge \vec{H} = \sigma \vec{E} + i\omega\epsilon \vec{E}$$

Combine with $\nabla \wedge \vec{E} = -\dot{\vec{B}} \Rightarrow \vec{k} \wedge \vec{E} = \omega\mu\vec{H}$

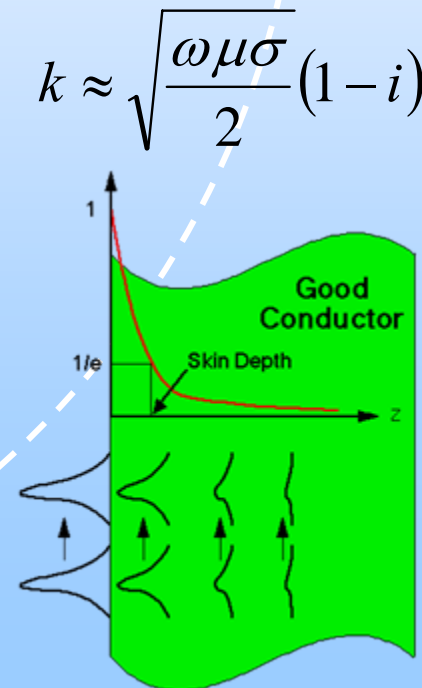
$$\Rightarrow \vec{k} \wedge (\vec{k} \wedge \vec{E}) = \omega\mu\vec{k} \wedge \vec{H} = -\omega\mu(-i\sigma + \omega\epsilon)\vec{E}$$

$$\Rightarrow k^2 = \omega\mu(-i\sigma + \omega\epsilon)$$

For a good conductor $D \gg 1$, $\sigma \gg \omega\epsilon$ $k^2 \approx -i\omega\mu\sigma \Rightarrow k \approx \sqrt{\frac{\omega\mu\sigma}{2}}(1-i)$

Wave form is $\exp\left[i\left(\omega t - \frac{x}{\delta}\right)\right]\exp\left(-\frac{x}{\delta}\right)$

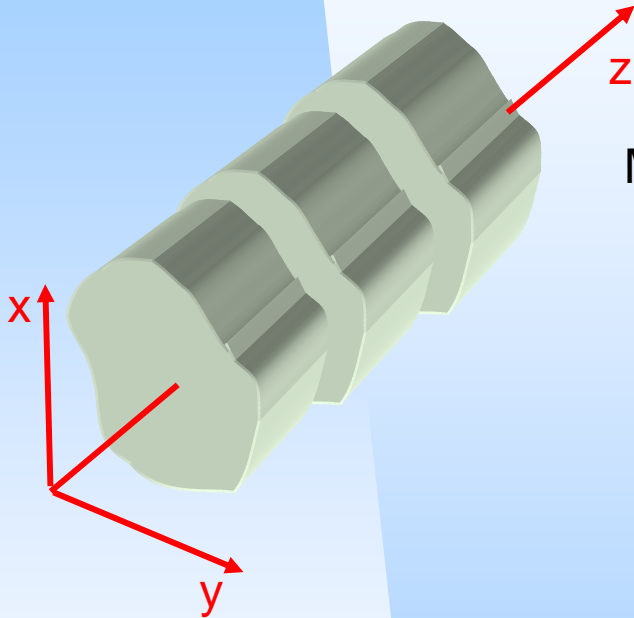
where $\delta = \sqrt{\frac{2}{\omega\mu\sigma}}$ is the skin - depth



[copper.mov](#)

[water.mov](#)

Maxwell's Equations in a uniform perfectly conducting guide



Hollow metallic cylinder with perfectly conducting boundary surfaces

Maxwell's equations with time dependence $\exp(j\omega t)$ are:

$$\begin{aligned} \nabla \wedge \vec{E} &= -\frac{\partial \vec{B}}{\partial t} = -i\omega\mu\vec{H} & \nabla^2 \vec{E} &= \nabla(\nabla \cdot \vec{E}) - \nabla \wedge (\nabla \wedge \vec{E}) \\ & & &= i\omega\mu \nabla \wedge \vec{H} \\ \nabla \wedge \vec{H} &= \frac{\partial \vec{D}}{\partial t} = i\omega\varepsilon\vec{E} & &= -\omega^2\varepsilon\mu\vec{E} \end{aligned} \Rightarrow$$

$$\left(\nabla^2 + \omega^2 \mu \varepsilon \right) \begin{Bmatrix} \vec{E} \\ \vec{H} \end{Bmatrix} = 0$$

Assume $\vec{E}(x, y, z, t) = \vec{E}(x, y)e^{(i\omega t - \gamma z)}$
 $\vec{H}(x, y, z, t) = \vec{H}(x, y)e^{(i\omega t - \gamma z)}$

γ is the propagation constant

Then $\left[\nabla_t^2 + (\omega^2 \varepsilon \mu + \gamma^2) \right] \begin{Bmatrix} \vec{E} \\ \vec{H} \end{Bmatrix} = 0$

Can solve for the fields completely in terms of E_z and H_z

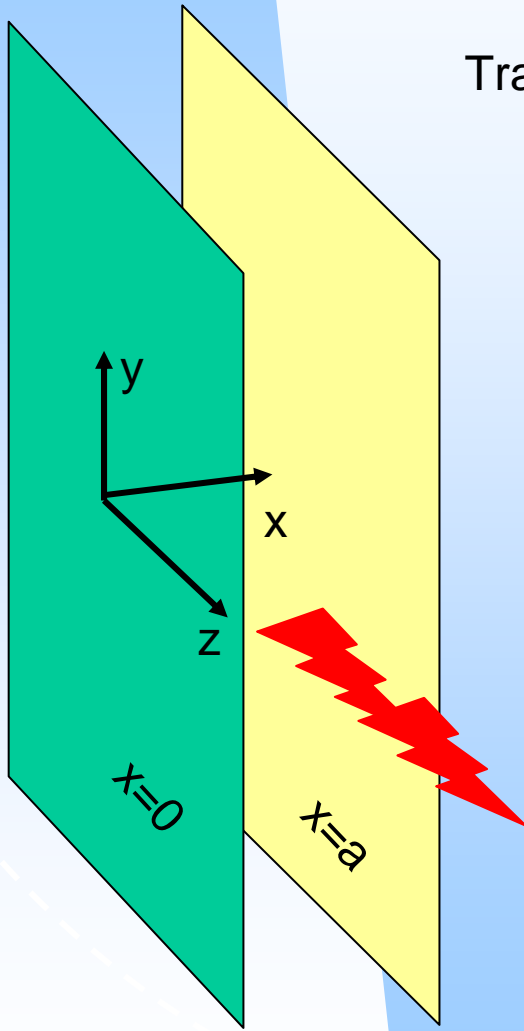
Special cases

- Transverse magnetic (TM modes):
 - $H_z=0$ everywhere, $E_z=0$ on cylindrical boundary

- Transverse electric (TE modes):
 - $E_z=0$ everywhere, $\frac{\partial H_z}{\partial n} = 0$ on cylindrical boundary

- Transverse electromagnetic (TEM modes):
 - $E_z=H_z=0$ everywhere
 - requires $\gamma^2 + \omega^2 \epsilon\mu = 0$ or $\gamma = \pm i\omega\sqrt{\epsilon\mu}$

A simple mode: Parallel Plate Waveguide



Transport between two infinite conducting plates (TE₀₁ mode):

$$\vec{E} = (0,1,0)E(x) e^{(i\omega t - \gamma z)} \quad \text{where } E(x) \text{ satisfies}$$

$$\nabla_t^2 E = \frac{d^2 E}{dx^2} = -K^2 E, \quad K^2 = \omega^2 \epsilon \mu + \gamma^2$$

$$\text{i.e. } E = A \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} Kx$$

To satisfy boundary conditions, $E=0$ on $x=0$ and $x=a$, so

$$E = A \sin Kx, \quad K = K_n = \frac{n\pi}{a}, \quad n \text{ integer}$$

$$\text{Propagation constant is } \gamma = \sqrt{K_n^2 - \omega^2 \epsilon \mu}$$

$$= \frac{n\pi}{a} \sqrt{1 - \left(\frac{\omega}{\omega_c}\right)^2} \quad \text{where } \omega_c = \frac{K_n}{\sqrt{\epsilon \mu}}$$

Cut-off frequency, ω_c

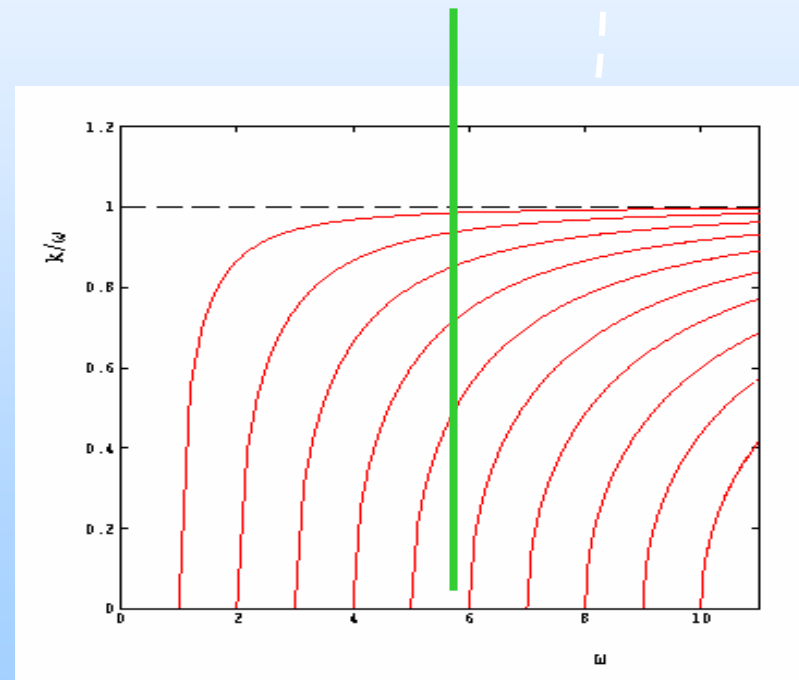
$$\gamma = \frac{n\pi}{a} \sqrt{1 - \left(\frac{\omega}{\omega_c}\right)^2}, \quad E = A \sin \frac{n\pi x}{a} e^{i\omega t - \gamma z}, \quad \omega_c = \frac{n\pi}{a\sqrt{\epsilon\mu}}$$

- $\omega < \omega_c$ gives real solution for γ , so attenuation only. No wave propagates: cut-off modes.
- $\omega > \omega_c$ gives purely imaginary solution for γ , and a wave propagates without attenuation.

$$\gamma = ik, \quad k = \sqrt{\epsilon\mu} \left(\omega^2 - \omega_c^2\right)^{1/2} = \omega\sqrt{\epsilon\mu} \left(1 - \frac{\omega_c^2}{\omega^2}\right)^{1/2}$$

- For a given frequency ω only a finite number of modes can propagate.

$$\omega > \omega_c = \frac{n\pi}{a\sqrt{\epsilon\mu}} \Rightarrow n < \frac{a\omega}{\pi} \sqrt{\epsilon\mu}$$

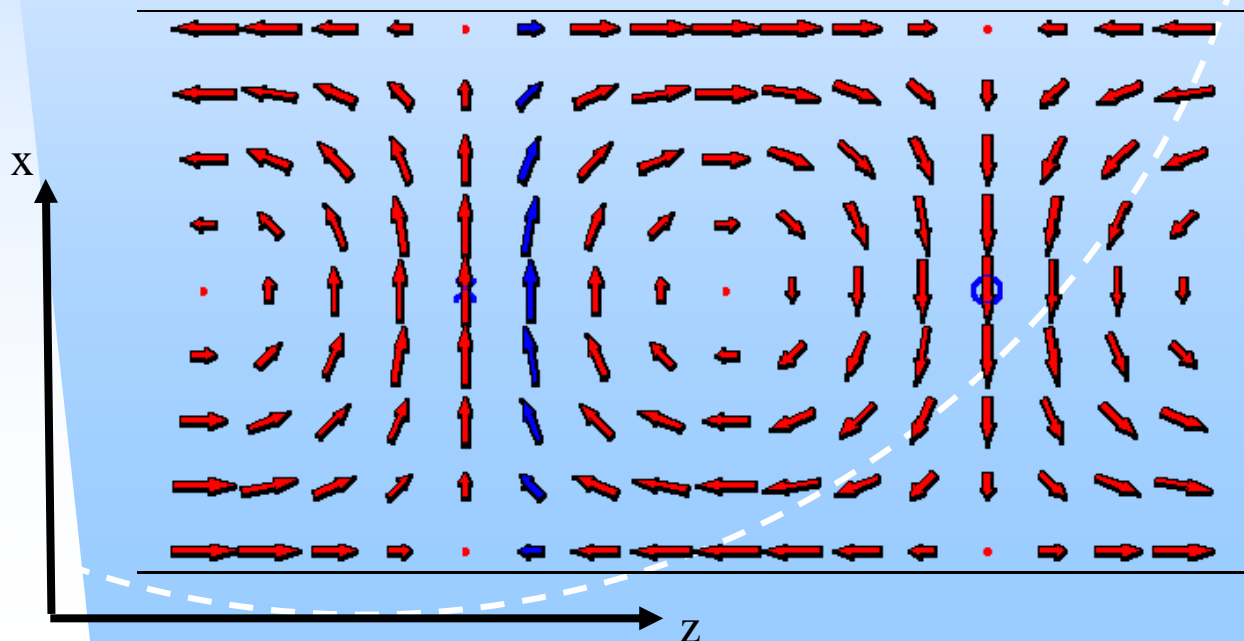


For given frequency, convenient to choose a s.t. only $n=1$ mode occurs.

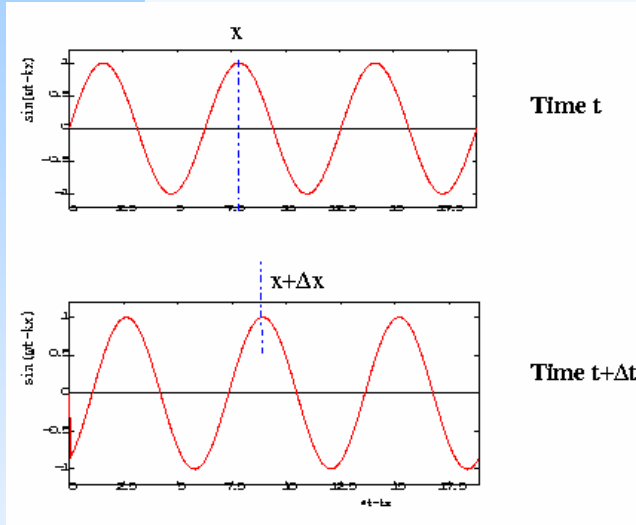
Propagated electromagnetic fields

□ From $\nabla \wedge \vec{E} = -\frac{\partial \vec{B}}{\partial t}$, (assuming A is real)

$$\vec{H} = \frac{i}{\omega\mu} \nabla \wedge \vec{E} \Rightarrow \begin{cases} H_x = -\frac{Ak}{\omega\mu} \sin\left(\frac{n\pi x}{a}\right) \cos(\omega t - kz) \\ H_y = 0 \\ H_z = -\frac{A}{\omega\mu} \frac{n\pi}{a} \cos\left(\frac{n\pi x}{a}\right) \sin(\omega t - kz) \end{cases}$$



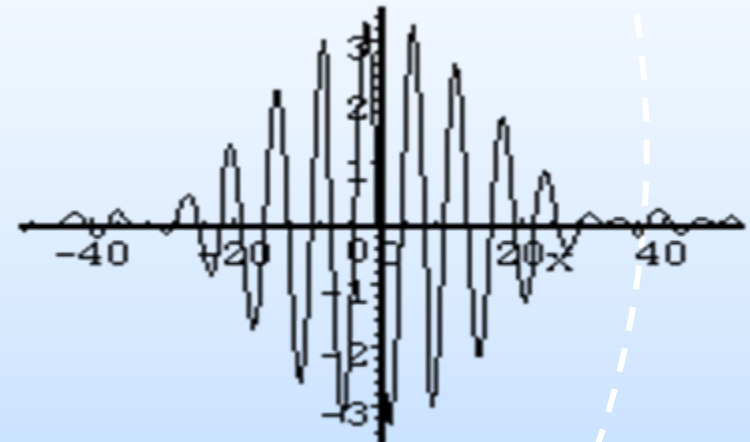
Phase and group velocities



Plane wave $\exp j(\omega t - kx)$ has constant phase $\omega t - kx$ at peaks

$$\omega \Delta t - k \Delta x = 0$$

$$\Leftrightarrow v_p = \frac{\Delta x}{\Delta t} = \frac{\omega}{k}$$

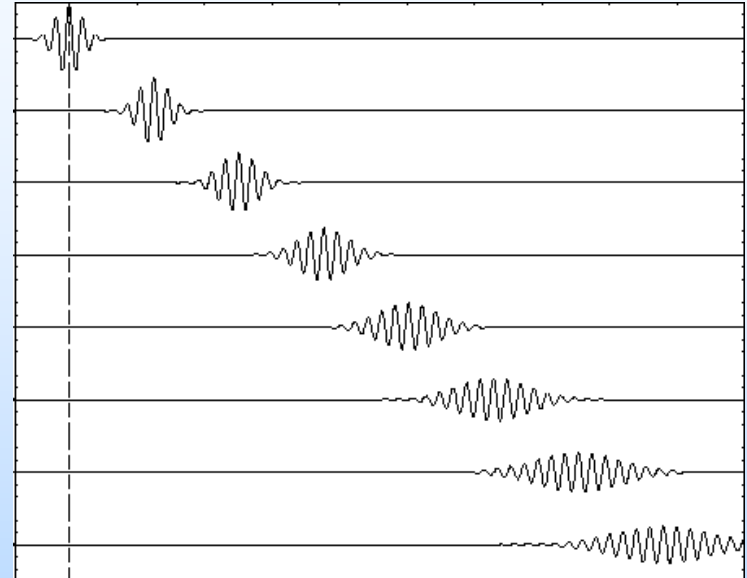
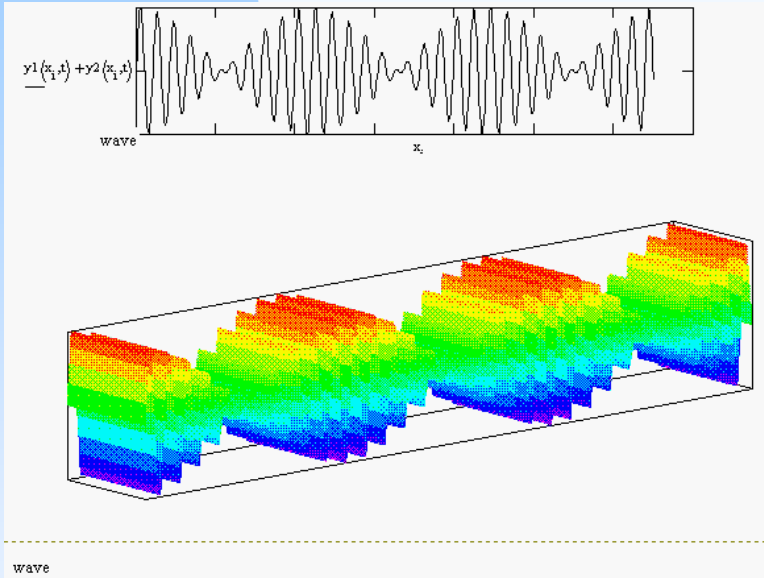


$$\int_{-\infty}^{\infty} A(k) e^{i[\omega(k)t - kx]} dk$$

Superposition of plane waves. While shape is relatively undistorted, pulse travels with the group velocity

$$v_g = \frac{d\omega}{dk}$$

Wave packet structure



- ❑ Phase velocities of individual plane waves making up the wave packet are different,
- ❑ The wave packet will then disperse with time

Phase and group velocities in the simple wave guide

- Wave number is $k = \sqrt{\epsilon\mu}(\omega^2 - \omega_c^2)^{1/2} < \omega\sqrt{\epsilon\mu}$
so wavelength in guide $\lambda = \frac{2\pi}{k} > \frac{2\pi}{\omega\sqrt{\epsilon\mu}}$, the free-space wavelength
- Phase velocity is $v_p = \frac{\omega}{k} > \frac{1}{\sqrt{\epsilon\mu}}$, larger than free-space velocity
- Group velocity is less than infinite space value

$$k^2 = \epsilon\mu(\omega^2 - \omega_c^2) \Rightarrow v_g = \frac{d\omega}{dk} = \frac{k}{\omega\epsilon\mu} < \frac{1}{\sqrt{\epsilon\mu}}$$

Calculation of wave properties

- If $a=3$ cm, cut-off frequency of lowest order mode is

$$f_c = \frac{\omega_c}{2\pi} = \frac{1}{2a\sqrt{\epsilon\mu}} \cong 5 \text{ GHz}$$

- At 7 GHz, only the $n=1$ mode propagates and

$$k = \sqrt{\epsilon\mu}(\omega^2 - \omega_c^2)^{1/2} \approx 103 \text{ m}^{-1}$$

$$\lambda = \frac{2\pi}{k} \approx 6 \text{ cm}$$

$$v_p = \frac{\omega}{k} \approx 4.3 \times 10^8 \text{ ms}^{-1}$$

$$v_g = \frac{k}{\omega\epsilon\mu} = 2.1 \times 10^8 \text{ ms}^{-1}$$

Waveguide animations

- TE1 mode above cut-off [ppwg_1-1.mov](#)
- TE1 mode, smaller ω [ppwg_1-2.mov](#)
- TE1 mode at cut-off [ppwg_1-3.mov](#)
- TE1 mode below cut-off [ppwg_1-4.mov](#)
- TE1 mode, variable ω [ppwg_1_vf.mov](#)
- TE2 mode above cut-off [ppwg_2-1.mov](#)
- TE2 mode, smaller [ppwg_2-2.mov](#)
- TE2 mode at cut-off [ppwg_2-3.mov](#)
- TE2 mode below cut-off [ppwg_2-4.mov](#)