Lattice Design II: Insertions

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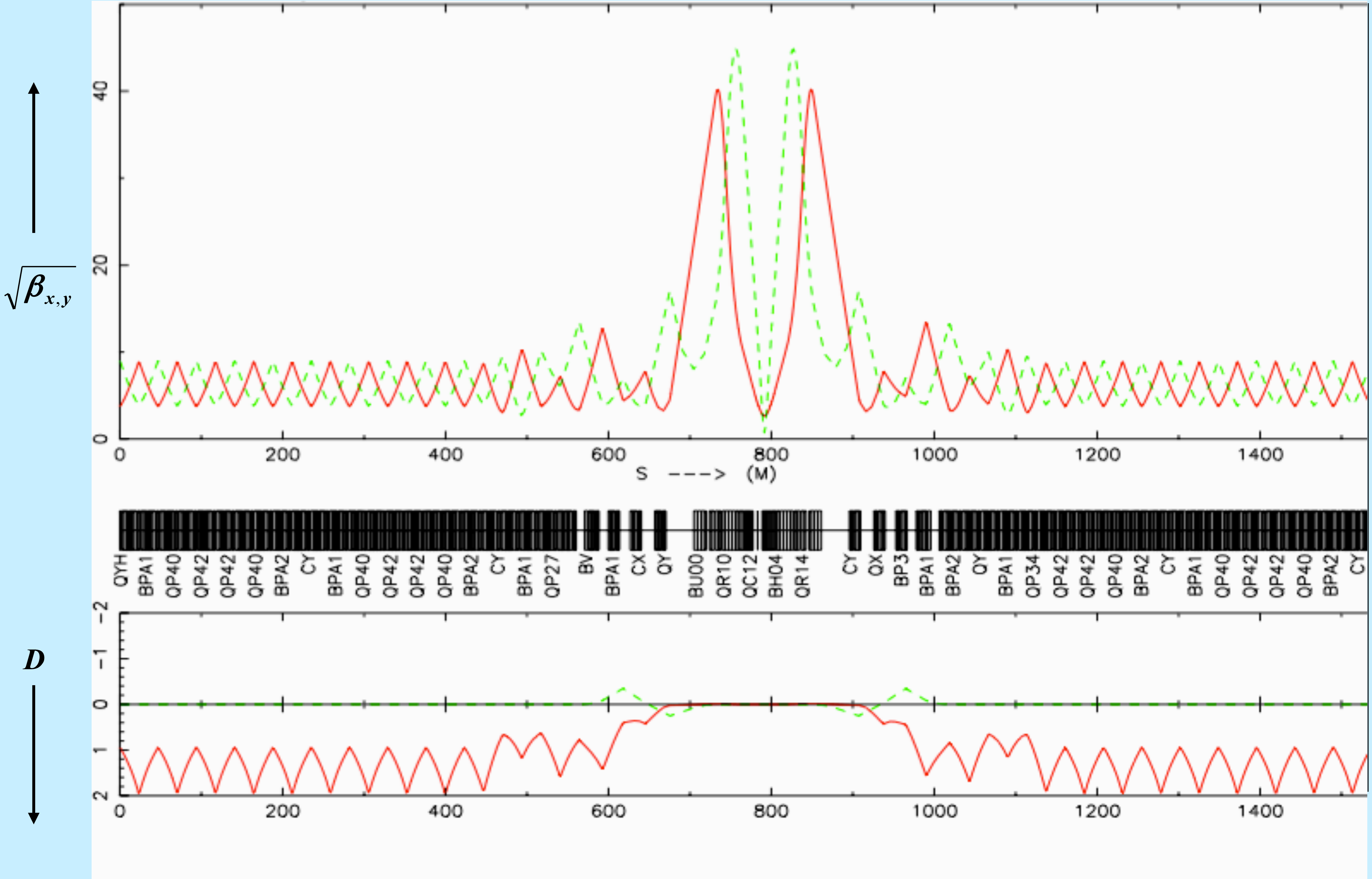
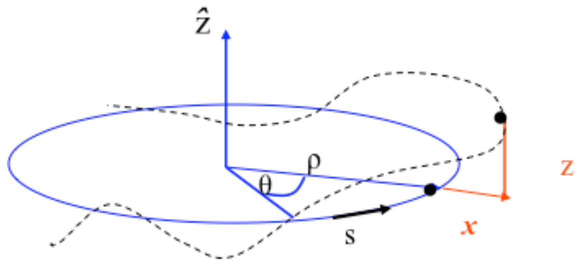


1.) Reminder:

equation of motion



$$x'' + K(s) * x = 0 \qquad K = -k + \frac{1}{\rho^2}$$

single particle trajectory considering both planes

$$\begin{pmatrix} x(s) \\ x'(s) \\ y(s) \\ y'(s) \end{pmatrix} = M * \begin{pmatrix} x(s_0) \\ x'(s_0) \\ y(s_0) \\ y'(s_0) \end{pmatrix}$$

APS Light Source

e.g. matrix for a quadrupole lens:

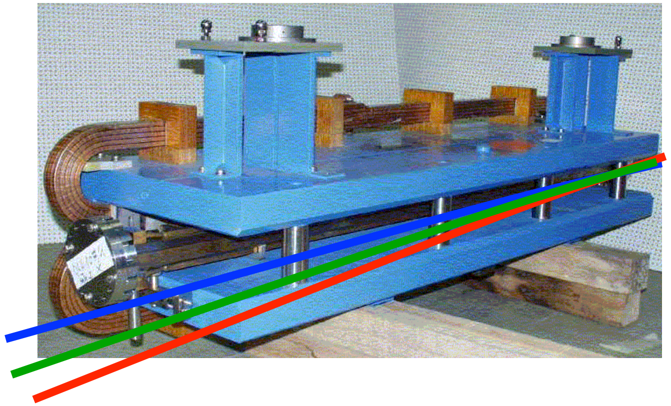
$$M_{foc} = \begin{pmatrix} \cos(\sqrt{|k|}s & \frac{1}{\sqrt{|k|}}\sin(\sqrt{|k|}s & 0 & 0 \\ -\sqrt{|k|}\sin(\sqrt{|k|}s & \cos(\sqrt{|k|}s & 0 & 0 \\ 0 & 0 & \cosh(\sqrt{|k|}s & \frac{1}{\sqrt{|k|}}\sinh(\sqrt{|k|}s \\ 0 & 0 & \sqrt{|k|}\sinh(\sqrt{|k|}s & \cosh(\sqrt{|k|}s \end{pmatrix} = \begin{pmatrix} C_x & S_x & 0 & 0 \\ C'_x & S'_x & 0 & 0 \\ 0 & 0 & C_y & S_y \\ 0 & 0 & C'_y & S'_y \end{pmatrix}$$

2.) Dispersion

momentum error:

$$\Delta p / p \neq 0$$

$$x'' + x\left(\frac{1}{\rho^2} - k\right) = \frac{\Delta p}{p} \cdot \frac{1}{\rho}$$



general solution:

$$x(s) = x_h(s) + x_i(s)$$

$$D(s) = \frac{x_i(s)}{\Delta p / p}$$

$$x(s) = x_\beta(s) + D(s) \cdot \frac{\Delta p}{p}$$

$$\begin{pmatrix} x \\ x' \\ \frac{\Delta p}{p} \end{pmatrix}_s = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} x \\ x' \\ \frac{\Delta p}{p} \end{pmatrix}_0$$

Dispersion

the dispersion function D(s) is (...obviously) defined by the focusing properties of the lattice and is given by:

$$D(s) = S(s) * \int \frac{1}{\rho(\tilde{s})} C(\tilde{s}) d\tilde{s} - C(s) * \int \frac{1}{\rho(\tilde{s})} S(\tilde{s}) d\tilde{s}$$

! weak dipoles → large bending radius → small dispersion

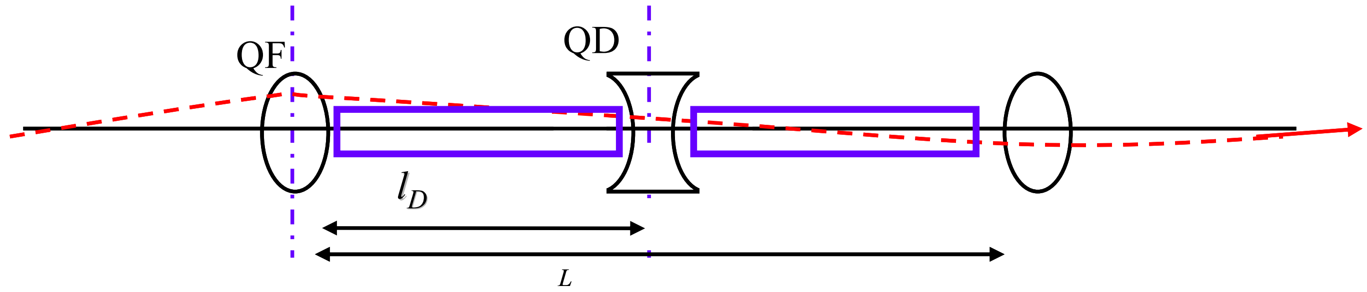
Example: Drift

$$M_D = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} \qquad D(s) = S(s) * \int \underbrace{\frac{1}{\rho(\tilde{s})}}_{=0} C(\tilde{s}) d\tilde{s} - C(s) * \int \underbrace{\frac{1}{\rho(\tilde{s})}}_{=0} S(\tilde{s}) d\tilde{s}$$

$$\rightarrow M_D = \begin{pmatrix} 1 & l & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

...in similar way for quadrupole matrices,
!!! in a quite different way for dipole matrix (see appendix)

Dispersion in a FoDo Cell:



!! we have now introduced dipole magnets in the FoDo:

→ we still neglect the weak focusing contribution $1/\rho^2$

→ but take into account $1/\rho$ for the dispersion effect

assume: length of the dipole = l_D

Calculate the matrix of the FoDo half cell in thin lens approximation:

in analogy to the derivations of $\hat{\beta}, \check{\beta}$

*** thin lens approximation**: $f = \dfrac{1}{k l_Q} >> l_Q$

*** length of quad negligible** $l_Q \approx 0, \rightarrow l_D = \dfrac{1}{2} L$

*** start at half quadrupole** $\dfrac{1}{\tilde{f}} = \dfrac{1}{2f}$

Matrix of the half cell

$$M_{HalfCell} = M_{\frac{QD}{2}} * M_B * M_{\frac{QF}{2}}$$

$$M_{HalfCell} = \begin{pmatrix} 1 & 0 \\ \frac{1}{\tilde{f}} & 1 \end{pmatrix} * \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} 1 & 0 \\ \frac{-1}{\tilde{f}} & 1 \end{pmatrix}$$

$$M_{HalfCell} = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} 1 - \frac{1}{\tilde{f}} & 1 \\ \frac{-1}{\tilde{f}} & 1 + \frac{1}{\tilde{f}} \end{pmatrix}$$

calculate the dispersion terms D, D´ from the matrix elements

$$\boldsymbol{D(s) = S(s) * \int \frac{1}{\rho(\tilde{s})} C(\tilde{s}) d\tilde{s} - C(s) * \int \frac{1}{\rho(\tilde{s})} S(\tilde{s}) d\tilde{s}}$$

$$D(\ell)=\underbrace{\ell}_{\mathbf{S(s)}}*\frac{1}{\rho}\int_0^{\ell}\underbrace{(1-\frac{s}{\tilde{f}})}_{\mathbf{C(s)}}ds-\underbrace{(1-\frac{\ell}{\tilde{f}})}_{\mathbf{C(s)}}\frac{1}{\rho}\underbrace{\int_0^{\ell}s\,ds}_{\mathbf{S(s)}}$$

$$D(\ell)=\frac{\ell}{\rho}(\ell-\frac{\ell^2}{2\tilde{f}})-(1-\frac{\ell}{\tilde{f}})*\frac{1}{\rho}*\frac{\ell^2}{2}\quad=\quad\frac{\ell^2}{\rho}-\frac{\ell^3}{2\tilde{f}\rho}-\frac{\ell^2}{2\rho}+\frac{\ell^3}{2\tilde{f}\rho}$$

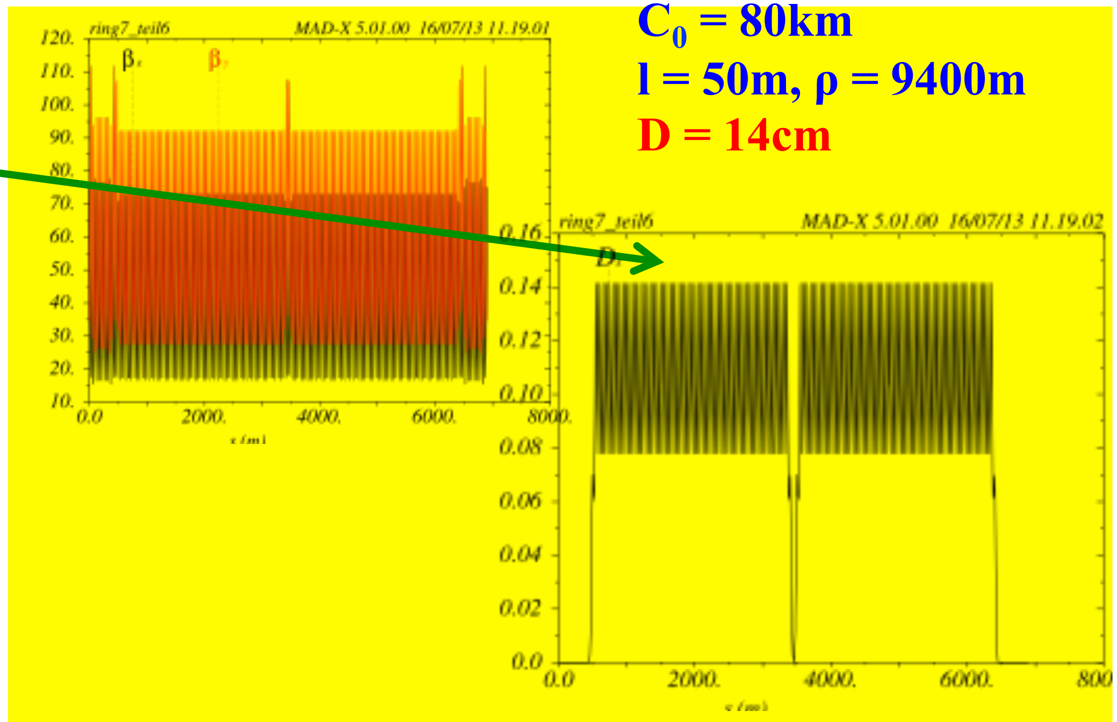
latest news: TLEP **E=175 GeV**

C_0 = 80km

l = 50m, ρ = 9400m

D = 14cm

$$D(\ell)=\frac{\ell^2}{2\rho}$$



in full analogy on derives for D´:

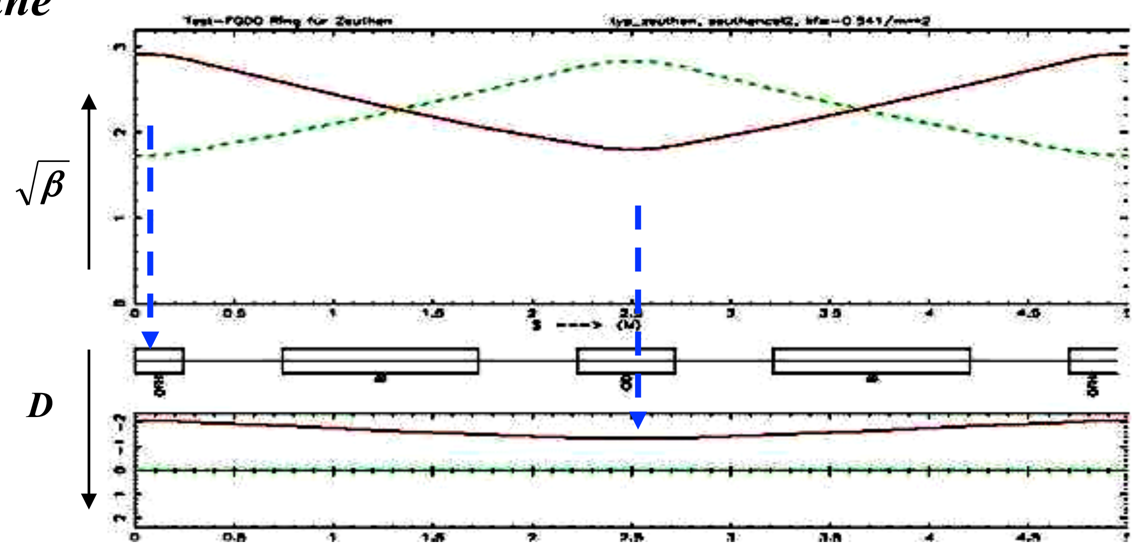
$$D'(s)=\frac{\ell}{\rho}\left(1+\frac{\ell}{2\tilde{f}}\right)$$

and we get the complete matrix including the dispersion terms D, D´

$$M_{halfCell} = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-\dfrac{\ell}{\tilde{f}} & \ell & \dfrac{\ell^2}{2\rho} \\ \dfrac{-\ell}{\tilde{f}^2} & 1+\dfrac{\ell}{\tilde{f}} & \dfrac{\ell}{\rho}\left(1+\dfrac{\ell}{2\tilde{f}}\right) \\ 0 & 0 & 1 \end{pmatrix}$$

boundary conditions for the transfer from the center of the foc. to the center of the defoc. quadrupole

$$\begin{pmatrix} \check{D} \\ 0 \\ 1 \end{pmatrix} = M_{1/2} * \begin{pmatrix} \hat{D} \\ 0 \\ 1 \end{pmatrix}$$



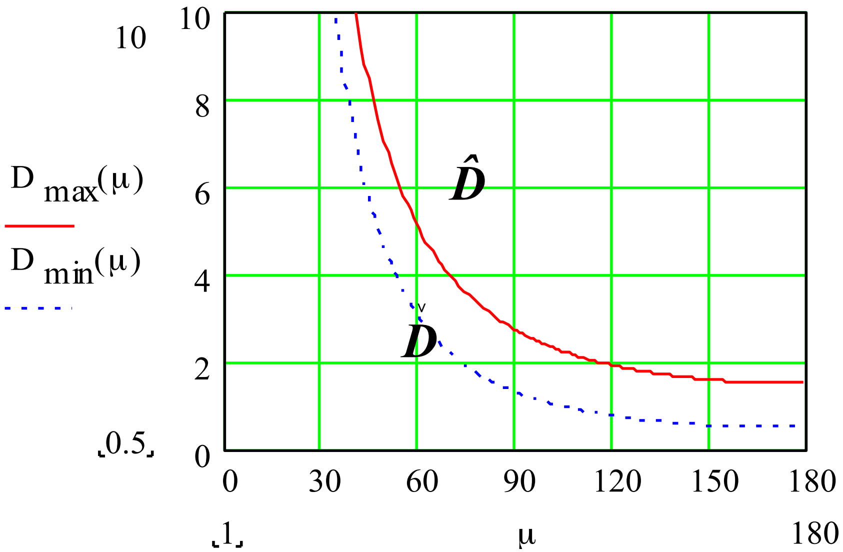
Dispersion in a FoDo Cell

$$\rightarrow \quad \breve{D} = \hat{D}\left(1 - \frac{\ell}{\tilde{f}}\right) + \frac{\ell^2}{2\rho}$$

$$\rightarrow \quad 0 = -\frac{\ell}{\tilde{f}^2} * \hat{D} + \frac{\ell}{\rho}\left(1 + \frac{\ell}{2\tilde{f}}\right)$$

$$\hat{D} = \frac{\ell^2}{\rho} * \frac{\left(1 + \frac{1}{2}\sin\frac{\psi_{cell}}{2}\right)}{\sin^2\frac{\psi_{cell}}{2}} \qquad \check{D} = \frac{\ell^2}{\rho} * \frac{\left(1 - \frac{1}{2}\sin\frac{\psi_{cell}}{2}\right)}{\sin^2\frac{\psi_{cell}}{2}}$$

where ψ_{cell} denotes the phase advance of the full cell and $l/f = \sin(\psi/2)$



Nota bene:

! small dispersion needs strong focusing
→ large phase advance

!! ↔ there is an optimum phase for small β

!!! ...do you remember the stability criterion?
½ trace = cos ψ ↔ ψ < 180°

!!!! … life is not easy

3.) Lattice Design: Insertions

... the most complicated one: the drift space

Question to the auditorium: what will happen to the beam parameters α, β, γ if we stop focusing for a while …?

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_S = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC'+S'C & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} * \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_0$$

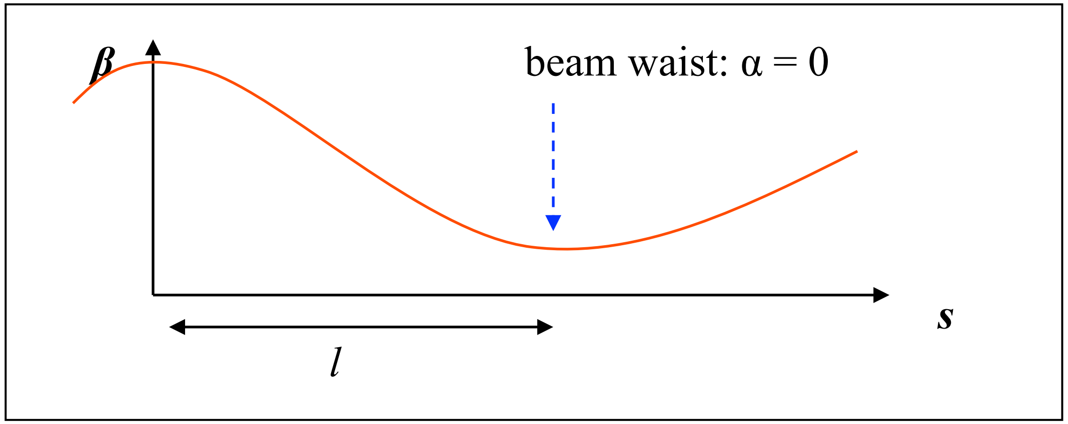
transfer matrix for a drift:

$$M = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$$

$$\beta(s) = \beta_0 - 2\alpha_0 s + \gamma_0 s^2$$
$$\alpha(s) = \alpha_0 - \gamma_0 s$$
$$\gamma(s) = \gamma_0$$

„0" refers to the position of the last lattice element
„s" refers to the position in the drift

location of the waist:



given the initial conditions $\alpha_{0,}\ \beta_{0,}\ \gamma_0$ **:** ***where is the point of smallest beam dimension in the drift … or at which location occurs the beam waist ?***

beam waist:

$$\alpha(s) = 0 \quad \rightarrow \quad \alpha_0 = \gamma_0 * s$$

$$l = \frac{\alpha_0}{\gamma_0}$$

beam size at that position:

$$\left.\begin{array}{l}\gamma(l) = \gamma_0 \\ \alpha(l) = 0\end{array}\right\} \rightarrow \gamma(l) = \frac{1+\alpha^2(l)}{\beta(l)} = \frac{1}{\beta(l)}$$

$$\beta(l) = 1/\gamma_0$$

β-Function in a Drift:

let ‘s assume we are at a symmetry point in the center of a drift.

$$\beta(s) = \beta_0 - 2\alpha_0 s + \gamma_0 s^2$$

as $\alpha_0 = 0, \quad \rightarrow \quad \gamma_0 = \frac{1+\alpha_0^{\;2}}{\beta_0} = \frac{1}{\beta_0}$

and we get for the β function in the neighborhood of the symmetry point

$$\beta(s) = \beta_0 + \frac{s^2}{\beta_0}$$

! ! !

Nota bene:

1.) this is very bad !!!

2.) this is a direct consequence of the conservation of phase space density (... in our words: ε = const) … and there is no way out.

3.) Thank you, Mr. Liouville !!!

Joseph Liouville, 1809-1882

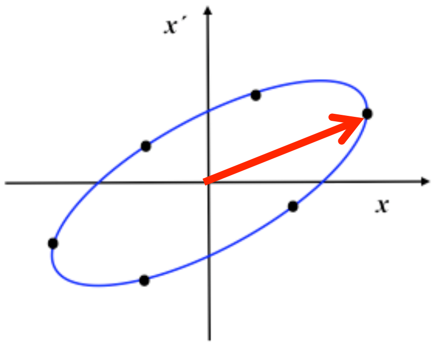
Alternative description: The Beam Matrix

„Once more unto the breach dear friends:"

Transformation of Twiss parameters

just because it is mathematical more elegant ...

let 's define a beam matrix: $B_0 = \begin{pmatrix} \beta_0 & -\alpha_0 \\ \alpha_0 & \gamma_0 \end{pmatrix}$



and a orbit vector: $X_0 = \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}, \quad X^T = \begin{pmatrix} x_0, & x'_0 \end{pmatrix}$

the product $\underline{X_0^{\ T} * B_0^{-1} * X_0} = (x_0, x'_0) * \begin{pmatrix} \gamma_0 & \alpha_0 \\ \alpha_0 & \beta_0 \end{pmatrix} * \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix} = \gamma_0 x_0^2 + 2\alpha_0 x_0 x'_0 + \beta_0 x_0'^2 = \varepsilon$

... is constant

transformation of the orbit vector: $X_1 = M * X_0$

and so we get: $\varepsilon = X_0^{\ T} * B_0^{-1} * X_0 = X_0^T \ \underline{M^T (M^T)^{-1}} \ B_0^{-1} \ \underline{M^{-1} M} \ X_0$

$$= X_0^T \ M^T \left\{ (M^T)^{-1} \ B_0^{-1} \ M^{-1} \right\} M \ X_0$$

Transformation of Twiss parameters

and using $A^T B^T = (BA)^T$ ***and*** $A^{-1}B^{-1} = (BA)^{-1}$

$$\varepsilon = X_0^T \; M^T \left\{ (M^T)^{-1} (MB_0)^{-1} \right\} M \; X_0$$

$$= X_0^T \; M^T \left\{ MB_0 M^T \right\}^{-1} M \; X_0$$

$$= \underbrace{(MX_0)^T} \left\{ MB_0 M^T \right\}^{-1} \underbrace{MX_0}$$

$$= X_1^{\;T} \left\{ MB_0 M^T \right\}^{-1} \; X_1$$

but we know already that

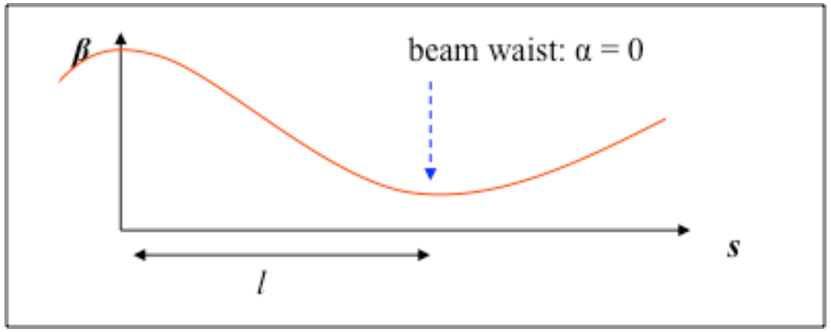
$$\varepsilon = const = \quad X_0^{\;T} * B_0^{-1} * X_0 \;=\; X_1^{\;T} * B_1^{-1} * X_1$$

and in the end and after all we learn that ... ***in full equivalence to ...***

$$B_1 = \begin{pmatrix} \beta_1 & -\alpha_1 \\ \alpha_1 & \gamma_1 \end{pmatrix} = M * B_0 * M^T$$

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_{s2} = \begin{pmatrix} m_{11}^2 & -2m_{11}m_{12} & m_{12}^2 \\ -m_{11}m_{21} & m_{12}m_{21} + m_{22}m_{11} & -m_{12}m_{22} \\ m_{12}^2 & -2m_{22}m_{21} & m_{22}^2 \end{pmatrix} * \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_{s1}$$

Transformation of Twiss parameters



Example again the drift space

... starting from $\alpha_0 = 0$

$$M = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$$

$$B_0 = \begin{pmatrix} \beta_0 & -\alpha_0 \\ -\alpha_0 & \gamma_0 \end{pmatrix} = \begin{pmatrix} \beta_0 & 0 \\ 0 & 1/\beta_0 \end{pmatrix}$$

Beam parameters after the drift: $B_1 = MB_0M^T = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} \beta_0 & 0 \\ 0 & 1/\beta_0 \end{pmatrix} * \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}$

$$= \begin{pmatrix} \beta_0 + \frac{s^2}{\beta_0} & \frac{s}{\beta_0} \\ \frac{s}{\beta_0} & \frac{1}{\beta_0} \end{pmatrix} \rightarrow \beta_1 = \beta_0 + \frac{s^2}{\beta_0}$$

A bit more in detail: β-Function in a Drift

If we cannot fight against Liouvuille theorem ... at least we can optimise

Optimisation of the beam dimension:

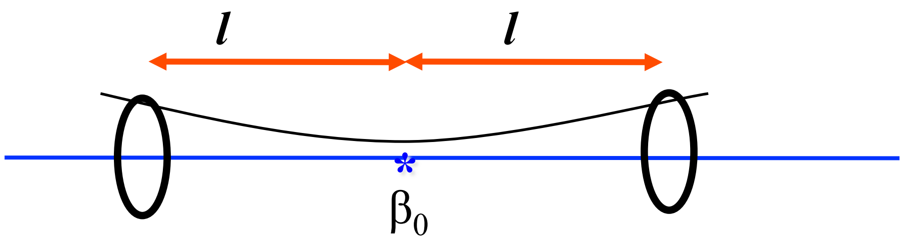
$$\beta(l) = \beta_0 + \frac{l^2}{\beta_0}$$

Find the β at the center of the drift that leads to the lowest maximum β at the end:

$$\frac{d\hat{\beta}}{d\beta_0} = 1 - \frac{l^2}{\beta_0^2} = 0$$

$$\rightarrow \beta_0 = l$$

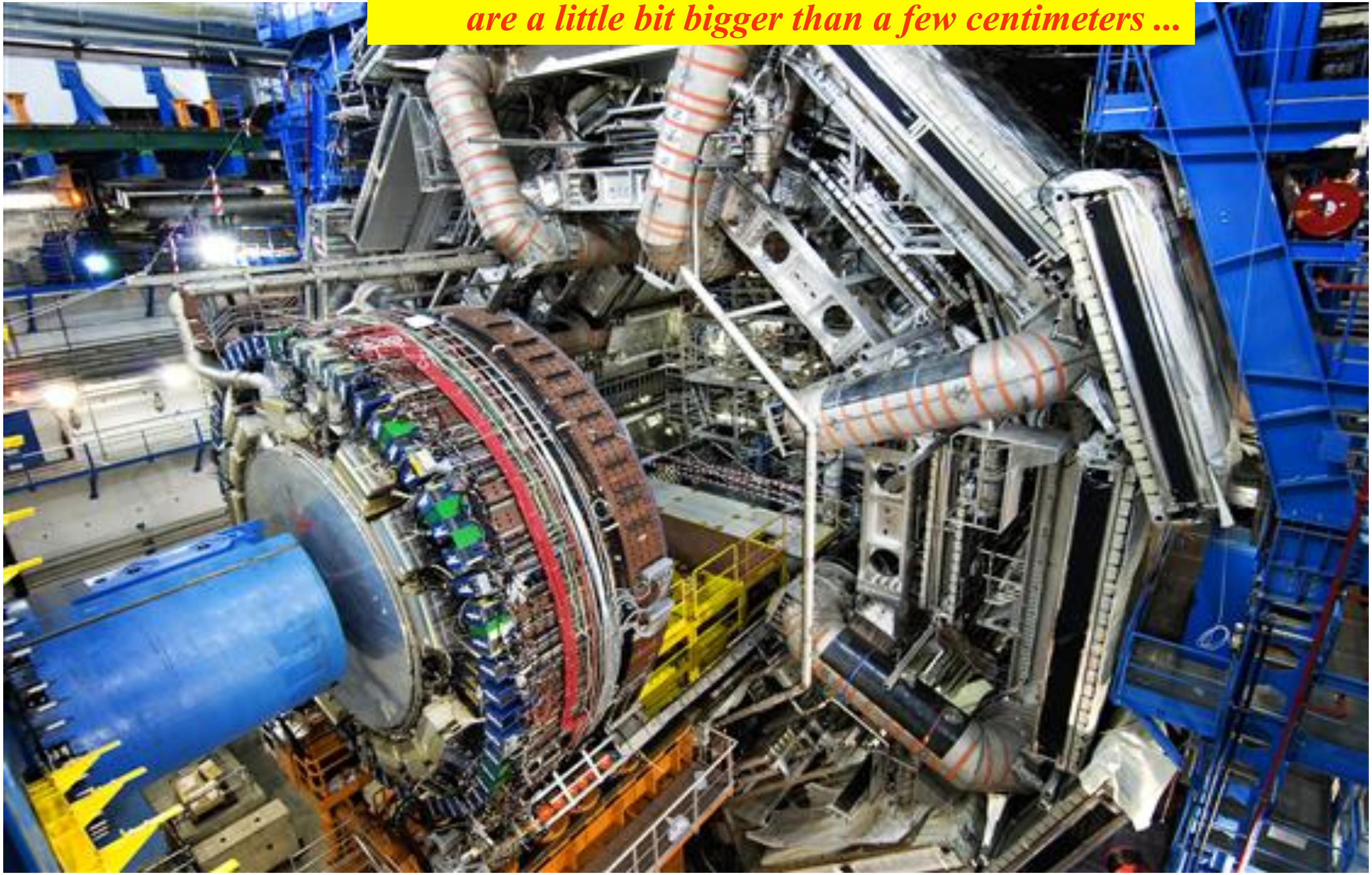
$$\rightarrow \hat{\beta} = 2\beta_0$$



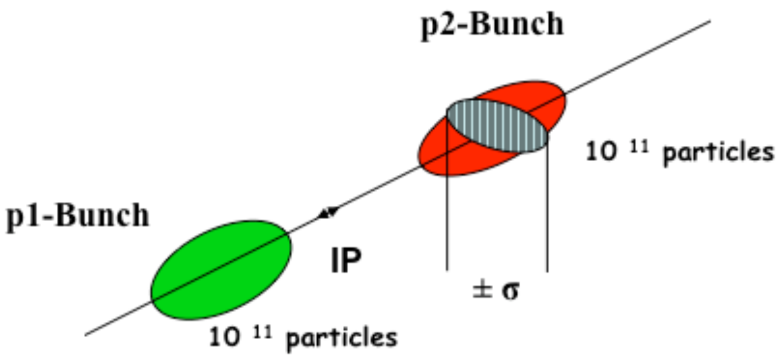
If we choose $\beta_0 = \ell$ we get the smallest β at the end of the drift and the maximum β is just twice the distance ℓ

... clearly there is an

***But:** ... unfortunately ... in general*
high energy detectors that are
installed in that drift spaces
are a little bit bigger than a few centimeters ...



Luminosity & Minibeta Insertion



$$R = L * \Sigma_{react}$$

$$L = \frac{1}{4\pi e^2 f_0 n_b} * \frac{I_{p1} I_{p2}}{\sigma_x \sigma_y}$$

Example: Luminosity run at LHC

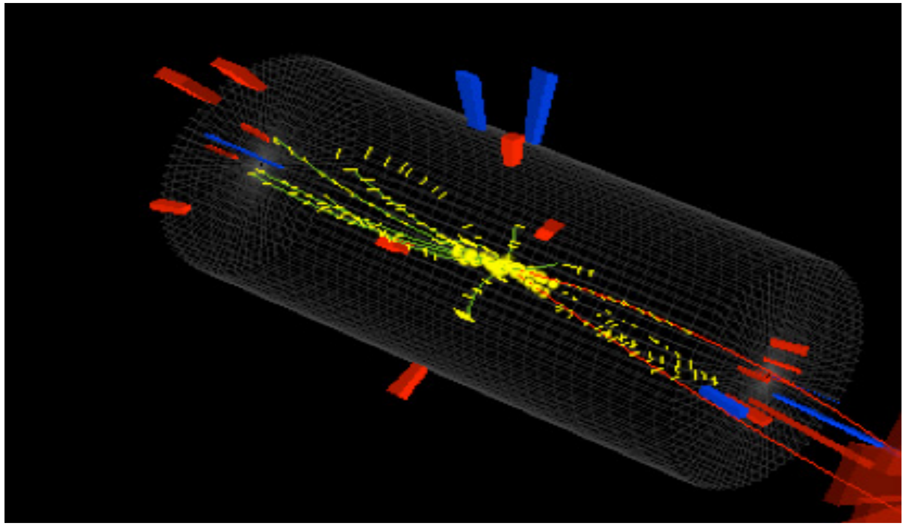
$$\beta_{x,y} = 0.55\ m \qquad f_0 = 11.245\ kHz$$

$$\varepsilon_{x,y} = 5 * 10^{-10}\ rad\ m \qquad n_b = 2808$$

$$\sigma_{x,y} = 17\ \mu m$$

$$I_p = 584\ mA$$

$$L = 1.0 * 10^{34}\ {}^{1}\!/_{cm^2 s}$$



production rate of events is determined by the cross section Σ_{react} and the luminosity that is given by the design of the accelerator

Mini-β Insertions: Betafunctions

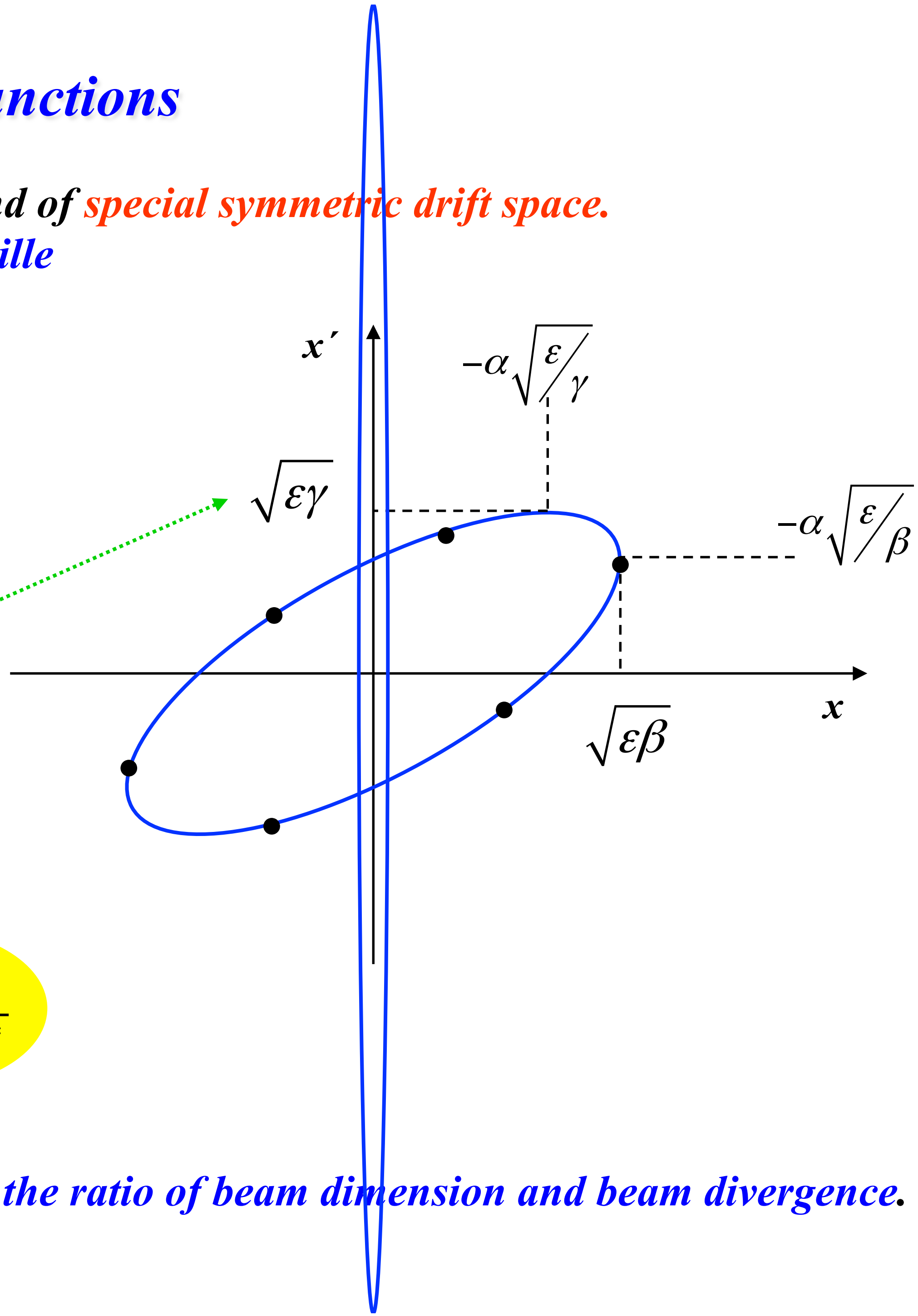
A mini-β insertion is always a kind of special symmetric drift space.

→greetings from Liouville

$$\alpha^* = 0$$

$$\gamma^* = \frac{1+\alpha^2}{\beta} = \frac{1}{\beta^*}$$

$$\sigma'^* = \sqrt{\frac{\varepsilon}{\beta^*}}$$

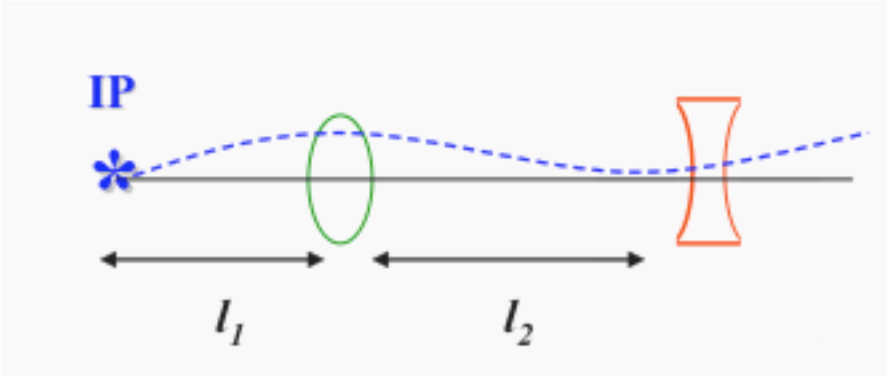


$$\beta^* = \frac{\sigma^*}{\sigma'^*}$$

at a symmetry point β is just the ratio of beam dimension and beam divergence.

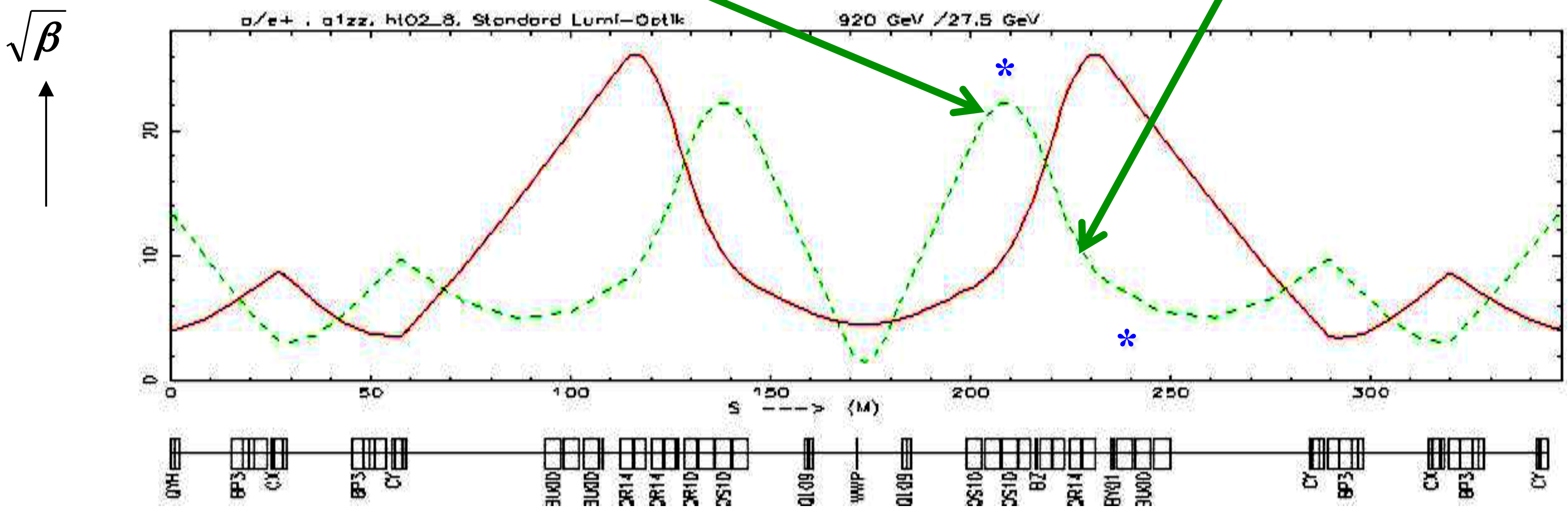
size of ***β at the second quadrupole lens*** ***(in thin lens approx):***

… ***after some transformations and a couple of beer*** **…**



$$\beta(l) = \beta_0 + \frac{l^2}{\beta_0}$$

$$\beta(s) = \left(1 + \frac{l_2}{f_1}\right)^2 * \beta^* + \frac{1}{\beta^*}\left(l_1 + l_2 + \frac{l_1 l_2}{f_1}\right)^2$$



Mini-β Insertions: Phase advance

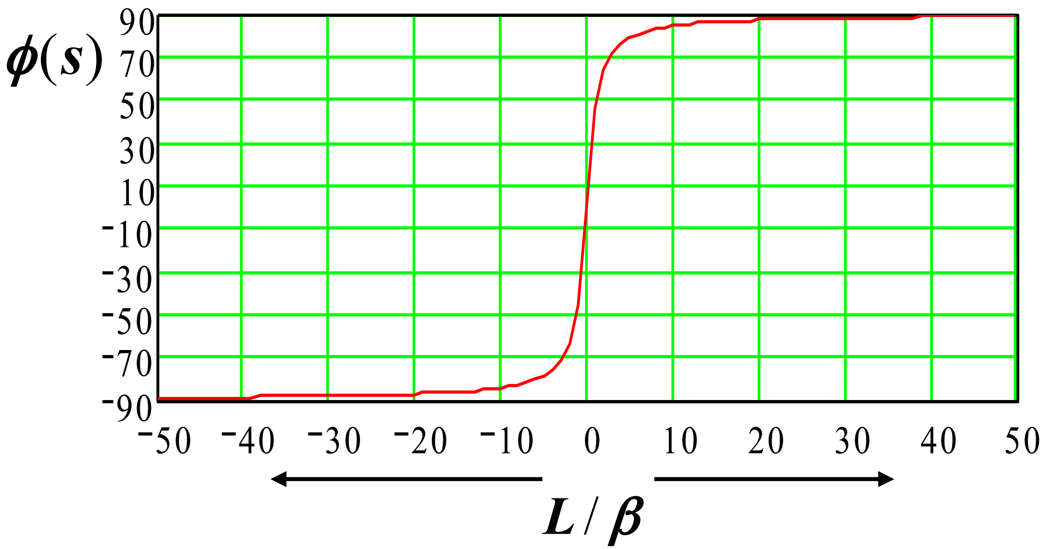
By definition the phase advance is given by:

$$\Phi(s) = \int \frac{1}{\beta(s)} ds$$

Now in a mini β insertion:

$$\beta(s) = \beta_0 \quad (1 + \frac{s^2}{\beta_0^2})$$

$$\rightarrow \Phi(s) = \frac{1}{\beta_0} \int_0^L \frac{1}{1 + s^2/\beta_0^2} ds = \arctan \frac{L}{\beta_0}$$



Consider the drift spaces on both sides of the IP: the phase advance of a mini β insertion is approximately π,
in other words: the tune will increase by half an integer.

Are there any problems ?

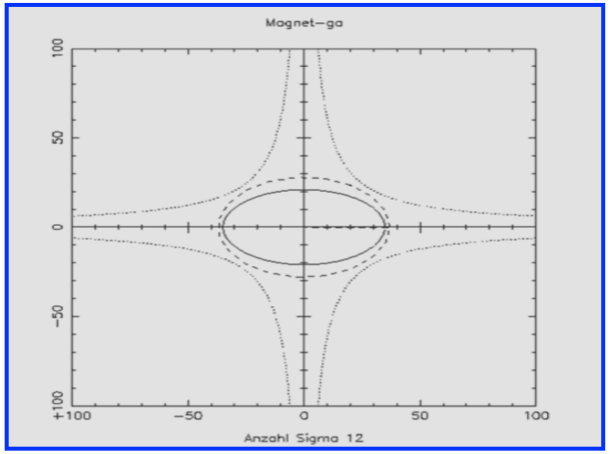
sure there are...

* ***large β values at the doublet quadrupoles → large contribution to chromaticity Q'   … and no local correction***

$$Q' = \frac{-1}{4\pi}\oint K(s)\beta(s)ds$$

* ***aperture of mini β quadrupoles limit the luminosity***

beam envelope at the first mini β quadrupole lens in the HERA proton storage ring



* ***field quality and magnet stability most critical at the high β sections***
effect of a quad error:

$$\Delta Q = \int_{s0}^{s0+l} \frac{\Delta K(s)\beta(s)ds}{4\pi}$$

→ keep distance „s" to the first mini β quadrupole as small as possible

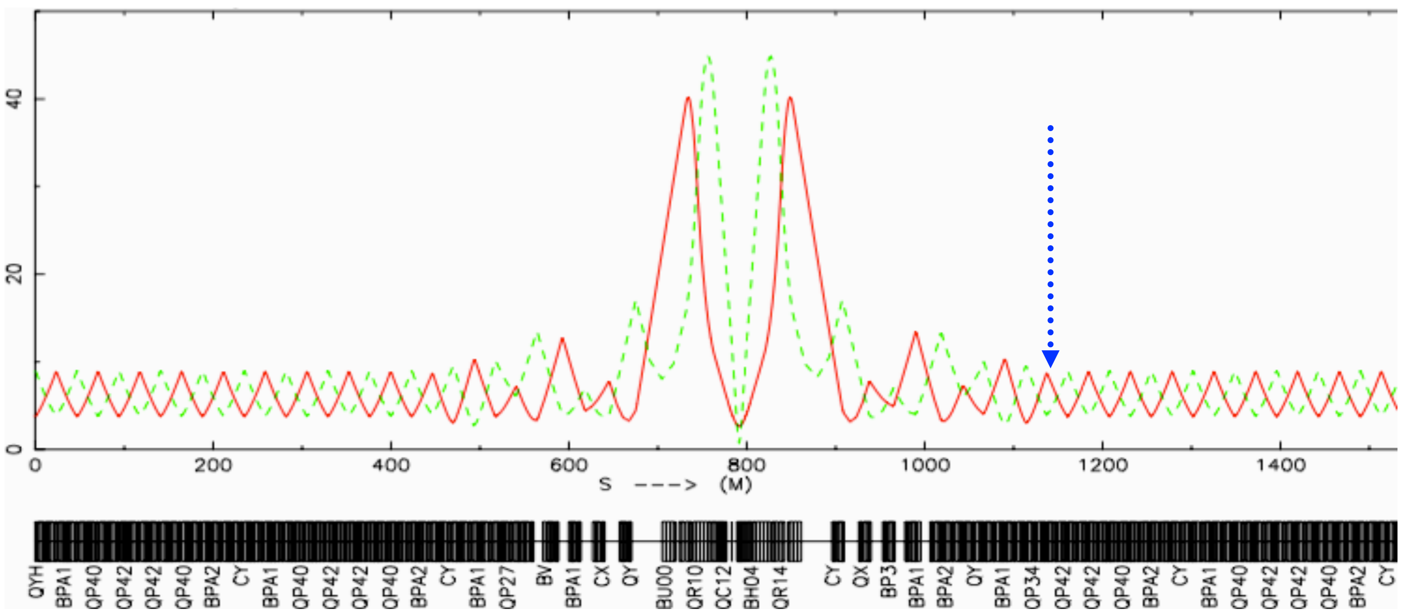
Mini-β Insertions: some guide lines

* *calculate the periodic solution in the arc*
* *introduce the drift space needed for the insertion device (detector ...)*
* *put a quadrupole doublet (triplet ?) as close as possible*
* *introduce additional quadrupole lenses to match the beam parameters to the values at the beginning of the arc structure*

parameters to be optimised & matched to the periodic solution:

$$\alpha_x,\ \beta_x \qquad D_x,\ D_x'$$
$$\alpha_y,\ \beta_y \qquad Q_x,\ Q_y$$

8 individually powered quad magnets are needed to match the insertion
(... at least)



5.) Dispersion Suppressors

There are two rules of paramount importance about dispersion:

! it is nasty
!! it is not easy to get rid of it.

remember: oscillation amplitude for a particle with momentum deviation

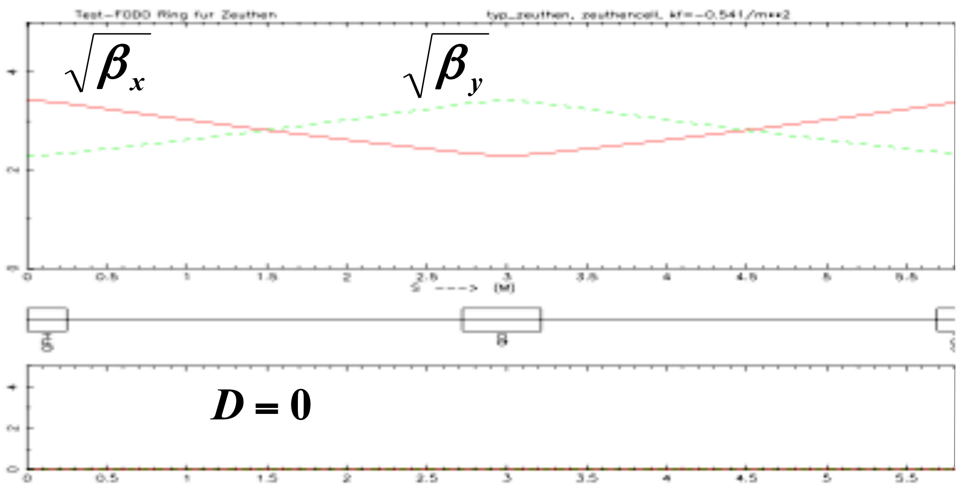
$$x(s) = x_{\beta}(s) + D(s) * \frac{\Delta p}{p}$$

beam size at the IP $\sigma^{*} = 17 \mu m$

dispersion trajectory $\left.\begin{array}{l} \overline{D} = 1.5 m \\ \frac{\Delta p}{p} \approx 1.1 * 10^{-4} \end{array}\right\} \quad x_D = 165 \mu m$

Dispersion Suppressors

$$D(s) = S(s) * \int \frac{1}{\rho(\tilde{s})} C(\tilde{s}) d\tilde{s} - C(s) * \int \frac{1}{\rho(\tilde{s})} S(\tilde{s}) d\tilde{s}$$

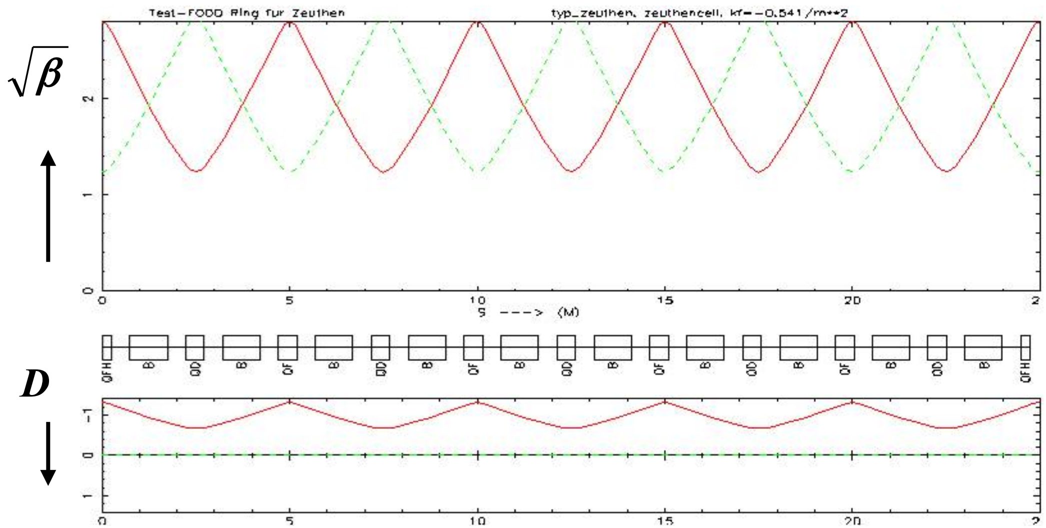


optical functions of a FoDo cell without dipoles: D = 0

Remember: Dispersion in a FoDo cell including dipoles

$$\hat{D} = \frac{\ell^2}{\rho} * \frac{\left(1 + \frac{1}{2}\sin\frac{\psi_{cell}}{2}\right)}{\sin^2\frac{\psi_{cell}}{2}} \qquad \breve{D} = \frac{\ell^2}{\rho} * \frac{\left(1 - \frac{1}{2}\sin\frac{\psi_{cell}}{2}\right)}{\sin^2\frac{\psi_{cell}}{2}}$$

FoDo cell including the effect of the bending magnets



Dispersion Suppressor Schemes

1.) The straight forward one: ***use additional quadrupole lenses to match the optical parameters ... including the D(s), D´(s) terms***

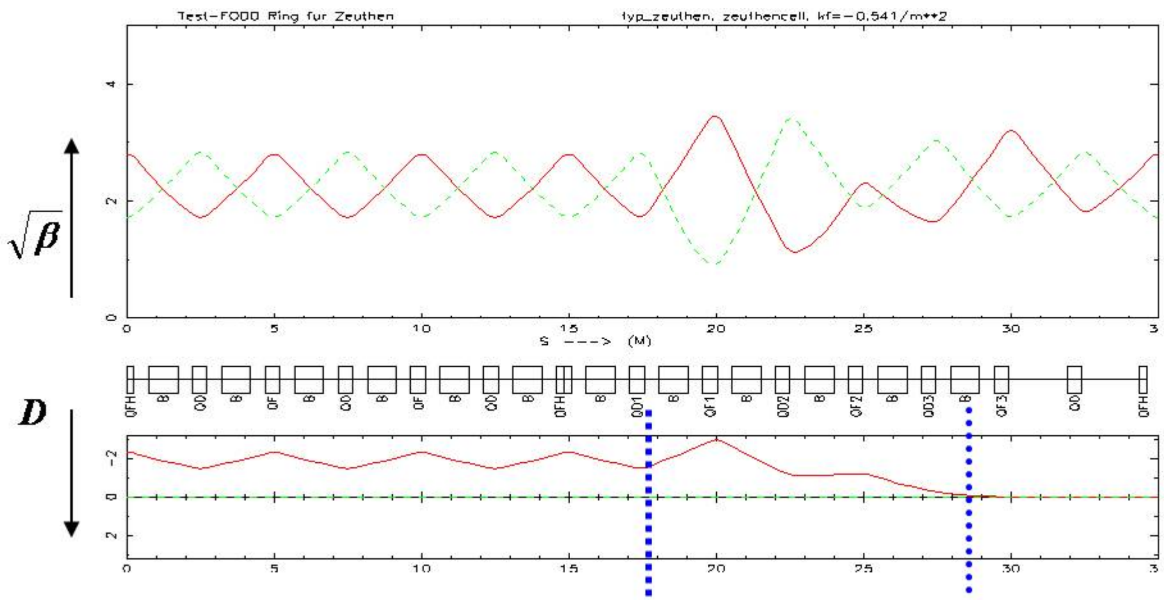
**** Dispersion suppressed by 2 quadrupole lenses,***

**** β and α restored to the values of the periodic solution by 4 additional quadrupoles***

$$\left.\begin{array}{l} D(s), \quad D'(s) \\ \beta_x(s), \alpha_x(s) \\ \beta_y(s), \alpha_y(s) \end{array}\right\} \rightarrow$$

6 additional quadrupole lenses required

Dispersion Suppressor Quadrupole Scheme



periodic FoDo structure

matching section including 6 additional quadrupoles

dispersion free section, regular FoDo without dipoles

Advantage:

- ***easy,***
- ***flexible: it works for any phase advance per cell***
- ***does not change the geometry of the storage ring,***
- ***can be used to match between different lattice structures (i.e. phase advances)***

Disadvantage:

- ***additional power supplies needed (→ expensive)***
- ***requires stronger quadrupoles***
- ***due to higher β values: more aperture required***

2.) The Missing Bend Dispersion Suppressor

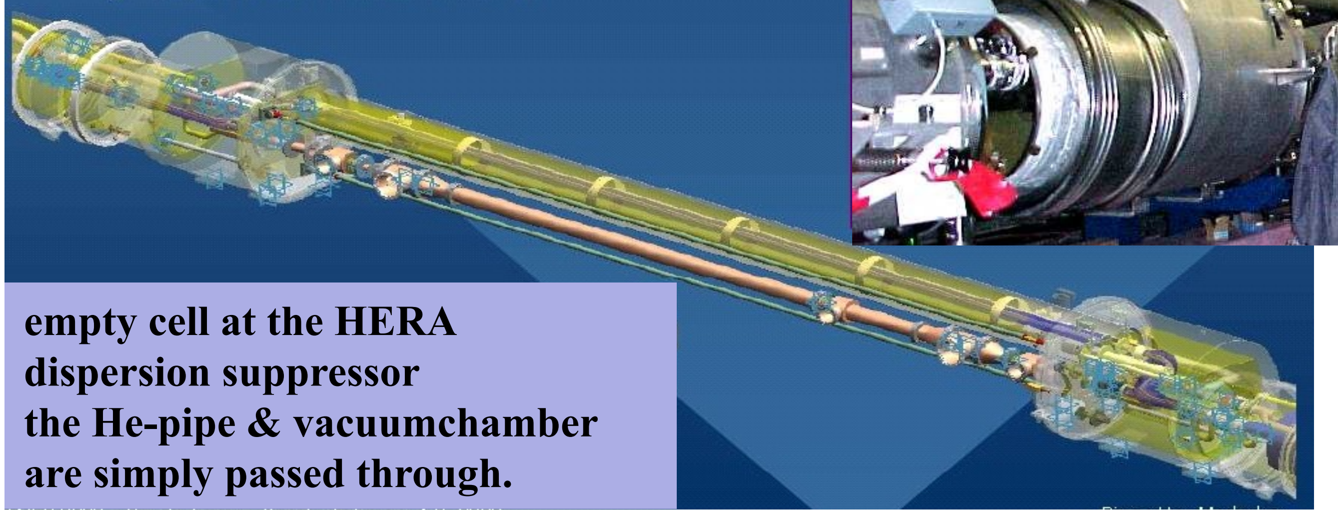
… turn it the other way round:

Start with $$D(s)=0, \quad D'(s)=0$$

and create dispersion – using dipoles - in such a way, that it fits exactly the conditions at the centre of the first regular quadrupoles:

$$D(s)=S(s)*\int\frac{1}{\rho(\tilde{s})}C(\tilde{s})d\tilde{s}-C(s)*\int\frac{1}{\rho(\tilde{s})}S(\tilde{s})d\tilde{s} \quad \rightarrow \quad \hat{D}=\frac{\ell^2}{\rho}*\frac{\left(1+\frac{1}{2}\sin\frac{\psi_{cell}}{2}\right)}{\sin^2\frac{\psi_{cell}}{2}}, \quad D'=0$$

at the end of the arc: add m cells without dipoles followed by n regular arc cells.



empty cell at the HERA dispersion suppressor
the He-pipe & vacuumchamber are simply passed through.

2.) The Missing Bend Dispersion Suppressor

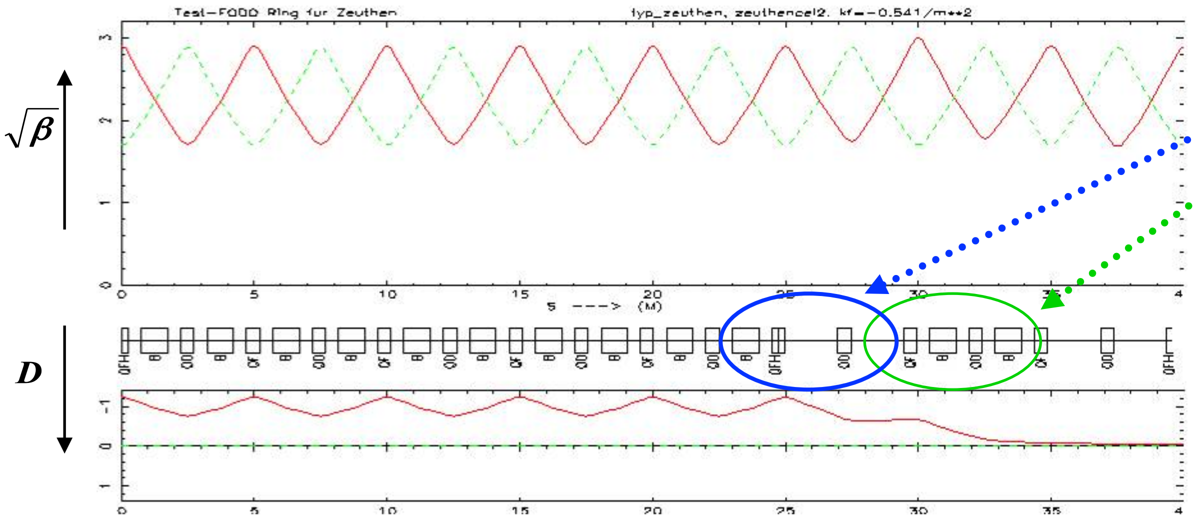
conditions for the (missing) dipole fields:

$$\frac{2m+n}{2}\Phi_C = (2k+1)\frac{\pi}{2}$$

$$\sin\frac{n\Phi_C}{2} = \frac{1}{2}\ , \ k = 0,2\ \dots\ or$$

$$\sin\frac{n\Phi_C}{2} = \frac{-1}{2}\ , \ k = 1,3\ \dots$$

m = number of cells without dipoles
followed by n regular arc cells.



Example:

phase advance in the arc $\Phi_C = 60°$
number of suppr. cells $m = 1$
number of regular cells $n = 1$

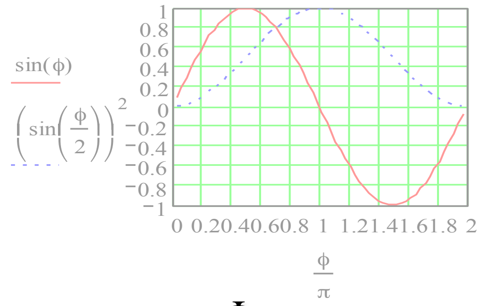
3.) The Half Bend Dispersion Suppressor

condition for vanishing dispersion: $$2*\delta_{\text{supr}}*\sin^2\left(\frac{n\Phi_c}{2}\right)=\delta_{\text{arc}}$$

so if we require $$\delta_{\text{supr}}=\frac{1}{2}*\delta_{\text{arc}}$$

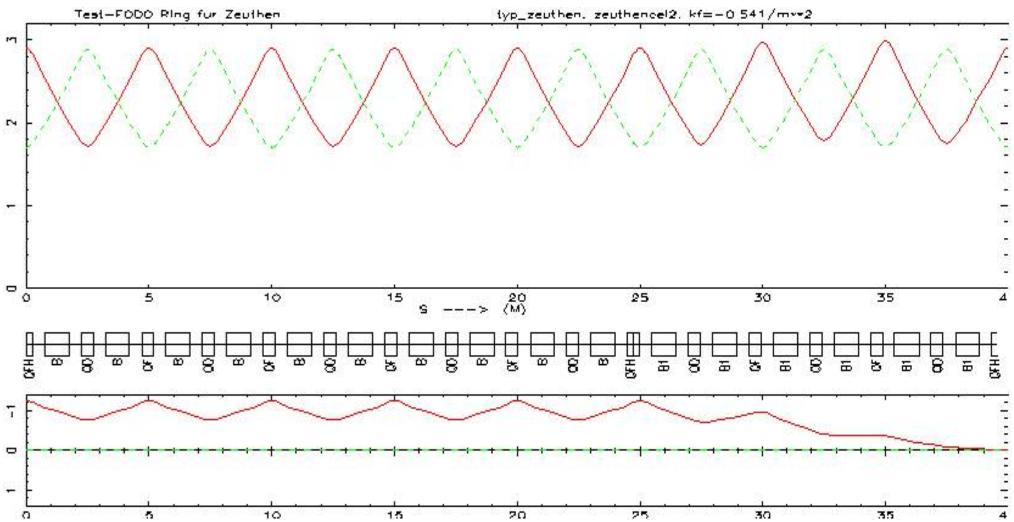
we get $$\sin^2\left(\frac{n\Phi_c}{2}\right)=1$$

$\sin(\Phi)$
$\sin^2(\Phi/2)$



Φ

and equivalent for D ‘=0 $$\sin(n\Phi_c)=0 \qquad n\Phi_c=k*\pi, \qquad k=1,3,\ldots$$

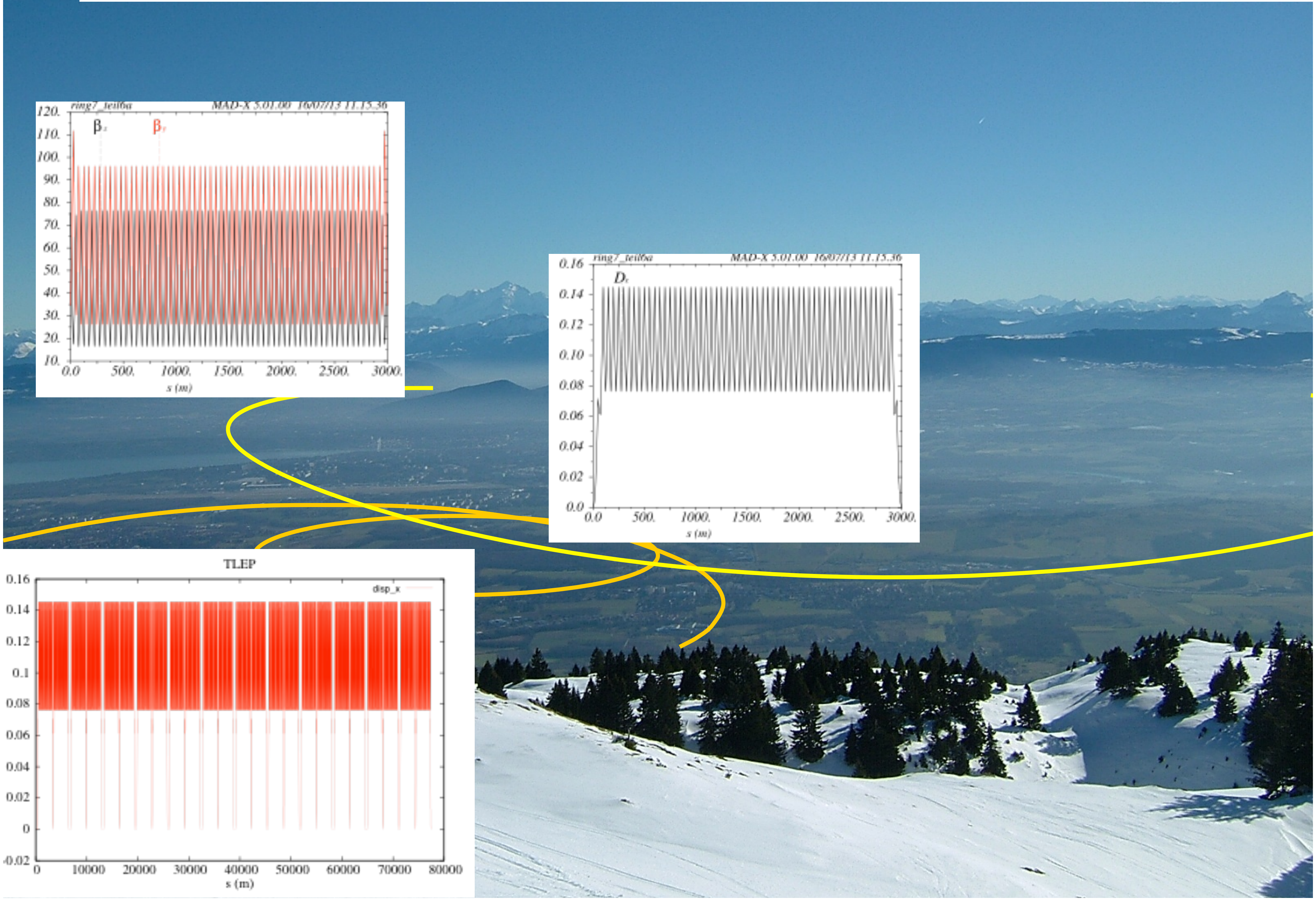


in the n suppressor cells the phase advance has to accumulate to a odd multiple of π

strength of suppressor dipoles is half as strong as that of arc dipoles, $\delta_{suppr}=1/2\ \delta_{arc}$

Example: ***phase advance in the arc***
$\Phi_C = 60°$
number of suppr. cells $n = 3$

TLEP: 100 km, 24 Arcs, **12 dispersion free straight sections**, half bend dispersion suppr



6.) Resume ‘

1.) Dispersion in a FoDo cell:

small dispersion ↔ large bending radius
short cells
strong focusing

$$\hat{D} = \frac{\ell^2}{\rho} * \frac{\left(1 + \frac{1}{2}\sin\frac{\psi_{cell}}{2}\right)}{\sin^2\frac{\psi_{cell}}{2}} \qquad \breve{D} = \frac{\ell^2}{\rho} * \frac{\left(1 - \frac{1}{2}\sin\frac{\psi_{cell}}{2}\right)}{\sin^2\frac{\psi_{cell}}{2}}$$

2.) Chromaticity of a cell:

small Q’ ↔ weak focusing
small β

$$Q'_{total} = \frac{-1}{4\pi}\oint\{K(s) - mD(s)\}\beta(s)ds$$

3.) Position of a waist at the cell end:

α_0, β_0 = values at the end of the cell

$$l = \frac{\alpha_0}{\gamma_0} \qquad \beta(l) = 1/\gamma_0$$

4.) β function in a drift

$$\beta(s) = \beta_0 - 2\alpha_0 s + \gamma_0 s^2$$

5.) Mini β insertion

small β↔ short drift space required
phase advance ≈ 180 °

$$\beta(s) = \beta_0 + \frac{s^2}{\beta_0}$$

Appendix I: Dispersion

… solution of the inhomogenious equation of motion

the dispersion function is given by

$$D(s) = S(s) * \int \frac{1}{\rho(\tilde{s})} C(\tilde{s}) d\tilde{s} - C(s) * \int \frac{1}{\rho(\tilde{s})} S(\tilde{s}) d\tilde{s}$$

proof:

$$D'(s) = S'(s) * \int \frac{1}{\rho(\tilde{s})} C(\tilde{s}) d\tilde{s} + S(s) * \frac{C(\tilde{s})}{\rho(\tilde{s})} - C'(s) * \int \frac{1}{\rho(\tilde{s})} S(\tilde{s}) d\tilde{s} - C(s) \frac{S(\tilde{s})}{\rho(\tilde{s})}$$

$$D'(s) = S'(s) * \int \frac{C}{\rho} d\tilde{s} - C'(s) * \int \frac{S}{\rho} d\tilde{s}$$

$$D''(s) = S''(s) * \int \frac{C}{\rho} d\tilde{s} + S' \frac{C}{\rho} - C''(s) * \int \frac{S}{\rho} d\tilde{s} - C' \frac{S}{\rho}$$

$$D''(s) = S''(s) * \int \frac{C}{\rho} d\tilde{s} - C''(s) * + \frac{1}{\rho}) \underbrace{(CS' - SC')}_{= \det(M) = 1}$$

$$D''(s) = S''(s) * \int \frac{C}{\rho} d\tilde{s} - C''(s) * \int \frac{S}{\rho} d\tilde{s} + \frac{1}{\rho}$$

now the principal trajectories S and C fulfill the homogeneous equation

$$S''(s) = -K * S \quad , \qquad C''(s) = -K * C$$

and so we get: $$D''(s) = -K * S(s) * \int \frac{C}{\rho} d\widetilde{s} + K * C(s) * \int \frac{S}{\rho} d\widetilde{s} + \frac{1}{\rho}$$

$$D''(s) = -K * D(s) + \frac{1}{\rho}$$

$$D''(s) + K * D(s) = \frac{1}{\rho}$$

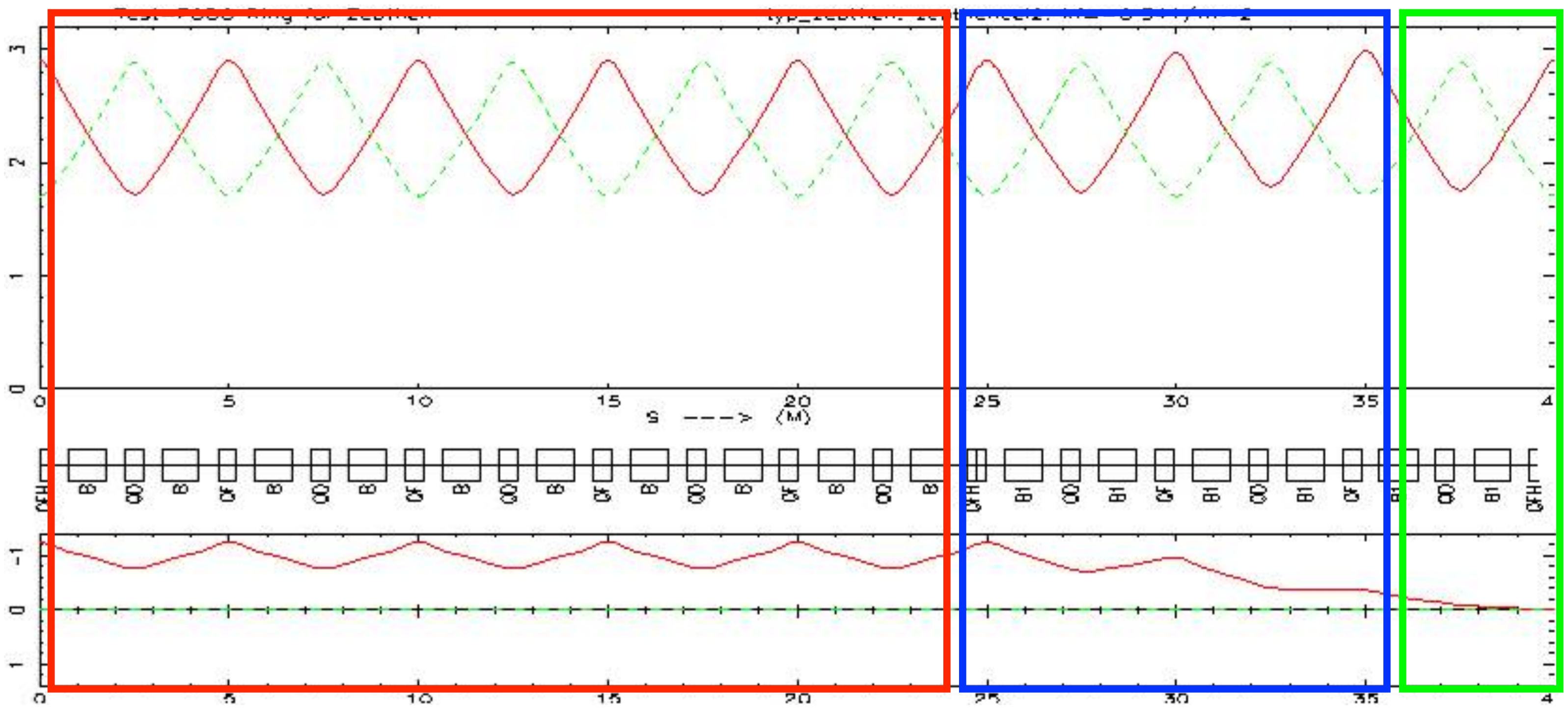
qed.

Appendix II: Dispersion Suppressors

... the calculation of the half bend scheme in full detail (for purists only)

1.) the lattice is split into 3 parts: ***(Gallia divisa est in partes tres)***

* periodic solution of the arc	periodic β, periodic dispersion D
* section of the dispersion suppressor	periodic β, dispersion vanishes
* FoDo cells without dispersion	periodic β, D = D′ = 0



2.) calculate the dispersion D in the periodic part of the lattice

transfer matrix of a periodic cell:

$$M_{0\to S} = \begin{pmatrix} \sqrt{\dfrac{\beta_S}{\beta_0}}(\cos\phi + \alpha_0 \sin\phi) & \sqrt{\beta_S\beta_0}\,\sin\phi \\ \dfrac{(\alpha_0 - \alpha_S)\cos\phi - (1+\alpha_0\alpha_S)\sin\phi}{\sqrt{\beta_S\beta_0}} & \sqrt{\dfrac{\beta_S}{\beta_0}}(\cos\phi - \alpha_S \sin\phi) \end{pmatrix}$$

for the transformation from one symmetriy point to the next (i.e. one cell) we have:
Φ_C = phase advance of the cell, α = 0 at a symmetry point. The index "*c*" refers to the periodic solution of one cell.

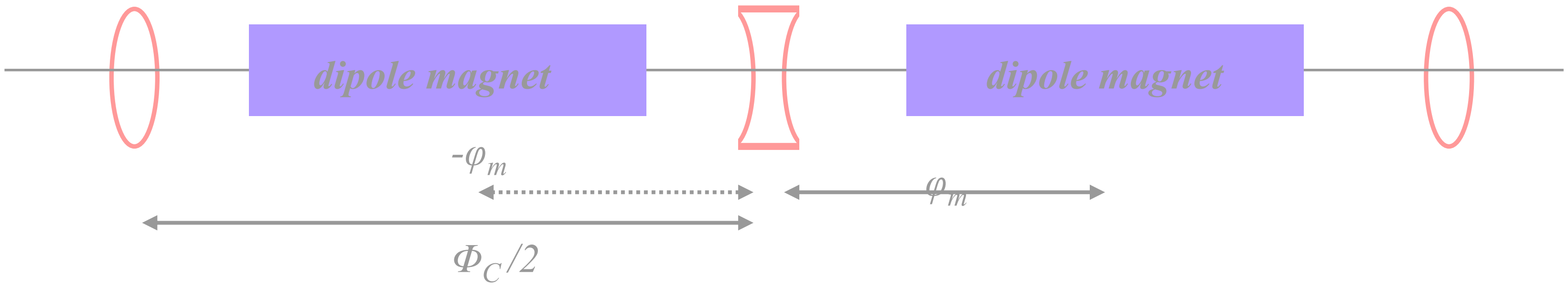
$$M_{Cell} = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\Phi_C & \beta_C \sin\Phi_C & D(l) \\ \dfrac{-1}{\beta_C}\sin\Phi_C & \cos\Phi_C & D'(l) \\ 0 & 0 & 1 \end{pmatrix}$$

The matrix elements D and D ‘ are given by the C and S elements in the usual way:

$$D(l) = S(l) * \int_0^l \frac{1}{\rho(\tilde{s})} C(\tilde{s})\, d\tilde{s} - C(l) * \int_0^l \frac{1}{\rho(\tilde{s})} S(\tilde{s})\, d\tilde{s}$$

$$D'(l) = S'(l) * \int_0^l \frac{1}{\rho(\tilde{s})} C(\tilde{s})\, d\tilde{s} - C'(l) * \int_0^l \frac{1}{\rho(\tilde{s})} S(\tilde{s})\, d\tilde{s}$$

here the values C(l) and S(l) refer to the symmetry point of the cell (middle of the quadrupole) and the integral is to be taken over the dipole magnet where $\rho \neq 0$. For ρ = const the integral over C(s) and S(s) is approximated by the values in the middle of the dipole magnet.



Transformation of C(s) from the symmetry point to the center of the dipole:

$$C_m = \sqrt{\frac{\beta_m}{\beta_C}} \cos \Delta\Phi = \sqrt{\frac{\beta_m}{\beta_C}} \cos(\frac{\Phi_C}{2} \pm \varphi_m) \qquad S_m = \beta_m \beta_C \sin(\frac{\Phi_C}{2} \pm \varphi_m)$$

where β_C is the periodic β function at the beginning and end of the cell, β_m its value at the middle of the dipole and φ_m the phase advance from the quadrupole lens to the dipole center.

Now we can solve the intergal for D and D':

$$D(l) = S(l) * \int_0^l \frac{1}{\rho(\tilde{s})} C(\tilde{s})\, d\tilde{s} - C(l) * \int_0^l \frac{1}{\rho(\tilde{s})} S(\tilde{s})\, d\tilde{s}$$

$$D(l) = \beta_C \sin\Phi_C * \frac{L}{\rho} * \sqrt{\frac{\beta_m}{\beta_C}} * \cos(\frac{\Phi_C}{2} \pm \varphi_m) - \cos\Phi_C * \frac{L}{\rho} \sqrt{\beta_m \beta_C} * \sin(\frac{\Phi_C}{2} \pm \varphi_m)$$

$$D(l) = \delta\sqrt{\beta_m\beta_C}\left\{\sin\Phi_C\left[\cos(\frac{\Phi_C}{2}+\varphi_m)+\cos(\frac{\Phi_C}{2}-\varphi_m)\right]-\right.$$

$$\left.-\cos\Phi_C\left[\sin(\frac{\Phi_C}{2}+\varphi_m)+\sin(\frac{\Phi_C}{2}-\varphi_m)\right]\right\}$$

I have put δ = L/ρ for the strength of the dipole

remember the relations $\cos x+\cos y = 2\cos\frac{x+y}{2}*\cos\frac{x-y}{2}$

$\sin x+\sin y = 2\sin\frac{x+y}{2}*\cos\frac{x-y}{2}$

$$D(l) = \delta\sqrt{\beta_m\beta_C}\left\{\sin\Phi_C*2\cos\frac{\Phi_C}{2}*\cos\varphi_m-\cos\Phi_C*2\sin\frac{\Phi_C}{2}*\cos\varphi_m\right\}$$

$$D(l) = 2\delta\sqrt{\beta_m\beta_C}*\cos\varphi_m\left\{\sin\Phi_C*\cos\frac{\Phi_C}{2}*-\cos\Phi_C*\sin\frac{\Phi_C}{2}\right\}$$

remember: $\sin 2x = 2\sin x*\cos x$

$\cos 2x = \cos^2 x-\sin^2 x$

$$D(l) = 2\delta\sqrt{\beta_m\beta_C}*\cos\varphi_m\left\{2\sin\frac{\Phi_C}{2}*\cos^2\frac{\Phi_C}{2}-(\cos^2\frac{\Phi_C}{2}-\sin^2\frac{\Phi_C}{2})*\sin\frac{\Phi_C}{2}\right\}$$

$$D(l) = 2\delta\sqrt{\beta_m\beta_C} * \cos\varphi_m * \sin\frac{\Phi_C}{2}\left\{2\cos^2\frac{\Phi_C}{2} - \cos^2\frac{\Phi_C}{2} + \sin^2\frac{\Phi_C}{2}\right\}$$

$$D(l) = 2\delta\sqrt{\beta_m\beta_C} * \cos\varphi_m * \sin\frac{\Phi_C}{2}$$

in full analogy one derives the expression for D ‘:

$$\boldsymbol{D'(l)} = 2\boldsymbol{\delta}\sqrt{\boldsymbol{\beta_m} / \boldsymbol{\beta_c}} * \cos\boldsymbol{\varphi_m} * \cos\frac{\boldsymbol{\Phi_c}}{2}$$

As we refer the expression for D and D ‘ to a periodic struture, namly a FoDo cell we require periodicity conditons:

$$\begin{pmatrix} D_C \\ D'_C \\ 1 \end{pmatrix} = M_C * \begin{pmatrix} D_C \\ D'_C \\ 1 \end{pmatrix}$$

and by symmetry: $D'_C = 0$

With these boundary conditions the Dispersion in the FoDo is determined:

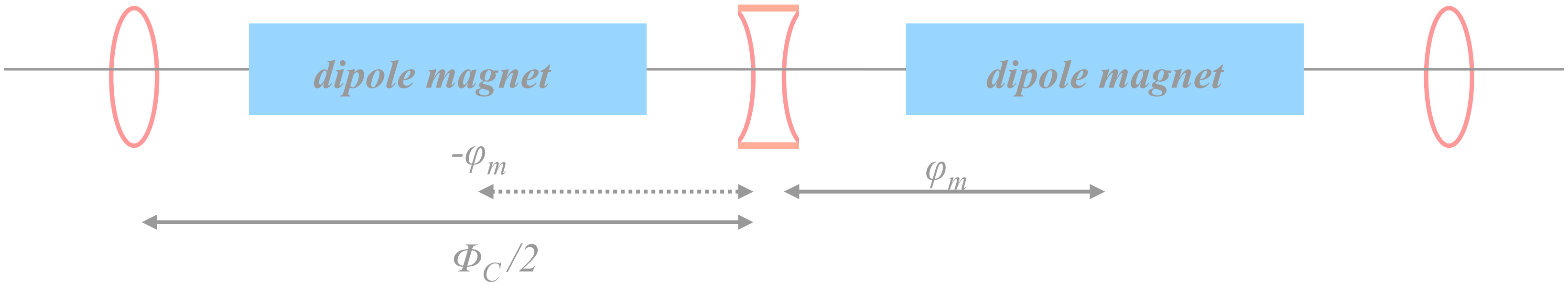
$$D_C * \cos\Phi_C + \delta\sqrt{\beta_m\beta_C} * \cos\varphi_m * 2\sin\frac{\Phi_C}{2} = D_C$$

(A1) $$D_C = \delta\sqrt{\beta_m \beta_C} * \cos\varphi_m / \sin\frac{\Phi_C}{2}$$

This is the value of the periodic dispersion in the cell evaluated at the position of the dipole magnets.

3.) Calculate the dispersion in the suppressor part:

We will now move to the second part of the dispersion suppressor: The section where ... starting from D=D ‘=0 the dispesion is generated ... or turning it around where the Dispersion of the arc is reduced to zero.
The goal will be to generate the dispersion in this section in a way that the values of the periodic cell that have been calculated above are obtained.



The relation for D, generated in a cell still holds in the same way:

$$D(l) = S(l) * \int_0^l \frac{1}{\rho(\tilde{s})} C(\tilde{s})\, d\tilde{s} - C(l) * \int_0^l \frac{1}{\rho(\tilde{s})} S(\tilde{s})\, d\tilde{s}$$

as the dispersion is generated in a number of n cells the matrix for these n cells is

$$M_n = M_C^n = \begin{pmatrix} \cos n\Phi_C & \beta_C \sin n\Phi_C & D_n \\ \frac{-1}{\beta_C}\sin n\Phi_C & \cos n\Phi_C & D'_n \\ 0 & 0 & 1 \end{pmatrix}$$

$$D_n = \beta_C \sin n\Phi_C * \delta_{\sup r} * \sum_{i=1}^{n} \cos(i\Phi_C - \frac{1}{2}\Phi_C \pm \varphi_m) * \sqrt{\frac{\beta_m}{\beta_C}} - \\ - \cos n\Phi_C * \delta_{\sup r} * \sum_{i=1}^{n} \sqrt{\beta_m \beta_C} * \sin(i\Phi_C - \frac{1}{2}\Phi_C \pm \varphi_m)$$

$$D_n = \sqrt{\beta_m \beta_C} * \sin n\Phi_C * \delta_{\sup r} * \sum_{i=1}^{n} \cos((2i-1)\frac{\Phi_C}{2} \pm \varphi_m) - \sqrt{\beta_m \beta_C} * \delta_{\sup r} * \cos n\Phi_C \sum_{i=1}^{n} \sin((2i-1)\frac{\Phi_C}{2} \pm \varphi_m)$$

remember: $\sin x + \sin y = 2\sin\frac{x+y}{2} * \cos\frac{x-y}{2}$ $\cos x + \cos y = 2\cos\frac{x+y}{2} * \cos\frac{x-y}{2}$

$$D_n = \delta_{\sup r} * \sqrt{\beta_m \beta_C} * \sin n\Phi_C * \sum_{i=1}^{n} \cos((2i-1)\frac{\Phi_C}{2}) * 2\cos\varphi_m - \\ -\delta_{\sup r} * \sqrt{\beta_m \beta_C} * \cos n\Phi_C \sum_{i=1}^{n} \sin((2i-1)\frac{\Phi_C}{2}) * 2\cos\varphi_m$$

$$D_n = 2\delta_{\sup r} * \sqrt{\beta_m \beta_C} * \cos\varphi_m \left\{ \sum_{i=1}^{n} \cos((2i-1)\frac{\Phi_C}{2}) * \sin n\Phi_C - \sum_{i=1}^{n} \sin((2i-1)\frac{\Phi_C}{2}) * \cos n\Phi_C \right\}$$

$$D_n = 2\delta_{\sup r} * \sqrt{\beta_m \beta_C} * \cos\varphi_m \left\{ \sin n\Phi_C \left\{ \frac{\sin\frac{n\Phi_C}{2} * \cos\frac{n\Phi_C}{2}}{\sin\frac{\Phi_C}{2}} \right\} - \cos n\Phi_C * \left\{ \frac{\sin\frac{n\Phi_C}{2} * \sin\frac{n\Phi_C}{2}}{\sin\frac{\Phi_C}{2}} \right\} \right\}$$

$$D_n = \frac{2\delta_{\sup r} * \sqrt{\beta_m \beta_C} * \cos\varphi_m}{\sin\frac{\Phi_C}{2}} \left\{ \sin n\Phi_C * \sin\frac{n\Phi_C}{2} * \cos\frac{n\Phi_C}{2} - \cos n\Phi_C * \sin^2\frac{n\Phi_C}{2} \right\}$$

set for more convenience x = nΦ_C/2

$$D_n = \frac{2\delta_{\sup r} * \sqrt{\beta_m \beta_C} * \cos\varphi_m}{\sin\frac{\Phi_C}{2}} \left\{ \sin 2x * \sin x * \cos x - \cos 2x * \sin^2 x \right\}$$

$$D_n = \frac{2\delta_{\sup r} * \sqrt{\beta_m \beta_C} * \cos\varphi_m}{\sin\frac{\Phi_C}{2}} \left\{ 2\sin x \cos x * \cos x \sin x - (\cos^2 x - \sin^2 x)\sin^2 x \right\}$$

(A2)

$$D_n = \frac{2\delta_{\sup r} * \sqrt{\beta_m \beta_C} * \cos\varphi_m}{\sin\frac{\Phi_C}{2}} * \sin^2\frac{n\Phi_C}{2}$$

and in similar calculations:

$$D'_n = \frac{2\delta_{\sup r} * \sqrt{\beta_m \beta_C} * \cos\varphi_m}{\sin\frac{\Phi_C}{2}} * \sin n\Phi_C$$

This expression gives the dispersion generated in a certain number of *n* cells as a function of the dipole kick δ in these cells.
At the end of the dispersion generating section the value obtained for D(s) and D ‘(s) has to be equal to the value of the periodic solution:

→equating *(A1)* and *(A2)* gives the conditions for the matching of the periodic dispersion in the arc to the values D = D ‘= 0 afte the suppressor.

$$D_n = \frac{2\delta_{\sup r} * \sqrt{\beta_m \beta_C} * \cos\varphi_m}{\sin\frac{\Phi_C}{2}} * \sin^2\frac{n\Phi_C}{2} = \delta_{arc}\sqrt{\beta_m \beta_C} * \frac{\cos\varphi_m}{\sin\frac{\Phi_C}{2}}$$

$$\left.\begin{array}{l}\rightarrow 2\delta_{\sup r}\sin^2\left(\frac{n\Phi_C}{2}\right)=\delta_{arc}\\ \rightarrow \sin(n\Phi_C)=0\end{array}\right\}\ \delta_{\sup r}=\frac{1}{2}\delta_{arc}$$

and at the same time the phase advance in the arc cell has to obey the relation:

$$n\Phi_C = k*\pi,\ \ k=1,3,\ \ldots$$