Non-linear dynamics

Phenomenology, applications and examples

Yannis PAPAPHILIPPOU

Accelerator and Beam Physics group

Beams Department

CERN

CERN Accelerator School

Advanced Accelerator Physics Course 2015

Warsaw, Poland

2-3 October 2015

Bibliography

The CERN Accelerator School

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- **Books on beam dynamics**
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- **Lectures on non-linear beam dynamics**
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Purpose of the Lectures

- Introduce "historical" approaches of non-linear dynamics (i.e. classical perturbation theory)
  - Show their usefulness
  - Demonstrate their practical limitation especially in beam dynamics
  - Connect naturally with the lectures of W.Herr on "Mathematical and Numerical methods", providing concrete examples
- Describe the phenomenology of non-linear dynamics (resonance, chaos, diffusion)
- Describe some tools for studying non-linear dynamics and detecting chaos (especially Frequency Map Analysis)
- Give a number of application examples

Non-linear effects

- Non-linear magnets, such as chromaticity sextupoles (especially in low emittance rings), octupoles,…
- Magnet imperfections and misalignments
- Insertion devices (wigglers, undulators) for synchrotron radiation storage rings
- Injection elements
- Magnet fringe fields (especially in high-intensity machines)
- Power supply ripple
- Ground motion (for e+/e-)
- Electron (Ion) cloud
- Beam-beam effect (for colliders)
- Space-charge effect (especially in high-intensity machines)

Non-linear effects affect performance

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- Performance impact
  - Reduced injection efficiency (especially in low emittance rings)
  - Particle losses causing
    - Reduced intensity and/or beam lifetime
    - Radio-activation (hands-on maintenance, equipment lifetime, super-conducting magnet quench)
    - Reduced machine availability
  - Emittance increase
  - Reduced number of bunches and/or increased crossing angle, affecting luminosity (for colliders)
  - Allow to damp instabilities (see W.Herr lecture on "Landau damping")
  - Can be used for beam extraction

…but also cost

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  - Emittance increase
  - Reduced number of bunches and/or increased crossing angle, affecting luminosity (for colliders)
  - Allow to damp instabilities (see W.Herr lecture on "Landau damping")
  - Can be used for beam extraction
- Cost issues
  - Magnetic field quality and alignment tolerances
  - Number of magnet correctors and families (power convertors)
  - Design of collimation system (for colliders and high-intensity machines)

Reminder of Hamiltonian formalism

Reminder of Hamiltonian formalism

- The **Hamiltonian** of the system is defined as the **Legendre** transformation of the Lagrangian

$$H(\mathbf{q},\mathbf{p},t) = \sum_i \dot{q}_i p_i - L(\mathbf{q},\dot{\mathbf{q}},t)$$

where the **generalised momenta** are $p_i = \dfrac{\partial L}{\partial \dot{q}_i}$

- The generalised velocities can be expressed as a function of the generalised momenta, if the previous equation is invertible, and thereby define the Hamiltonian of the system
- Example: consider $L(\mathbf{q},\dot{\mathbf{q}}) = \dfrac{1}{2}\sum_i m_i \dot{q}_i^2 - V(q_1,\ldots,q_n)$
- From this the momentum can be determined as $p_i = \dfrac{\partial L}{\partial \dot{q}_i} = m\dot{q}_i$

which can be trivially inverted to provide the Hamiltonian

$$H(\mathbf{q},\mathbf{p}) = \sum_i \frac{p_i^2}{2m_i} + V(q_1,\ldots,q_n)$$

Hamilton's equations

- The equations of motion can be derived from the Hamiltonian following the same variational principle as for the Lagrangian ("least" action) but also by simply taking the differential of the Hamiltonian

$$dH = \sum_i \cancel{p_i d\dot{q}_i} + \dot{q}_i dp_i - \cancel{\underbrace{\frac{\partial L}{\partial \dot{q}_i}}_{p_i} d\dot{q}_i} - \underbrace{\frac{\partial L}{\partial q_i}}_{\dot{p}_i} dq_i - \frac{\partial L}{\partial t} dt$$

or

$$dH = \sum_i \dot{q}_i dp_i - \dot{p}_i dq_i - \frac{\partial L}{\partial t} dt = \sum_i \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial t} dt$$

- By equating terms, **Hamilton's equations** are derived

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q}, \quad \frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t}$$

- These are indeed $2n+2$ equations describing the motion in the "extended" phase space $(q_i, \ldots, q_n, p_1, \ldots, p_n, t, -H)$

Properties of Hamiltonian flow

- The variables $(q_i, \ldots, q_n, p_1, \ldots, p_n, t, -H)$ are called **canonically conjugate** (or canonical) and define the evolution of the system in phase space
- These variables have the special property that they preserve volume in phase space, i.e. satisfy the well-known **Liouville's theorem**
- The variables used in the Lagrangian do not necessarily have this property
- Hamilton's equations can be written in vector form $\dot{\mathbf{z}} = \mathbf{J} \cdot \nabla H(\mathbf{z})$ with $\mathbf{z} = (q_i, \ldots, q_n, p_1, \ldots, p_n)$ and $\nabla = (\partial q_i, \ldots, \partial q_n, \partial p_1, \ldots, \partial p_n)$
- The $2n \times 2n$ matrix $\mathbf{J} = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{pmatrix}$ is called the **symplectic** matrix

Poisson brackets

- Crucial step in study of Hamiltonian systems is identification of integrals of motion
- Consider a time dependent function of phase space. Its time evolution is given by

$$\frac{d}{dt} f(\mathbf{p}, \mathbf{q}, t) = \sum_{i=1}^{n} \left(\frac{dq_i}{dt} \frac{\partial f}{\partial q_i} + \frac{dp_i}{dt} \frac{\partial f}{\partial p_i} \right) + \frac{\partial f}{\partial t}$$

$$= \sum_{i=1}^{n} \left(\frac{\partial H}{\partial p_i} \frac{\partial f}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial f}{\partial p_i} \right) + \frac{\partial f}{\partial t} = [H, f] + \frac{\partial f}{\partial t}$$

where $[H, f]$ is the **Poisson bracket** of f with H

- If a quantity is explicitly time-independent and its Poisson bracket with the Hamiltonian vanishes (i.e. commutes with the H), it is a **constant** (or **integral**) of motion (as an **autonomous** Hamiltonian itself)

Canonical tranformations

Canonical Transformations

- Find a function for transforming the Hamiltonian from variable $(\mathbf{q}, \mathbf{p})$ to $(\mathbf{Q}, \mathbf{P})$ so system becomes simpler to study
- This transformation should be **canonical** (or **symplectic**), so that the Hamiltonian properties of the system are preserved
- These "mixed variable" **generating** functions are derived by

$$F_1(\mathbf{q},\mathbf{Q}) : p_i = \frac{\partial F_1}{\partial q_i}, \quad P_i = -\frac{\partial F_1}{\partial Q_i} \qquad F_3(\mathbf{Q},\mathbf{p}) : q_i = -\frac{\partial F_3}{\partial p_i}, \quad P_i = -\frac{\partial F_3}{\partial Q_i}$$

$$F_2(\mathbf{q},\mathbf{P}) : p_i = \frac{\partial F_2}{\partial q_i}, \quad Q_i = \frac{\partial F_2}{\partial P_i} \qquad F_4(\mathbf{p},\mathbf{P}) : q_i = -\frac{\partial F_4}{\partial p_i}, \quad Q_i = \frac{\partial F_4}{\partial P_i}$$

- A general non-autonomous Hamiltonian is transformed to

$$H(\mathbf{Q},\mathbf{P},t) = H(\mathbf{q},\mathbf{p},t) + \frac{\partial F_j}{\partial t}, \quad j = 1, 2, 3, 4$$

- One generating function can be constructed by the other through Legendre transformations, e.g.

$$F_2(\mathbf{q},\mathbf{P}) = F_1(\mathbf{q},\mathbf{Q}) - \mathbf{Q}\cdot\mathbf{P}, \quad F_3(\mathbf{Q},\mathbf{p}) = F_1(\mathbf{q},\mathbf{Q}) - \mathbf{q}\cdot\mathbf{p}, \quad \ldots$$

with the inner product define as $\mathbf{q}\cdot\mathbf{p} = \sum_i q_i p_i$

Preservation of Phase Volume

- A fundamental property of canonical transformations is the preservation of phase space volume
- This volume preservation in phase space can be represented in the old and new variables as

$$\int \prod_{i=1}^{n} dp_i dq_i = \int \prod_{i=1}^{n} dP_i dQ_i$$

- The volume element in old and new variables are related through the Jacobian

$$\prod_{i=1}^{n} dp_i dq_i = \frac{\partial(P_1, \ldots, P_n, Q_1, \ldots, Q_n)}{\partial(p_1, \ldots, p_n, q_1, \ldots, q_n)} \prod_{i=1}^{n} dP_i dQ_i$$

- These two relationships imply that the Jacobian of a canonical transformation should have determinant equal to 1

$$\frac{\partial(P_1, \ldots, P_n, Q_1, \ldots, Q_n)}{\partial(p_1, \ldots, p_n, q_1, \ldots, q_n)} = \frac{\partial(p_1, \ldots, p_n, q_1, \ldots, q_n)}{\partial(P_1, \ldots, P_n, Q_1, \ldots, Q_n)} = 1$$

Examples of transformations

- The transformation $Q = -p \,, \quad P = q$, which interchanges conjugate variables is area preserving, as the Jacobian is

$$\frac{\partial(P,Q)}{\partial(p,q)} = \begin{vmatrix} \frac{\partial P}{\partial p} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial Q}{\partial q} \end{vmatrix} = \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} = 1$$

- On the other hand the transformation from polar to Cartesian coordinates $q = P\cos Q \,, \quad p = P\sin Q$ is not, since

$$\frac{\partial(q,p)}{\partial(Q,P)} = \begin{vmatrix} -P\sin Q & P\cos Q \\ \cos Q & \sin Q \end{vmatrix} = -P$$

- There are actually "polar" coordinates that are canonical, given by $q = -\sqrt{2P}\cos Q \,, \quad p = \sqrt{2P}\sin Q$ for which

$$\frac{\partial(q,p)}{\partial(Q,P)} = \begin{vmatrix} \sqrt{2P}\sin Q & \sqrt{2P}\cos Q \\ -\frac{\cos Q}{\sqrt{2P}} & \frac{\sin Q}{\sqrt{2P}} \end{vmatrix} = 1$$

The Relativistic Hamiltonian for electromagnetic fields

Single-particle relativistic Hamiltonian

- Neglecting self fields and radiation, motion can be described by a "single-particle" Hamiltonian

$$H(\mathbf{x},\mathbf{p},t) = c\sqrt{\left(\mathbf{p}-\frac{e}{c}\mathbf{A}(\mathbf{x},t)\right)^2 + m^2c^2} + e\Phi(\mathbf{x},t)$$

  - $\mathbf{x} = (x, y, z)$ Cartesian positions
  - $\mathbf{p} = (p_x, p_y, p_z)$ conjugate momenta
  - $\mathbf{A} = (A_x, A_y, A_z)$ magnetic vector potential
  - Φ electric scalar potential

- The ordinary kinetic momentum vector is written

$$\mathbf{P} = \gamma m \mathbf{v} = \mathbf{p} - \frac{e}{c}\mathbf{A}$$

with $\mathbf{v}$ the velocity vector and $\gamma = (1 - v^2/c^2)^{-1/2}$ the relativistic factor

Single-particle relativistic Hamiltonian

- It is generally a 3 degrees of freedom one plus time (i.e. 4 degrees of freedom)
- The Hamiltonian represents the total energy

$$H \equiv E = \gamma m c^2 + e\Phi$$

- The total kinetic momentum is

$$P = \left(\frac{H^2}{c^2} - m^2 c^2 \right)^{1/2}$$

- Using Hamilton's equations

$$(\dot{\mathbf{x}}, \dot{\mathbf{p}}) = [(\mathbf{x}, \mathbf{p}), H]$$

it can be shown that motion is governed by **Lorentz equations**

- Making a series of canonical transformations and approximations (see appendix)
  - From Cartesian to Frenet-Serret (rotating) coordinate system (bending in the horizontal plane)
  - Changing the independent variable from time to the path length s
  - Electric field set to zero, as longitudinal (synchrotron) motion is much slower then transverse (betatron) one
  - Consider static and transverse magnetic fields
  - Rescale the momentum and move the origin to the periodic orbit
  - For the ultra-relativistic limit $\beta_0 \to 1$, $\frac{1}{\beta_0^2\gamma^2} \to 0$ the Hamiltonian becomes

$$\mathcal{H}(x, y, l, p_x, p_y, \delta) = (1+\delta) - e\hat{A}_s - \left(1 + \frac{x}{\rho(s)}\right)\sqrt{(1+\delta)^2 - p_x^2 - p_y^2}$$

with $\frac{P_t - P_0}{P_0} \equiv \delta$

High-energy, large ring approximation

- Note that the Hamiltonian is non-linear even in the absence of any field component (i.e. for a drift)!
- Last approximation: transverse momenta (rescaled to the reference momentum) are considered to be much smaller than 1, i.e. the square root can be expanded. Considering also the large machine approximation $x << \rho$, (dropping cubic terms), the Hamiltonian is simplified to

$$\mathcal{H} = \frac{p_x^2 + p_y^2}{2(1+\delta)} - \frac{x(1+\delta)}{\rho(s)} - e\hat{A}_s$$

- This expansion may not be a good idea, especially for low energy, small size rings

Magnetic multipole expansion

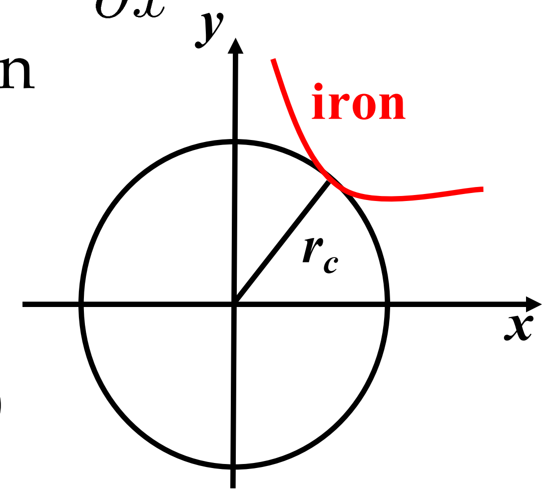
- From Gauss law of magnetostatics, a vector potential exist

$$\nabla \cdot \mathbf{B} = 0 \quad \rightarrow \quad \exists \mathbf{A} : \quad \mathbf{B} = \nabla \times \mathbf{A}$$

- Assuming transverse 2D field, vector potential has only one component A_s. The Ampere's law in vacuum (inside the beam pipe) $\nabla \times \mathbf{B} = 0 \quad \rightarrow \quad \exists V : \quad \mathbf{B} = -\nabla V$
- Using the previous equations, the relations between field components and potentials are

$$B_x = -\frac{\partial V}{\partial x} = \frac{\partial A_s}{\partial y}, \quad B_y = -\frac{\partial V}{\partial y} = -\frac{\partial A_s}{\partial x}$$

i.e. Riemann conditions of an analytic function



Exists complex potential of $z = x + iy$ with power series expansion convergent in a circle with radius $|z| = r_c$ (distance from iron yoke)

$$\mathcal{A}(x+iy) = A_s(x,y) + iV(x,y) = \sum_{n=1}^{\infty} \kappa_n z^n = \sum_{n=1}^{\infty} (\lambda_n + i\mu_n)(x+iy)^n$$

Multipole expansion II

- From the complex potential we can derive the fields

$$B_y + iB_x = -\frac{\partial}{\partial x}(A_s(x,y) + iV(x,y)) = -\sum_{n=1}^{\infty} n(\lambda_n + i\mu_n)(x+iy)^{n-1}$$

- Setting $b_n = -n\lambda_n \;, \quad a_n = n\mu_n$

$$B_y + iB_x = \sum_{n=1}^{\infty}(b_n - ia_n)(x+iy)^{n-1}$$

- Define normalized coefficients

$$b'_n = \frac{b_n}{10^{-4}B_0}r_0^{n-1} \;, \; a'_n = \frac{a_n}{10^{-4}B_0}r_0^{n-1}$$

on a reference radius r_0, 10^{-4} of the main field to get

$$B_y + iB_x = 10^{-4}B_0\sum_{n=1}^{\infty}(b'_n - ia'_n)(\frac{x+iy}{r_0})^{n-1}$$

- **Note:** $n' = n - 1$ is the US convention

Linear magnetic fields

- Assume a simple case of linear transverse magnetic fields,

$$B_x = b_2(s)y$$
$$B_y = -b_1(s) + b_2(s)x \quad ,$$

  - main bending field $-B \equiv b_1(s) = \frac{P_0 c}{e\rho(s)}$ [T]
  - normalized quadrupole gradient $K(s) = b_2(s)\frac{e}{cP_0} = \frac{b_2(s)}{B\rho}$ [1/m^2]
  - magnetic rigidity $B\rho = \frac{P_0 c}{e}$ [T $\cdot$ m]

- The vector potential has only a longitudinal component which in curvilinear coordinates is

$$B_x = -\frac{1}{1+\frac{x}{\rho(s)}}\frac{\partial A_s}{\partial y} \;, \quad B_y = \frac{1}{1+\frac{x}{\rho(s)}}\frac{\partial A_s}{\partial x}$$

- It can be integrated to give

$$A_s(x,y,s) = \frac{P_0 c}{e}\left[-\frac{x}{\rho(s)} - \left(\frac{1}{\rho(s)^2} + K(s)\right)\frac{x^2}{2} + K(s)\frac{y^2}{2}\right] = P_0 c\, \hat{A}_s(x,y,s)$$

The integrable Hamiltonian

- The Hamiltonian for linear fields can be finally written as

$$\mathcal{H} = \frac{p_x^2+p_y^2}{2(1+\delta)} - \frac{x\delta}{\rho(s)} + \frac{x^2}{2\rho(s)^2} + \frac{K(s)}{2}(x^2-y^2)$$

- Hamilton's equation are

$$\frac{dx}{ds} = \frac{p_x}{1+\delta}\,, \quad \frac{dp_x}{ds} = \frac{\delta}{\rho(s)} - \left(\frac{1}{\rho^2(s)} + K(s)\right)x$$

$$\frac{dy}{ds} = \frac{p_y}{1+\delta}\,, \quad \frac{dp_y}{ds} = K(s)y$$

and they can be written as two second order uncoupled differential equations, i.e. Hill's equations

$$x'' + \frac{1}{1+\delta}\overbrace{\left(\frac{1}{\rho(s)^2} + K(s)\right)}^{K_x}x = \frac{\delta}{\rho(s)}$$

$$y'' - \frac{1}{1+\delta}\underbrace{K(s)}_{K_y}y = 0$$

with the usual solution for $\delta = 0$ and $u = x, y$

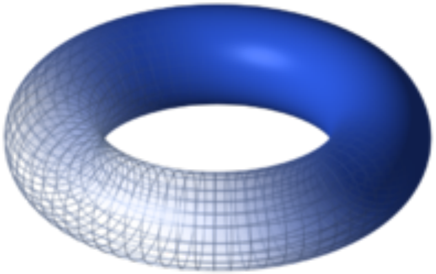
$$u(s) = \sqrt{\epsilon\beta(s)}\cos(\psi(s)+\psi_0)$$

$$u'(s) = \sqrt{\frac{\epsilon}{\beta(s)}}\left(\sin(\psi(s)+\psi_0) + \alpha(s)\cos(\psi(s)+\psi_0)\right)$$

Action-angle variables

- There is a canonical transformation to some optimal set of variables which can simplify the phase-space motion
- This set of variables are the **action-angle** variables
- The action vector is defined as the integral $\mathbf{J} = \oint \mathbf{p} d\mathbf{q}$ over closed paths in phase space.
- An integrable Hamiltonian is written as a function of only the actions, i.e. $H_0 = H_0(\mathbf{J})$. Hamilton's equations give

$$\dot{\phi}_i = \frac{\partial H_0(\mathbf{J})}{\partial J_i} = \omega_i(\mathbf{J}) \Rightarrow \phi_i = \omega_i(\mathbf{J})t + \phi_{i0}$$



$$\dot{J}_i = -\frac{\partial H_0(\mathbf{J})}{\partial \phi_i} = 0 \Rightarrow J_i = \text{const.}$$

i.e. the actions are integrals of motion and the angles are evolving linearly with time, with constant frequencies which depend on the actions

- The actions define the surface of an invariant torus, topologically equivalent to the product of n circles

Accelerator Hamiltonian in action-angle variables

- Considering on-momentum motion, the Hamiltonian can be written as

$$\mathcal{H} = \frac{p_x^2+p_y^2}{2} + \frac{K_x(s)x^2-K_y(s)y^2}{2}$$

- The generating function from the original to action angle variables is

$$F_1(x,y,\phi_x,\phi_y;s) = -\frac{x^2}{2\beta_x(s)}\left[\tan\phi_x(s)+a_x(s)\right] - \frac{y^2}{2\beta_y(s)}\left[\tan\phi_y(s)+a_y(s)\right]$$

- The old variables with respect to actions and angles are

$$u(s) = \sqrt{2\beta_u(s)J_u}\cos\phi_u(s)\ , \quad p_u(s) = -\sqrt{\frac{2J_u}{\beta_u(s)}}\left(\sin\phi_u(s)+\alpha_u(s)\cos\phi_u(s)\right)$$

and the Hamiltonian takes the form

$$\mathcal{H}_0(J_x,J_y,s) = \frac{J_x}{\beta_x(s)} + \frac{J_y}{\beta_y(s)}$$

- The "time" (path length) dependence can be eliminated by the transformation to normalized coordinates

$$\begin{pmatrix}\mathcal{U}\\ \mathcal{U}'\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{\beta}} & 0\\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta}\end{pmatrix}\begin{pmatrix}u\\ u'\end{pmatrix} \text{ or } \begin{pmatrix}\mathcal{U}\\ \mathcal{U}'\end{pmatrix} = \sqrt{2J}\begin{pmatrix}\cos(\nu\phi)\\ \sin(\nu\phi)\end{pmatrix} \text{ with } \nu = \frac{1}{2\pi}\oint\frac{du}{\beta(s)}$$

General non-linear Accelerator Hamiltonian

- Considering the general expression of the the longitudinal component of the vector potential is
  - In curvilinear coordinates (curved elements)

$$A_s = (1 + \frac{x}{\rho(s)})B_0 \Re e \sum_{n=1}^{\infty} \frac{b_n + i a_n}{n}(x+iy)^n$$

  - In Cartesian coordinates $A_s = B_0 \Re e \sum_{n=1}^{\infty} \frac{b_n + i a_n}{n}(x+iy)^n$

with the multipole coefficients being written as

$$a_n = \frac{1}{B_0\, n!} \frac{\partial^n B_x}{\partial x^n}\Big|_{x=y=0} \quad \text{and} \quad b_n = \frac{1}{B_0\, n!} \frac{\partial^n B_y}{\partial x^n}\Big|_{x=y=0}$$

- The general non-linear Hamiltonian can be written as

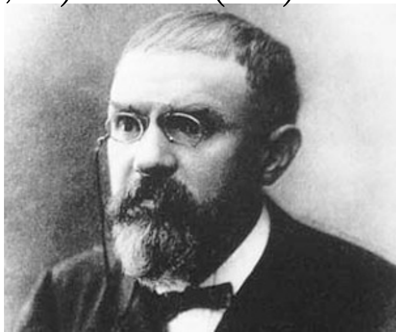
$$\mathcal{H}(x,y,p_x,p_y,s) = \mathcal{H}_0(x,y,p_x,p_y,s) + \sum_{k_x,k_y} h_{k_x,k_y}(s) x^{k_x} y^{k_y}$$

with the periodic functions $h_{k_x,k_y}(s) = h_{k_x,k_y}(s+C)$

Canonical perturbation theory

Canonical perturbation theory

- Consider a general Hamiltonian with n degrees of freedom
  $$H(\boldsymbol{J},\boldsymbol{\varphi},\theta) = H_0(\mathbf{J}) + \epsilon H_1(\boldsymbol{J},\boldsymbol{\varphi},\theta) + \mathcal{O}(\epsilon^2)$$
  where the non-integrable part $H_1(\boldsymbol{J},\boldsymbol{\varphi},\theta)$ is 2π-periodic on the angles $\boldsymbol{\varphi}$ and the "time" θ
- Provided that ϵ is sufficiently small, tori should still exist but they are distorted
- We seek a canonical transformation that could "straighten up" the tori, i.e. it could transform the non-integrable part of the Hamiltonian (at first order in ϵ) to a function only of some new actions $\bar{H}(\bar{\boldsymbol{J}})$ plus higher orders in ϵ
- This can be performed by a mixed variable close to identity generating function $S(\bar{\boldsymbol{J}},\boldsymbol{\varphi},\theta) = \bar{\boldsymbol{J}}\cdot\boldsymbol{\varphi} + \epsilon S_1(\bar{\boldsymbol{J}},\boldsymbol{\varphi},\theta) + \mathcal{O}(\epsilon^2)$ for transforming old variables to new ones $(\bar{\boldsymbol{J}},\bar{\boldsymbol{\varphi}})$
- In principle, this procedure can be carried to arbitrary powers of the perturbation



Canonical perturbation theory

- By the canonical transformation equations (slide 12), the old action and new angle can be also represented by a power series in ϵ

$$\boldsymbol{J} = \bar{\boldsymbol{J}} + \epsilon \frac{\partial S_1(\bar{\boldsymbol{J}}, \boldsymbol{\varphi}, \theta)}{\partial \boldsymbol{\varphi}} + \mathcal{O}(\epsilon^2) \qquad \boldsymbol{J} = \bar{\boldsymbol{J}} + \epsilon \frac{\partial S_1(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\varphi}}, \theta)}{\partial \bar{\boldsymbol{\varphi}}} + \mathcal{O}(\epsilon^2)$$

$$\bar{\boldsymbol{\varphi}} = \boldsymbol{\varphi} + \epsilon \frac{\partial S_1(\bar{\boldsymbol{J}}, \boldsymbol{\varphi}, \theta)}{\partial \bar{\boldsymbol{J}}} + \mathcal{O}(\epsilon^2) \quad \text{or} \quad \boldsymbol{\varphi} = \bar{\boldsymbol{\varphi}} - \epsilon \frac{\partial S_1(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\varphi}}, \theta)}{\partial \bar{\boldsymbol{J}}} + \mathcal{O}(\epsilon^2)$$

- The previous equations expressing the old as a function of the new variables assume that there is possibility to invert the equation on the left, so that $S_1(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\varphi}}, \theta)$ becomes a function of the new variables
- The new Hamiltonian is then

$$\bar{H}(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\varphi}}, \theta) = H(\boldsymbol{J}(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\varphi}}), \boldsymbol{\varphi}(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\varphi}}), \theta) + \epsilon \frac{\partial S_1(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\varphi}}, \theta)}{\partial \theta} + \mathcal{O}(\epsilon^2)$$

- The second term is appearing because of the "time" dependence through θ

Form of the generating function

- The question is what is the form of the generating function that eliminates the angle dependence
- The procedure is cumbersome (see appendix for details), but here is the final result,

$$S(\bar{\boldsymbol{J}},\bar{\boldsymbol{\varphi}}) = \bar{\boldsymbol{J}}\cdot\bar{\boldsymbol{\varphi}} + \epsilon i \sum_{\mathbf{k}\neq\mathbf{0}} \frac{H_{1\mathbf{k}}(\bar{\mathbf{J}})}{\boldsymbol{k}\cdot\boldsymbol{\nu}(\bar{\boldsymbol{J}})+p} e^{i(\boldsymbol{k}\cdot\bar{\boldsymbol{\varphi}}+p\theta)} + \mathcal{O}(\epsilon^2)$$

with the frequency vector $\boldsymbol{\nu}(\bar{\boldsymbol{J}}) = \dfrac{\partial H_0(\bar{\boldsymbol{J}})}{\partial \bar{\boldsymbol{J}}}$
and the integers $\boldsymbol{k}, p \neq \mathbf{0}$

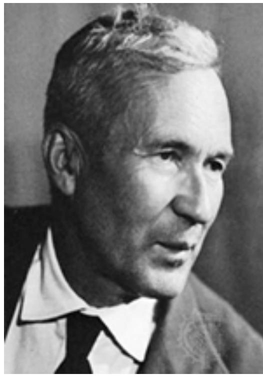
- If the denominator vanishes, i.e. for the **resonance condition** $\boldsymbol{k}\cdot\boldsymbol{\nu}(\bar{\boldsymbol{J}})+p=0$, the Fourier series coefficients (**driving terms**) become infinite
- It actually implies that even at first order in the perturbation parameter and in the vicinity of a resonance, it is impossible to construct a generating function for seeking some approximate integrals of motion

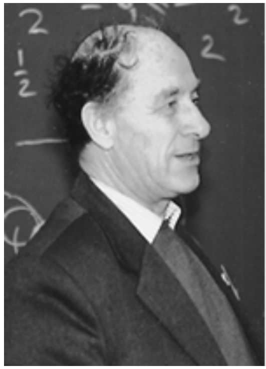
Problem of small denominators

- In principle, the technique works for arbitrary order, but the disentangling of variables becomes difficult even to 2nd order!!!
- The solution was given in the late 60s by introducing the **Lie transforms** (e.g. see Deprit 1969), which are algorithmic for constructing generating functions and were adapted to beam dynamics by Dragt and Finn (1976)
- The problem of **small denominators** due to resonances is not just a mathematical one. The inability to construct solutions close to a resonance has to do with the un-predictable nature of motion and the onset of chaos
- **KAM theory** developed the mathematical framework into which local solutions could be constructed provided some general conditions on the size of the perturbation and the distance of the system from resonances are satisfied

KAM theory

- Original idea of **Kolmogorov** (1954) (super-convergent series expansion) later proved by **Arnold** (1963) and **Moser** (1962)
- If a Hamiltonian system is subjected to weak nonlinear perturbation, some invariant tori are deformed and survive
- Trajectories starting on one of these tori remain on it thereafter, executing quasi-periodic motion with a fixed frequency vector depending only on the torus.
- The family of tori is parameterized over a Cantor set of frequency vectors, while in the gaps of the Cantor set chaotic behavior can occur
- The **KAM theorem** specifies quantitatively the size of the perturbation for this to be true.
- The **KAM tori** that survive are those that have "sufficiently irrational" frequencies
- The conditions of the KAM theorem become increasingly difficult to satisfy for systems with more degrees of freedom. As the number of dimensions of the system increases, the volume occupied by the tori decreases
- A complement of KAM theory for the stability of dynamical systems were given by **Nekhoroshev** (1971) who proved that if the density of tori is large all solutions will stay close to the tori for exponentially long times showing practical stability of motion





Perturbation treatment for a sextupole

- Consider the simple case of a periodic sextupole perturbation and restrict the study only to one plane. The Hamiltonian is written as,

$$H(x,p_x,s) = \frac{p_x^2+k_1(s)x^2}{2} + \frac{k_2(s)x^3}{6}$$

where $k_1(s)$ and $k_2(s)$ are periodic functions.

- We proceed to the transformation in action angle variables to write the Hamiltonian in the form

$$H = H_0(J) + H_1(\phi,J) = \frac{J}{\beta(s)} + \frac{\sqrt{2}k_2(s)}{3}(J\beta(s))^{3/2}\cos^3\phi = \frac{J}{\beta(s)} + \frac{\sqrt{2}k_2(s)}{12}(J\beta(s))^{3/2}(\cos 3\phi + 3\cos\phi)$$

- The perturbation procedure implies to split the perturbation in an average part over the angles and an oscillating part

$$H_1 = \langle H_1\rangle_\phi + \{H_1\} = \frac{\sqrt{2}k_2(s)}{12}(J\beta(s))^{3/2}(\cos 3\phi + 3\cos\phi)$$

Tuneshift from a sextupole

- The average part should be only a function of the action
- Its derivative with respect to the action should provide the frequency shift (tune-shift) due to the non-linearity
- It can be shown that this quantity vanishes for a sextupole perturbation

$$\langle\frac{\partial H_1(\phi, J)}{\partial J}\rangle_\phi = \frac{k_2(s)\beta(s)}{8\sqrt{2}\pi}\,(J\beta(s))^{1/2}\int_0^{2\pi}(\cos 3\phi + 3\cos\phi)d\phi = 0$$

- Sextupoles do not provide any tune-shift at **first order**
- But we know by experience that this is not true, i.e. first order perturbation theory fails to give the correct answer
- One has to go to **higher order** (see appendix)

Perturbation treatment for a sextupole

- By rescaling the independent variable, the close to identity generating function is written as

$$S(\bar{J},\phi,\theta) = \bar{J}\cdot\phi + S_1(\bar{J},\phi,\theta) + \dots$$

- Following the perturbation steps, the generating function has to be chosen such that the following relationship is satisfied $\frac{\partial S_1(\bar{J},\phi,\theta)}{\partial\theta} + \nu(\bar{J})\cdot\frac{\partial S_1(\bar{J},\phi,\theta)}{\partial\phi} = -\{H_1(\bar{J},\phi,\theta)\}$ with

$$\{H_1\} = H_1 = \frac{\sqrt{2}k_2(s)}{12}\left(\bar{J}\beta(s)\right)^{3/2}(\cos 3\phi + 3\cos\phi)$$

- Following the canonical perturbation procedure the generating function is

$$S(\bar{J},\phi) = \bar{J}\cdot\phi + i\sum_{k,p\neq 0}\frac{H_{1k}(\bar{J})}{k\cdot\nu(\bar{J}) + p}e^{i(k\cdot\phi+p\theta)} + \dots$$

- The only non-zero coefficients are for $k = 1, 3$ and

$$S(\bar{J},\phi) = \bar{J}\cdot\phi + i\frac{K_s(s)}{6\sqrt{2}}\left(\bar{J}\beta(s)\right)^{3/2}\sum_{p=-\infty}^{\infty}\left(\frac{e^{i(3\phi+p\theta)}}{3\nu+p} + \frac{3e^{i(\phi+p\theta)}}{\nu+p}\right)$$

Perturbation treatment for more sextupoles

- The previous formula can be generalised for more sextupoles
- First, expand both perturbation and generating function in Fourier series of the form

$$S_1(\bar{J},\phi,\theta)=\sum_k S_{1k}(\bar{J},\theta)e^{ik\phi} \quad \text{and} \quad \{H_1(\bar{J},\phi,\theta)\}=\sum_k H_{1k}(\bar{J},\theta)e^{ik\phi}$$

- The equation relating the amplitudes is now

$$i\ k\ \nu\ S_{1k}+\frac{\partial S_{1k}}{\partial\theta}=-H_{1k}$$ and can be solved yielding

$$S_{1k}=\frac{i}{2\sin(\pi k\nu)}\int_\theta^{\theta+2\pi}H_{1k}e^{ik\nu(\theta'-\theta-\pi)}d\theta'$$

- Using these as Fourier coefficients, the generating function is

$$S_1=\sum_k\frac{i}{2\sin(\pi k\nu)}\int_\theta^{\theta+2\pi}H_{1k}e^{ik[\phi+\nu(\theta'-\theta-\pi)]}d\theta'$$

- For sextupoles, and letting $\psi(s)=\int_0^s\frac{ds'}{\beta(s')}$ we have

$$S_1=-\frac{\bar{J}^{3/2}}{4\sqrt{2}}\int_s^{s+C}k_2(s')\beta(s')^{3/2}\left[\frac{\sin(\phi+\psi(s')-\psi(s)-\pi\nu)}{\sin(\pi\nu)}+\frac{\sin 3(\phi+\psi(s')-\psi(s)-\pi\nu)}{3\sin(3\pi\nu)}\right]ds'$$

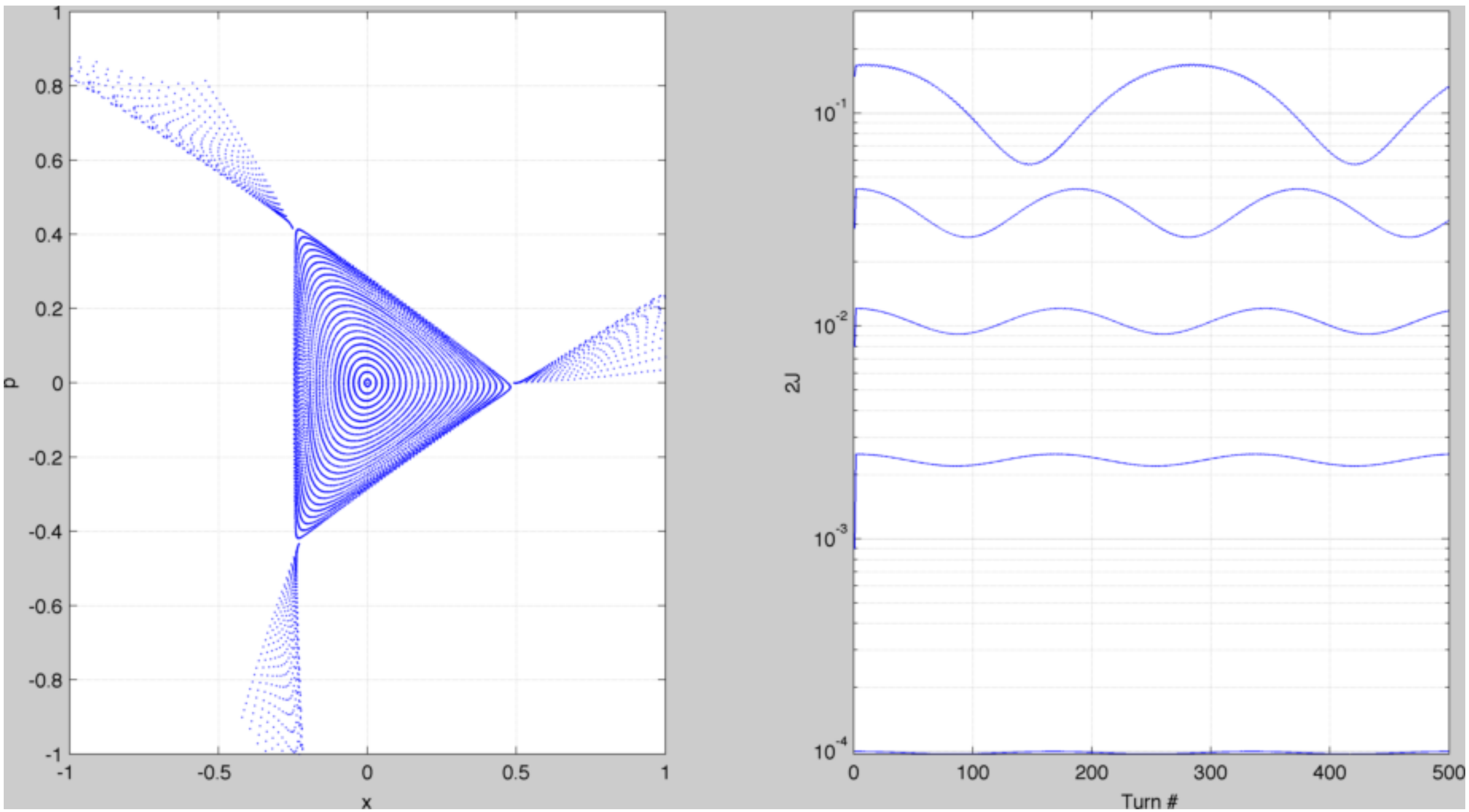
Perturbation treatment for a sextupole

- We derived (with a lot of effort) the common result that sextupoles **at first order** excite **integer** and **third integer** resonances
- Again this is not generally true! It is known that sextupoles can drive **any resonance** (either if they are large enough, or if the particle is far away from the closed orbit)
- This can be shown again by pursuing the perturbation approach to **second order** (as for the tune-shift)
- A useful application is to use the generating function for computing the correction to the original invariant, as the new one should be an integral of motion (at first order)

$$J \approx \bar{J} + \frac{\partial S_1(\bar{J}, \varphi, \theta)}{\partial \varphi}$$

Phase space for sextupole perturbation

- For small perturbations, the new action variable is almost an invariant but for larger ones phase space gets deformed
- Close to the integer or third integer resonance, canonical perturbation theory cannot be applied
- The solution is provided by **secular perturbation theory**



General accelerator Hamiltonian

- The general accelerator Hamiltonian is written as

$$\mathcal{H}(x,y,p_x,p_y,s) = \mathcal{H}_0(x,y,p_x,p_y,s) + \sum_{k_x,k_y} h_{k_x,k_y}(s)x^{k_x}y^{k_y}$$

- The transverse coordinated can be expressed in action-angle variables as

$$u(s) = \sqrt{\frac{J_u\beta_u(s)}{2}}\left(e^{i(\phi_u(s)+\theta_u(s))} + e^{-i(\phi_u(s)+\theta_u(s))}\right)$$

- The Hamiltonian in action-angle variables is

$$\mathcal{H}'(J_x,J_y,\phi_x,\phi_y) = H_0(J_x,J_y) + H_1(J_x,J_y,\phi_x,\phi_y)$$

  - The integrable part $H_0(J_x,J_y) = \frac{1}{R}(\nu_x J_x + \nu_y J_y)$
  - The perturbation

$$H_1(J_x,J_y,\phi_x,\phi_y;s) = \sum_{k_x,k_y} J_x^{k_x/2} J_y^{k_y/2} \sum_j^{k_x}\sum_l^{k_y} g_{j,k,l,m}(s)e^{i[(j-k)\phi_x+(l-m)\phi_y]}$$

- The coefficients $g_{j,k,l,m}(s) = \frac{h_{k_x,k_y}(s)}{2^{\frac{j+k+l+m}{2}}}\binom{k_x}{j}\binom{k_y}{l}\beta_x^{k_x/2}(s)\beta_y^{k_y/2}(s)e^{i[(j-k)\theta_x(s)+(l-m)\theta_y(s)]}$ depend on the optics, with the indexes $k_x = j+k\ ,\quad k_y = l+m$

Resonance driving terms

- As the coefficients $h_{k_x,k_y}(s)$ are periodic, the perturbation can be expanded in Fourier series

$$H_1(J_x, J_y, \phi_x, \phi_y; \theta) = \sum_{k_x,k_y} J_x^{k_x/2} J_y^{k_y/2} \sum_j^{k_x} \sum_l^{k_y} \sum_{p=-\infty}^{\infty} g_{j,k,l,m;p} e^{i[(j-k)\phi_x+(l-m)\phi_y-p\theta]}$$

with the **resonance driving terms**

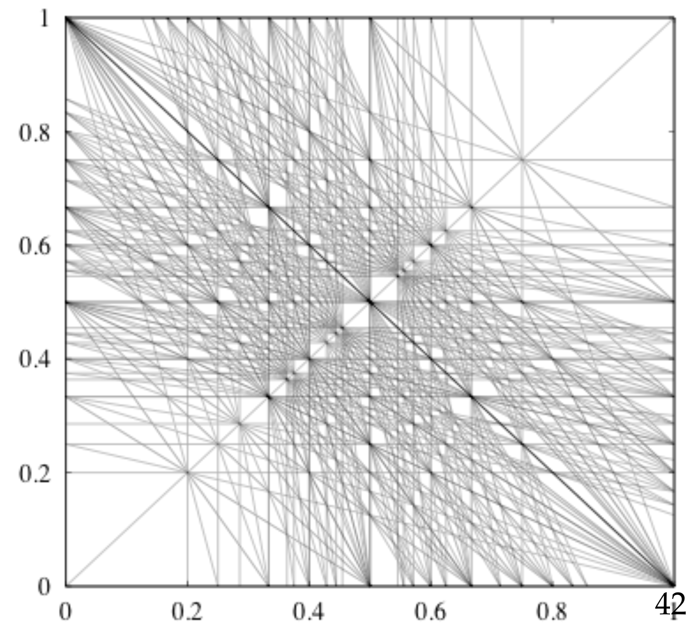
$$g_{j,k,l,m;p} = \binom{k_x}{j}\binom{k_y}{l} \frac{1}{2^{\frac{j+k+l+m}{2}}} \frac{1}{2\pi} \oint h_{k_x,k_y}(s)\beta_x^{k_x/2}(s)\beta_y^{k_y/2}(s) e^{i[(j-k)\phi_x(s)+(l-m)\phi_y(s)+p\theta]}$$

- For $n_x = j - k$, $n_y = l - m$, resonance conditions appear for $n_x\nu_x + n_y\nu_y = p$
- Goal of accelerator design and correction systems is to minimize the resonance driving terms
  - Change magnet design so that $h_{k_x,k_y}(s)$ become smaller
  - Introduce magnetic elements capable of creating a cancelling effect
  - Sort magnets or non-linear elements in a way that phase terms are minimised

General resonance conditions

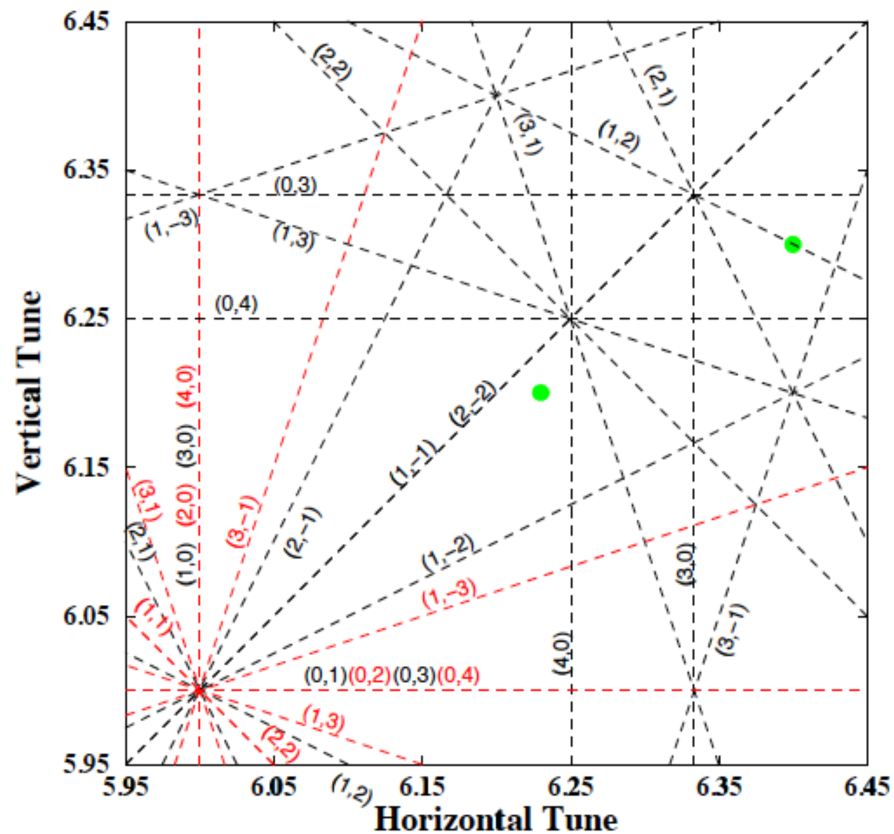
- The general resonance conditions is $n_x\nu_x + n_y\nu_y = p$ with order $n_x + n_y$
- For all the polynomial field terms of a $2m$-pole, the excited resonances (**at first order**) satisfy the condition $n_x + n_y = m$ but there are also **sub-resonances** for which $n_x + n_y < m$
- For **normal** (erect) multi-poles, the resonances (**at first order**) are $(n_x, n_y) = (m, 0), \ (m-2, \pm 2), \ldots$ whereas for **skew** multi-poles $(n_x, n_y) = (m-1, \pm 1), \ (m-3, \pm 3), \ldots$

- If perturbation is large, **all** resonances can be potentially excited
- The resonance conditions form lines in the frequency space and fill it up as the order grows (the rational numbers form a dense set inside the real numbers)



Systematic and random resonances

- If lattice is made out of N identical cells, and the perturbation follows the same periodicity, resulting in a reduction of the resonance conditions to the ones satisfying $n_x\nu_x + n_y\nu_y = jN$
- These are called **systematic** resonances
- Practically, any (linear) lattice perturbation breaks super-periodicity and any **random** resonance can be excited
- Careful choice of the working point is necessary



Tune-shift and tune-spread

- First order correction to the tunes is computed by the derivatives with respect to the action of the average part of perturbation. For a given term, $h_{k_x,k_y}(s)x^{k_x}y^{k_y}$ the leading order correction to the tunes are

$$\delta\nu_x = \frac{J_x^{k_x/2-1} J_y^{k_y/2}}{4\pi^2} \sum_j^{k_x} \sum_l^{k_y} \bar{g}_{j,k,l,m} \oint e^{i[(j-k)\phi_x+(l-m)\phi_y]}$$

$$\delta\nu_y = \frac{J_x^{k_x/2} J_y^{k_y/2-1}}{4\pi^2} \sum_j^{k_x} \sum_l^{k_y} \bar{g}_{j,k,l,m} \oint e^{i[(j-k)\phi_x+(l-m)\phi_y]}$$

where $\bar{g}_{j,k,l,m}$ is the average of $g_{j,k,l,m}(s)$ around the ring.

- In the accelerator jargon if $\delta\nu_{x,y}$ is independent of the action, it is referred to as tune-shift, whereas, if it depends on the action, it is called tune-spread (or amplitude detuning)
- At **first order**, $\delta\nu_{x,y} = 0$, for odd normal and all skew multi-pole (trigonometric functions give zero averages)

Lie Transformations and normal forms

Reminder: Symplectic maps

- Consider two sets of canonical variables $\mathbf{z}$, $\bar{\mathbf{z}}$ which may be even considered as the evolution of the system between two points in phase space
- A transformation from the one to the other set can be constructed through a map $\mathcal{M} : \mathbf{z} \mapsto \bar{\mathbf{z}}$
- The Jacobian matrix of the map $M = M(\mathbf{z}, t)$ is composed by the elements $M_{ij} \equiv \frac{\partial \bar{z}_i}{\partial z_j}$
- The map is symplectic if $M^T J M = J$ where $J = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{pmatrix}$
- It can be shown that the variables defined through a symplectic map $[\bar{z}_i, \bar{z}_j] = [z_i, z_j] = J_{ij}$ which is a known relation satisfied by canonical variables
- In other words, symplectic maps **preserve** Poisson brackets
- Symplectic maps provide a very useful framework to represent and analyze motion through an accelerator

Reminder: Lie formalism

- The Poisson bracket properties satisfy what is mathematically called a **Lie** algebra
- They can be represented by (Lie) operators of the form $:f:g = [f,g]$ and $:f:^2 g = [f,[f,g]]$ etc.
- For a Hamiltonian system $H(\mathbf{z},t)$ there is a **formal solution** of the equations of motion $\frac{d\mathbf{z}}{dt} = [H,\mathbf{z}] =: H : \mathbf{z}$ written as $\mathbf{z}(t) = \sum_{k=0}^{\infty} \frac{t^k :H:^k}{k!}\mathbf{z}_0 = e^{t:H:}\mathbf{z}_0$ with a symplectic map $\mathcal{M} = e^{:H:}$
- The 1-turn accelerator map can be represented by the composition of the maps of each element $\mathcal{M} = e^{:f_2:}\, e^{:f_3:}\, e^{:f_4:} \ldots$ where f_i (called the generator) is the Hamiltonian for each element, a polynomial of degree m in the variables $z_1, \ldots, z_n$

Example: The $-I$ transformer

- Consider two identical sextupoles in a beam line represented by a map $\mathcal{R}$
- The sextupole map can be represented at second order as

$$\mathcal{S}_2 = e^{-\frac{1}{2}L_s:H_d:}e^{-L_s:H_s:}e^{-\frac{1}{2}L_s:H_d:}$$

with the sextupole effective Hamiltonian $H_s = \frac{1}{6}k_2(x^3 - 3xy^2)$
and H_d the drift Hamiltonian

- The total map can be approximated at 2nd order by

$$\mathcal{M} = \mathcal{SRS} \approx \mathcal{RS}_2\mathcal{R} = e^{-\frac{1}{2}L_s:H_d:}e^{-L_s:H_s:}\bar{\mathcal{R}}e^{-L_s:H_s:}e^{-\frac{1}{2}L_s:H_d:}$$

with the map $\bar{\mathcal{R}} = e^{-\frac{1}{2}L_s:H_d:}\mathcal{R}e^{-\frac{1}{2}L_s:H_d:}$

- Taking into account that $\bar{\mathcal{R}}\bar{\mathcal{R}}^{-1} = \mathcal{I}$ and the similarity transformation $\bar{\mathcal{R}}^{-1}e^{-L_s:H_s:}\bar{\mathcal{R}} = e^{-L_s:\bar{\mathcal{R}}^{-1}H_s:}$ the map can be rewritten as

$$\mathcal{M} \approx e^{-\frac{1}{2}L_s:H_d:}\bar{\mathcal{R}}e^{-L_s:\bar{\mathcal{R}}^{-1}H_s:}e^{-L_s:H_s:}e^{-\frac{1}{2}L_s:H_d:}$$

Example: The $-I$ transformer

- If the map $\bar{\mathcal{R}}$ is chosen such that $\bar{\mathcal{R}}^{-1} H_s = -H_s$ then the sextupole map Lie operators

$$e^{-L_s:\bar{\mathcal{R}}^{-1} H_s:} e^{-L_s:H_s:} = e^{L_s:H_s:} e^{-L_s:H_s:} = \mathcal{I}$$

- In that way, the sextupole non-linearity is getting eliminated in the final map

$$\mathcal{M} \approx e^{-\frac{1}{2} L_s:H_d:} \bar{\mathcal{R}} e^{-\frac{1}{2} L_s:H_d:} = e^{-L_s:H_d:} \mathcal{R} e^{-L_s:H_d:}$$

- Inspecting the form of H_s this can be achieved if

$$\bar{\mathcal{R}} x = -x, \qquad \bar{\mathcal{R}} p_x = -p_x, \qquad \bar{\mathcal{R}} y = \pm y, \qquad \bar{\mathcal{R}} p_y = \pm p_y$$

or in matrix form

$$\bar{\mathcal{R}} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & \pm 1 & 0 \\ 0 & 0 & 0 & \pm 1 \end{pmatrix} = \begin{pmatrix} \cos\mu_x + a_x \sin\mu_x & b_x \sin\mu_x & 0 & 0 \\ -c_x \sin\mu_x & \cos\mu_x - a_x \sin\mu_x & 0 & 0 \\ 0 & 0 & \cos\mu_y + a_y \sin\mu_y & b_y \sin\mu_y \\ 0 & 0 & -c_y \sin\mu_y & \cos\mu_y - a_y \sin\mu_y \end{pmatrix}$$

- The horizontal part of the matrix is $-\mathcal{I}_2$ and the vertical part is $\pm\mathcal{I}_2$, which is obtained for phase advances

$$\mu_x = (2n_x + 1)\pi, \qquad \mu_y = n_y \pi$$

Normal forms

- Normal forms consists of finding a canonical transformation of the 1-turn map, so that it becomes simpler to analyze
- In the linear case, the Floquet transformation is a kind a normal form as it turns **ellipses** into **circles**
- The transformation can be written formally as

$$
\begin{array}{ccc}
\mathbf{z} & \xrightarrow{\mathcal{M}} & \mathbf{z}' \\
\Phi^{-1}\downarrow & & \downarrow\Phi^{-1} \\
\boldsymbol{u} & \xrightarrow[\mathcal{N}]{} & \boldsymbol{u}'
\end{array}
$$

with the original map $\mathcal{M} = \Phi^{-1} \circ \mathcal{N} \circ \Phi$ and its normal form

$$\mathcal{N} = \Phi \circ \mathcal{M} \circ \Phi^{-1} = e^{:h_{eff}:}$$

- The transformation $\Phi = e^{:F:}$ is better suited in action angle variables, i.e. $\zeta = e^{-:F_r:}\,\mathbf{h}$ taking the system from the original action-angle $h^{\pm}_{x,y} = \sqrt{2J_{x,y}}\, e^{\mp i\phi_{x,y}}$ to a new set $\zeta^{\pm}_{x,y}(N) = \sqrt{2I_{x,y}}\, e^{\mp i\psi_{x,y}(N)}$ with the angles being just simple rotations, $\psi_{x,y}(N) = 2\pi N\nu_{x,y} + \psi_{x,y_0}$ and the new effective Hamiltonian depends only on the new actions

Effective Hamiltonian

- The generating function can be written as a polynomial in the new actions, i.e.

$$F_r = \sum_{jklm} f_{jklm} {\zeta_x^+}^j {\zeta_x^-}^k {\zeta_y^+}^l {\zeta_y^-}^m = f_{jklm} (2I_x)^{\frac{j+k}{2}} (2I_y)^{\frac{l+m}{2}} e^{-i\psi_{jklm}}$$

- There are **software tools** that built this transformation
- Once the "new" effective Hamiltonian is known, all interesting quantities can be derived
- This Hamiltonian is a function only of the new actions, and to 3rd order it is obtained as

$$\begin{aligned} h_{eff} =& \nu_x I_x + \nu_y I_y \\ & + \frac{1}{2}\alpha_c \delta^2 + c_{x1} I_x \delta + c_{y1} I_y \delta + c_3 \delta^3 \\ & + c_{xx} I_x^2 + c_{xy} I_x I_y + c_{yy} I_y^2 + c_{x2} I_x \delta^2 + c_{y2} I_y \delta^2 c_4 \delta^4 \end{aligned}$$

Effective Hamiltonian

- The correction of the tunes is given by

$$Q_x = \frac{1}{2\pi}\frac{\partial h_{eff}}{\partial I_x} = \frac{1}{2\pi}\left(\nu_x + 2c_{xx}I_x + c_{xy}I_y + c_{x1}\delta + c_{x2}\delta^2\right)$$

$$Q_y = \frac{1}{2\pi}\frac{\partial h_{eff}}{\partial I_y} = \frac{1}{2\pi}\left(\nu_y + 2c_{yy}I_y + c_{xy}I_x + c_{y1}\delta + c_{y2}\delta^2\right)$$

tunes **tune-shift with amplitude** **1st and 2nd order chromaticity**

- The correction to the path length is

$$\Delta s = \frac{\partial h_{eff}}{\partial \delta} = \alpha_c\delta + c_3\delta^2 + 4c_4\delta^3 + c_{x1}I_x + c_{y1}I_y + 2c_{x2}I_x\delta + 2c_{y2}I_y\delta$$

1st, 2nd and 3rd momentum compaction

Normal form for perturbation

- Using the BCH formula, one can prove that the composition of two maps with g small can be written as

$$e^{:f:}e^{:g:} = \exp\left[:f + \left(\frac{:f:}{1-e^{-:f:}}\right) g + \mathcal{O}(g^2):\right]$$

- Consider a linear map (rotation) followed by a small perturbation $\mathcal{M} = e^{:f_2:}e^{:f_3:}$
- We are seeking for transformation such that

$$\mathcal{N} = \Phi\mathcal{M}\Phi^{-1} = e^{:F:}e^{:f_2:}e^{:f_3:}e^{:-F:}$$

- This can be written as

$$\begin{aligned}\mathcal{N} &= e^{:f_2:}e^{-:f_2:}e^{:F:}e^{:f_2:}e^{:f_3:}e^{:-F:} \\ &= e^{:f_2:}e^{:e^{-:f_2:}F + f_3 - F:} + \dots \\ &= e^{:f_2:}e^{:(e^{-:f_2:}-1)F + f_3:} + \dots\end{aligned}$$

$$F = \frac{f_3}{1-e^{-:f_2:}}$$

- This will transform the new map to a rotation to leading order

Example: Octupole

- Consider a linear map followed by an octupole

$$\mathcal{M} = e^{-\frac{\nu}{2}:x^2+p^2:}e^{:\frac{x^4}{4}:} = e^{:f_2:}e^{:\frac{x^4}{4}:}$$

- The generating function has to be chosen such as to make the following expression simpler

$$(e^{-:f_2:}-1)F+\frac{x^4}{4}$$

- The simplest expression is the one that the angles are eliminated and there is only dependence on the action
- We pass to the resonance basis variables

$$h^{\pm} = \sqrt{2J}\,e^{\mp i\phi} = x \mp ip$$

- The perturbation is

$$x^4 = (h_+ + h_-)^4 = h^{\pm} = h_+^4 + 4h_+^3h_- + 6h_+^2h_-^2 + 4h_+h_-^3 + h_-^4$$

Example: Octupole

- The term $6h_+^2 h_-^2 = 24J^2$ is independent on the angles. Thus we may choose the generating functions such that the other terms are eliminated. It takes the form

$$F = \frac{1}{16}\left(\frac{h_+^4}{1-e^{4i\nu}} + \frac{4h_+^3 h_-}{1-e^{2i\nu}} + \frac{4h_+ h_-^3}{1-e^{2i\nu}} + \frac{h_-^4}{1-e^{4i\nu}}\right)$$

- The map is now written as

$$\mathcal{M} = e^{-:F:} e^{:\nu J + \frac{3}{8}J^2:} e^{:F:}$$

- The new effective Hamiltonian is depending only on the actions and contains the tune-shift terms
- The generator in the original variables is written as

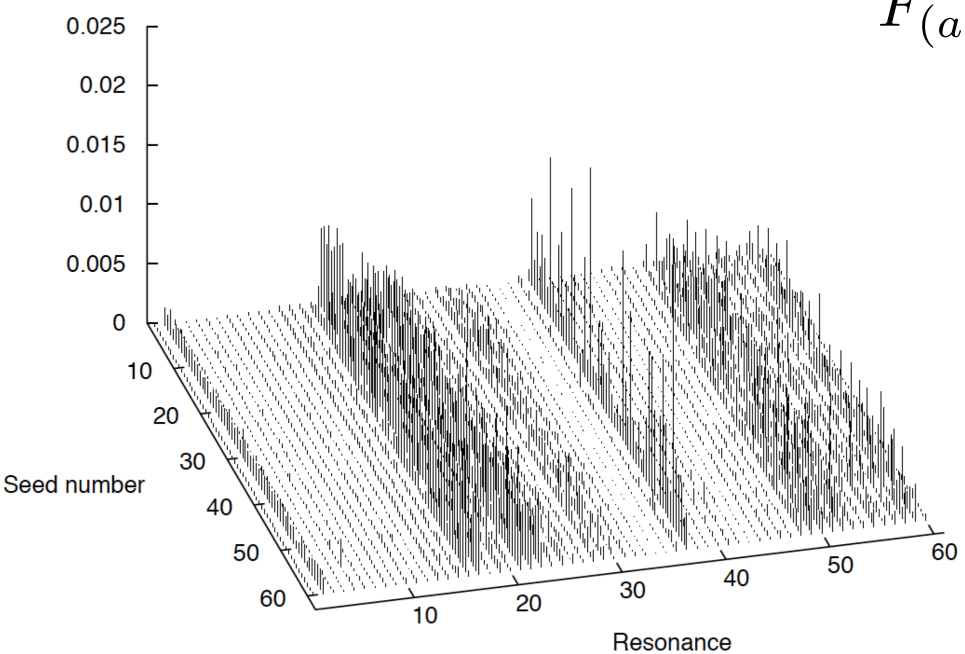
$$F = -\frac{1}{64}\left[-5x^4 + 3p^4 + 6x^2p^2 + 4x^3p(2\cot(\nu) + \cot(2\nu)) + 4xp^3(2\cot(\nu) - \cot(2\nu))\right]$$

- Constant values of the generator describe the trajectories in phase space

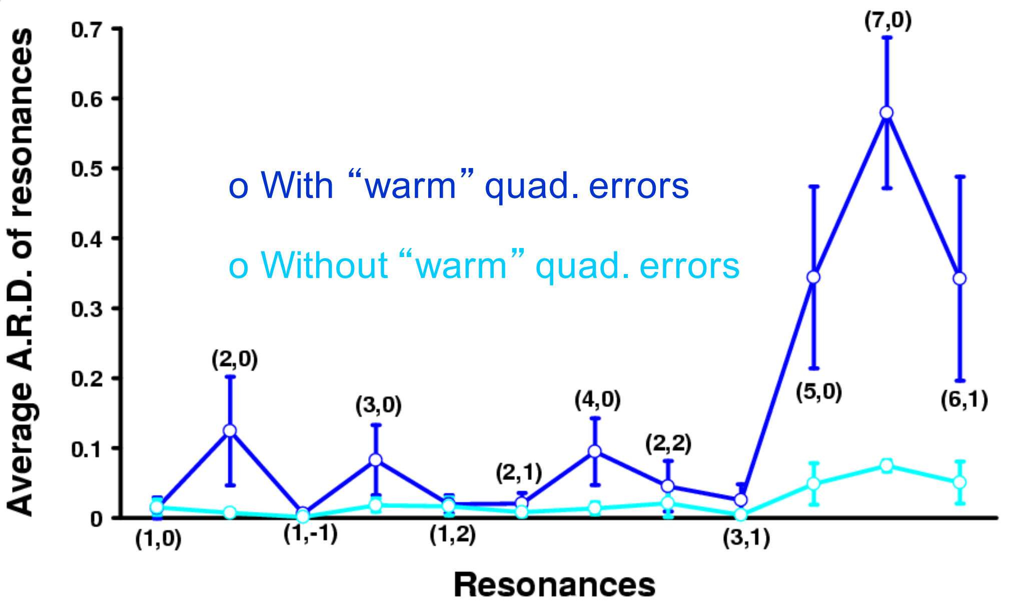
Graphical resonance representation

- It is possible by constructing the one turn map to built the generating (sometimes called "**distortion**") function $F_r \approx \sum_{jklm} f_{jklm}\, J_x^{\frac{j+k}{2}}\, J_y^{\frac{l+m}{2}}\, e^{-i\psi_{jklm}}$
- For any resonance $a\nu_x + bq_y = c$, and setting $\psi_{jklm} = 0$, the associated part of the functions is

$$F_{(a,b)} \approx \sum_{\substack{jklm \\ j+k+l+m\leq n \\ j+k=a\ ,\ l+m=b}} f_{jklm}\, J_x^{\frac{a}{2}}\, J_y^{\frac{b}{2}}$$



Normal forms for LHC models



Phase	Type	DA (σ)	LHC Version		
			4	5	
				Nominal	Target
15°	Warm Quads switched ON	Average	10.0	9.1	10.4
		Minimum	8.5	7.4	8.6
	Warm Quads switched OFF	Average	10.7	11.6	12.4
		Minimum	9.6	10.3	11.3
45°	Warm Quads switched ON	Average	11.1	11.3	12.8
		Minimum	9.5	9.2	11.4
	Warm Quads switched OFF	Average	11.4	12.4	13.8
		Minimum	10.1	10.7	12.3

- In the LHC at injection (450 GeV), beam stability is necessary over a very large number of turns (10^7)
- Stability is reduced from random multi-pole imperfections mainly in the super-conducting magnets
- Area of stability (Dynamic aperture - DA) computed with particle tracking for a large number of random magnet error distributions
- Numerical tool based on normal form analysis (GRR) permitted identification of DA reduction reason (errors in the "warm" quadrupoles)

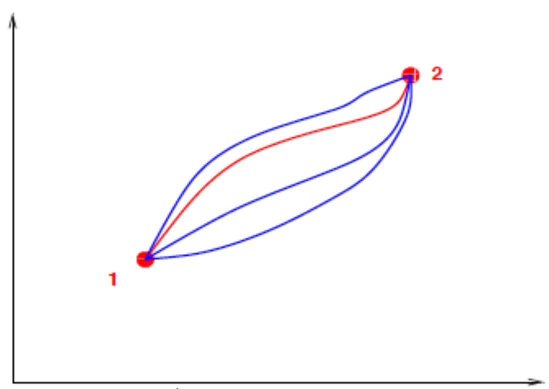
Summary of the 1st lecture

- **Hamiltonian formalism** provides the **natural framework** to analyse (linear and non-linear) beam dynamics
- **Canonical (symplectic) transformations** enable to move from variables describing a distorted phase space to something simpler (ideally circles)
- The **generating functions** passing from the old to the new variables are bounded to **diverge** in the vicinity of **resonances** (emergence of chaos, see 2nd lecture)
- Calculating this generating function with **canonical perturbation theory** becomes **hopeless** for higher orders
- Representing the accelerator (or beam line) like a composition of maps (through **Lie transformations**) enables derivation of the generating functions in an **algorithmic way**, in principle to **arbitrary order**

Appendix

Lagrangian formalism

- Describe the motion of particles in q_n coordinates (n degrees of freedom from time t_1 to time t_2
- Describe motion by the Lagrangian function $L(q_1,\ldots,q_n,\dot{q}_1,\ldots,\dot{q}_n,t)$ with $(q_1,\ldots,q_n)$ the generalized coordinates and $(\dot{q}_1,\ldots,\dot{q}_n)$ the generalized velocities
- The Lagrangian function defined as $L = T - V$, i.e. difference between kinetic and potential energy
- The integral $I = \int L(q_i,\dot{q}_i,t)dt$ defines the action
- **Hamilton's principle**: system evolves so as the action becomes extremum (principle of **stationary action**)



Lagrange equations

- The variation of the action can be written as

$$\delta W = \int_{t_1}^{t_2} \left(L(q+\delta q, \dot{q}+\delta\dot{q}, t) - L(q,\dot{q},t)\right) dt = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta\dot{q}\right) dt$$

- Taking into account that $\delta\dot{q} = \frac{d\delta q}{dt}$, the 2nd part of the integral can be integrated by parts giving

$$\delta W = \left|\frac{\partial L}{\partial \dot{q}}\delta q\right|_{t_1}^{t_2} + \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\right)\delta q dt = 0$$

- The first term is zero because $\delta q(t_1) = \delta q(t_2) = 0$ so the second integrant should also vanish providing the following differential equations for each degree of freedom, the **Lagrange equations**

$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0}$$

Lagrangian mechanics

- For a simple force law contained in a potential function, governing motion among interacting particles, the Lagrangian is (or as Landau-Lifshitz put it "experience has shown that…")

$$L = T - V = \sum_{i=1}^{n} \frac{1}{2} m_i q_i^2 - V(q_1, \ldots, q_n)$$

- For velocity independent potentials, Lagrange equations become

$$m_i \ddot{q}_i = -\frac{\partial V}{\partial q_i}$$

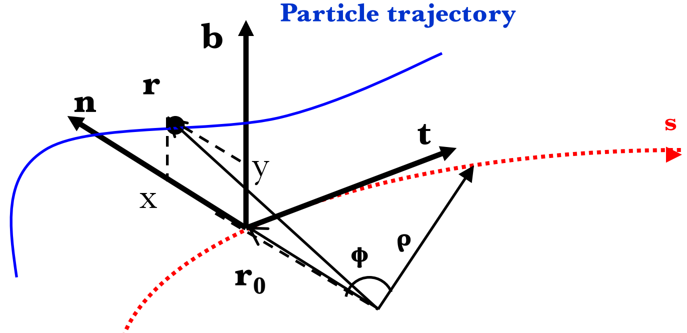
i.e. Newton's equations.

From Lagrangian to Hamiltonian

- Some disadvantages of the Lagrangian formalism:
  - Not uniqueness: different Lagrangians can lead to same equations
  - Physical significance not straightforward (even its basic form given more by "experience" and the fact that it actually works that way!)
- Lagrangian function provides in general n second order differential equations (coordinate space)
- We already observed the advantage to move to a system of $2n$ first order differential equations, which are more straightforward to solve (phase space)
- These equations can be derived by the Hamiltonian of the system

From Cartesian to "curved" coordinates

- It is useful (especially for rings) to transform the Cartesian coordinate system to the Frenet-Serret system moving to a closed curve, with path length s



- The position coordinates in the two systems are connected by $\mathbf{r} = \mathbf{r_0}(s) + X\mathbf{n}(s) + Y\mathbf{b}(s) = x\mathbf{u_x} + y\mathbf{u_y} + z\mathbf{u_z}$
- The Frenet-Serret unit vectors and their derivatives are defined as $(\mathbf{t}, \mathbf{n}, \mathbf{b}) = (\frac{d}{ds}\mathbf{r_0}(s), -\rho(s)\frac{d^2}{ds^2}\mathbf{r_0}(s), \mathbf{t} \times \mathbf{n})$

$$\frac{d}{ds}\begin{pmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{\rho(s)} & 0 \\ 0 & 0 & \tau(s) \\ \frac{1}{\rho(s)} & 0 & -\tau(s)\end{pmatrix}\begin{pmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{pmatrix}$$

with $\rho(s)$ the radius of curvature and $\tau(s)$ the torsion which vanishes in case of planar motion

From Cartesian to "curved" variables

- We are seeking a canonical transformation between

$$\begin{array}{rcl}(\mathbf{q},\mathbf{p}) & \mapsto & (\mathbf{Q},\mathbf{P}) \text{ or} \\ (x,y,z,p_x,p_y,p_z) & \mapsto & (X,Y,s,P_x,P_y,P_s)\end{array}$$

- The generating function is

$$(\mathbf{q},\mathbf{P}) = -\left(\frac{\partial F_3(\mathbf{p},\mathbf{Q})}{\partial \mathbf{p}}, \frac{\partial F_3(\mathbf{p},\mathbf{Q})}{\partial \mathbf{Q}}\right)$$

- By using the relationship between the coordinates, the generating function is

$$F_3(\mathbf{p},\mathbf{Q}) = -\mathbf{p}\cdot\mathbf{r} + \overline{F_3}(\mathbf{Q}) = -\mathbf{p}\cdot\mathbf{r}$$

and, for planar motion, the momenta are

$$\mathbf{P} = (P_X, P_Y, P_s) = \mathbf{p}\cdot\left(\mathbf{n}, \mathbf{b}, (1+\frac{X}{\rho})\mathbf{t}\right)$$

- Finally, the new Hamiltonian is given by

$$\mathcal{H}(\mathbf{Q},\mathbf{P},t) = c\sqrt{(P_X - \frac{e}{c}A_X)^2 + (P_Y - \frac{e}{c}A_Y)^2 + \frac{(P_s - \frac{e}{c}A_s)^2}{(1+\frac{X}{\rho(s)})^2} + m^2c^2} + e\Phi(\mathbf{Q})$$

Changing of the independent variable

- It is more convenient to use s, instead of the time as the independent variable
- First, note that the Hamiltonian can be considered as a 4 degree of freedom, where the 4th coordinate is time and its conjugate momentum is $P_t = -\mathcal{H}$
- In the same way the new Hamiltonian with the path length as the independent variable is just $P_s = -\tilde{\mathcal{H}}(X, Y, t, P_X, P_Y, P_t, s)$ with

$$\tilde{\mathcal{H}} = -\frac{e}{c}A_s - \left(1 + \frac{X}{\rho(s)}\right)\sqrt{(\frac{P_t + e\Phi}{c})^2 - m^2c^2 - (P_x - \frac{e}{c}A_X)^2 - (P_Y - \frac{e}{c}A_Y)^2}$$

- It can be proved that this is indeed a canonical transformation
- Note the existence of the reference orbit for zero vector potential, for which $(X, Y, P_X, P_Y, P_s) = (0, 0, 0, 0, P_0)$

Neglecting electric fields

- Due to the fact that longitudinal (synchrotron) motion is much slower then the transverse (betatron) one, the electric field can be set to zero and the Hamiltonian is written as

$$\tilde{\mathcal{H}} = -\frac{e}{c}A_s - \left(1+\frac{X}{\rho(s)}\right)\sqrt{\underbrace{(\frac{\mathcal{H}}{c})^2 - m^2c^2}_{P^2} - (P_x - \frac{e}{c}A_X)^2 - (P_Y - \frac{e}{c}A_Y)^2}$$

- The Hamiltonian is then written as

$$\tilde{\mathcal{H}} = -\frac{e}{c}A_s - \left(1+\frac{X}{\rho(s)}\right)\sqrt{(P^2 - (P_x - \frac{e}{c}A_X)^2 - (P_Y - \frac{e}{c}A_Y)^2}$$

- If **static** magnetic fields are considered, the time dependence is also dropped, and the system is 2 degrees of freedom + “time” (path length)

Momentum rescaling

- Due to the fact that total momentum is much larger then the transverse ones, another transformation may be considered, where the transverse momenta are rescaled

$$(\mathbf{Q},\mathbf{P}) \quad \mapsto \quad (\bar{\mathbf{q}},\bar{\mathbf{p}}) \;\; \text{or}$$

$$(X,Y,t,P_X,P_Y,P_t) \quad \mapsto \quad (\bar{x},\bar{y},\bar{t},\bar{p}_x,\bar{p}_y,\bar{p}_t) = (X,Y,-c\,t,\frac{P_X}{P_0},\frac{P_Y}{P_0},-\frac{P_t}{P_0 c})$$

- The new variables are indeed canonical if the Hamiltonian is also rescaled and written as

$$\bar{\mathcal{H}}(\bar{x},\bar{y},\bar{t},\bar{p}_x,\bar{p}_y,\bar{p}_t) = \frac{\tilde{\mathcal{H}}}{P_0} = -e\bar{A}_s - \left(1+\frac{\bar{x}}{\rho(s)}\right)\sqrt{\bar{p}_t^2 - \frac{m^2c^2}{P_0} - (\bar{p}_x - e\bar{A}_x)^2 - (\bar{p}_y - e\bar{A}_y)^2}$$

with $\quad (\bar{A}_x,\bar{A}_y,\bar{A}_z) = \dfrac{1}{P_0\,c}(A_x,A_y,A_s)$

and $\quad \dfrac{m^2c^2}{P_0} = \dfrac{1}{\beta_0^2\gamma_0^2}$

Moving the reference frame

- Along the reference trajectory $\bar{p}_{t0} = \frac{1}{\beta_0}$ and
$$\frac{d\bar{t}}{ds}\Big|_{P=P_0} = \frac{\partial \bar{H}}{\partial \bar{p}_t}\Big|_{P=P_0} = -\bar{p}_{t0} = -\frac{1}{\beta_0}$$
- It is thus useful to move the reference frame to the reference trajectory for which another canonical transformation is performed
$$(\bar{\mathbf{q}}, \bar{\mathbf{p}}) \quad \mapsto \quad (\hat{\mathbf{q}}, \hat{\mathbf{p}}) \;\; \text{or}$$
$$(\bar{x}, \bar{y}, \bar{t}, \bar{p}_x, \bar{p}_y, \bar{p}_t) \quad \mapsto \quad (\hat{x}, \hat{y}, \hat{t}, \hat{p}_x, \hat{p}_y, \hat{p}_t) = (\hat{x}, \hat{y}, \bar{t} + \frac{s - s_0}{\beta_0}, \hat{p}_x, \hat{p}_y, \bar{p}_t - \frac{1}{\beta_0})$$
- The mixed variable generating function is
$$(\hat{\mathbf{q}}, \bar{\mathbf{p}}) = (\frac{\partial F_2(\bar{\mathbf{q}}, \hat{\mathbf{p}})}{\partial \hat{\mathbf{p}}}, \frac{\partial F_2(\bar{\mathbf{q}}, \hat{\mathbf{p}})}{\partial \bar{\mathbf{q}}}) \;\; \text{providing}$$
$$F_2(\bar{\mathbf{q}}, \hat{\mathbf{p}}) = \bar{x}\hat{p}_x + \bar{y}\hat{p}_y + (\bar{t} + \frac{s - s_0}{\beta_0})(\hat{p}_t + \frac{1}{\beta_0})$$
- The Hamiltonian is then
$$\hat{\mathcal{H}}(\hat{x}, \hat{y}, \hat{t}, \hat{p}_x, \hat{p}_y, \hat{p}_t) = \frac{1}{\beta_0}(\frac{1}{\beta_0} + \hat{p}_t) - e\hat{A}_s - \left(1 + \frac{\hat{x}}{\rho(s)}\right)\sqrt{(\hat{p}_t + \frac{1}{\beta_0})^2 - \frac{1}{\beta_0^2\gamma_0^2} - (\hat{p}_x - e\hat{A}_x)^2 - (\hat{p}_y - e\bar{A}_y)^2}$$

Relativistic and transverse field approximations

- First note that $\hat{p}_t = \bar{p}_t - \frac{1}{\beta_0} = \bar{p}_t - \bar{p}_{t0} = \frac{P_t - P_0}{P_0} \equiv \delta$ and $l = \hat{t}$
- In the ultra-relativistic limit $\beta_0 \to 1$, $\frac{1}{\beta_0^2\gamma^2} \to 0$ and the Hamiltonian is written as

$$\mathcal{H}(x,y,l,p_x,p_y,\delta) = (1+\delta) - e\hat{A}_s - \left(1 + \frac{x}{\rho(s)}\right)\sqrt{(1+\delta)^2 - (p_x - e\hat{A}_x)^2 - (p_y - e\hat{A}_y)^2}$$

where the "hats" are dropped for simplicity

- If we consider only transverse field components, the vector potential has only a longitudinal component and the Hamiltonian is written as

$$\mathcal{H}(x,y,l,p_x,p_y,\delta) = (1+\delta) - e\hat{A}_s - \left(1 + \frac{x}{\rho(s)}\right)\sqrt{(1+\delta)^2 - p_x^2 - p_y^2}$$

- Note that the Hamiltonian is non-linear even in the absence of any field component (i.e. for a drift)!

Canonical perturbation theory

- Expand term by term the Hamiltonian $H(\boldsymbol{J}(\bar{\boldsymbol{J}},\bar{\boldsymbol{\varphi}}),\boldsymbol{\varphi}(\bar{\boldsymbol{J}},\bar{\boldsymbol{\varphi}}),\theta)$ to leading order in ϵ

$$H_0(\boldsymbol{J}(\bar{\boldsymbol{J}},\bar{\boldsymbol{\varphi}})) = H_0(\bar{\boldsymbol{J}}) + \epsilon \frac{\partial H_0(\bar{\boldsymbol{J}})}{\partial \bar{\boldsymbol{J}}} \frac{\partial S_1(\bar{\boldsymbol{J}},\bar{\boldsymbol{\varphi}},\theta)}{\partial \bar{\boldsymbol{\varphi}}} + \mathcal{O}(\epsilon^2)$$

$$\epsilon H_1(\boldsymbol{J}(\bar{\boldsymbol{J}},\bar{\boldsymbol{\varphi}}),\boldsymbol{\varphi}(\bar{\boldsymbol{J}},\bar{\boldsymbol{\varphi}}),\theta) = \epsilon H_1(\bar{\boldsymbol{J}},\bar{\boldsymbol{\varphi}}) + \mathcal{O}(\epsilon^2)$$

- The new Hamiltonian can also be expanded in orders of ϵ

$$\bar{H} = \bar{H}_0 + \epsilon \bar{H}_1 + \ldots$$

- Equating the terms of equal orders in ϵ , we obtain
  - Zero order $\bar{H}_0 = H_0(\bar{\boldsymbol{J}})$
  - First order $\bar{H}_1 = \frac{\partial S_1(\bar{\boldsymbol{J}},\bar{\boldsymbol{\varphi}},\theta)}{\partial \theta} + \boldsymbol{\omega}(\bar{\boldsymbol{J}}) \cdot \frac{\partial S_1(\bar{\boldsymbol{J}},\bar{\boldsymbol{\varphi}},\theta)}{\partial \bar{\boldsymbol{\varphi}}} + H_1(\bar{\boldsymbol{J}},\bar{\boldsymbol{\varphi}})$

  where the frequency vector is $\boldsymbol{\omega}(\bar{\boldsymbol{J}}) = \frac{\partial H_0(\bar{\boldsymbol{J}})}{\partial \bar{\boldsymbol{J}}}$

Canonical perturbation theory

- From the first order Hamiltonian, the angles have to be eliminated. For this purpose, it can be split in two parts:
  - Average part: $\langle H_1 \rangle_{\bar{\varphi}} = \left(\frac{1}{2\pi}\right)^n \oint H_1(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\varphi}}) d\bar{\boldsymbol{\varphi}}$
  - Oscillating part: $\{H_1\} = H_1 - \langle H_1 \rangle_{\bar{\varphi}}$
- The 1st order perturbation part of the Hamiltonian then becomes

$$\bar{H}_1 = \frac{\partial S_1(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\varphi}}, \theta)}{\partial \theta} + \boldsymbol{\omega}(\bar{\boldsymbol{J}}) \cdot \frac{\partial S_1(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\varphi}}, \theta)}{\partial \bar{\boldsymbol{\varphi}}} + \langle H_1(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\varphi}}) \rangle_{\bar{\varphi}} + \{H_1(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\varphi}})\}$$

- Thus, the generating function should be chosen such that the angle dependence is eliminated, for which

$$\bar{H}_1(\bar{\boldsymbol{J}}) = \langle H_1(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\varphi}}) \rangle_{\bar{\varphi}} \quad \text{and} \quad \frac{\partial S_1(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\varphi}}, \theta)}{\partial \theta} + \boldsymbol{\omega}(\bar{\boldsymbol{J}}) \cdot \frac{\partial S_1(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\varphi}}, \theta)}{\partial \bar{\boldsymbol{\varphi}}} = -\{H_1(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\varphi}})\}$$

- The new Hamiltonian is a function of the new actions $\bar{H}(\bar{\boldsymbol{J}}) = H_0(\bar{\boldsymbol{J}}) + \epsilon \langle H_1(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\varphi}}) \rangle_{\bar{\varphi}} + \mathcal{O}(\epsilon^2)$ with the new frequency vector

$$\bar{\boldsymbol{\omega}}(\bar{\boldsymbol{J}}) = \frac{\partial \bar{H}(\bar{\boldsymbol{J}})}{\partial \bar{\boldsymbol{J}}} = \boldsymbol{\omega}(\bar{\boldsymbol{J}}) + \epsilon \frac{\partial \langle H_1(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\varphi}}) \rangle_{\bar{\varphi}}}{\partial \bar{\boldsymbol{J}}} + \mathcal{O}(\epsilon^2)$$

Form of the generating function

- The question that remains to be answered is whether a generating function can be found that eliminates the angle dependence
- The oscillating part of the perturbation and the generating function can be expanded in Fourier series

$$\{H_1(\bar{\boldsymbol{J}},\bar{\boldsymbol{\varphi}})\} = \sum_{\boldsymbol{k},p} H_{1\mathbf{k}}(\bar{\boldsymbol{J}})e^{i(\boldsymbol{k}\cdot\bar{\boldsymbol{\varphi}}+p\theta)} \quad S_1(\bar{\boldsymbol{J}},\bar{\boldsymbol{\varphi}},\theta) = \sum_{\boldsymbol{k},p} S_{1\mathbf{k}}(\bar{\boldsymbol{J}})e^{i(\boldsymbol{k}\cdot\bar{\boldsymbol{\varphi}}+p\theta)}$$

with
$$\boldsymbol{k}\cdot\bar{\boldsymbol{\varphi}} = k_1\bar{\varphi}_1 + \cdots + k_n\bar{\varphi}_n$$

- Following the relationship for the angle elimination, the Fourier coefficients of the generating function should satisfy

$$S_{1k}(\bar{\boldsymbol{J}}) = i\frac{H_{1k}(\bar{\boldsymbol{J}})}{\boldsymbol{k}\cdot\boldsymbol{\omega}(\bar{\boldsymbol{J}})+p} \qquad \text{with} \qquad \boldsymbol{k},p\neq\mathbf{0}$$

- Then, the generating function can be written as

$$S(\bar{\boldsymbol{J}},\bar{\boldsymbol{\varphi}}) = \bar{\boldsymbol{J}}\cdot\bar{\boldsymbol{\varphi}} + \epsilon i\sum_{\mathbf{k}\neq\mathbf{0}}\frac{H_{1\mathbf{k}}(\bar{\mathbf{J}})}{\boldsymbol{k}\cdot\boldsymbol{\omega}(\bar{\boldsymbol{J}})+p}e^{i(\boldsymbol{k}\cdot\bar{\boldsymbol{\varphi}}+p\theta)} + \mathcal{O}(\epsilon^2)$$

Second order sextupole tune-shift

- It can be shown that at second order in perturbation theory the Hamiltonian depending only on the actions can be written $\bar{H}_2(\bar{J}) = \langle \frac{1}{2}\frac{\partial^2 H_0}{\partial \bar{J}^2}\left(\frac{\partial S_1}{\partial \phi}\right)^2 + \frac{\partial H_1}{\partial \bar{J}}\frac{\partial S_1}{\partial \phi}\rangle_\phi$
- This can be simplified to $\bar{H}_2(\bar{J}) = \langle \frac{\partial H_1}{\partial \bar{J}}\frac{\partial S_1}{\partial \phi}\rangle_\phi$
- The two terms are $\frac{\partial H_1}{\partial \bar{J}} = \frac{K_s(s)}{2\sqrt{2}}\bar{J}^{1/2}\beta(s)^{3/2}(\cos 3\phi + 3\cos\phi)$

$$\frac{\partial S_1}{\partial \phi} = -\frac{\bar{J}^{3/2}}{2\sqrt{2}}\int_s^{s+C} K_s(s')\beta(s')^{3/2}\left[\frac{\cos(\phi+\psi(s')-\psi(s)-\pi\nu)}{\sin(\pi\nu)} + \frac{\cos 3(\phi+\psi(s')-\psi(s)-\pi\nu)}{\sin(3\pi\nu)}\right]ds'$$

- The 2nd order Hamiltonian is given by the angle-averaged product of the last two terms.
- It is quadratic in the sextupole strength and the new action. The 2nd order tune-shift is the derivative in the action

$$\nu(\bar{J}) = \langle\frac{\partial H_2}{\partial \bar{J}}\rangle_{\phi,s} = -\frac{\bar{J}}{16\pi}\int_0^C ds K_s(s)\beta(s)^{3/2}\int_s^{s+C} K_s(s')\beta(s')^{3/2}$$
$$\times\left[\frac{\cos(\phi+\psi(s')-\psi(s)-\pi\nu)}{\sin(\pi\nu)} + \frac{\cos 3(\phi+\psi(s')-\psi(s)-\pi\nu)}{\sin(3\pi\nu)}\right]ds'$$

Single resonance for accelerator Hamiltonian

- The single resonance accelerator Hamiltonian (Hagedorn (1957), Schoch (1957), Guignard (1976, 1978))

$$H(J_x, J_y, \phi_x, \phi_y, s) = \frac{1}{R}(\nu_x J_x + \nu_y J_y) + g_{n_x,n_y} \frac{2}{R} J_x^{\frac{k_x}{2}} J_y^{\frac{k_y}{2}} \cos(n_x\phi_x + n_y\phi_y + \phi_0 - p\theta)$$

with $\quad g_{n_x,n_y} e^{i\phi_0} = g_{j,k,l,m;p}$

- From the generating function

  $F_r(\phi_x, \phi_y, \hat{J}_x, \hat{J}_y, s) = (n_x\phi_x + n_y\phi_y - p\theta)\hat{J}_x + \phi_y \hat{J}_y$

  the relationships between old and new variables are

$$\hat{\phi}_x = (n_x\phi_x + n_y\phi_y - p\theta) \;, \quad J_x = n_x \hat{J}_x$$

$$\hat{\phi}_y = \phi_y \;, \qquad J_y = n_y \hat{J}_x + \hat{J}_y$$

- The following Hamiltonian is obtained

$$\hat{H}(\hat{J}_x, \hat{J}_y, \hat{\phi}_x) = \frac{(n_x\nu_x + n_y\nu_y - p)\hat{J}_x + \hat{J}_y}{R} + g_{n_x,n_y} \frac{2}{R} (n_x \hat{J}_x)^{\frac{k_x}{2}} (n_y \hat{J}_x + \hat{J}_y)^{\frac{k_y}{2}} \cos(\hat{\phi}_x + \phi_0)$$

Resonance widths

- There are two integrals of motion
  - The Hamiltonian, as it is independent on "time"
  - The new action $\hat{J}_y$ as the Hamiltonian is independent on $\hat{\phi}_y$
- The two invariants in the old variables are written as:

$$c_1 = \frac{J_x}{n_x} - \frac{J_y}{n_y}$$

$$c_2 = (\nu_x - \frac{p}{n_x + n_y})J_x + (\nu_y - \frac{p}{n_x + n_y})J_y + 2g_{n_x,n_y} J_x^{\frac{k_x}{2}} J_y^{\frac{k_y}{2}} \cos(n_x\phi_x + n_y\phi_y + \phi_0 - p\theta)$$

- Two cases can be distinguished
  - n_x , n_y have **opposite** sign, i.e. **difference** resonance, the motion is the one of an ellipse, so bounded
  - n_x , n_y have the **same** sign, i.e. **sum** resonance, the motion is the one of an hyperbola, so **not** bounded
- These are **first order** perturbation theory considerations
- The distance from the resonance is obtained as

$$\Delta = \frac{g_{n_x,n_y}}{R} J_x^{\frac{k_x-2}{2}} J_y^{\frac{k_y-2}{2}} (k_x n_x J_x + k_y n_y J_y)$$

Magnetic element Hamiltonians

- Dipole:

$$H = \frac{x\delta}{\rho} + \frac{x^2}{2\rho^2} + \frac{p_x^2 + p_y^2}{2(1+\delta)}$$

- Quadrupole:

$$H = \frac{1}{2}k_1(x^2 - y^2) + \frac{p_x^2 + p_y^2}{2(1+\delta)}$$

- Sextupole:

$$H = \frac{1}{3}k_2(x^3 - 3xy^2) + \frac{p_x^2 + p_y^2}{2(1+\delta)}$$

- Octupole:

$$H = \frac{1}{4}k_3(x^4 - 6x^2y^2 + y^4) + \frac{p_x^2 + p_y^2}{2(1+\delta)}$$

Lie operators for simple elements

Element	Map	Lie Operator
Drift space	$x = x_0 + Lp_0$ $p = p_0$	$\exp(:-\frac{1}{2}Lp^2:)$
Thin-lens Quadrupole	$x = x_0$ $p = p_0 - \frac{1}{f}x_0$	$\exp(:-\frac{1}{2f}x^2:)$
Thin-lens Multipole	$x = x_0$ $p = p_0 + \lambda n x^{n-1}$	$\exp(:\lambda x^n:)$
Thin-lens kick	$x = x_0$ $p = p_0 + f(x)$	$\exp(:\int_0^x f(x')dx':)$
Thick focusing quad	$x = x_0 \cos kL + \frac{p_0}{k}\sin kL$ $p = -kx_0 \sin kL + p_0 \cos kL$	$\exp[:-\frac{1}{2}L(k^2x^2 + p^2):]$
Thick defocusing quad	$x = x_0 \cosh kL + \frac{p_0}{k}\sinh kL$ $p = kx_0 \sinh kL + p_0 \cosh kL$	$\exp[:\frac{1}{2}L(k^2x^2 - p^2):]$
Coordinate shift	$x = x_0 - b$ $p = p_0 + a$	$\exp(:ax + bp:)$
Coordinate rotation	$x = x_0 \cos\mu + p_0 \sin\mu$ $p = -x_0 \sin\mu + p_0 \cos\mu$	$\exp[:-\frac{1}{2}\mu(x^2 + p^2):]$
Scale change	$x = e^{-\lambda}x_0$ $p = e^{\lambda}p_0$	$\exp(:\lambda xp:)$

Formulas for Lie operators

$$:a: = 0, \quad e^{:a:} = 1$$

$$:f:a = 0, \quad e^{:f:}a = a$$

$$:f:f = 0, \quad e^{:f:}f = f$$

$$\{:f:, :g:\} = :[f,g]:$$

$$e^{:f:}g(X) = g(e^{:f:}X)$$

$$e^{:\tilde{C}X:}g(X) = g(X - SC)$$

$$e^{:f:}G(:g:)e^{-:f:} = G(:e^{:f:}g:)$$