Physics of Landau Damping

An introduction (to a controversial topic)

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http://cern.ch/Werner.Herr/CAS2015_LECTURES/Otwock_Landau-Damping.pdf

Landau damping - a mystery ?

- First publication in 1946
- Applied to longitudinal oscillations of an electron plasma
  - Was not believed for $\approx$ 20 years
    (but worked in simulations and experiment)
  - Still plenty of papers every year ($\approx$ 6000 in 2012)

    (and many attempts to teach it ...)
  - Many applications: plasma physics, accelerators
  - Physical interpretation often unclear
  - Many mathematical subtleties ...

Recommended Bibliography (physics):

[LD] L.D. Landau, *J. Phys. USSR* **10** (1946) 26.

[VL] A.A. Vlasov, *J. Phys. USSR* **9** (1945) 25.

[WH] W. Herr, *Introduction to Landau Damping* in Proc. of CAS: **Advanced Accelerator Physics**, Trondheim, Norway, August 2013, CERN-2014-009 (CERN, Geneva, 2014), pp. 377-404.

[AH] A. Hofmann, *Introduction to Landau Damping*, in Proceedings of the CERN Accelerator School.

[DS] D. Sagan, On the physics of Landau damping, CLNS 93/1185 (1993).

[AC] A. Chao, *Theory of Collective Beam Instabilities in Accelerators* (Wiley, New York, 1993).

[EK] E. Keil and W. Schnell, Concerning longitudinal stability in the ISR, CERN-ISR-TH-RH/69-48 (1969).

[HV] W. Herr and L. Vos, Tune distributions and effective tune spread from beam–beam interactions and the consequences for Landau damping in the LHC, LHC Project Note 316 (2003).

[AH] Y. Alexahin, W. Herr *et al.*, Coherent beam–beam effects, Proc. HEACC 2001, Tsukuba, Japan, 2001.

Recommended Bibliography (mathematics):

[LD] L.D. Landau, *J. Phys. USSR* **10** (1946) 26.

[VK] N.G. Van Kampen, *Physica* **21** (1955) 949.

[VL] A.A. Vlasov, *J. Phys. USSR* **9** (1945) 25.

[MV] C. Mouhot and C. Villani, *arXiv:0904.2760* (2009).

[RB] R.W.B. Best, *Physica* **64** (1973) 387.

[BG] D. Bohm and E. Gross, *Phys. Rev.* **75** (1949) 1851 and 1864.

[RB] K.M. Case, *Annals of Physics* **7** (1959) 349.

[JD] J. Dawson, *Physics of Fluids* Vol.4, **7** (1961) 869.

[DR] D.D. Ryutov, *Plasma Phys.* **41** (1999) A1.

Additional material also in handout of the lecture

Why is Landau damping confusing ?

- In a plasma:
  - Landau damping damps collective oscillations
  - Leads to exponentially decaying oscillations

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- In an accelerator:
  - Landau damping does not damp anything !!

Why is Landau damping confusing ?

- In a plasma:
  - Landau damping damps collective oscillations
  - Leads to exponentially decaying oscillations
- In an accelerator:
  - Landau damping does not damp anything !!
  - We do not want exponentially decaying oscillations

    "Landau damping" is confused with decoherence
  - Landau damping stabilizes the beam, i.e.

    "Landau damping" is the absence of oscillations !!!

Why is Landau damping confusing ?

The non-trivial part:

- In a beam (any plasma) particles interact via Coulomb forces (binary collisions)
- For Landau damping: particles "interact" with the beam (collective field)

Must distinguish:

- Binary interactions (collisions) of particles
- Interactions of particles with a collective field (mode)

→ **Landau damping does not involve collisions !!!**

(If you want to remember something, remember that !)

- Often confused with "decoherence"
  - Landau damping does not lead to emittance growth
  - Decoherence does !
- Different treatment (and results !) for
  - Bunch and unbunched beams
  - Transverse and longitudinal motion

Landau damping - the menu

- Sketch Landau's treatment for plasmas
- Mechanisms of stabilization - physical origin
- Conditions for stabilization - beam transfer function and stability diagrams
- Collective motion, physics and description
- Example: how it is used, limits, problems ...
- Do not go through all formal mathematics (found in many places, or discussed in the bar), rather intuitive approach to touch the concepts, give hints ..

Why an intuitive approach ?

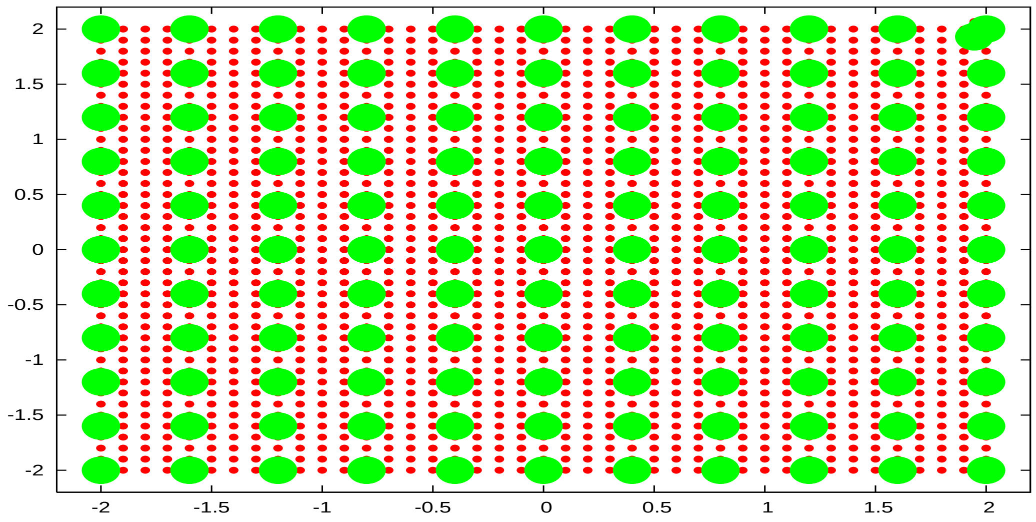
A lot of attention is often paid to interpretation of subtle (mathematical and philosophical) problems:

- Singularities
- Reversibility versus Irreversibility
- Linearity versus Non-linearity

The truth is:

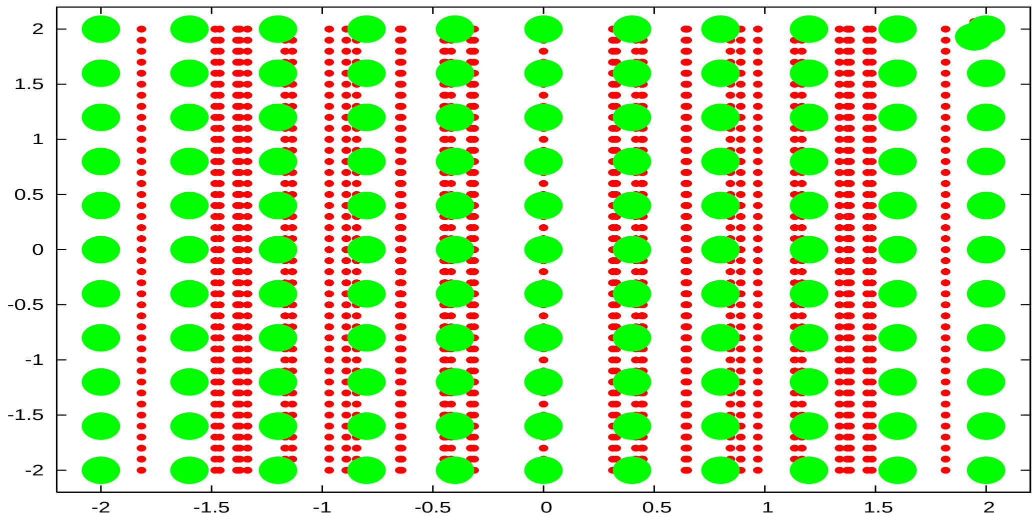
- Most "problems" are fictitious
- Not coming from the physics of the process
- Appear in specific mathematical treatment and versions of theory
- Make publications

Plasma oscillations



- Plasma without disturbance: ions (●) and electrons (•)

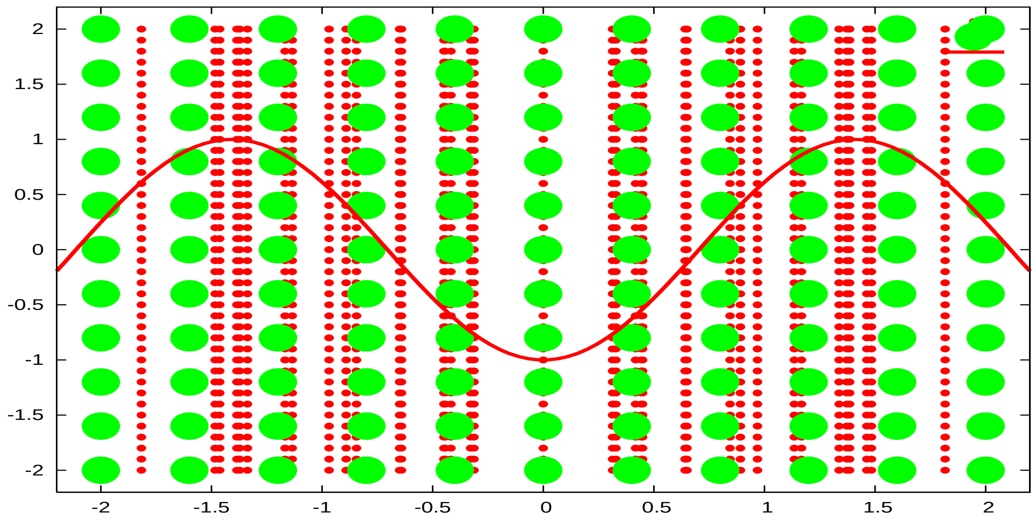
Plasma oscillations



- Plasma: stationary ions (green dot) with displaced electrons (red dot)
- Restoring force: oscillate at plasma frequency $\omega^2 = \frac{ne^2}{m\epsilon_0}$

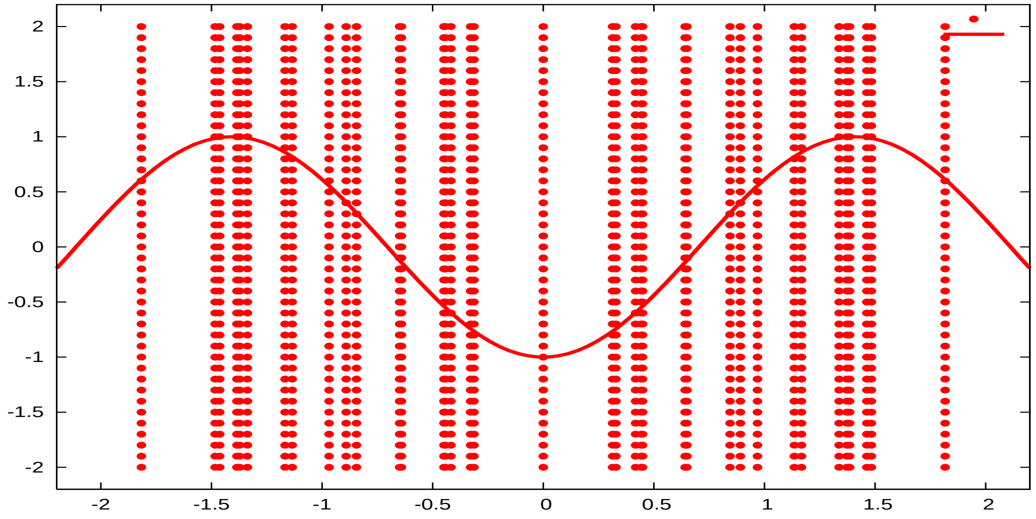
i.e. a stationary plane wave solution (Langmuir, 1929)

Plasma oscillations



- Restoring force: oscillate at plasma frequency $\omega^2 = \frac{ne^2}{m\epsilon_0}$
- Produces field (mode) of the form:
  $E(x,t) = E_0 \sin(kx - \omega t)$ (or $E(x,t) = E_0 e^{i(kx-\omega t)}$)

Plasma oscillations



- Electrons interact with the field they produce
- Field (mode) of the form:
  $E(x,t) = E_0 \sin(kx - \omega t)$ (or $E(x,t) = E_0 e^{i(kx-\omega t)}$)

Plasma oscillations

- Individual particles interact with the field produced by all particles
  - → Changes behaviour of the particles
  - → Can change the field producing the forces
  - → Particles may have different velocities !
- Self-consistent treatment required

If we allow ω to be complex ($\omega = \omega_r + i\omega_i$):

$$E(x,t) = E_0 e^{i(kx - \omega t)} \;\Rightarrow\; E(x,t) = E_0 e^{i(kx - \omega_r t)} \cdot e^{\omega_i t}$$

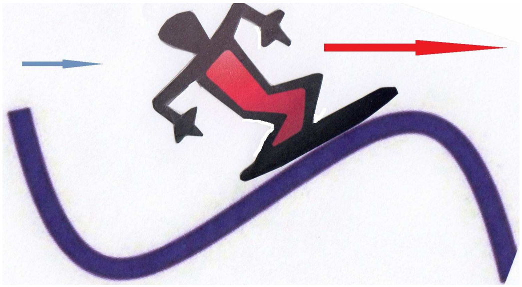
we can have a damped oscillation for $\omega_i < 0$

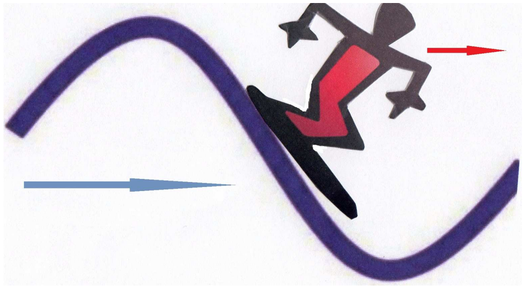
Resonance damping

- Interaction with a "mode" -

- Surfer gains energy from the mode (wave)

Interaction with a "mode"





- If Surfer faster than wave:

  mode gains energy from the surfer

- If Surfer slower than wave:

  mode loses energy to the surfer

- Does that always work like that ?

NO, consider two extreme cases:

- Surfer very fast: "jumps" across the wave crests, little interaction with the wave (water skiing)
- Surfer not moving: "oscillates" up and down with the waves

→ Wave velocity and Surfer velocity must be similar ... !!

→ Surfer is "trapped" by the wave

Interaction with a "mode"

- Remember: particles may have different velocities !
- If more particles are moving slower than the wave:
  - Net absorption of energy <u>from</u> the wave
  - Wave is damped !
- If more particles are moving faster than the wave:
  - Net absorption of energy <u>by</u> the wave
  - Wave is anti-damped !
- Always: the <u>slope</u> of the particle distribution at the wave velocity is important !

→ Have to show that now (with some theory)

Liouville theorem

- Consider an ensemble of N particles
- Described by a density distribution function $\psi(\vec{x}, \vec{p}, t)$:

$\int \psi(\vec{x}, \vec{p}, t) d\vec{x} d\vec{p} = \int \psi(\vec{x}, \vec{p}, t) dx dy dz dp_x dp_y dp_z = N$

($\vec{x}$ and $\vec{p}$ are 3·N-dimensional vectors)

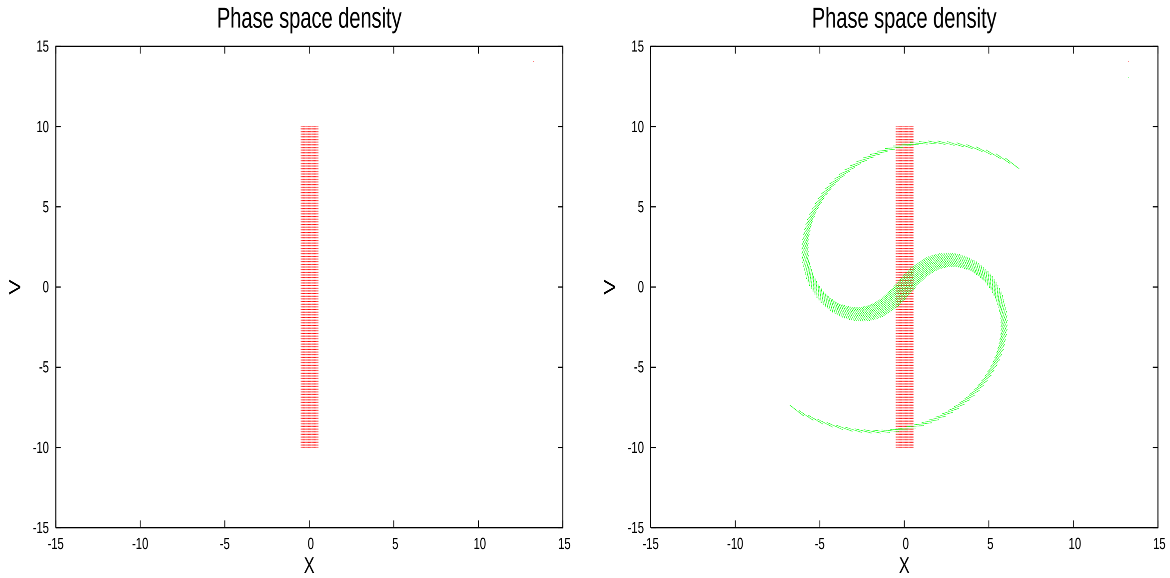
- If the distribution function is stationary $\rightarrow$

$$\boxed{\frac{\partial \psi}{\partial t} = [\psi, H] = 0}$$

→ Flashback to last week, → this is the Liouville's Theorem:

Density is conserved and in phase space moves like incompressible fluid for a Hamiltonian system

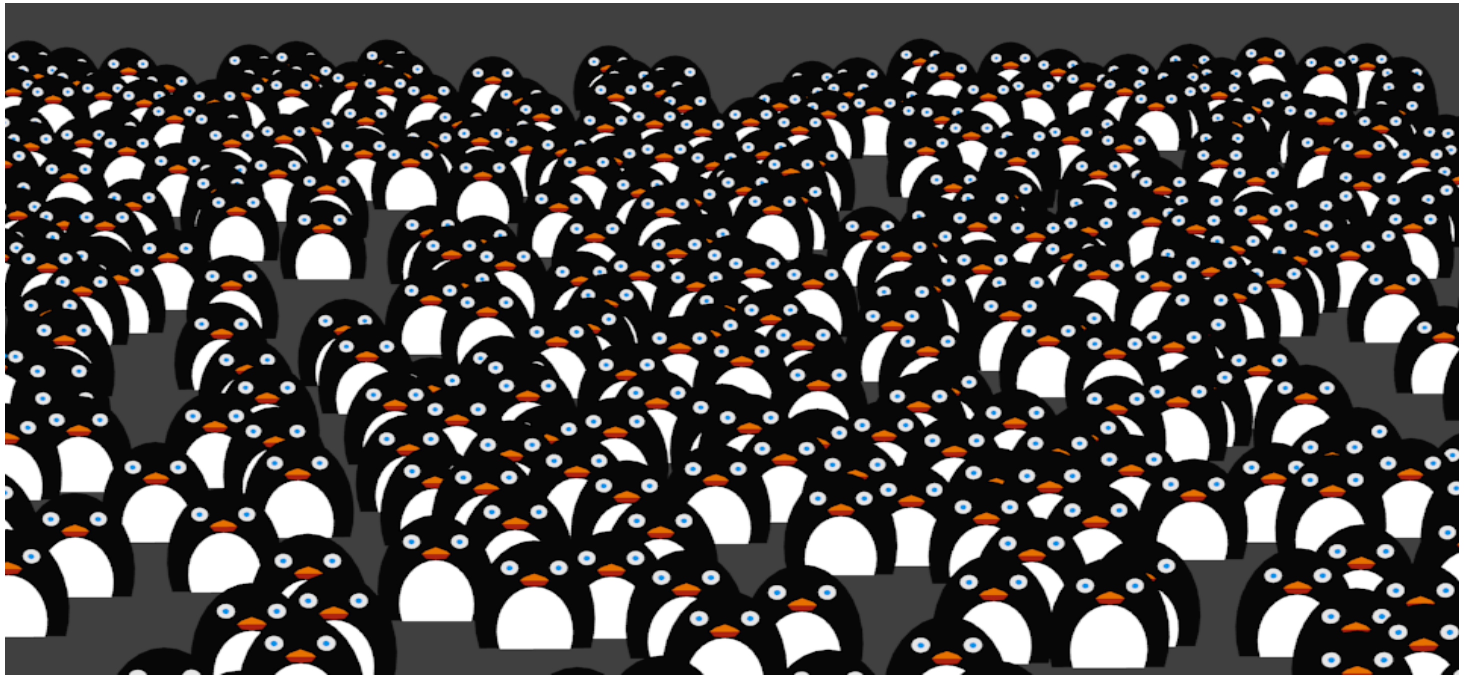
Example: "motion" of particle distribution in Phase Space



- Form of phase space distorted by non-linear motion

  (How would the picture look like for linear motion ?)

- Local phase space density is conserved
- Global density is changed (e.g. beam size)



- Local phase space density is conserved (number and distance of neighbours)
- How do we describe the evolution of the distribution ?

Despite better knowledge: move from $(\vec{x}, \vec{p})$ to $(\vec{x}, \vec{v})$

Boltzmann equation

Time evolution of $\psi(\vec{x}, \vec{v}, t)$:

$$\frac{d\psi}{dt} = \underbrace{\frac{\partial\psi}{\partial t}}_{\text{time change}} + \underbrace{\vec{v} \cdot \frac{\partial\psi}{\partial\vec{x}}}_{\text{space change}} + \underbrace{\frac{1}{m}\vec{F}(\vec{x}, t) \cdot \frac{\partial\psi}{\partial\vec{v}}}_{\text{v change, force F}} + \underbrace{\Omega(\psi)}_{\text{collision}}$$

Without collisions and stationary, it becomes Vlasov-equation:

$$\frac{d\psi}{dt} = \frac{\partial\psi}{\partial t} + \vec{v} \cdot \frac{\partial\psi}{\partial\vec{x}} + \frac{1}{m}\vec{F}(\vec{x}, t) \cdot \frac{\partial\psi}{\partial\vec{v}} = 0$$

$$\left(\text{Note}: \quad \frac{\partial\psi}{\partial\vec{x}} = \nabla\psi\right)$$

Why is the Vlasov equation useful ?

$$\frac{d\psi}{dt} \quad = \quad \frac{\partial\psi}{\partial t} + \vec{v}\cdot\frac{\partial\psi}{\partial\vec{x}} + \frac{1}{m}\vec{F}(\vec{x},t)\cdot\frac{\partial\psi}{\partial\vec{v}} \quad = \quad 0$$

$\vec{F}(\vec{x},t)$ is force of the field (mode) on the particles

Can be due to impedances, beam-beam effects, etc.

It is the basis for treatment of collective effects

From the solution one can determine whether a disturbance is:
growing: instability, negative imaginary part of frequency
decaying: stability, positive imaginary part of frequency

Strictly speaking: $\vec{F}(\vec{x},t)$ are given by external forces. When a particle interacts strongly with the collective forces produced by the other particles ($\vec{F}(\vec{x},t) \longrightarrow \vec{F}(\psi,\vec{x},t)$), they can be treated the same as external forces.

Back to Plasma Oscillations

For our problem we need:

for the force $\vec{F}$ (depending on field $\vec{E}$):

$$\vec{F} = e \cdot \vec{E}$$

for the field $\vec{E}$ (depending on potential Φ):

$$\vec{E} = -\nabla\Phi$$

for the potential Φ (depending on distribution ψ):

$$\Delta\Phi = -\frac{\rho}{\epsilon_0} = -\frac{e}{\epsilon_0}\int \psi dv$$

Therefore:

$$\frac{d\psi}{dt} = \frac{\partial\psi}{\partial t} + \vec{v}\cdot\frac{\partial\psi}{\partial\vec{x}} + \frac{1}{m}\vec{E}(\vec{x},t)\cdot\frac{\partial\psi}{\partial\vec{v}} = 0$$

and:

$$\Delta\Phi = \frac{e}{\epsilon_0}\int\psi dv$$

Coupled equations: perturbation produces field which acts back on perturbation.

Do we find a solution ?

Plasma oscillations

Assume a small non-stationary perturbation ψ_1 on the stationary distribution $\psi_0(\vec{v})$:

$$\psi(\vec{x},\vec{v},t) \;=\; \psi_0(\vec{v}) + \psi_1(\vec{x},\vec{v},t)$$

Then we get:

$$\frac{d\psi}{dt} \;=\; \frac{\partial\psi_1}{\partial t} + \vec{v}\cdot\frac{\partial\psi_1}{\partial\vec{x}} + \frac{1}{m}\vec{E}(\vec{x},t)\cdot\frac{\partial\psi_0}{\partial\vec{v}} \;=\; 0$$

and:

$$\Delta\Phi = -\frac{\rho}{\epsilon_0} = -\frac{e}{\epsilon_0}\int\psi_1 dv$$

Plasma oscillations

$$\psi_1(\vec{x}, \vec{v}, t) \implies \vec{E}(\vec{x}, t) \implies \psi_1(\vec{x}, \vec{v}, t) \implies \ldots$$

- Density perturbation produces electric field
- Electric field acts back and changes density perturbation
- Change with time ..
- How can we attack that ?

Plasma oscillations - Vlasov's approach

Expand as double Fourier transform:*)

$$\psi_1(\vec{x}, \vec{v}, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \tilde{\psi}_1(k, \vec{v}, \omega) e^{i(kx - \omega t)} dk d\omega$$

$$\Phi(\vec{x}, \vec{v}, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \tilde{\Phi}(k, \vec{v}, \omega) e^{i(kx - \omega t)} dk d\omega$$

and apply to Vlasov equation

*) Remember: we assumed the field (mode) of the form:
$E(x, t) = E_0 e^{i(kx - \omega t)}$

Plasma oscillations

Assuming a perturbation as above, the condition for a solution is:

$$1 + \frac{e^2}{\epsilon_0 m k} \int \frac{\partial \psi_0 / \partial v}{(\omega - kv)} dv = 0$$

This is the Dispersion Relation for plasma waves

i.e. relation between frequency (ω) and wavelength (k)

Plasma oscillations

Looking at this relation:

- It depends on the (velocity) distribution ψ
- It depends on the slope of the distribution $\partial\psi_0/\partial v$
- The effect is strongest for velocities close to the wave velocity, i.e. $v \approx \frac{\omega}{k}$

→ There seems to be a complication (singularity) at $v \equiv \frac{\omega}{k}$

Can we deal with this problem ?

Dealing with the singularity

- Hand waving argument [VL]:
  - In practice ω is never real (collisions !)
- Optimistic argument [BG]:
  - $\partial\psi_0/\partial v \;=\; 0$ where $v \equiv \frac{\omega}{k}$
- Alternative approach [VK]:
  - Search for stationary solutions (normal mode expansion)
  - Continous versus discrete modes (not treated here)
- Better argument (with 20/20 hindsight) [LD]:
  - Initial value problem with perturbation $\psi_1(\vec{x}, \vec{v}, t)$ at $t \;=\; 0$, (time dependent solution with complex ω)
  - Procedure: in time domain use Laplace transformation

    in space domain use Fourier transformation

Plasma oscillations - Landau's approach

Fourier transform in space domain:

$$\tilde{\psi}_1(k,\vec{v},t) \;=\; \frac{1}{2\pi}\int_{-\infty}^{+\infty}\psi_1(\vec{x},\vec{v},t)e^{i(kx)}dx$$

$$\tilde{E}(k,t) \;=\; \frac{1}{2\pi}\int_{-\infty}^{+\infty}E(\vec{x},t)e^{i(kx)}dx$$

and Laplace transform in time domain:

$$\Psi_1(k,\vec{v},p) \;=\; \int_{0}^{+\infty}\tilde{\psi}_1(k,\vec{v},t)e^{(-pt)}dt$$

$$\mathcal{E}(k,p) \;=\; \int_{0}^{+\infty}\tilde{E}(k,t)e^{(-pt)}dt$$

Plasma oscillations

In Vlasov equation and after some algebra (see books) this leads to the modified dispersion relation:

$$1 + \frac{e^2}{\epsilon_0 m k}\left[P.V. \int \frac{\partial\psi_0/\partial v}{(\omega - kv)} dv - \frac{i\pi}{k}\left(\frac{\partial\psi_0}{\partial v}\right)_{v=\omega/k}\right] = 0$$

P.V. refers to "Cauchy Principal Value" (see mathbooks or ask a tutor)

Second term only in Landau's treatment

→ responsible for damping, appears only in the Initial value problem

Plasma oscillations

Evaluating the term:

$$-\frac{i\pi}{k}\left(\frac{\partial\psi}{\partial v}\right)_{v=\omega/k}$$

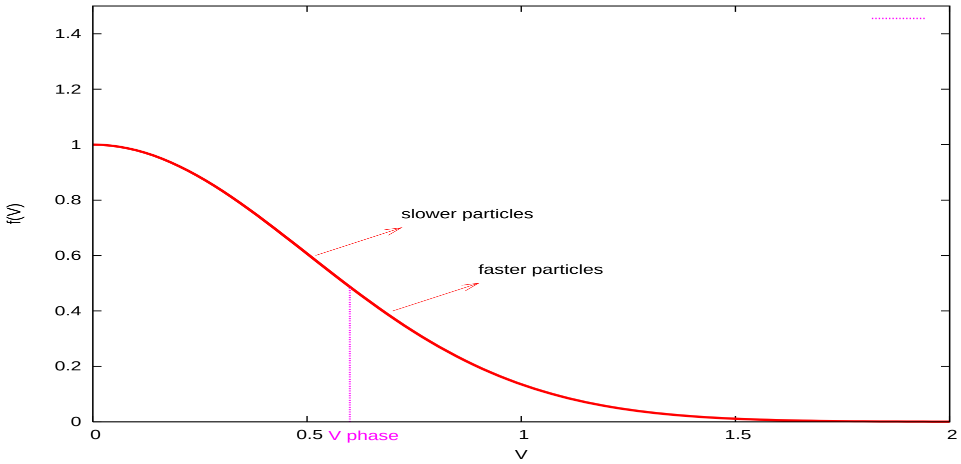
- ω is complex and the imaginary part becomes:

$$Im(\omega) = \omega_i = \frac{\pi}{2}\frac{\omega_p e^2}{\epsilon_0 m k^2}\left(\frac{\partial\psi}{\partial v}\right)_{v=\omega/k}$$

- Get a damping (without collisions) if: $\left(\frac{\partial\psi}{\partial v}\right)_{v=\omega/k} < 0$

→ Landau Damping

Velocity distribution



- Distribution of particle velocities (e.g. Maxwellian distribution) relative to wave velocity
- More "slower" than "faster" particles ➡ damping
- More "faster" than "slower" particles ➡ anti-damping

➡ Therefore: slope is important !

Warning: a paradox

- For a bar discussion (or a question tomorrow) →

- Lets consider a Lorentz transformation (which must not change the physics):

  It is possible to go to a Lorentz frame which is moving relative to the particles faster than the wave (phase-) velocity !

  Then ALL particles are faster than the wave velocity !!

  In this frame we always have anti-damping !!!

- Is this true ???

Now what about accelerators ???

- Landau damping in plasmas, all right
- Physical origin rather simple
- How to apply it in accelerators ?
- We have:
  - No plasmas but beams
  - No distribution of velocity, but tune
  - No electrons, but ions (e.g. p)
  - Also transverse oscillations

Now what about accelerators ???

- How to apply it in accelerators ?
- Can be formally solved using Vlasov equation, but physical interpretation very fuzzy (and still debated ..)
- Different (more intuitive) treatment ([AC], [AH],[DS])
- Look now at:
  - Beam response to excitation
  - Beam transfer function and stability diagrams
  - Phase mixing
  - Conditions and tools for stabilization, problems

Response of a beam to excitations

How does a beam respond to an external excitation?

Consider a harmonic, linear oscillator with frequency ω driven by an external sinusoidal force $f(t)$ with frequency Ω: The equation of motion is:

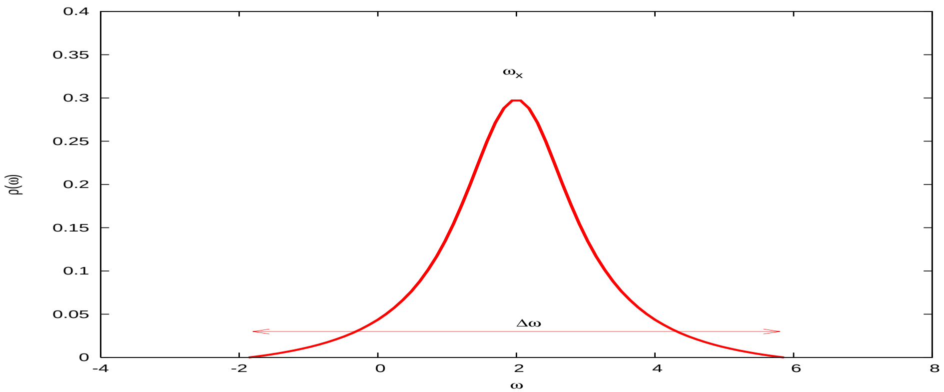
$$\ddot{x} + \omega^2 x = A\ cos \Omega t = f(t)$$

for initial conditions $x(0) = 0$ and $\dot{x}(0) = 0$ the solution is:

$$x(t) = -\frac{A}{(\Omega^2 - \omega^2)}(cos\ \Omega t \quad \underbrace{-\ cos\ \omega t}_{x(0)=0, \dot{x}(0)=0}\ )$$

In general a beam consists of an ensemble of oscillators with <u>different</u> frequencies ω with a distribution $\rho(\omega)$ and a spread $\Delta\omega$.
Number of particles per frequency band:

$\rho(\omega) \;=\; \frac{1}{N} dN/d\omega$ with $\int_{-\infty}^{\infty} \rho(\omega) d\omega = 1$



reminder: for a transverse (betatron motion) ω_x is the tune !

IMPORTANT MESSAGE !

- $\rho(\omega)$ is distribution of external focusing frequencies !
  - Transverse, bunched and unbunched beams: betatron tune
  - Longitudinal, bunched beams: synchrotron tune
  - Longitudinal, unbunched beams: ??? (see later !)
- $\Delta\omega$ is spread of external focusing frequencies !

Given the frequency distribution $\rho(\omega) = \frac{1}{N} dN/d\omega$
and the single particle response:

$$x(t) = -\frac{A}{(\Omega^2 - \omega^2)}(cos\ \Omega t \quad \underbrace{- \ cos\ \omega t}_{x(0)=0, \dot{x}(0)=0} \ )$$

The average beam response (centre of mass) is then:

$$< x(t) >= \int_{-\infty}^{\infty} x(t)\rho(\omega)d\omega =$$

$$< x(t) >= -\int_{-\infty}^{\infty}\left[\frac{A}{(\Omega^2 - \omega^2)}(cos\ \Omega t\ -\ cos\ \omega t)\right]\rho(\omega)d\omega$$

We can re-write (simplify) the expression

$$< x(t) >= -\int_{-\infty}^{\infty} \left[\frac{A}{(\Omega^2 - \omega^2)} (cos\ \Omega t\ -\ cos\ \omega t) \right] \rho(\omega) d\omega$$

for a narrow beam spectrum around a frequency ω_x (tune) and the driving force near this frequency $\Omega \approx \omega_x$ *)

$$< x(t) >= -\frac{A}{2\omega_x} \int_{-\infty}^{\infty} \left[\frac{1}{(\Omega - \omega)} (cos\ \Omega t\ -\ cos\ \omega t) \right] \rho(\omega) d\omega$$

For the further evaluation we transform variables from ω to $u = \omega - \Omega$, and assume that Ω is complex: $\Omega = \Omega_r + i\Omega_i$

*) **justified later ... (but you may already guess !)**

We get now two contributions to the integral:

$$
\begin{aligned}
< x(t) > \quad = \quad & -\frac{A}{2\omega_x} cos(\Omega t) \int_{-\infty}^{\infty} du \;\rho(u+\Omega) \frac{1-cos(ut)}{u} \\
& + \frac{A}{2\omega_x} sin(\Omega t) \int_{-\infty}^{\infty} du \;\rho(u+\Omega) \frac{sin(ut)}{u}
\end{aligned}
$$

This avoids singularities for $u = 0$

We are interested in long term behaviour,

i.e. $t \to \infty$, so we use:

$$\lim_{t\to\infty} \frac{sin(ut)}{u} = \pi\delta(u)$$

$$\lim_{t\to\infty} \frac{1-cos(ut)}{u} = P.V. \left(\frac{1}{u}\right)$$

and obtain for the asymptotic behaviour (back to ω, Ω)$^{*)}$:

$$< x(t) >= \frac{A}{2\omega_x}\left[\pi\rho(\Omega)sin(\Omega t) + cos(\Omega t)P.V. \int_{-\infty}^{\infty} d\omega \frac{\rho(\omega)}{(\omega - \Omega)}\right]$$

The response or Beam Transfer Function has a:

Resistive part: absorbs energy from oscillation → damping (would not be there without the term $-cos\ \omega t$)

Reactive part: "capacitive" or "inductive", depending on sign of term relative to driving force

$^{*)}$ Assuming Ω is complex, we integrate around the pole and obtain a 'principal value P.V.' and a 'residuum'

Response of a beam to excitations

What do we see:

- The "damping" part only appeared because of the initial conditions $x(0) = 0$ and $\dot{x}(0) = 0$!!!

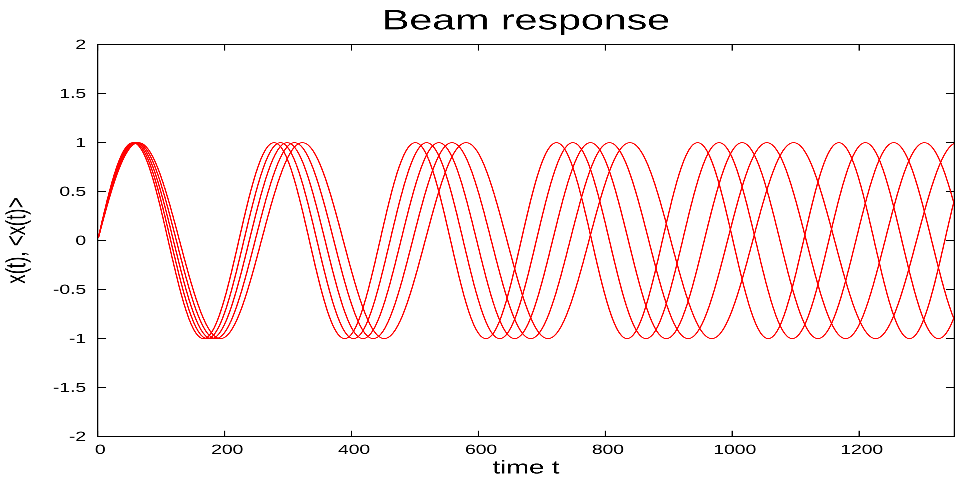
- With other initial conditions, we get additional terms in the beam response

- I.e. for $x(0) \neq 0$ and $\dot{x}(0) \neq 0$ we may add:

$$x(0)\int d\omega\rho(\omega)cos(\omega t) + \dot{x}(0)\int d\omega\rho(\omega)\frac{sin(\omega t)}{\omega}$$

➤ Do not participate in the dynamics, what do they do ?

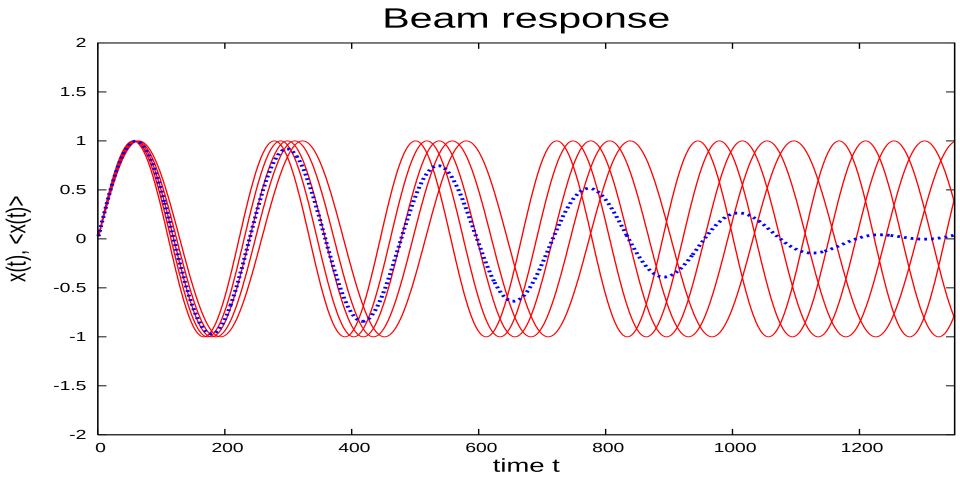
Particle motion



- Oscillation of particles with different tunes

Initial conditions: $x(0) = 0$ and $\dot{x}(0) \neq 0$

Particle motion

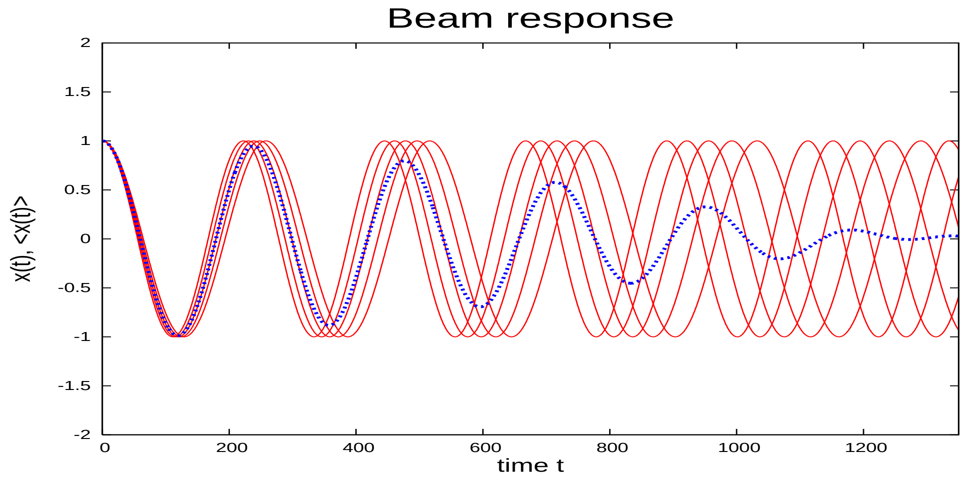


— Oscillation of particles with different tunes

Initial conditions: $x(0) = 0$ and $\dot{x}(0) \neq 0$

— Average over particles, centre of mass motion

Particle motion



- Oscillation of particles with different tunes

  Initial conditions: $x(0) \neq 0$ and $\dot{x}(0) = 0$

- Average over particles, centre of mass motion

This is NOT Landau Damping !!

End of Part 1, to remember:

- Landau Damping is not to be confused with Decoherence
- It relies on interactions with collective fields, collisionless
- Initial conditions provide the "damping part" in the (dispersion) equations:

  Stable beam at the beginning: Landau Damping inhibits the instability

  "Landau damping" is the absence of oscillations !!!

Physics of Landau Damping

Part 2

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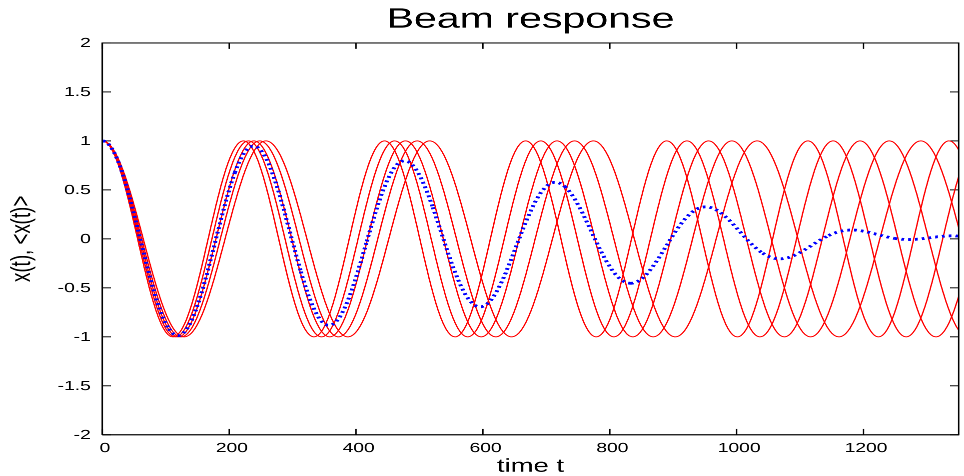
http://cern.ch/Werner.Herr/CAS2015/lectures/Otwock/Landau-damping.pdf

Interpretation of Landau Damping

- Initial conditions: $x(0) = 0$ and $\dot{x}(0) = 0$, beam is quiet
- Spread of frequencies $\rho(\omega)$
- When an excitation is applied:
  - Particles cannot organize into collective response (phase mixing)
  - Average response is zero
  - The beam is kept stable, i.e. stabilized

Particle motion



- Oscillation of particles with different tunes

  Initial conditions: $x(0) \neq 0$ and $\dot{x}(0) = 0$

- Average over particles, centre of mass motion

This is NOT Landau Damping !!

Interpretation of Landau Damping

- Initial conditions: $x(0) = 0$ and $\dot{x}(0) = 0$, beam is quiet
- Spread of frequencies $\rho(\omega)$
- When an excitation is applied:
  - Particles cannot organize into collective response (phase mixing)
  - Average response is zero
  - The beam is kept stable, i.e. stabilized

→ Next : quantitative analysis

Response of a beam to excitations

For this, we re-write (simplify) the response in complex notation:

$$< x(t) >= \frac{A}{2\omega_x} \left[\pi\rho(\Omega) sin(\Omega t) + cos(\Omega t) P.V. \int_{-\infty}^{\infty} d\omega \frac{\rho(\omega)}{(\omega - \Omega)} \right]$$

becomes:

$$< x(t) >= \frac{A}{2\omega_x} e^{-i\Omega t} \left[P.V. \int d\omega \frac{\rho(\omega)}{(\omega - \Omega)} + i\pi\rho(\Omega) \right]$$

First part describes oscillation with complex frequency Ω

Response of a beam to excitations

Reminds us a few things ....

Since we know the collective motion is described as $e^{(-i\Omega t)}$

For an oscillating solution Ω must fulfill the relation

$$1 + \frac{1}{2\omega_x}\left[P.V. \int d\omega \frac{\rho(\omega)}{(\omega - \Omega)} + i\pi\rho(\Omega)\right] = 0$$

This is again a dispersion relation, i.e. condition for oscillating solution.

What do we do with that ??

Well, look where $\Omega_i < 0$ provides damping !!

Note: no contribution to damping when Ω outside spectrum !!

Simplify by moving to normalized parametrization.

Following Chao's proposal, in the expression:

$$< x(t) >= \frac{A}{2\omega_x} e^{-i\Omega t} \left[P.V. \int d\omega \frac{\rho(\omega)}{(\omega - \Omega)} + i\pi\rho(\Omega) \right]$$

we use again u, but normalized to frequency spread $\Delta\omega$:

$$u \ = (\omega_x - \Omega) \qquad \overbrace{\Longrightarrow}^{normalize} \qquad u \ = \frac{(\omega_x - \Omega)}{\Delta\omega}$$

and introduce two functions $f(u)$ and $g(u)$:

$$\boxed{f(u) = \Delta\omega P.V. \int d\omega \frac{\rho(\omega)}{\omega - \Omega}}$$

$$\boxed{g(u) = \pi\Delta\omega\rho(\omega_x - u\Delta\omega) = \pi\Delta\omega\rho(\Omega)}$$

remember: ω_x $\approx$ tune, Ω is the driving frequency

The response with the driving force reads now:

$$< x(t) >= \frac{A}{2\omega_x \ \Delta\omega} \ e^{-i\Omega t} \left[f(u) + i \cdot g(u) \right]$$

where $\Delta\omega$ is the frequency spread of the distribution.

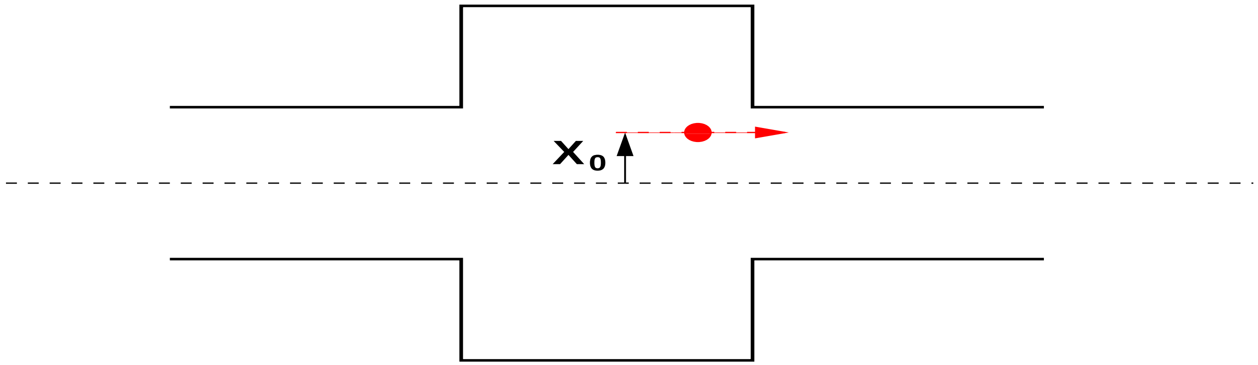
The expression $f(u) + i \cdot g(u)$ is the Beam Transfer Function
Easier with this to evaluate the different cases and examples ....

For important distributions $\rho(\omega)$ analytical functions f(u) and g(u) exist (see e.g. Chao, Tigner, "Handbook ..")

Will lead us to stability diagrams.

Response of a beam in presence of wake fields

Example: the driving force comes from the displacement of the beam as a whole, i.e. $< x > = X_0$! For example driven by a wake field or impedance.



The equation of motion for a particle is then something like:

$$\ddot{x} + \omega^2 x = f(t) = K \cdot < x >$$

where K is a "coupling coefficient"

Coupling coefficient K depends on nature of wake field:

- Purely real:
  - Force in phase with the displacement
  - e.g. image space charge in perfect conductor
- Purely imaginary:
  - Force in phase with the velocity
- In practice, have both and we can write:

$$K = 2\omega_x (U - iV)$$

Response of a beam in presence of wake fields

Interpretation:

- A beam travelling off centre through an impedance induces transverse fields
- Transverse fields kick back on all particles in the beam, via:

$$\ddot{x} + \omega^2 x = f(t) = K \cdot < x >$$

- If beam moves as a whole (in phase, collectively !) this can grow for $V > 0$
- The coherent frequency Ω becomes complex and shifted by $(\Omega - \omega_x)^{*)}$

$^{*)}$ without impedance: $\Omega = \omega_x$ (betatron frequency, i.e.tune)

For a beam without frequency spread ($\rho(\omega) = \delta(\omega - \omega_x)$) we can easily sum over all particles and for the centre of mass motion $< x >$ we get:

$$< \ddot{x} > + \Omega^2 < x > = f(t) = -2\omega_x(U - iV) \cdot < x >$$

- For the original coherent motion with frequency Ω this means
  - In-phase component U changes the frequency
  - Out-of-phase component V creates growth ($V > 0$) or damping ($V < 0$)

For any $V > 0$ the beam is unstable (even if very small) !!

Response of a beam in presence of wake fields

What happens for a beam with an frequency spread ?

The response (and therefore the driving force) was:

$$< x(t) > = \frac{A}{2\omega_x \ \Delta\omega} \ e^{-i\Omega t} \left[f(u) + i \cdot g(u) \right]$$

Response of a beam in presence of wake fields

The (complex) frequency Ω is now determined by the condition:

$$-\frac{(\Omega - \omega_x)}{\Delta\omega} = \frac{1}{(f(u) + ig(u))}$$

All information about stability contained in this relation !

- The (complex) frequency difference $(\Omega - \omega_x)$ contains impedance, intensity, γ, ... (see lecture by G. Rumolo).
- The right hand side contains information about the frequency spectrum (see definitions for $f(u)$ and $g(u)$).

Without Landau damping (no frequency spread):

- If $\Im(\Omega - \omega_x) \; < \; 0$ beam is stable
- If $\Im(\Omega - \omega_x) \; > \; 0$ beam is unstable (growth rate τ^{-1} !)

With Landau damping we have a condition for stability:

$$-\frac{(\Omega - \omega_x)}{\Delta\omega} \quad = \quad \frac{1}{(f(u) + ig(u))}$$

How to proceed to find limits ?

Could find the complex Ω at edge of stability ($\tau^{-1} \; = \; 0$!)

→ Can do a bit more ...

Stability diagram

Look at the right hand side first.

Take the (real) parameter u in

$$D_1 \quad = \quad \frac{1}{(f(u) + ig(u))}$$

1 Scan u from $-\infty$ to $+\infty$

2 Plot the real and imaginary part of D_1 in complex plane

Why is this formulation interesting ???

The expression:

$$(f(u) + ig(u))$$

is actually the Beam Transfer Function,

i.e. it can be measured !!

- With its knowledge (more precise: its inverse) we have conditions on $(\Omega - \omega_x)$ for stability
- Intensities, impedances, ...

Example: rectangular distribution:

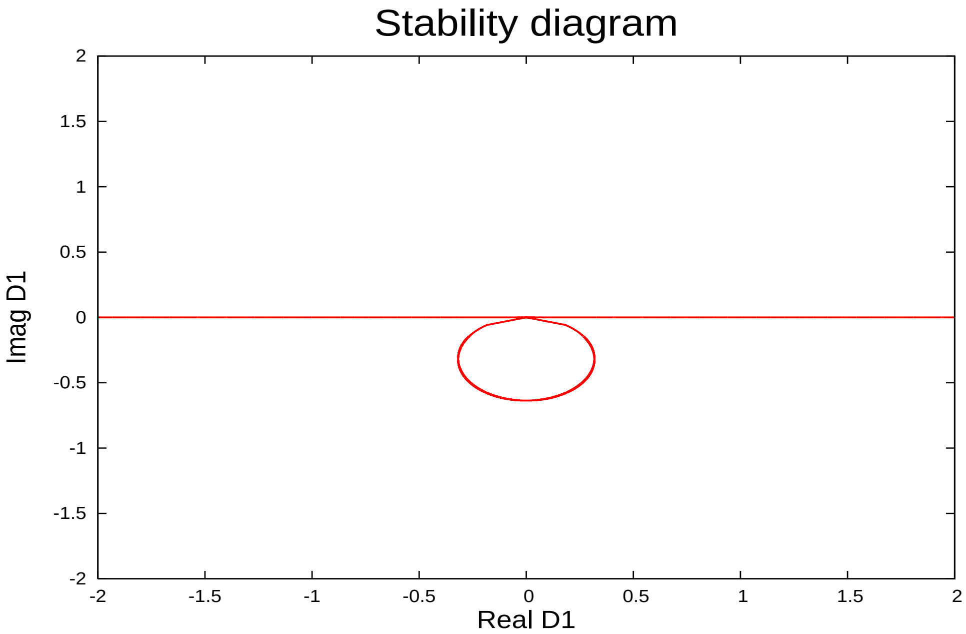
$$\rho(\omega) \;=\; \begin{cases} \frac{1}{2\Delta\omega} & \text{for} |\omega - \omega_x| \leq \Delta\omega \\ 0 & \text{otherwise} \end{cases}$$

Step 1: Compute f(u) and g(u) (or look it up, e.g. Chao, Tigner, "Handbook of ...")

$$f(u) = \frac{1}{2}\ln\left|\frac{u+1}{u-1}\right| \qquad\qquad g(u) = \frac{\pi}{2} \cdot H(1-|u|)$$

Step 2: Plot the real and imaginary part of D_1

Stability diagram



- Real(D_1) versus Imag(D_1) for rectangular $\rho(\omega)$
- This is a Stability Boundary Diagram
- Separates stable from unstable regions (stability limit)

Stability diagram

Take the (real) parameter u in

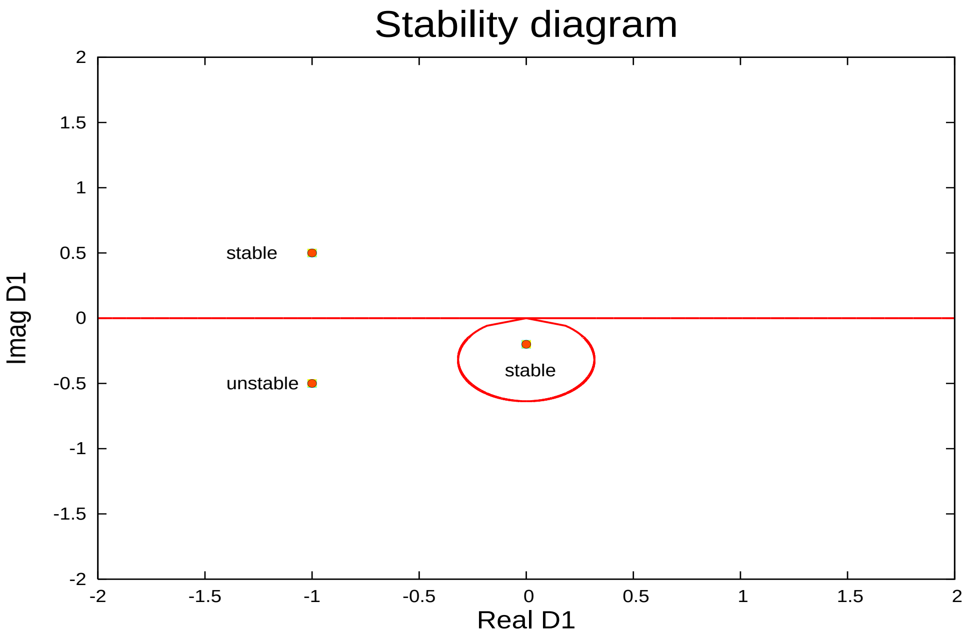
$$D_1 \quad = \quad \frac{1}{(f(u) + ig(u))}$$

1 Scan u from $-\infty$ to $+\infty$

2 Plot the real and imaginary part of D_1 in complex plane

➡ Plot the complex expression of $-\frac{(\Omega - \omega_x)}{\Delta\omega}$ in the same plane as a point (this point depends on impedances, intensities ..)

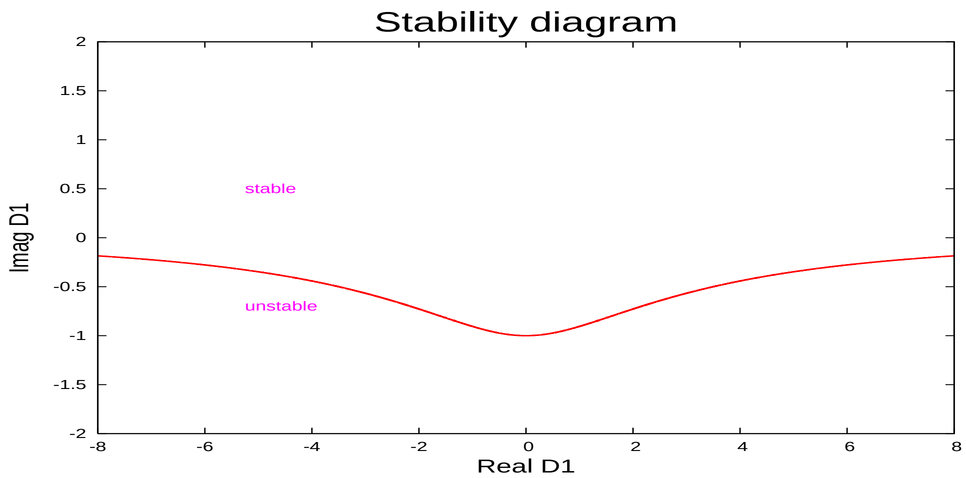
Stability diagram



- This is a **Stability Boundary Diagram**
- Separates stable from unstable regions

Stability diagram

For other types of frequency distributions, example:



Real(D_1) versus Imag(D_1) for bi-Lorentz distribution $\rho(\omega)$

In all cases: half of the complex plane is stable without Landau Damping

Now: transverse instability of unbunched beams

The technique applies directly. Frequency (tune) spread from:

- Change of revolution frequency with energy spread (momentum compaction)
- Change of betatron frequency with energy spread (chromaticity)

but oscillation depends on mode number n (number of oscillations around the circumference C):

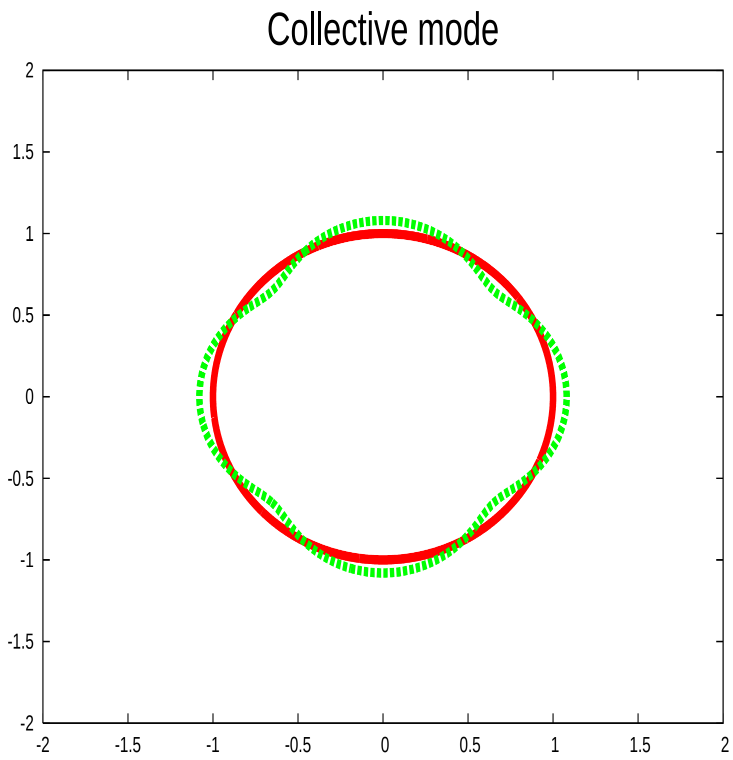
$$\propto \quad exp(-i\Omega t + in(s/C))$$

and the variable u should be written:

$$u \; = \; (\omega_x + n \cdot \omega_0 - \Omega)/\Delta\omega$$

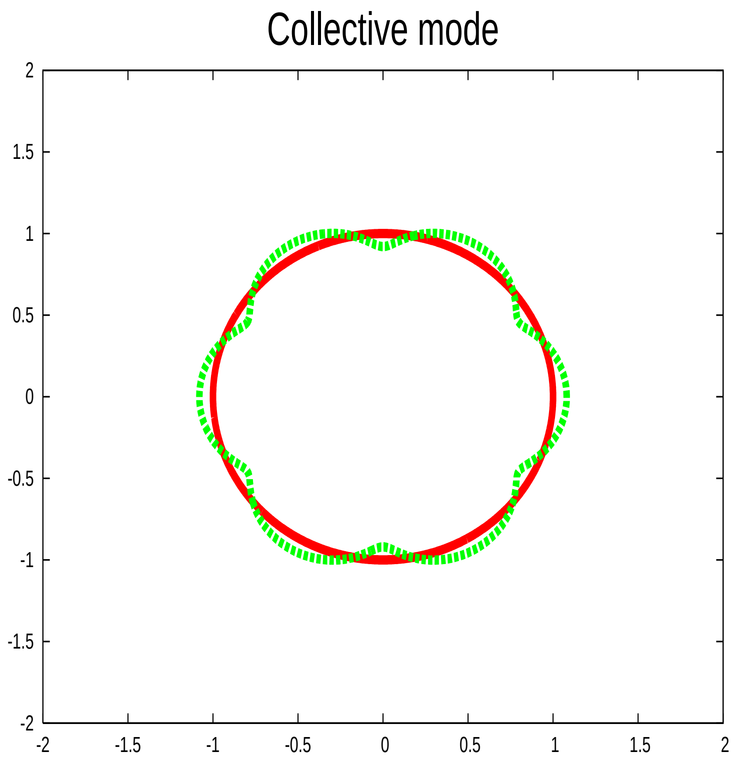
the rest is the same treatment.

Examples: transverse instability of unbunched beams



- Transverse collective mode with mode index $n = 4$

Examples: transverse instability of unbunched beams



- Transverse collective mode with mode index $n = 6$

What about longitudinal instability of unbunched beams

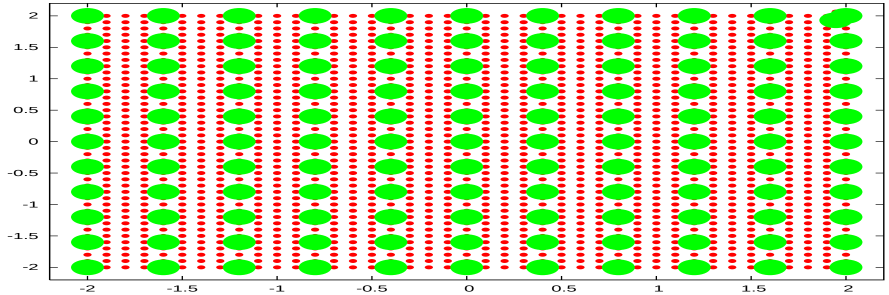
- No external focusing !
- No spread $\Delta\omega$ of focusing frequencies !
  - Spread in revolution frequency: related to energy
  - Energy excitations directly affect frequency spread

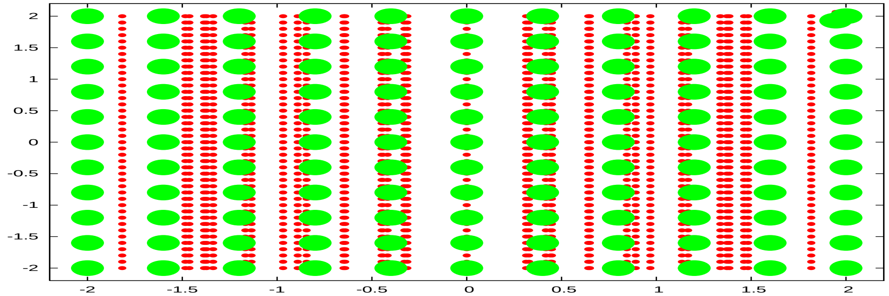
$$\frac{\Delta\omega_{rev}}{\omega_0} = -\frac{\eta}{\beta^2}\frac{\Delta E}{E_0}$$

Frequency distribution by:

$$\rho(\omega_{rev}) \quad \text{and} \quad \Delta\omega_{rev}$$

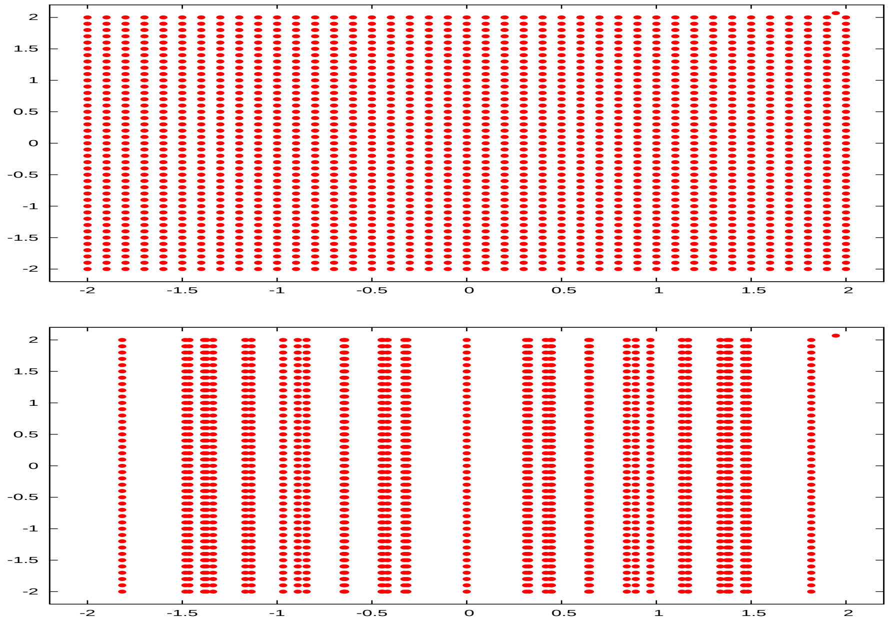
What about longitudinal instability of unbunched beams





- With and without perturbation in a plasma

What about longitudinal instability of unbunched beams



- With and without longitudinal modulation in a beam

What about longitudinal instability of unbunched beams

No external focusing ($\omega_x = 0$):

$$u = \frac{(\omega_x + n \cdot \omega_0 - \Omega)}{\Delta\omega} \quad \Rightarrow \quad u = \frac{(n \cdot \omega_0 - \Omega)}{n \cdot \Delta\omega}$$

$$-\frac{(\Omega - n \cdot \omega_0)^2}{n^2 \Delta\omega^2} = \frac{1}{(F(u) + iG(u))} = D_1$$

and introduce two new functions $F(u)$ and $G(u)$:

$$F(u) = n \cdot \Delta\omega^2 P.V. \int d\omega_0 \frac{\rho'(\omega_0)}{n \cdot \omega_0 - \Omega}$$

$$G(u) = \pi \Delta\omega^2 \rho'(\Omega/n)$$

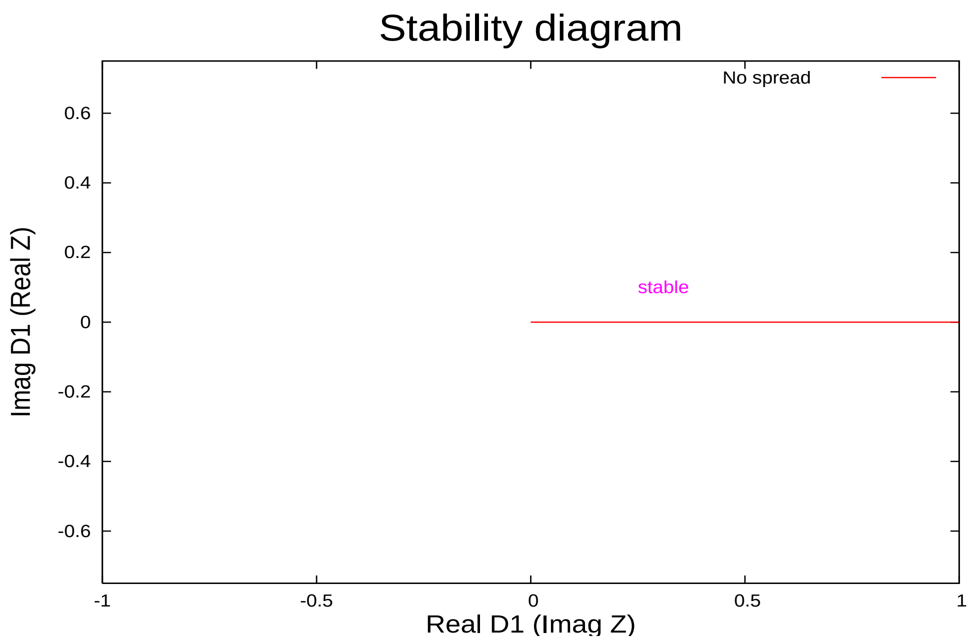
IMPORTANT MESSAGE !

$$-\frac{(\Omega - n \cdot \omega_0)^2}{n^2 \Delta\omega^2} \quad = \quad \frac{1}{(F(u) + iG(u))} \quad = \quad D_1$$

- The impedance now related to the square of the complex frequency shift $(\Omega - n \cdot \omega_0)^2$
- Consequence: no more stable in one half of the plane !
- Landau damping always required

Longitudinal stability - unbunched beam

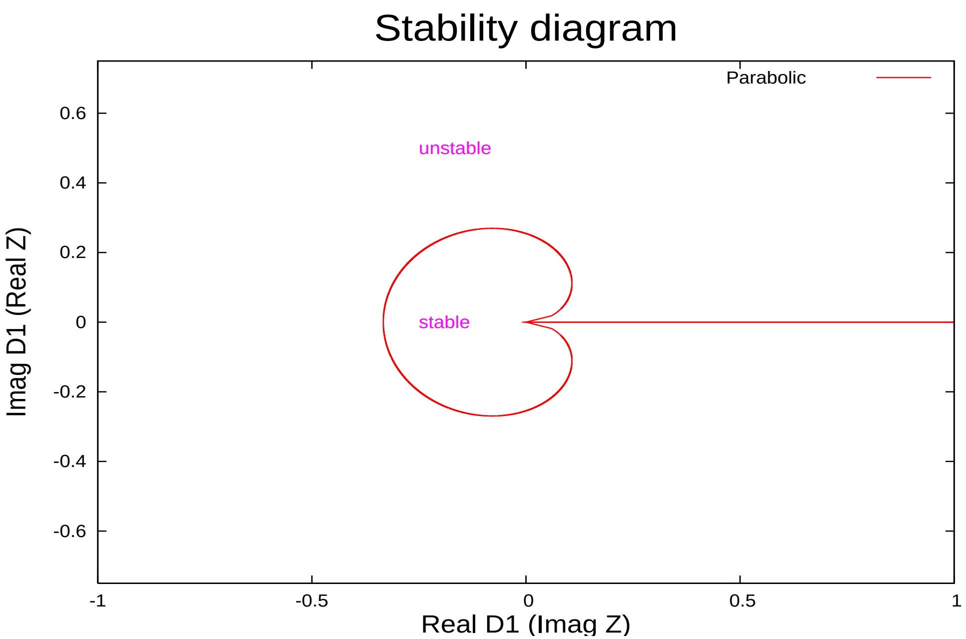
Stability diagram for unbunched beams, longitudinal, no spread:



Real(D_1) versus Imag(D_1) unbunched beam without spread

Longitudinal stability - unbunched beam

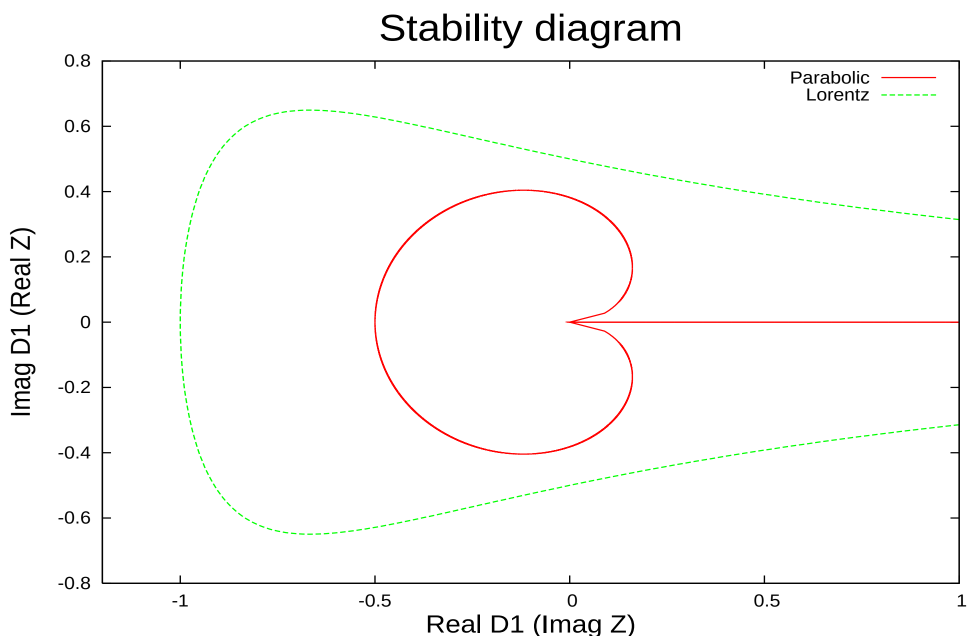
Stability diagram for unbunched beams, longitudinal:



Real(D_1) versus Imag(D_1) for parabolic $\rho(\omega)$ and unbunched beam

Longitudinal stability - unbunched beam

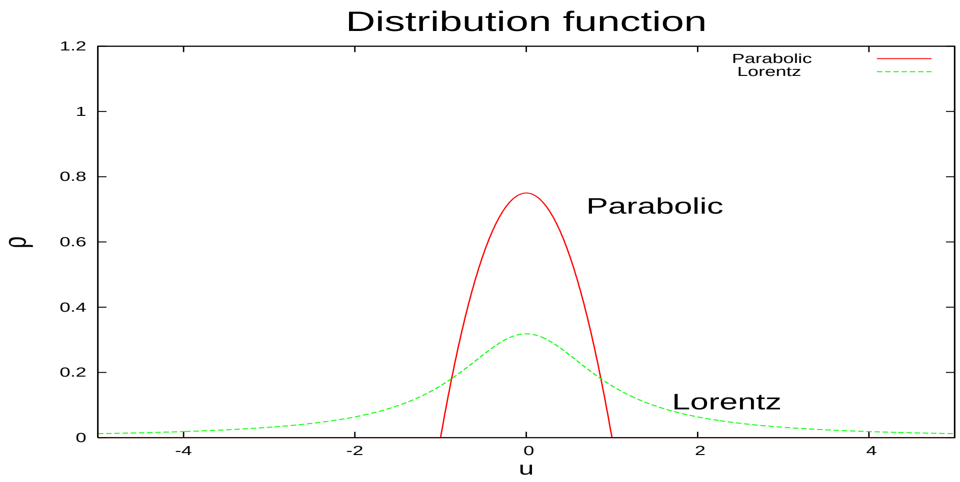
Stability diagram for unbunched beams, longitudinal:



Real(D_1) versus Imag(D_1) for parabolic and Lorentz distribution $\rho(\omega)$ and unbunched beam

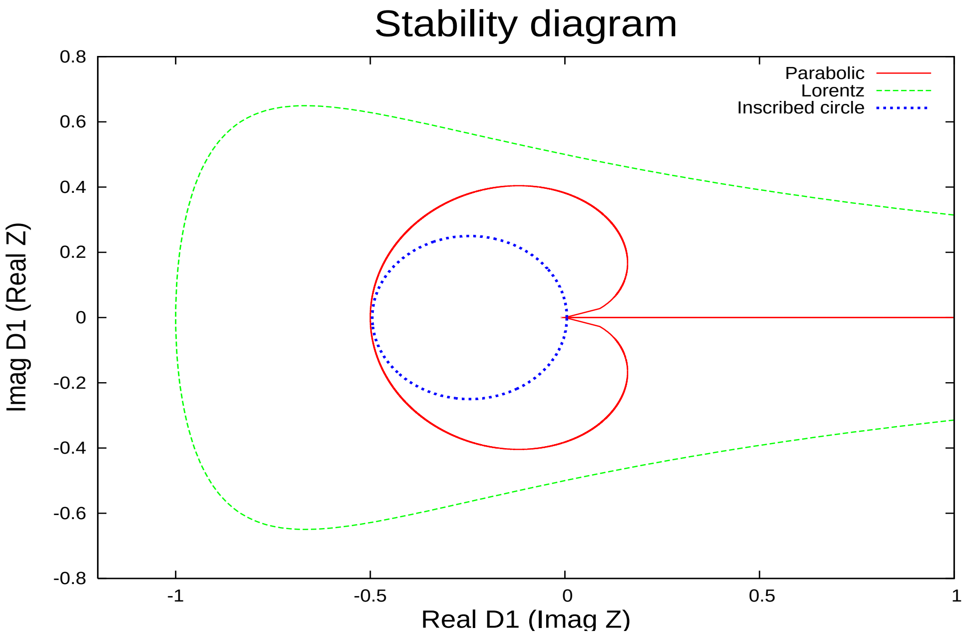
Longitudinal stability - unbunched beam

Why so different stability region:



- Larger stability provided by tail of frequency distribution ....
- What if we do not know exactly the distribution function ?

Longitudinal stability - unbunched beam



- Stability boundary relates Z, I, etc. with frequency spread
- Can derive criteria for stable or unstable beams
- Simplified criterion: inscribe $\approx$ circle as estimate

Longitudinal stability - unbunched beam

- For longitudinal stability/instability:

$$\frac{|Z_{\|}|}{n} \;\le\; F\,\frac{\beta^2 E_0 |\eta_c|}{qI}\left(\frac{\Delta p}{p}\right)^2$$

- This is the Keil-Schnell criterion [EK], frequency spread from momentum spread and momentum compaction η_c
- For given beam parameters define maximum impedance $\frac{|Z_{\|}|}{n}$
- Can derive similar criteria for other instabilities (see lecture by G. Rumolo)

Effect of the simplifications

- We have used a few simplifications in the derivation:
  - Oscillators are linear
  - Movement of the beam is rigid (i.e. beam shape and size does not change)
- What if we consider the "real" cases ?

  i.e. non-linear oscillators

The case of non-linear oscillators

Consider now a bunched beam, because of the synchrotron oscillation: revolution frequency and betatron spread (from chromaticity) average out !

- Source of frequency spread: non-linear force
  - Longitudinal: sinusoidal RF wave
  - Transverse: octupolar or high multipolar field components

Can we use the same considerations as for an ensemble of linear oscillators ?

The case of non-linear oscillators

NO !

The excited betatron oscillation will change the frequency distribution $\rho(\omega)$ (frequency depends on amplitude) !!

Complete derivation through Vlasov equation.

The equation:

$$< x(t) >= \frac{A}{2\omega_x} e^{-i\Omega t} \left[P.V. \int d\omega \frac{\rho(\omega)}{(\omega - \Omega)} + i\pi \rho(\Omega) \right]$$

becomes:

$$< x(t) >= \frac{A}{2\omega_x} e^{-i\Omega t} \left[P.V. \int d\omega \frac{\partial \rho(\omega)/\partial \omega}{(\omega - \Omega)} + i\pi \partial \rho(\Omega)/\partial \Omega \right]$$

Response in the presence of non-linear fields

Study this configuration for instabilities in the transverse plane

Since the frequency ω depends now on the particles amplitudes J_x and J_y *):

$$\omega_x(J_x, J_y) = \frac{\partial H}{\partial J_x}$$

is the amplitude dependent betatron tune (similar for ω_y).

We then have to write:

$$\rho(\omega) \quad \rightarrow \quad \rho(J_x, J_y)$$

*) see e.g. "Tools for Non-Linear Dynamics" (W.Herr, this school)

Response in the presence of non-linear fields

Assuming a periodic force in the horizontal (x) plane and using now the tune (normalized frequency) $Q = \frac{\omega}{\omega_0}$:

$$F_x = A \cdot exp(-i\omega_0 Qt)$$

the dispersion integral can be written as:

$$1 = -\Delta Q_{coh} \int_0^\infty \mathrm{d}J_x \int_0^\infty \mathrm{d}J_y \frac{J_x \frac{\partial \rho(J_x, J_y)}{\partial J_x}}{Q - Q_x(J_x, J_y)}$$

➡ Then proceed as before to get stability diagram ...

What happens when bunches are not rigid ?

If particle distribution changes (often as a function of time), obviously the frequency distribution $\rho(\omega)$ changes as well. :

- Examples:
  - Higher order modes
  - Coherent beam-beam modes
- Treatment requires solving the Vlasov equation (perturbation theory or numerical integration)

→ Pragmatic approach (20-20 hindsight): use
unperturbed stability region and perturbed complex tune shift ...

Landau damping as a cure

If the boundary of

$$D_1 \quad = \quad \frac{1}{(f(u) + ig(u))}$$

determines the stability, can we:

- Increase the stable region by:
  - Modifying the frequency distribution $\rho(\omega)$, i.e. $\rho(J_x, J_y)$
  - Introducing tune spread artificially (octupoles, other high order fields)

The tune dependence of an octupole (k_3) can be written as*):

$$Q_x(J_x, J_y) = Q_0 + a \cdot k_3 \cdot J_x + b \cdot k_3 \cdot J_y$$

*) see e.g. "Tools for Non-Linear Dynamics" (W.Herr, this school)

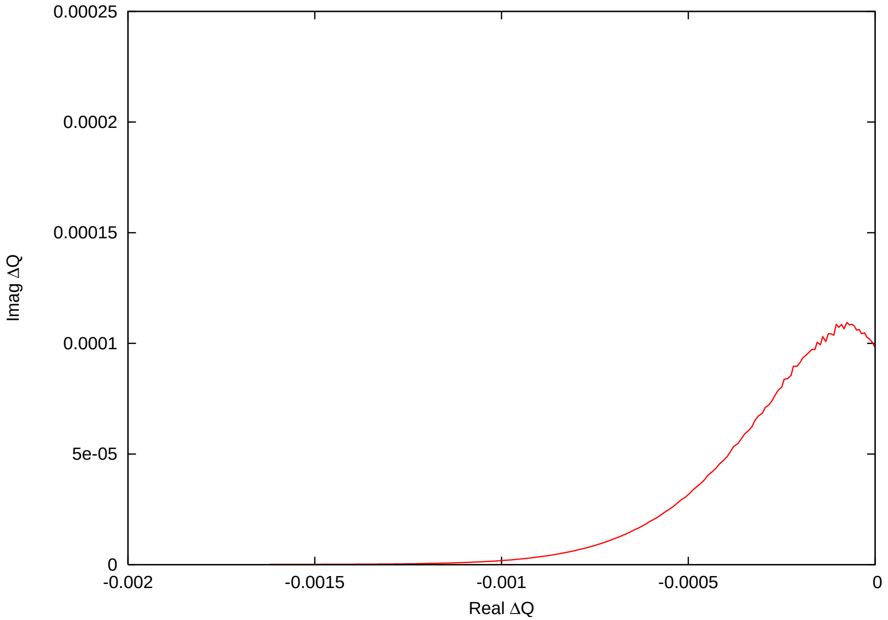
Landau damping as a cure

- Other sources to introduce tune spread:
  - Space charge
  - Chromaticity
  - High order multipole fields
  - Beam-beam effects (colliders only)

Landau damping as a cure

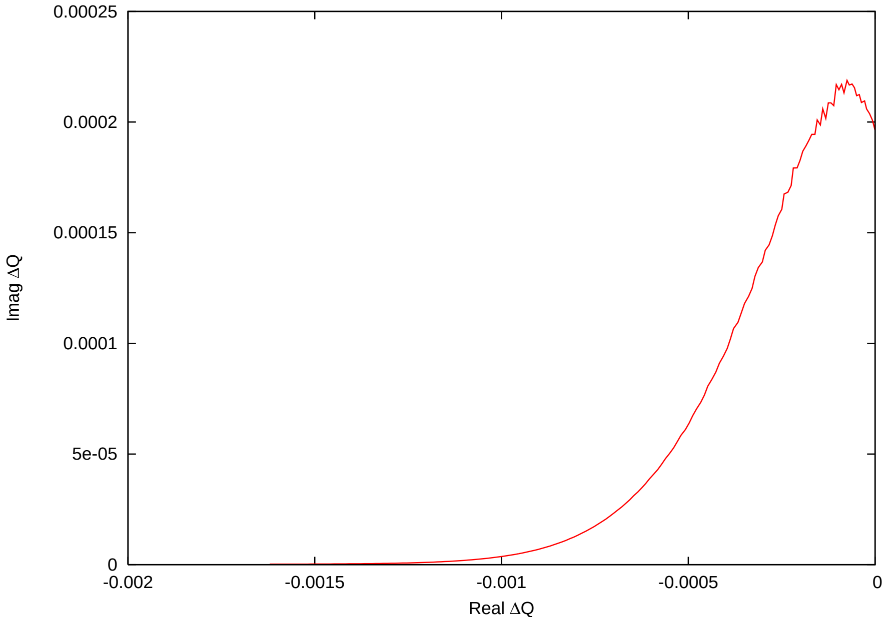
- Recipe for "generating" Landau damping:
  - For a multipole field, compute detuning $Q(J_x, J_y)$
  - Given the distribution $\rho(J_x, J_y)$
  - Compute the stability diagram by scanning frequency

Stability diagram with octupoles



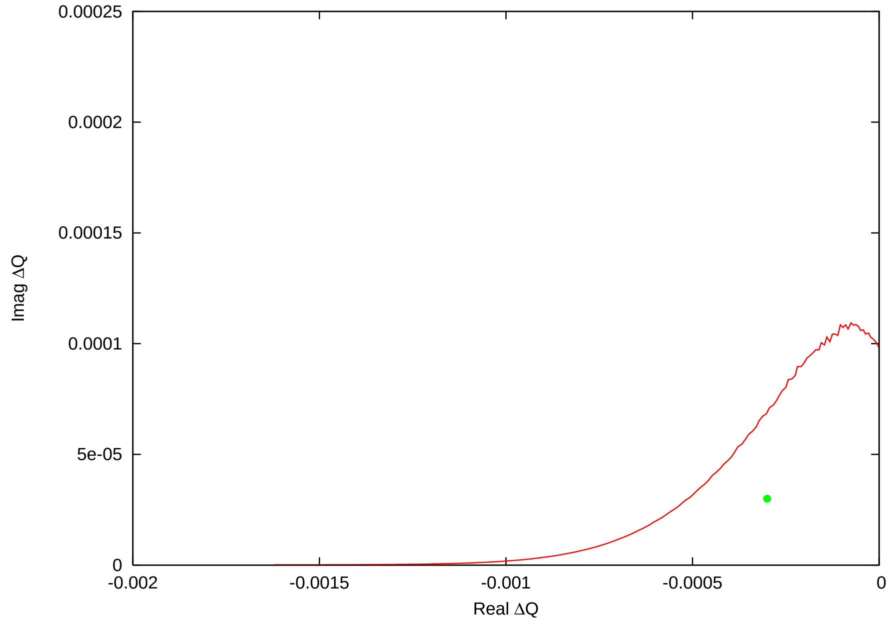
- Stabilization with octupoles

Stability diagram with octupoles



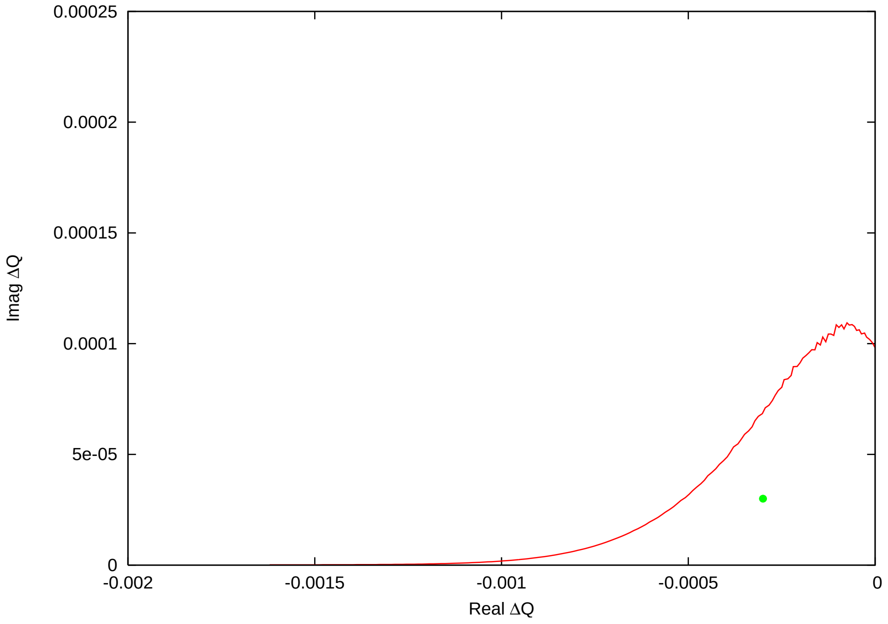
- Stabilization with octupoles, increased strengths

Stability diagram with octupoles



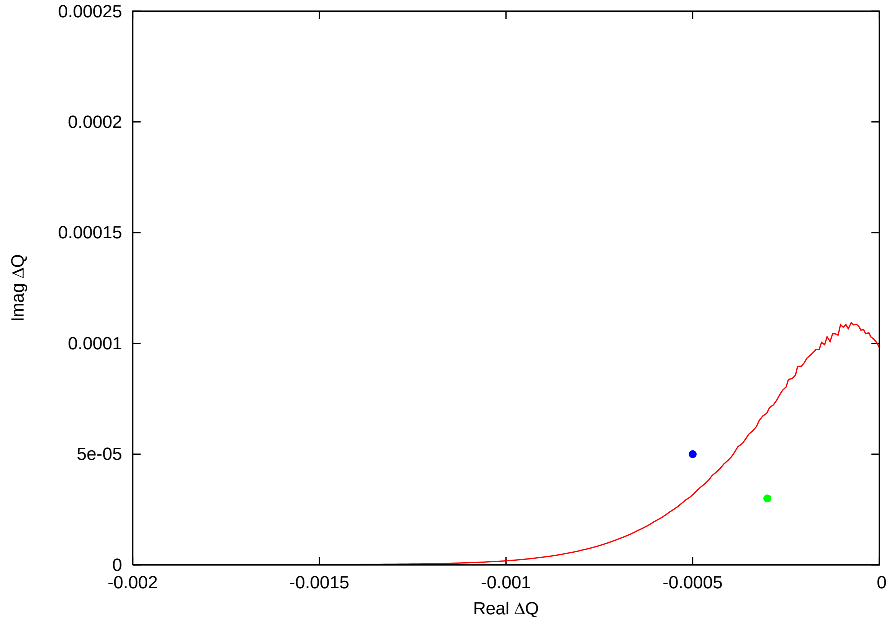
- Complex coherent tune of an unstable mode
- Now in the stable region

Stability diagram with octupoles



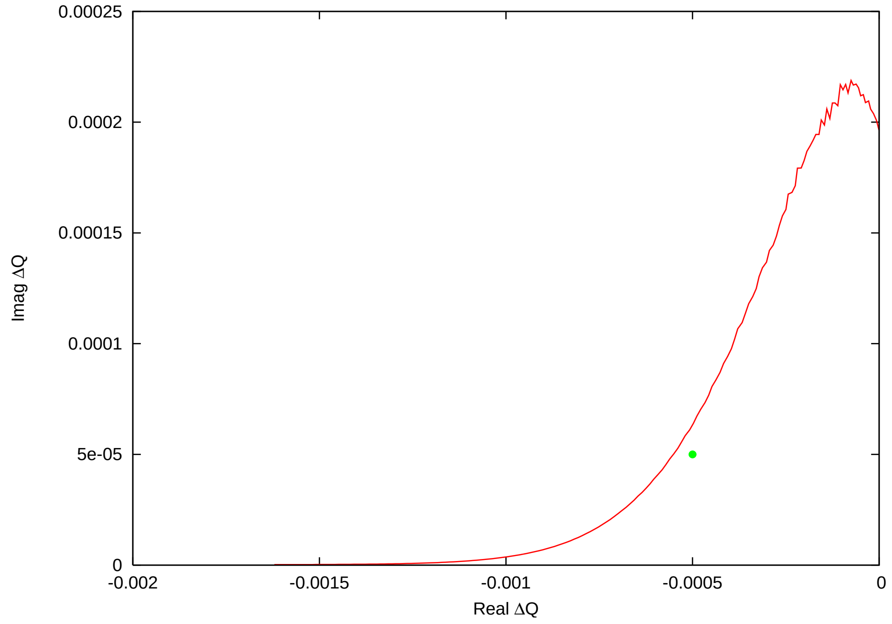
- Complex coherent tune of an unstable mode
- What if we increase the impedance (or intensity) ?

Stability diagram with octupoles



- Complex coherent tune of an unstable mode
- Now in the unstable region

Stability diagram with octupoles



- Complex coherent tune of an unstable mode
- Increased octupole strength makes it stable again

Stability diagram with octupoles

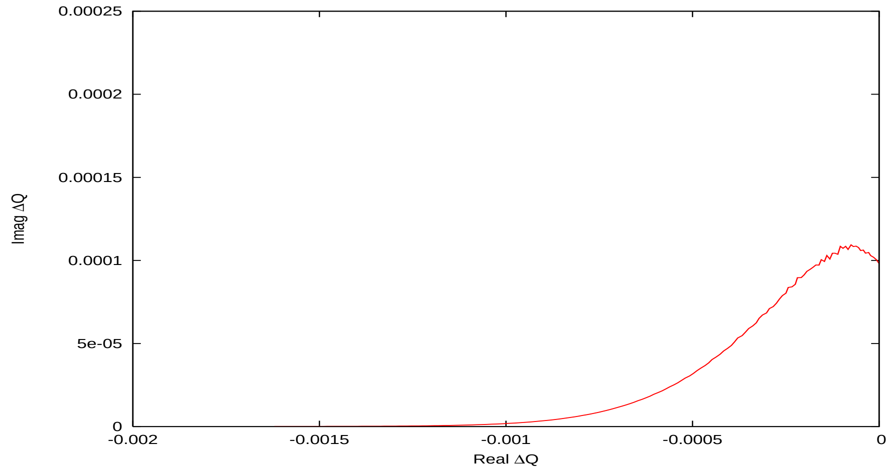
- Can we increase the octupole strength (current) as we like ??
- No, we get several problems:
  - Not many particles at large amplitudes: requires large strengths
  - Octupoles introduce strong non-linearities at large amplitudes
  - Can cause reduction of dynamic aperture and life time
  - They can change the chromaticity !
  - They can catch fire
- The lesson: use them if you have no choice (or run out of ideas) ....

Another example: Head-Tail modes

(see e.g. Lecture G. Rumolo)

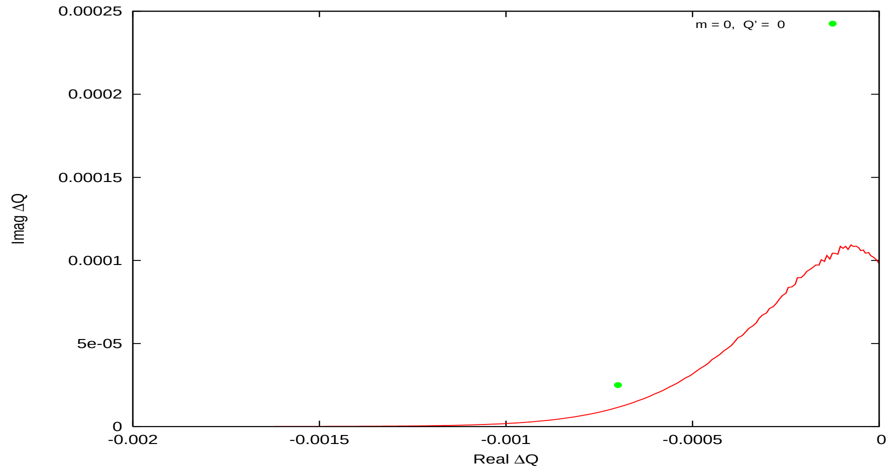
- For short range wake fields
- Broad band impedance
- Growth and damping controlled with chromaticity Q'
  - Some modes need positive Q'
  - Some modes need negative Q'
  - Some modes can be damped by feedback ($m = 0$)
- In the control room: juggle with octupoles and Q' ....

Stability diagram and head-tail modes



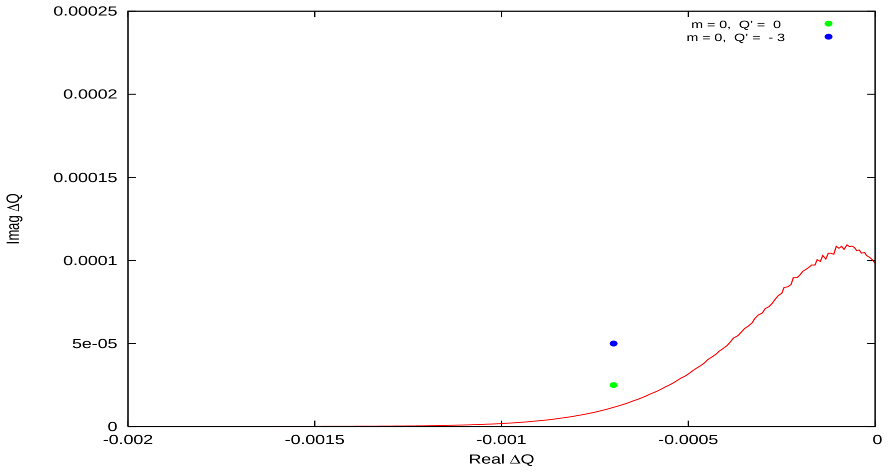
- Stability region from non-linear fields

Stability diagram and head-tail modes



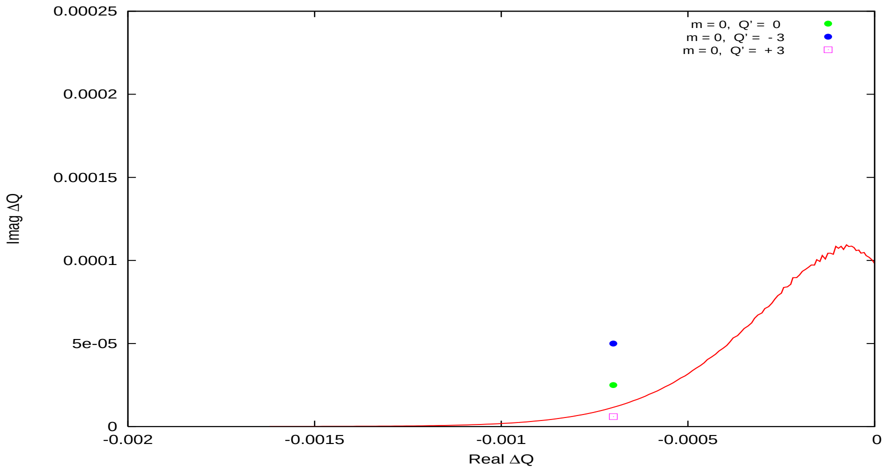
- Stability region from non-linear fields
- Head-tail mode (m = 0), unstable

Stability diagram and head-tail modes



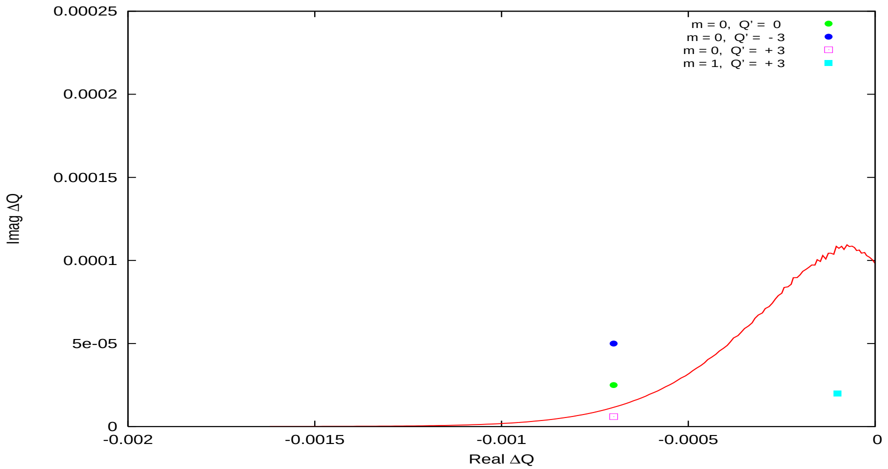
- Stability region from non-linear fields
- Head-tail mode (m = 0), unstable
- For two different chromaticities

Stability diagram and head-tail modes



- Stability region from non-linear fields
- Head-tail mode (m = 0), unstable
- For three different chromaticities ➡ one is stable
- What about higher order head-tail modes (m = 1, -1, ...) ?

Stability diagram and head-tail modes

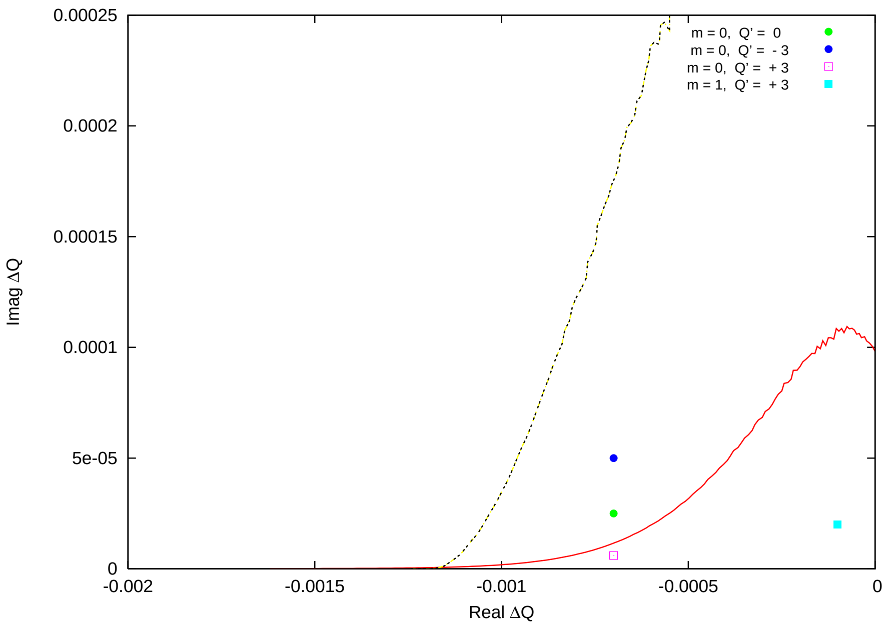


- Large chromaticity "moves" $m = 1$ mode to positive imaginary tune shift, need Landau damping to stabilize

Stability diagram with octupoles

- Would need very large octupole strength for stabilization
- The known problems:
  - Can cause reduction of dynamic aperture and life time
  - Life time important when beam stays in the machine for a long time
  - Colliders: life time more than 10 - 20 hours needed ...
- Is there another option ?

Stability diagram and head-tail modes



- Stability region and head-tail modes for different chromaticity
- Stabilization with octupoles or colliding beams [HV]

Colliding beams seem to have a very large stable region !

What makes the difference ... ?

The tune dependence of an octupole can be written as:

$$Q_x(J_x, J_y) = Q_0 + aJ_x + bJ_y$$

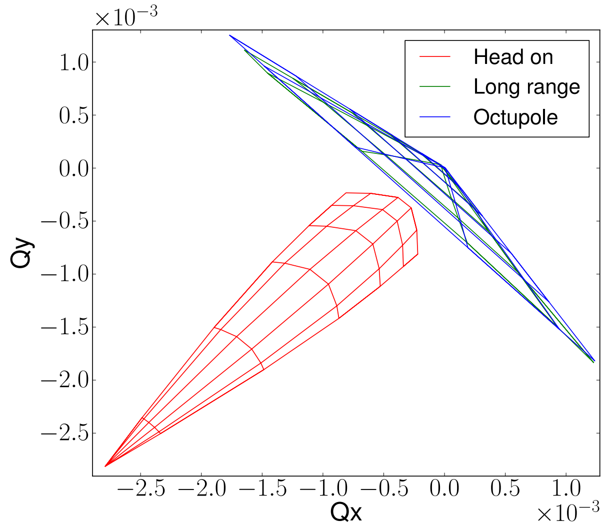
linear in the action (for coefficients, see Appendix).

The tune dependence of a head-on beam-beam collision can be written as$^{*)}$:

with $\alpha = \frac{x}{\sigma^*}$ we get $\Delta Q/\xi = \frac{4}{\alpha^2}\left[1 - I_0(\frac{\alpha^2}{4}) \cdot e^{\frac{-\alpha^2}{4}}\right]$

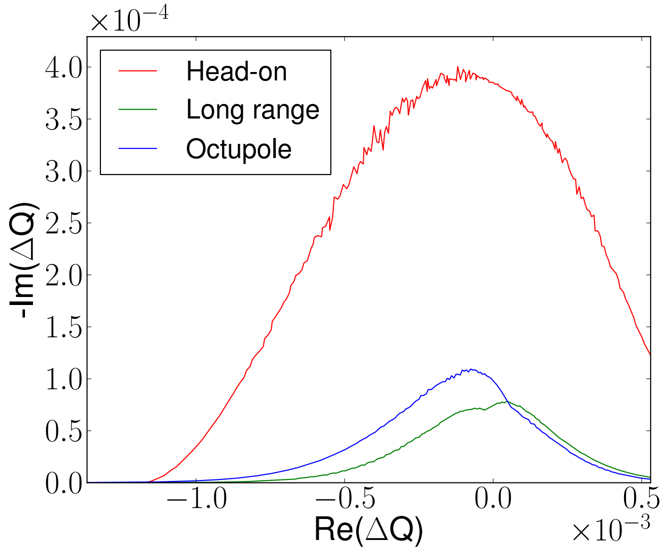
$^{*)}$ see e.g. "Beam-Beam effects" (Tatiana Pieloni, this school)

Response in the presence of non-linear fields



- Tune footprints for beam-beam and octupoles
- Overall tune spread the same, but:
- For octupoles largest effect for largest amplitudes
- For beam-beam largest effect for small amplitudes

Response in the presence of non-linear fields



- Stability diagrams for beam-beam and octupoles [HV]
- Stability region very different !

The Good, the Bad, and the Surprise ...

Landau Damping with non-linear fields: Are there any side effects ?

- The Good:
  - Stability region increased
- The Bad:
  - Non-linear fields introduced (resonances !)
  - Changes optical properties, e.g. chromaticity ... (feed-down !)
- The Surprise:
  - Non-linear effects for large amplitudes only (octupoles)
  - Much better: head-on beam-beam (but only in colliders ...)

Conditions for Landau "damping"

- Presence of an incoherent frequency (tune) spread
- Coherent mode must be inside this spread
- Coherent mode must be inside the stability diagramm

The SAME particles must be involved !!!

Summary

- Long history, heavily debated (still)
- Different approaches to the mathematical treatment, (needed for rigorous treatment of different configurations)
- Many applications (plasmas, accelerators, wind waves, bio-physics, astrophysics, ...)
- Very important for hadron accelerators, but should be used with care ...
- It works ! It is not a mystery !

APPENDIX:
Tune shift of an octupole:

The tune dependence of an octupole can be written as:

$$Q_x(J_x, J_y) = Q_0 + aJ_x + bJ_y$$

for the coefficients:

$$\Delta\ Q_x \;=\; \left[\frac{3}{8\pi}\int \beta_x^2 \frac{K_3}{B\rho} ds\right] J_x \;-\; \left[\frac{3}{8\pi}\int 2\beta_x\beta_y \frac{K_3}{B\rho} ds\right] J_y$$

$$\Delta\ Q_y \;=\; \left[\frac{3}{8\pi}\int \beta_y^2 \frac{K_3}{B\rho} ds\right] J_y \;-\; \left[\frac{3}{8\pi}\int 2\beta_x\beta_y \frac{K_3}{B\rho} ds\right] J_x$$