Resonances

- introduction: driven oscillators and resonance condition
- smooth approximation for motion in accelerators
- field imperfections and normalized field errors
- perturbation treatment
- Poincare section
- stabilization via amplitude dependent tune changes
- sextupole perturbation & slow extraction
- chaotic particle motion

Introduction: Damped Harmonic Oscillator

- equation of motion for a damped harmonic oscillator:

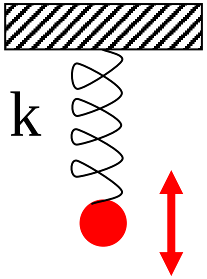
$$\frac{d^2}{dt^2} w(t) + \omega_0 \cdot Q^{-1} \cdot \frac{d}{dt} w(t) + \omega_0{}^2 \cdot w(t) = 0$$

Q is the damping coefficient

→ (amplitude decreases with time)

ω_0 is the Eigenfrequency of the HO

- example: weight on a spring ($Q = \infty$)



$$\frac{d^2}{dt^2} w(t) + k \cdot w(t) = 0 \quad \longrightarrow \quad w(t) = a \cdot \sin(\sqrt{k} \cdot t + \phi_0)$$

Introduction: Driven Oscillators

- an external driving force can ‘pump’ energy into the system:

$$\frac{d^2}{dt^2} w(t) + \omega_0 \cdot Q^{-1} \cdot \frac{d}{dt} w(t) + \omega_0{}^2 \cdot w(t) = \frac{F}{m} \cdot \cos(\omega \cdot t)$$

- general solution:

$$w(t) = w_{tr}(t) + w_{st}(t)$$

- stationary solution:

$$w_{st}(t) = W(\omega) \cdot \cos[\omega \cdot t - \alpha(\omega)]$$

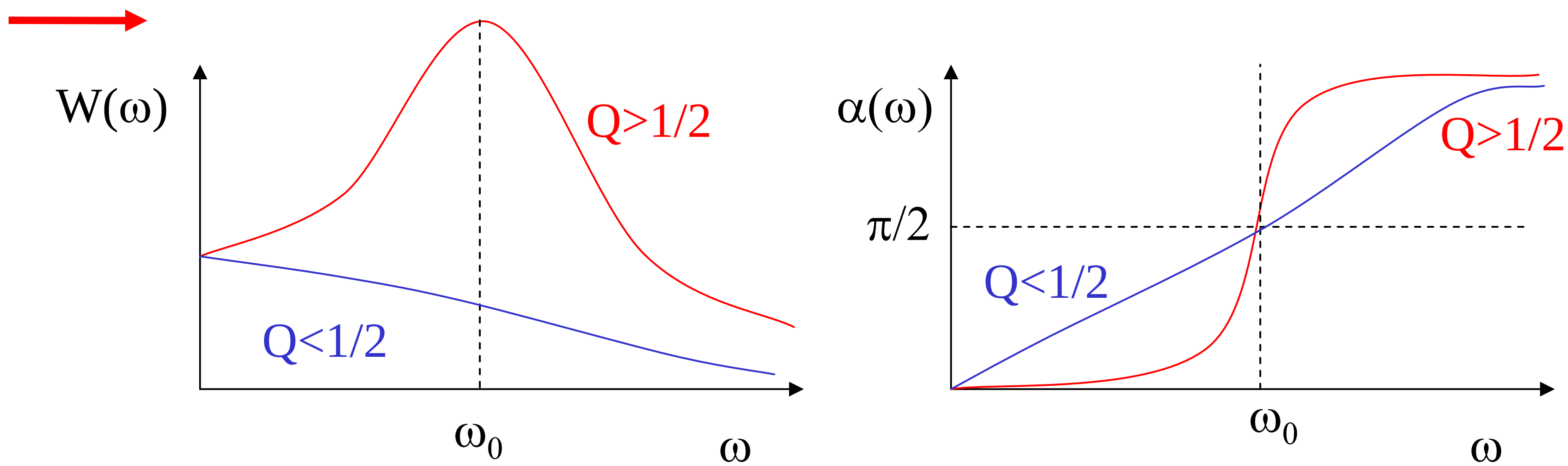
→ where ‘ω’ is the driving angular frequency!
and W(ω) can become large for certain frequencies!

Introduction: Driven Oscillators

- stationary solution

stationary solution follows the frequency of the driving force:

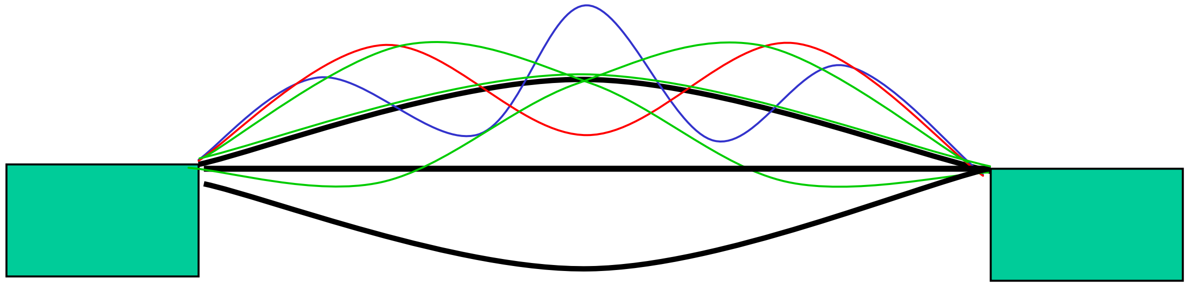
$$w_{st}(t) = W(\omega) \cdot \cos[\omega \cdot t - \alpha(\omega)]$$



- oscillation amplitude can become large for weak damping

Introduction: Pulsed Driven Resonances Example

- higher harmonics:



- example of a bridge:

[Bob Barrett; Messiah College]

2nd harmonic:

3nd harmonic:

5nd harmonic:

- peak amplitude depends on the excitation frequency and damping

Introduction: Instabilities

- resonance catastrophe without damping:

$$W(\omega) = W(0) \cdot \frac{1}{\sqrt{[1-(\frac{\omega}{\omega_0})^2]^2 + (\frac{\omega}{Q\omega_0})^2}}$$

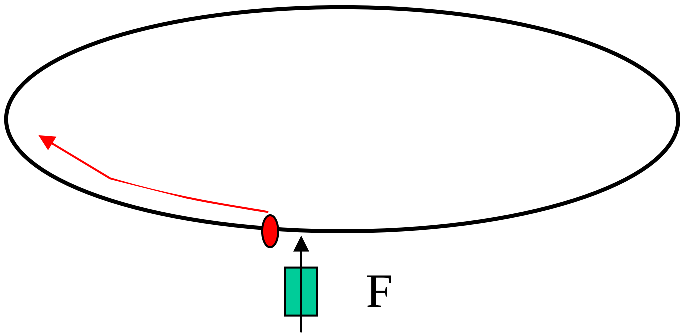
- weak damping: resonance condition: $\omega = \omega_0$

Tacoma Narrow bridge
1940

excitation by strong wind on the Eigenfrequencies

Smooth Approximation: Resonances in Accelerators

- revolution frequency:

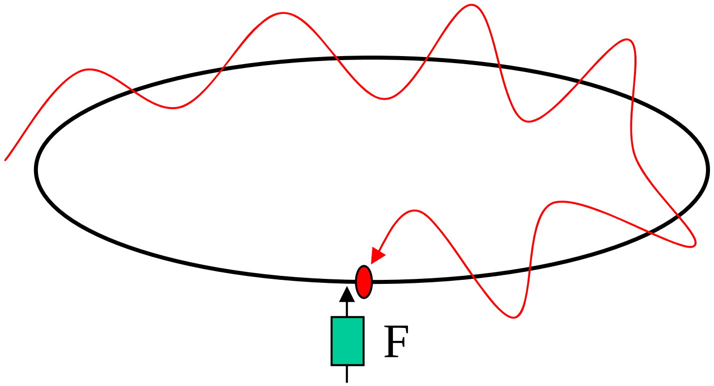


→ periodic kick

→ excitation with f_{rev}

($\omega_{rev} = 2\pi\ f_{rev}$)

- betatron oscillations:



Eigenfrequency: $\omega_0 = 2\pi\ f_\beta$

$$Q = \omega_0 / \omega_{rev}$$

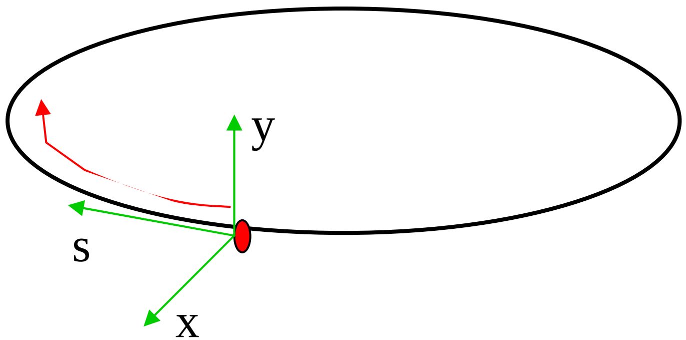
→ driven oscillator

→ weak or no damping!

(synchrotron radiation damping (single particle) or Landau damping distributions)

Smooth Approximation: Free Parameter

- co-moving coordinate system:



➔ choose the longitudinal coordinate as the free parameter for the equations of motion

- equations of motion:

$$\frac{d}{dt} = \frac{ds}{dt} \cdot \frac{d}{ds}$$

with:

$$\frac{ds}{dt} = v$$

➔

$$\frac{d^2}{dt^2} = v^2 \cdot \frac{d^2}{ds^2}$$

Smooth Approximation: Equation of Motion I

Smooth approximation for Hills equation:

$$\frac{d^2}{ds^2} w(s) + K(s) \cdot w(s) = 0 \xrightarrow{K(s) = \text{const}} \frac{d^2}{ds^2} w(s) + \omega_0{}^2 \cdot w(s) = 0$$

(constant β-function and phase advance along the storage ring)

$$\longrightarrow \quad w(s) = A \cdot \cos(\omega_0 \cdot s + \phi_0) \qquad \omega_0 = 2\pi \cdot Q_0 / L$$

(Q is the number of oscillations during one revolution)

perturbation of Hills equation:

$$\frac{d^2}{ds^2} w(s) + \omega_0{}^2 \cdot w(s) = F(w(s), s) / (v \cdot p)$$

in the following the force term will be the Lorenz force of a charged particle in a magnetic field:

$$F = q \cdot \vec{v} \times \vec{B}$$

Field Imperfections: Origins for Perturbations

- linear magnet imperfections: derivation from the design dipole and quadrupole fields due to powering and alignment errors

- time varying fields: feedback systems (damper) and *wake fields due to collective effects (wall currents)*

- non-linear magnets: sextupole magnets for chromaticity correction and octupole magnets for Landau damping

- beam-beam interactions: strongly non-linear field!

- non-linear magnetic field imperfections: particularly difficult to control for super conducting magnets where the field quality is entirely determined by the coil winding accuracy

Field Imperfections: Localized Perturbation

- periodic delta function:

$$\delta_L(s-s_0)=\begin{cases}1 & \text{for `s' = } s_0\\ 0 & \text{otherwise}\end{cases} \quad \text{and} \quad \oint \delta_L(s-s_0)\,ds=1$$

- equation of motion for a single perturbation in the storage ring:

$$\frac{d^2}{ds^2}w(s)+\omega_0{}^2\cdot w(s)=\delta_L(s-s_0)\cdot l\cdot F(w,s)/(v\cdot p)$$

Fourier expansion of the periodic delta function:

$$\frac{d^2}{ds^2}w(s)+\omega_0{}^2\cdot w(s)=\frac{l}{L}\sum_{r=-\infty}^{\infty}\cos(r\cdot 2\pi\cdot s/L)\cdot F(w,s)/(v\cdot p)$$

→ infinite number of driving frequencies

Field Imperfections: Constant Dipole

- normalized field error: $$\frac{F}{v \cdot p} = q \cdot \frac{\vec{v} \times \vec{B}}{v \cdot p} \xrightarrow{v \perp B} k_0 = q \cdot B / p$$

- equation of motion for single kick:

→ $$\frac{d^2}{ds^2} w(s) + \omega_0{}^2 \cdot w(s) = \frac{l k_0}{L} \sum_{r=-\infty}^{\infty} \cos(r \cdot 2\pi \cdot s / L)$$

→ resonance condition: $$\omega_0 = r \cdot 2\pi / L \xrightarrow{\omega_0 = 2\pi \cdot Q_0 / L} Q_0 = r$$

→ **avoid integer tunes!**

→ remember the example of a single dipole imperfection from the 'Linear Imperfection' lecture yesterday!

Field Imperfections: Constant Quadrupole

equations of motion:

$$\frac{d^2}{ds^2}x(s) + \omega_x{}^2 \cdot x(s) = k_1 \cdot x(s)$$

$$y(s) \equiv 0$$

with: $k_1 = \frac{q}{p} \cdot \frac{\partial B_y}{\partial x}$

$$\longrightarrow \quad \frac{d^2}{ds^2}x(s) + (\omega_x{}^2 - k_1) \cdot x(s) = 0$$

$\longrightarrow$ change of tune but no amplitude growth due to resonance excitations!

Field Imperfections: Single Quadrupole Perturbation

- assume y = 0 and B_x = 0:

$$F(s)/(v \cdot p) = \delta_L(s - s_0) \cdot l \cdot k_1 \cdot x$$

$$\longrightarrow \quad \frac{d^2}{ds^2} x(s) + \omega_{x,0}{}^2 \cdot x(s) = \frac{lk_1}{L} \sum_{r=-\infty}^{\infty} \cos(2\pi \cdot r \cdot s / L) \cdot x(s)$$

- resonance condition: $\omega_{x,0} = n \cdot 2\pi / L \pm \omega_{x,0} \xrightarrow{\omega_0 = 2\pi \cdot Q_0 / L} Q_0 = n/2$

avoid half integer tunes plus resonance width from tune modulation!

- exact solution: variation of constants ➔ see the lecture yesterday

Field Imperfections: Time Varying Dipole Perturbation

time varying perturbation:

$$F(t) = F_0 \cdot \cos(\omega_{kick} \cdot t) \xrightarrow{t \to s} F_0 \cdot \cos(2\pi \cdot \frac{\omega_{kick}}{\omega_{rev}} \cdot s / L) / (v \cdot p)$$

$$\frac{d^2}{ds^2} w(s) + \omega_0^{\;2} \cdot w(s) = \frac{lF_0}{2L} \sum_{r=-\infty}^{\infty} \cos(2\pi \cdot [r \pm \omega_{kick} / \omega_{rev}] \cdot s / L) / (v \cdot p)$$

resonance condition:

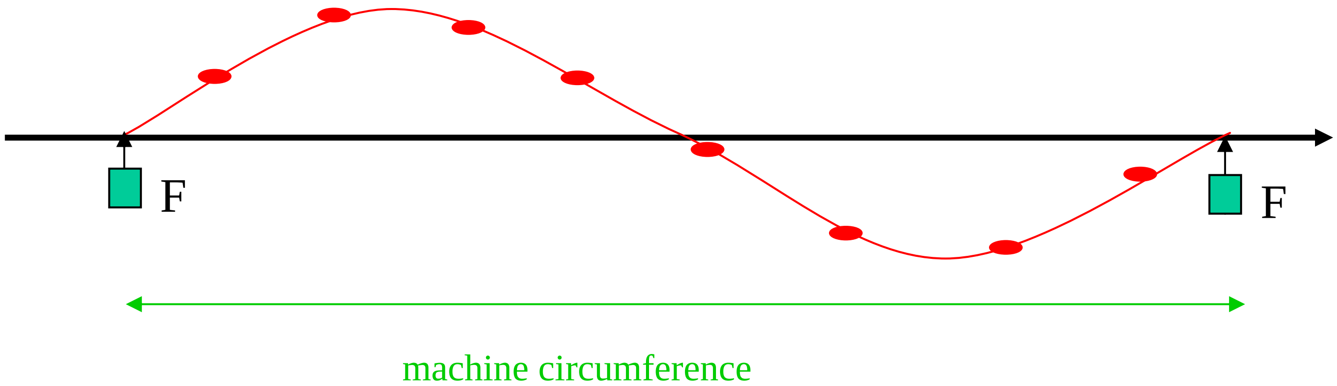
$$\omega_0 = 2\pi \cdot (r \pm \omega_{kick} / \omega_{rev}) / L \xrightarrow{\omega_0 = 2\pi \cdot Q_0 / L} f_{kick} = f_{rev} \cdot (Q_0 \pm r)$$

avoid excitation on the betatron frequency!

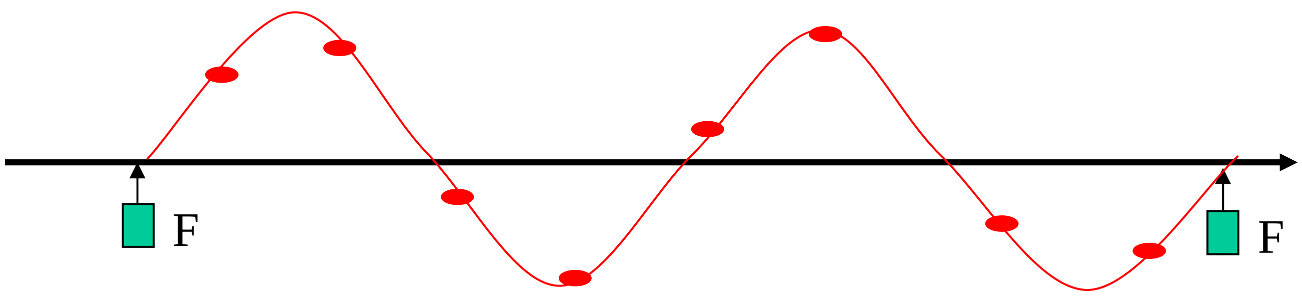
(the integer multiple of the revolution frequency corresponds to the modes of the bridge in the introduction example)

Field Imperfections: Several Bunches

$$F(t) = B \cdot \cos(\omega_{kick} \cdot t); \omega_{kick} \approx \omega_{rev} :$$



$$F(t) = B \cdot \cos(\omega_{kick} \cdot t); \omega_{kick} \approx 2 \cdot \omega_{rev} :$$



→ higher modes analogous to bridge illustration

Field Imperfections: Multipole Expansion

- Taylor expansion of the magnetic field:

$$B_y + iB_x = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot f_n \cdot (x+iy)^n$$

with:

$$f_n = \frac{\partial^{n+1} B_y}{\partial x^{n+1}}$$

multipole	order	B_x	B_y
dipole	0	0	B_0
quadrupole	1	$f_1 \cdot y$	$f_1 \cdot x$
sextupole	2	$f_2 \cdot x \cdot y$	$\frac{1}{2} \cdot f_2 \cdot (x^2 \cdot y^2)$
octupole	3	$\frac{1}{6} \cdot f_3 \cdot (3yx^2 - y^3)$	$\frac{1}{6} \cdot f_3 \cdot (x^3 - 3xy^2)$

- normalized multipole gradients:

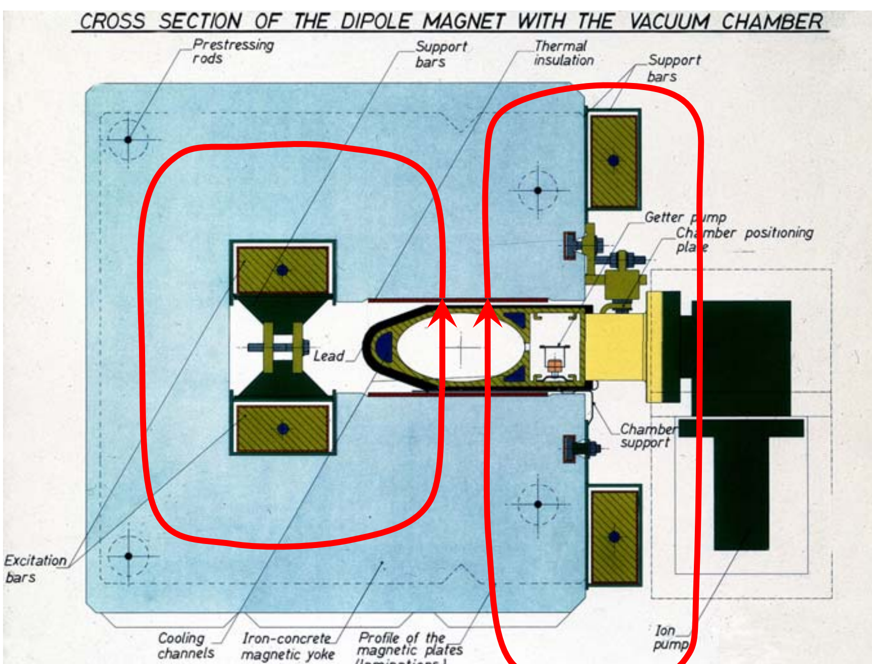
$$F(s)/(v \cdot p) = \frac{q \cdot (\vec{v} \times \vec{B})}{(v \cdot p)} \qquad k_n = \frac{q}{p} \cdot f_n \qquad k_n = 0.3 \cdot \frac{f_n [T/m^n]}{p[GeV/c]} \qquad [k_n] = \frac{1}{m^{n+1}}$$

Field Imperfections: Dipole Magnets

- dipole magnet designs:

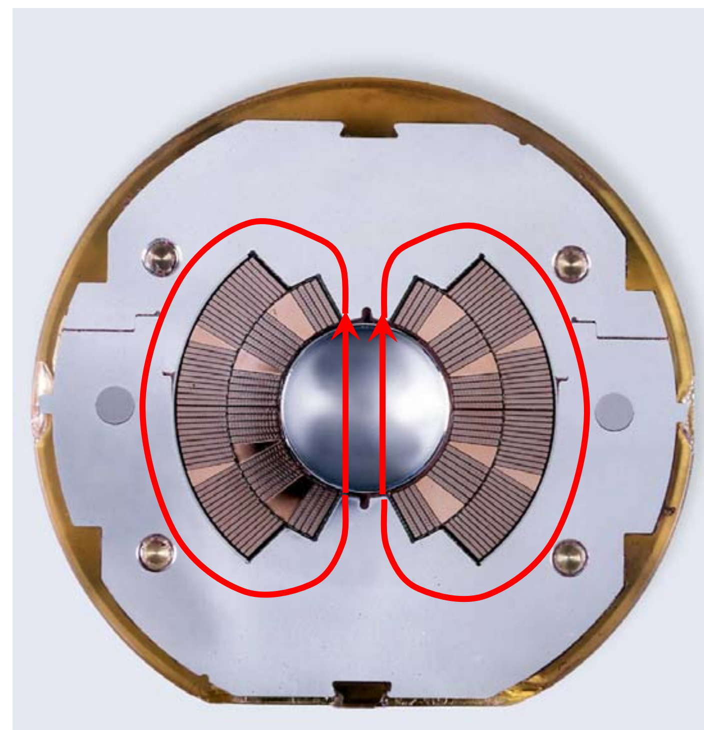
LEP dipole magnet:

conventional magnet design relying on pole face accuracy of a Ferromagnetic Yoke



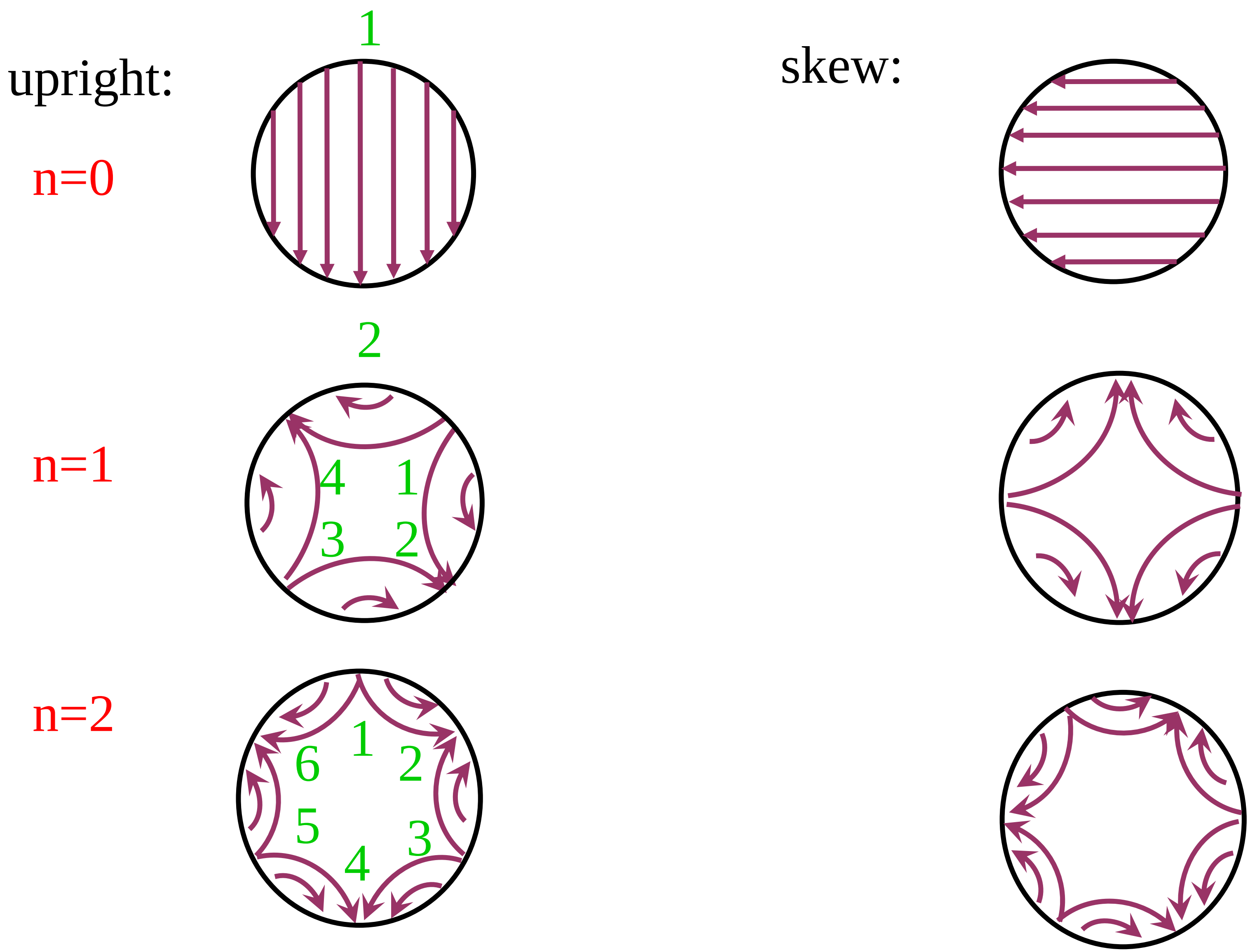
LHC dipole magnet:

air coil magnet design relying on precise current distribution



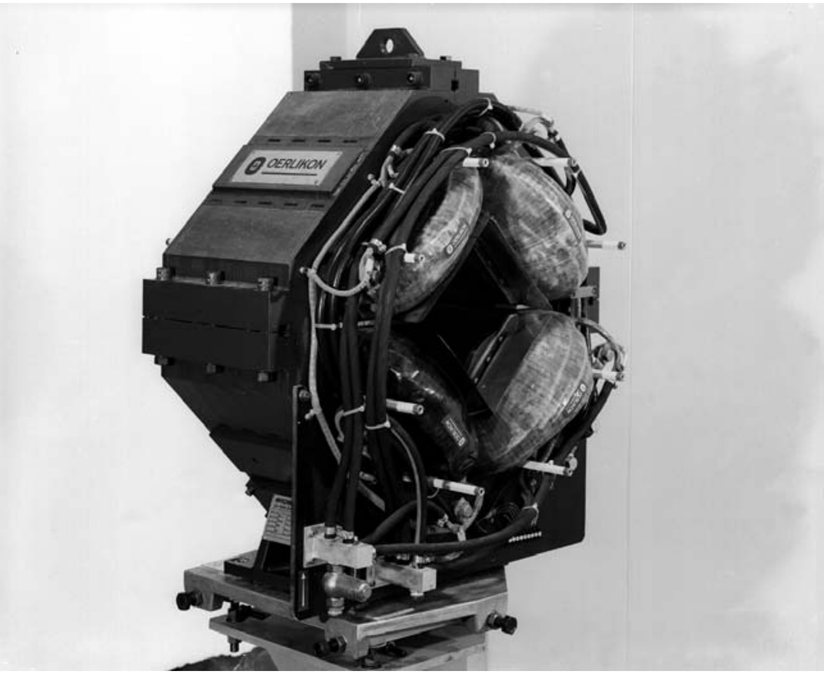
Field Imperfections: Multipole Illustration

upright and skew field errors



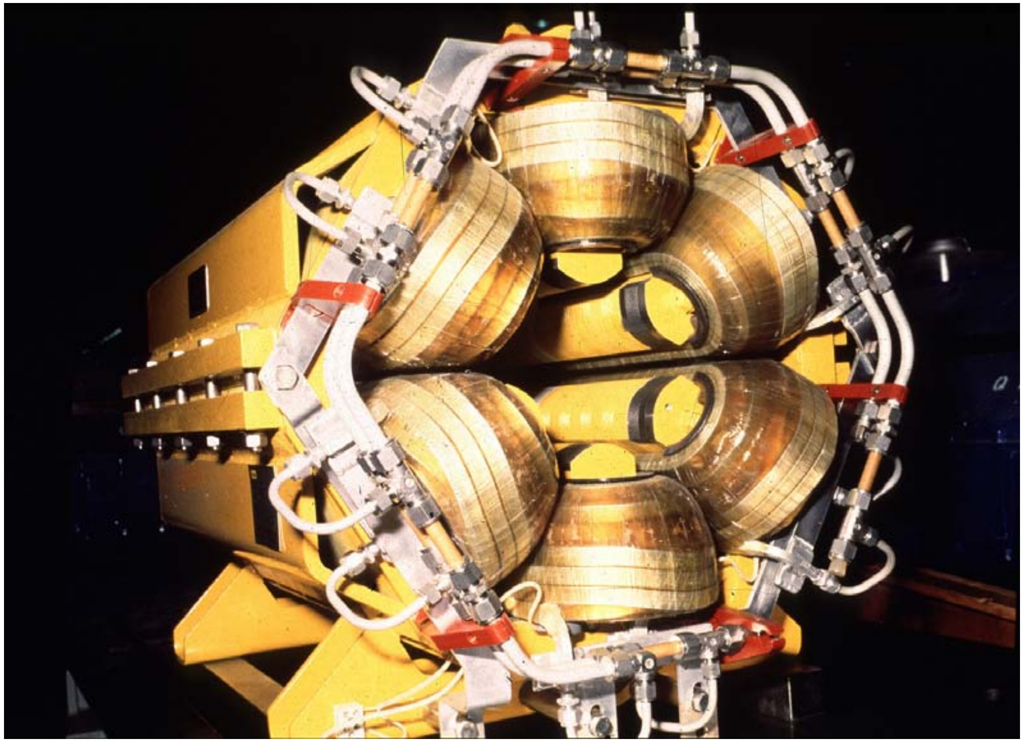
Field Imperfections: Multipole Illustrations

quadrupole and sextupole magnets



ISR quadrupole

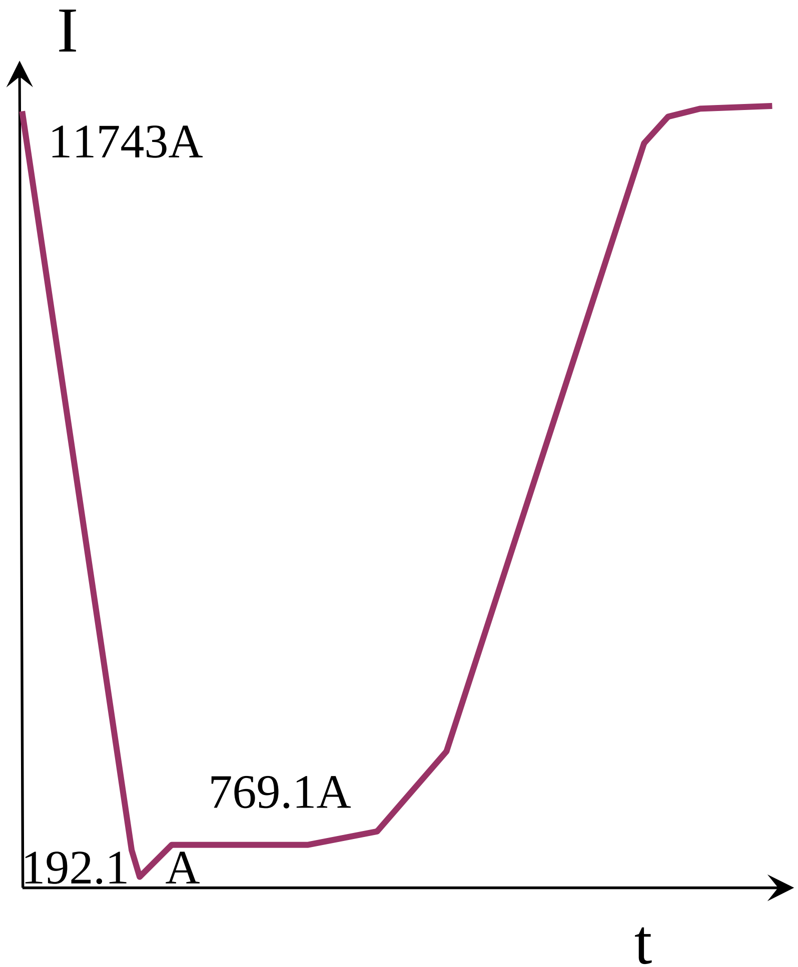
LEP Sextupole

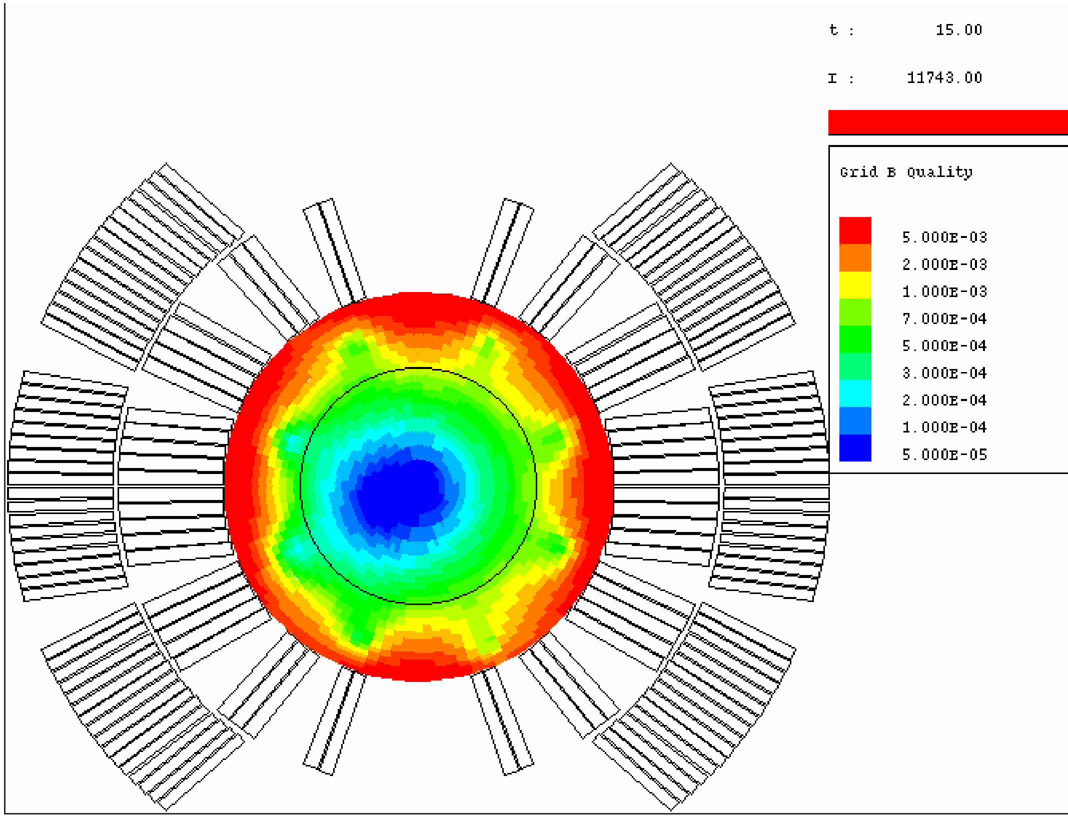


Field Imperfections: Super Conducting Magnets

- time varying field errors in super conducting magnets

Luca Bottura CERN, AT-MAS





Perturbation Treatment: Resonance Condition

- equations of motion:

(n^{th} order Polynomial in x and y for n^{th} order multipole)

$$\frac{d^2}{ds^2} w(s) + \omega_0{}^2 \cdot w(s) = \varepsilon \cdot \sum_{\substack{l+m<n,\\ r}} a_{n,m,r} \cdot x^l \cdot y^m \cdot \cos(2\pi \cdot r \cdot s / L)$$

with: $w = x, y$

- perturbation treatment:

$$w(s) = x_0 + \varepsilon \cdot x_1 + \varepsilon^2 \cdot x_2 + ... + O(\varepsilon^n) \qquad \omega_0 = \frac{2\pi}{L} Q_{x,y}$$

with: $x_0(s) = x_0 \cdot \cos(2\pi \cdot Q_{x,0} \cdot s / L + \phi_{x,0})$ [same for 'y(s)']

→

$$\frac{d^2}{ds^2} x_1 + \omega_0{}^2 \cdot x_1 = \sum_{\tilde{l}<l, \tilde{m}<m} a_{\tilde{n},\tilde{m},r} \cos\left(\frac{2\pi}{L} \cdot [\tilde{l}\, Q_{x,0} + \tilde{m} Q_{y,0} + r] \cdot s\right)$$

Perturbation Treatment: Tune Diagram I

resonance condition:

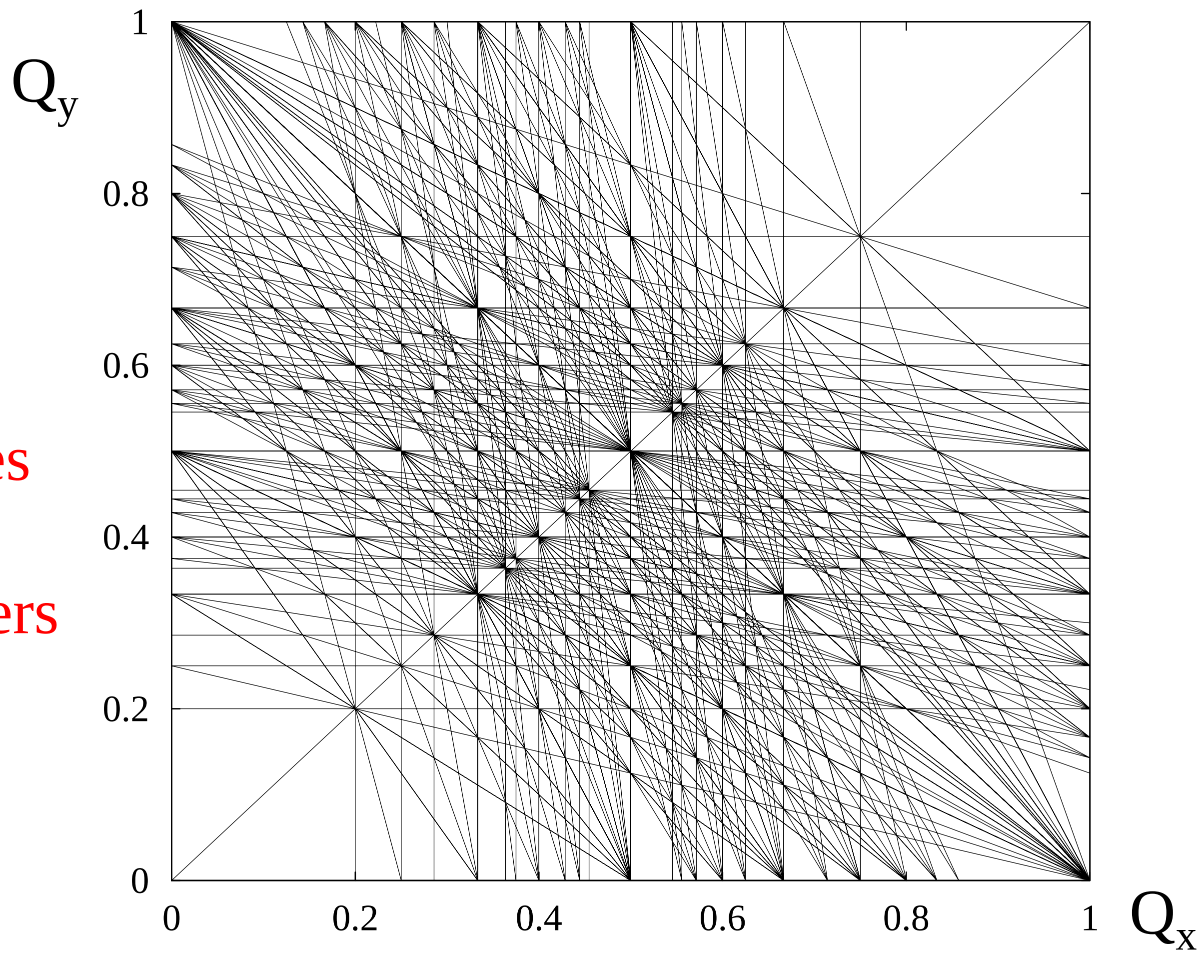
$$\frac{2\pi}{L}\cdot(\tilde{l}\cdot Q_x+\tilde{m}\cdot Q_y+r)=\frac{2\pi}{L}\cdot Q_{x,y}$$

→ $l\cdot Q_x + m\cdot Q_y = r$

avoid rational tune values!

tune diagram:

up to 11 order (p+l <12)



→

there are resonances everywhere!
(the rational numbers lie dens within the real number)

Perturbation Treatment: Tune Diagram II

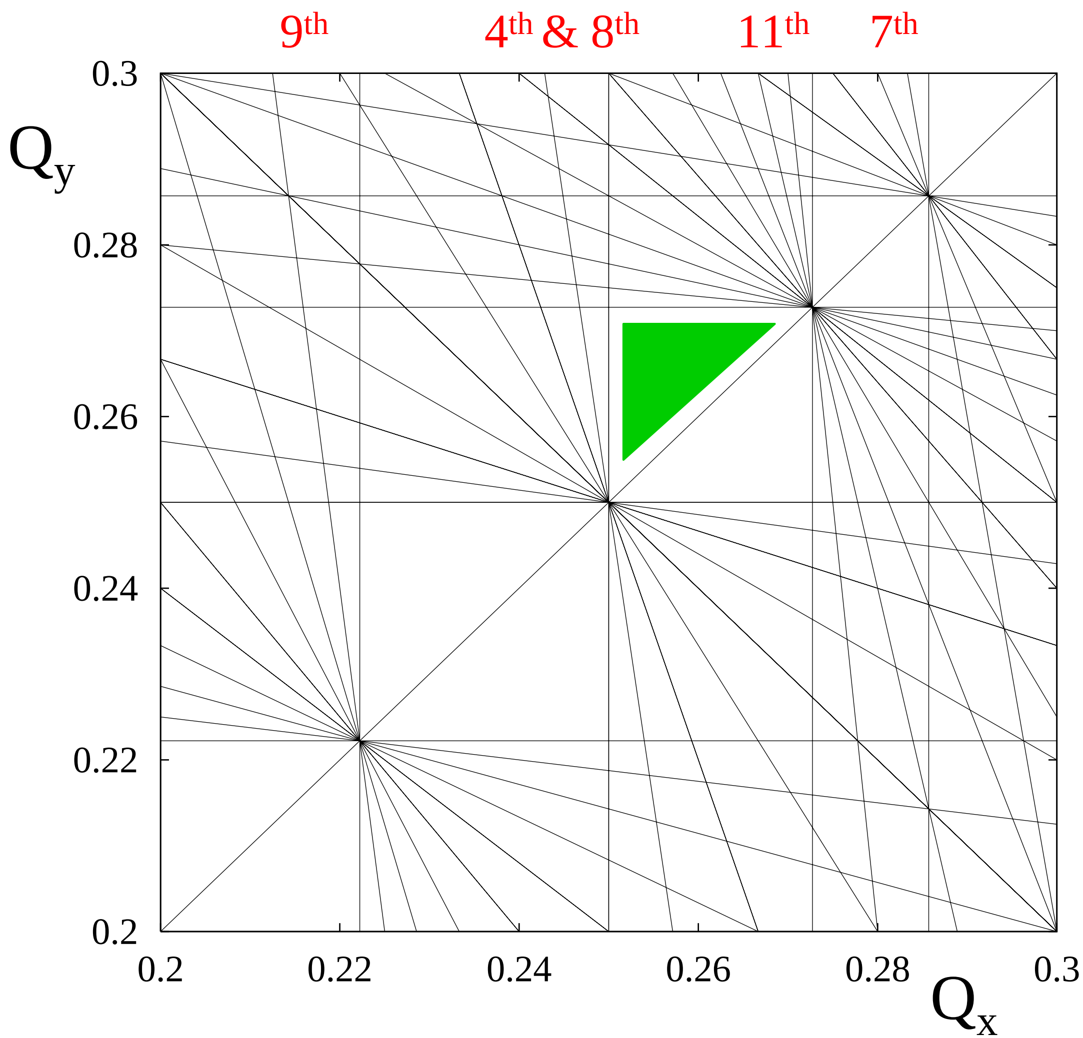
regions with few resonances:

$$l \cdot Q_x + m \cdot Q_y = r$$

avoid low order resonances!

→ < 12th order for a proton beam without damping

→ < 3rd ⇔ 5th order for electron beams with damping



coupling resonance:

regions without low order resonances are relatively small!

Perturbation Treatment: Single Sextupole Perturbation

- perturbed equations of motion: $F(s)/(v\cdot p)=\frac{1}{2}\cdot\delta_L(s-s_0)\cdot lk_2\cdot x^2$

→ $$\frac{d^2}{ds^2}x_1(s)+\omega_0{}^2\cdot x_1(s)=\frac{1}{2}\cdot lk_2\cdot x_0{}^2\cdot\frac{1}{L}\sum_{r=-\infty}^{\infty}\cos(2\pi\cdot r\cdot s/L)$$

with: $x_0(s)=A\cdot\cos(\omega_{0,x}\cdot s+\phi_0)$ and $\omega_{0,x}=2\pi\cdot Q_{x,0}/L$

→ $$\frac{d^2}{ds^2}x_1(s)+(2\pi Q_{x,0}/L)^2\cdot x_1(s)=\frac{lk_1}{2L}\cdot A^2\cdot\sum_{r=-\infty}^{\infty}\cos(2\pi\cdot r\cdot s/L)$$

$$+\frac{lk_1}{8L}\cdot A^2\cdot\sum_{r=-\infty}^{\infty}\cos(2\pi\cdot[r\pm 2Q_{x,0}]\cdot s/L)$$

Perturbation Treatment: Sextupole Perturbation

- resonance conditions:

$$2\pi Q_{x,o} = 2\pi \cdot (r) \longrightarrow Q_{x,0} = r$$

$$2\pi Q_{x,o} = 2\pi \cdot (r \pm 2Q_{x,0}) \xrightarrow{r-2Q_{x,0}} Q_{x,0} = r/3$$

$$\xrightarrow{r+2Q_{x,0}} Q_{x,0} = r$$

→ **avoid integer and r/3 tunes!**

- perturbation treatment:

contrary to the previous examples no exact solution exist!

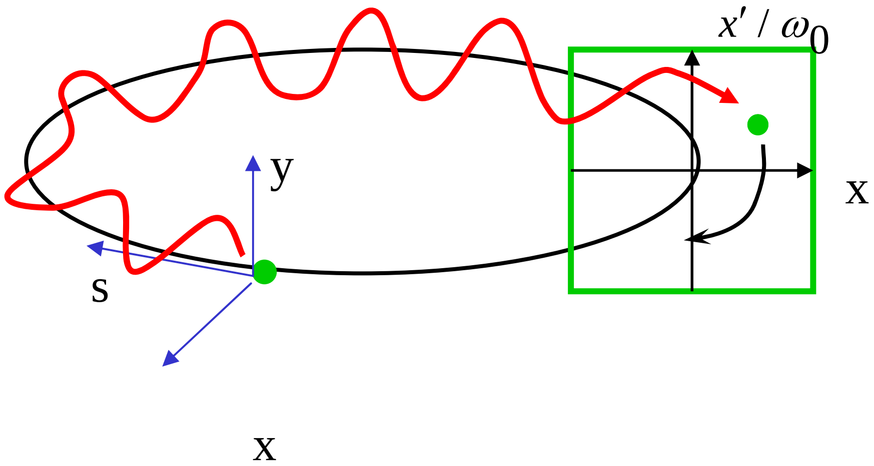
this is a consequence of the non-linear perturbation

(remember the 3 body problem?)

➔ graphic tools for analyzing the particle motion

Poincare Section: Definition

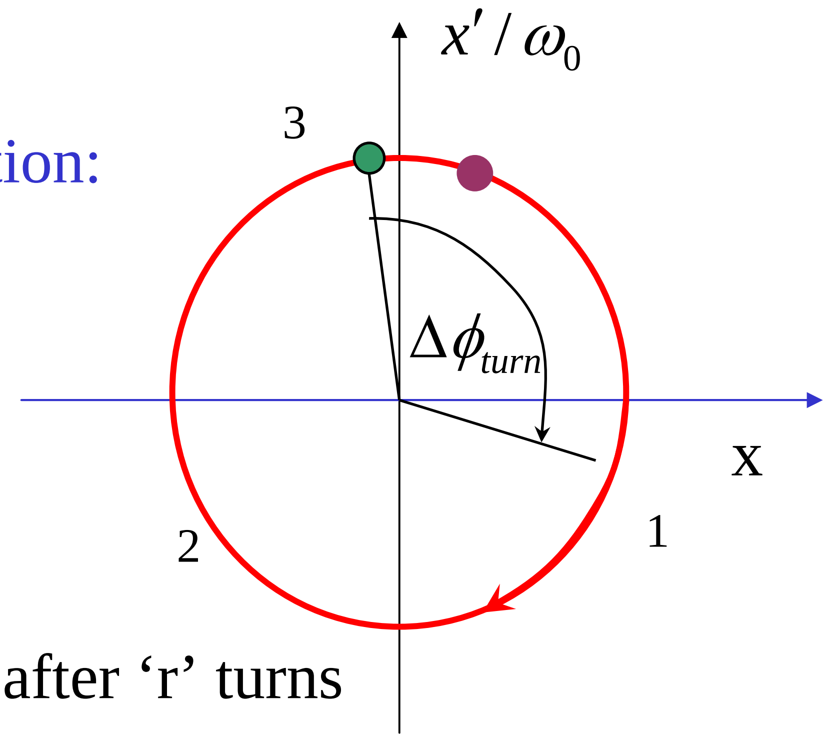
- Poincare Section:



➔ record the particle coordinates at one location in the storage ring

- resonance in the Poincare section:

➔ $$\Delta\phi_{turn} = 2\pi \cdot Q$$



fixed point condition: Q = n/r

points are mapped onto themselves after ‘r’ turns

Poincare Section: Linear Motion

- unperturbed solution:

$$x(s) = \sqrt{R} \cdot \cos(\phi) \qquad \text{with} \qquad \frac{d}{ds}\phi = \omega_0$$

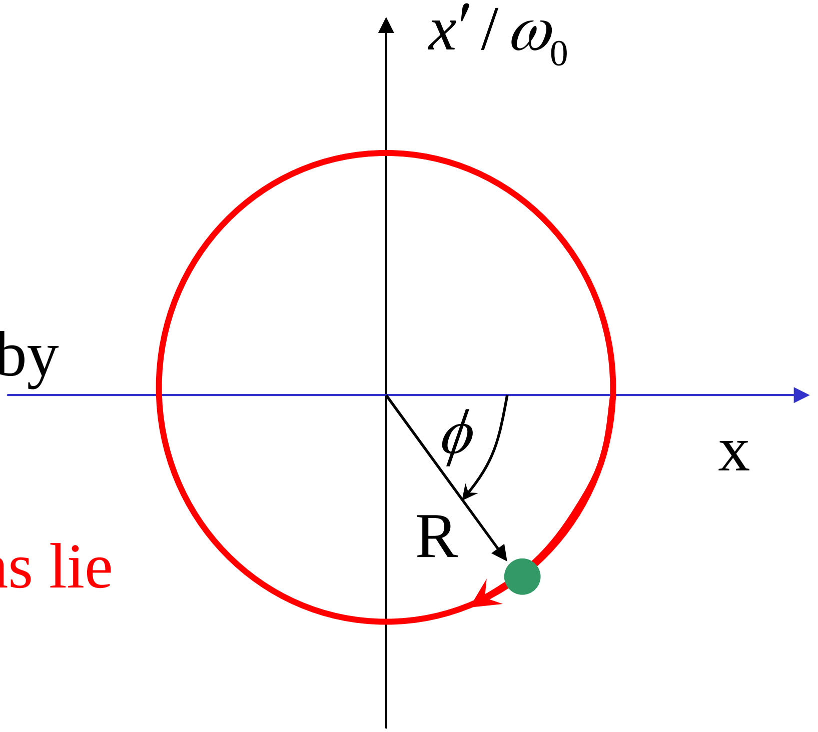
$$x' = \frac{d}{ds}x = -\sqrt{R} \cdot \omega_0 \cdot \sin(\phi)$$

- phase space portrait:

→ the motion lies on an ellipse

→ linear motion is described by a simple rotation

→ consecutive intersections lie on closed curves



Poincare Section: Non-Linear Motion

- momentum change due to perturbation: $$\Delta x' = \oint \frac{F(s)}{v \cdot p} \cdot ds$$

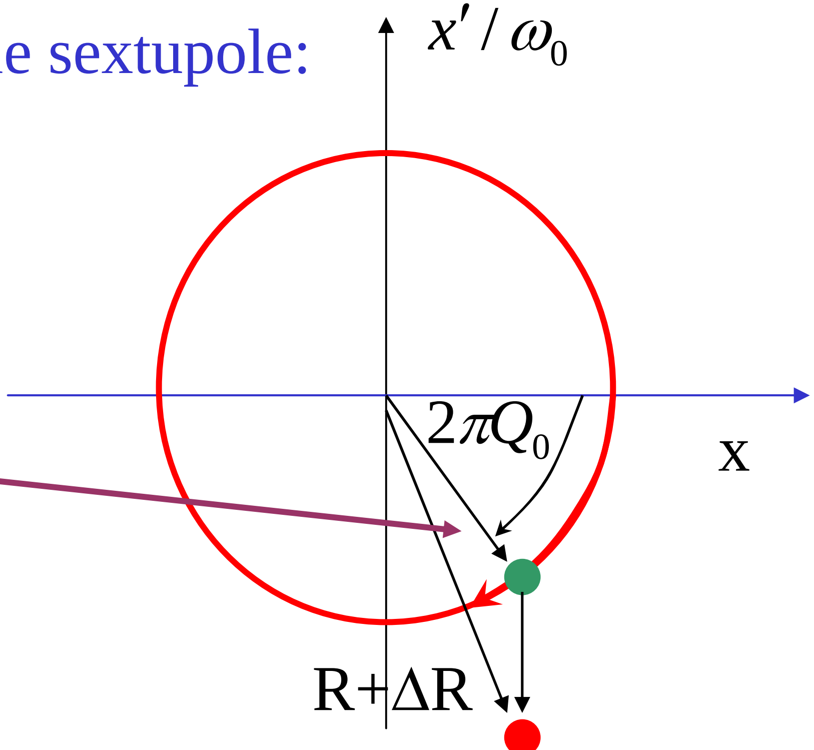
- single n-pole kick: $$\Delta x' = \frac{1}{n!} \cdot lk_n \cdot x^n$$

- phase space portrait with single sextupole:

→ $$\Delta x' = \frac{1}{2} \cdot lk_2 \cdot x^2$$

→ sextupole kick changes the amplitude and the phase advance per turn!

$$\Delta Q_{turn} \propto x^2$$



Poincare Section: Stability?

- instability can be fixed by 'detuning':

  ➔ overall stability depends on the balance between amplitude increase per turn and tune change per turn:

$\Delta Q_{turn}(x)$ ➔ motion moves eventually off resonance

$\Delta R_{turn}(x)$ ➔ motion becomes unstable

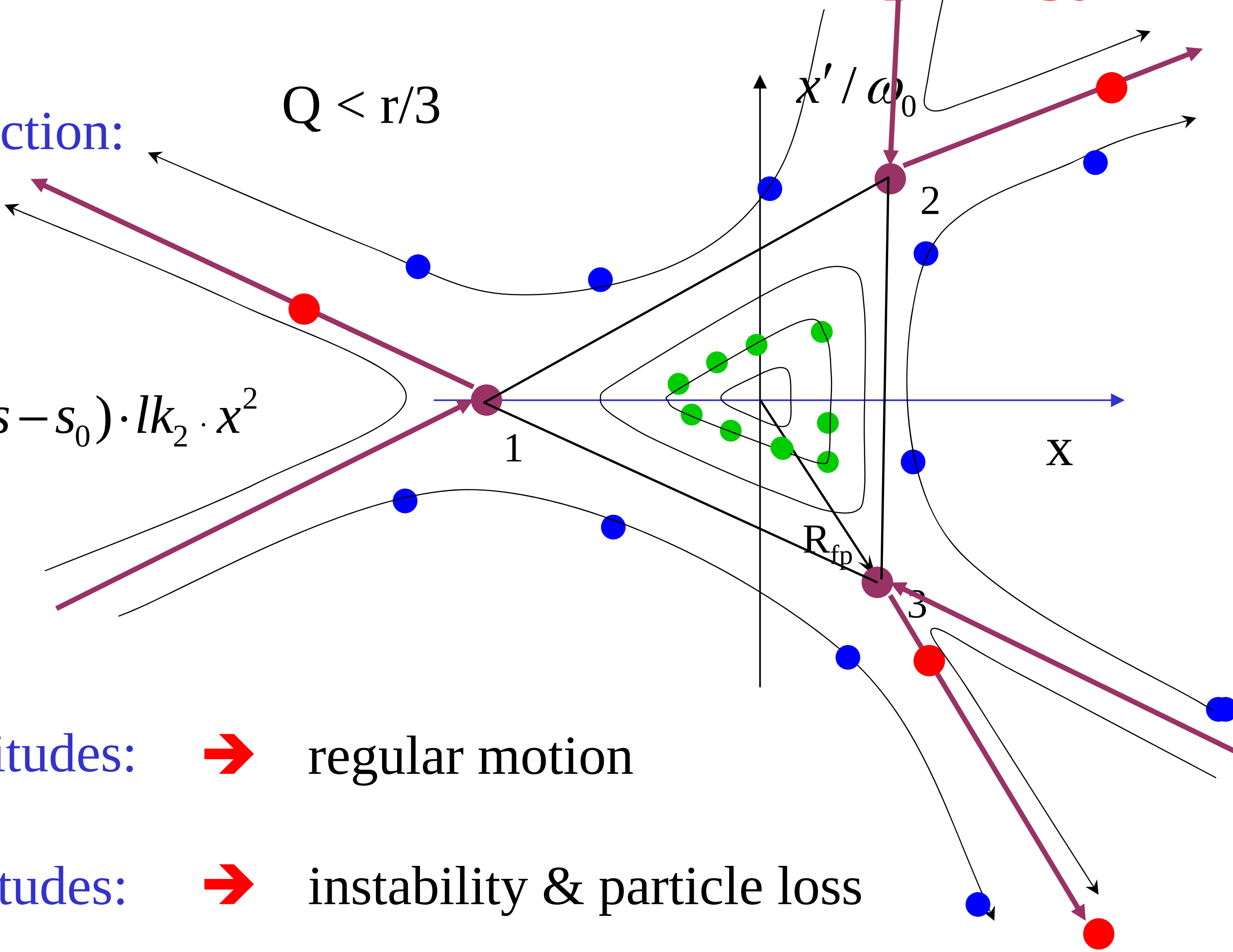
- sextupole kick:

amplitudes increases faster then the tune can change

➔ overall instability!

Poincare Section: Illustration of Topology

- Poincare section:



$$F(s)/(v\cdot p) = \frac{1}{2}\cdot\delta_L(s-s_0)\cdot lk_2\cdot x^2$$

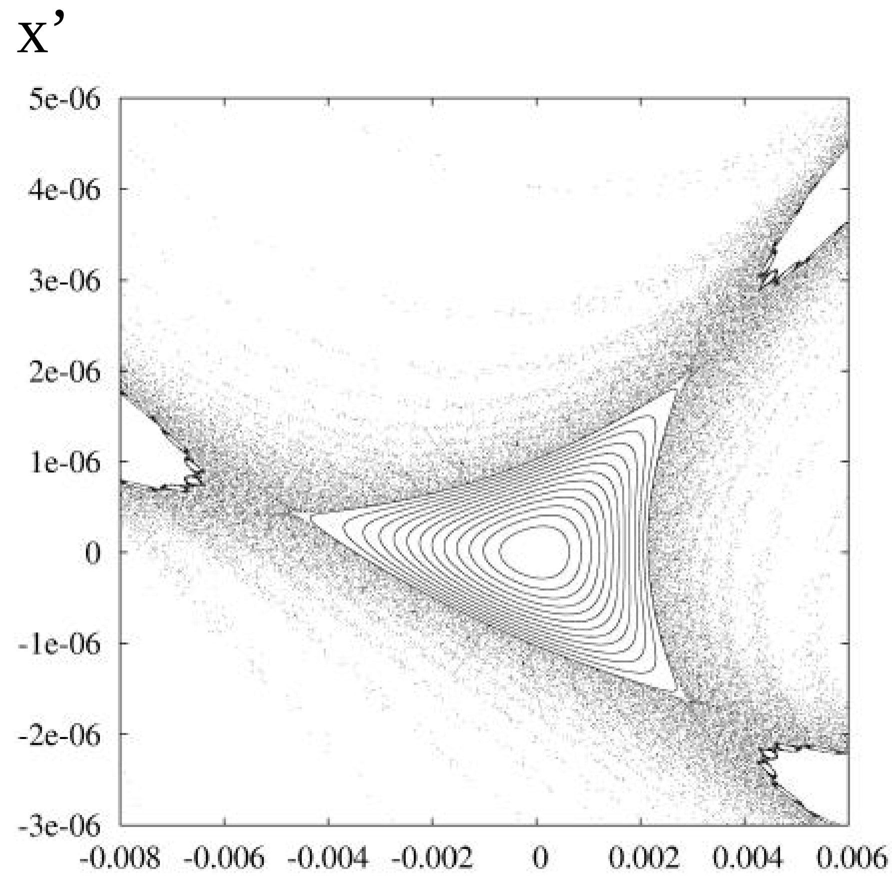
- small amplitudes: ➔ regular motion
- large amplitudes: ➔ instability & particle loss
- fixed points and seperatrix border between stable and unstable motion ➔ chaotic motion

Poincare Section: Simulatiosn for a Sextupole Perturbation

- Poincare Section right after the sextupole kick

→ for small amplitudes the intersections still lie on closed curves → regular motion!

→ separatrix location depends on the tune distance from the exact resonance condition ($Q < n/3$)



for large amplitudes and near the separatrix the intersections fill areas in the Poincare Section → chaotic motion;
→ no analytical solution exist!

Stabilization of Resonances

- instability can be fixed by stronger 'detuning':

  ➔ if the phase advance per turn changes uniformly with increasing R the motion moves off resonance and stabilizes

- octupole perturbation: $F(s)/(v \cdot p) = \frac{1}{6} \cdot lk_3 \cdot x^3$ → ε

- perturbation treatment: $x(2) = x_0(s) + \varepsilon \cdot x_1(2) + \ldots$

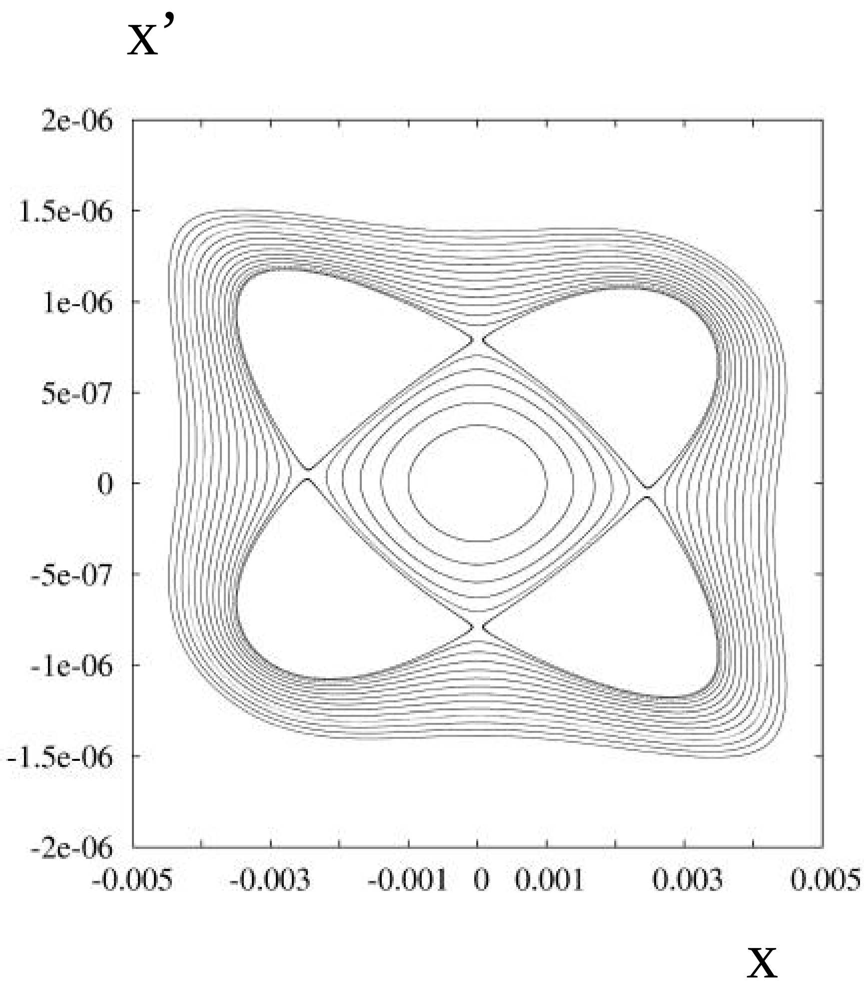
➔ $$\frac{d^2}{ds^2} x_1(s) + (2\pi Q_{x,0} / L)^2 \cdot x_1(s) = \frac{1}{6} \cdot lk_3 \cdot x_0^2 \cdot x_1$$

➔ $$x_0 = A \cdot \cos(\omega_0 \cdot s + \phi_0) \Rightarrow x_0^2 = \frac{A^2}{2} \cdot [1 + \cos(2\omega_0 \cdot s + 2\phi_0)]$$

$$\frac{d^2}{ds^2} x_1(s) + \left[(2\pi Q_{x,0} / L)^2 - \frac{A^2 \cdot lk_3}{2 \cdot 6}\right] \cdot x_1(s) = \frac{A^2 \cdot lk_3}{2 \cdot 6} \cdot \cos(2\omega_0 \cdot s) \cdot x_1$$

Stabilization of Resonances

- resonance stability for octupole:



➔ an octupole perturbation generate phase independent detuning and amplitude growth of the same order

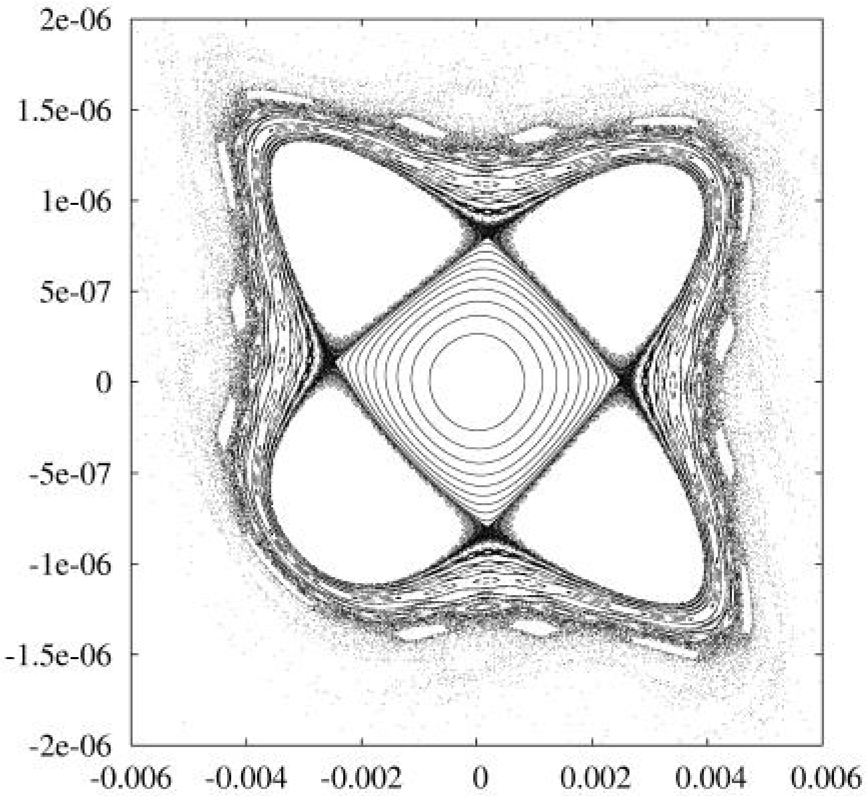
➔ amplitude growth and detuning are balanced and the overall motion is stable!

➔ this is not generally true in case of several resonance driving terms and coupling between the horizontal and vertical motion!

Chaotic Motion

octupole + sextupole perturbation:

➔ the interference of the octupole and sextupole perturbations generate additional resonances
➔additional island chains in the Poincare Section!

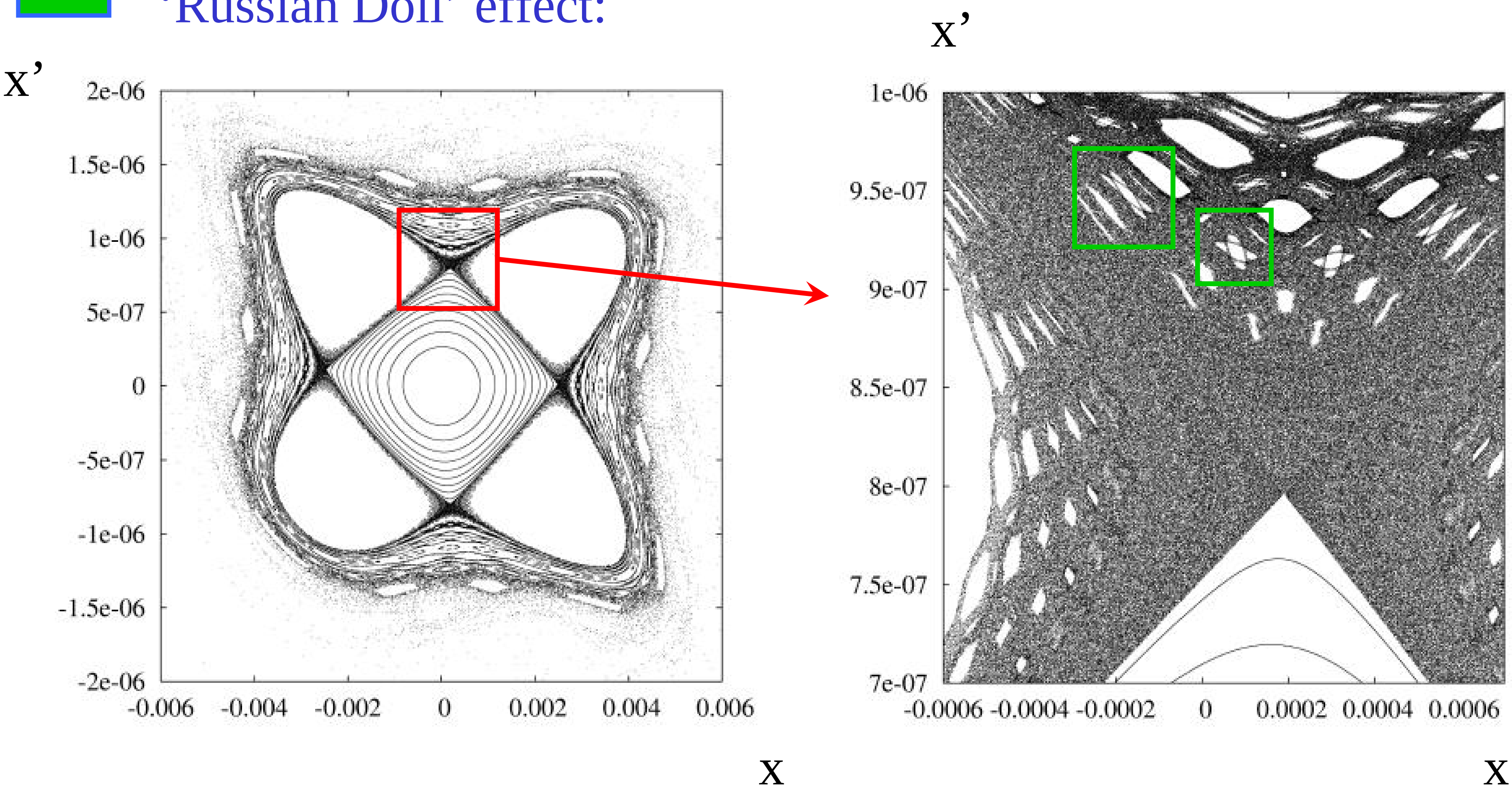


➔ intersections near the resonances lie no longer on closed curves ➔ local chaotic motion around the separatrix & instabilities
➔slow amplitude growth (Arnold diffusion)

➔ neighboring resonance islands start to 'overlap' for large amplitudes ➔ global chaos & fast instabilities

Chaotic Motion

‘Russian Doll’ effect:



➔ magnifying sections of the Poincare Section reveals always the same pattern on a finer scale ➔ renormalization theory!

Summary

- field imperfections drive resonances
- higher order than quadrupole field imperfections generate non-linear equations of motion (no closed analytical solution)

  (three body problem of Sun, Earth and Jupiter)

  → solutions only via perturbation treatment

- Poincare Section as a graphical tool for analyzing the stability
- slow extraction as example of resonance application in accelerator
- island chains as signature for non-linear resonances
- island overlap as indicator for globally chaotic & unstable motion