Special Relativity

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Overview

- The principle of special relativity
- Lorentz transformation and consequences
- Space-time
- 4-vectors: position, velocity, momentum, invariants, covariance.
- Derivation of $E=mc^2$
- Examples of the use of 4-vectors
- Inter-relation between β and γ, momentum and energy
- An accelerator problem in relativity
- Relativistic particle dynamics
- Lagrangian and Hamiltonian Formulation
- Radiation from an Accelerating Charge
- Photons and wave 4-vector
- Motion faster than speed of light

Reading

- W. Rindler: Introduction to Special Relativity (OUP 1991)
- D. Lawden: An Introduction to Tensor Calculus and Relativity
- N.M.J. Woodhouse: Special Relativity (Springer 2002)
- A.P. French: Special Relativity, MIT Introductory Physics Series (Nelson Thomes)
- Misner, Thorne and Wheeler: Relativity
- C. Prior: Special Relativity, CERN Accelerator School (Zeegse)

Historical background

- Groundwork of Special Relativity laid by Lorentz in studies of electrodynamics, with crucial concepts contributed by Einstein to place the theory on a consistent footing.
- Maxwell's equations (1863) attempted to explain electromagnetism and optics through wave theory
  - light propagates with speed $c = 3\times10^8$ m/s in "ether" but with different speeds in other frames
  - the ether exists solely for the transport of e/m waves
  - Maxwell's equations not invariant under Galilean transformations
  - To avoid setting e/m apart from classical mechanics, assume
    - light has speed c only in frames where source is at rest
    - the ether has a small interaction with matter and is carried along with astronomical objects

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Contradicted by:

- Aberration of star light (small shift in apparent positions of distant stars)
- Fizeau's 1859 experiments on velocity of light in liquids
- Michelson-Morley 1907 experiment to detect motion of the earth through ether
- Suggestion: perhaps material objects contract in the direction of their motion

$$L(v) = L_0\left(1 - \frac{v^2}{c^2}\right)^{1/2}$$

This was the last gasp of ether advocates and the germ of Special Relativity led by Lorentz, Minkowski and Einstein.

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  - Apparatus transferred from F to F', then from F' to F'', agrees with apparatus transferred directly from F to F''.
- *The Principle of Special Relativity states that all physical laws take equivalent forms in related inertial frames, so that we cannot distinguish between the frames.*

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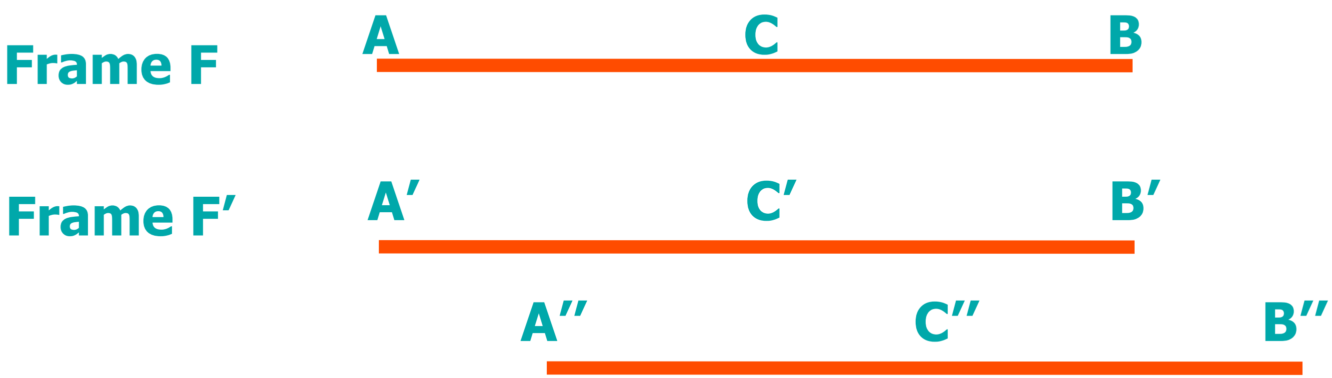
Simultaneity

- Two clocks A and B are synchronised if light rays emitted at the same time from A and B meet at the mid-point of AB



- Frame F' moving with respect to F. Events simultaneous in F cannot be simultaneous in F'.
- Simultaneity is **not** absolute but frame dependent.

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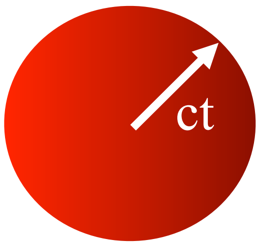

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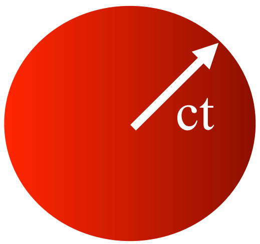

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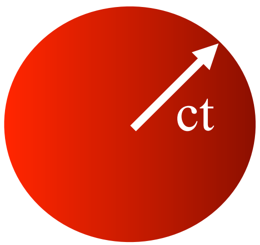

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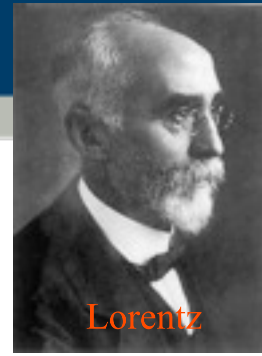

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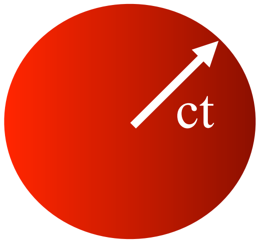

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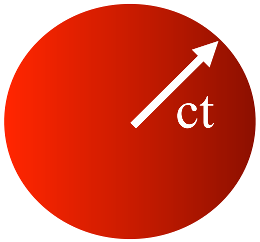

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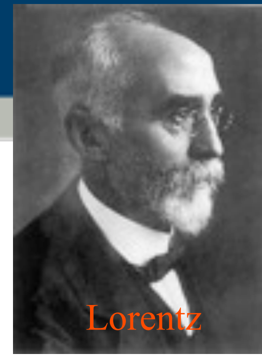

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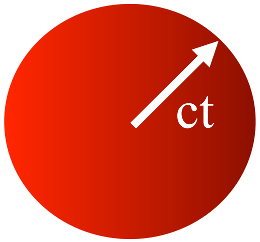

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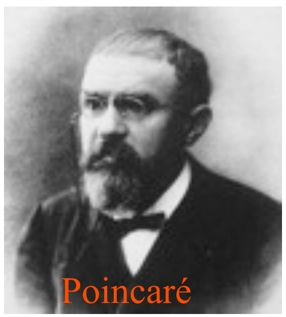

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Outline of Derivation

Set $t' = \alpha t + \beta x$

$x' = \gamma x + \delta t$

$y' = \varepsilon y$

$z' = \varsigma z$

Then $P = kQ$

$$\Leftrightarrow c^2 t'^2 - x'^2 - y'^2 - z'^2 = k\left(c^2 t^2 - x^2 - y^2 - z^2\right)$$

$$\Rightarrow c^2(\alpha t + \beta x)^2 - (\gamma x + \delta t)^2 - \varepsilon^2 y^2 - \varsigma^2 z^2 = k\left(c^2 t^2 - x^2 - y^2 - z^2\right)$$

Equate coefficients of x, y, z, t.

Isotropy of space $\Rightarrow$ $k = k(\vec{v}) = k(|\vec{v}|) = \pm 1$

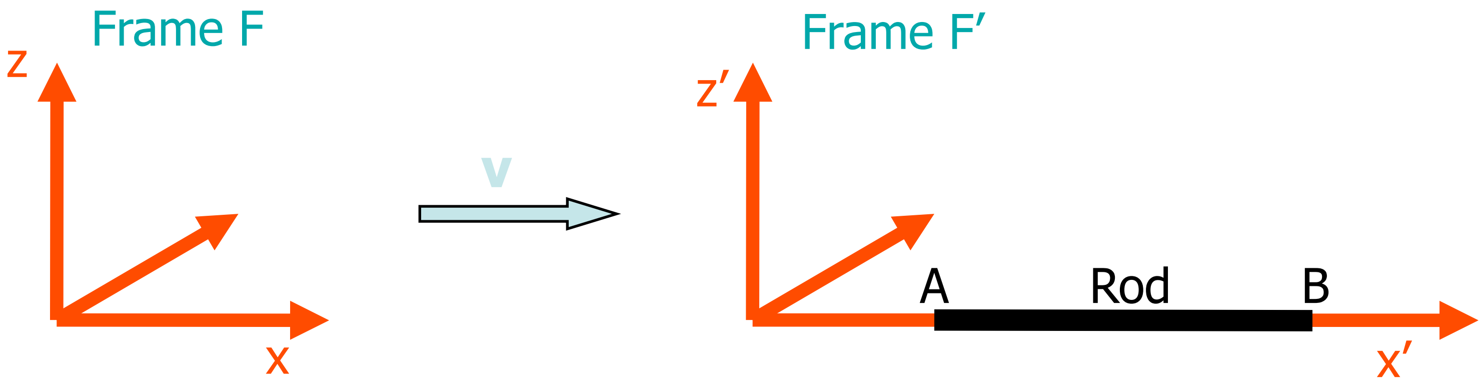
Apply some common sense (e.g. $\varepsilon, \varsigma, k = +1$ and not -1)

General 3D form of Lorentz Transformation:

$$\vec{x}' = \vec{x} + \vec{v}\left(\gamma t + (\gamma - 1)\frac{\vec{v}\cdot\vec{x}}{v^2}\right)$$

$$t' = \gamma\left(t + \frac{\vec{v}\cdot\vec{x}}{c^2}\right)$$

Consequences: length contraction



Rod AB of length L' fixed in F' at x'_A, x'_B. What is its length measured in F?

Must measure positions of ends in F at the same time, so events in F are (t,x_A) and (t,x_B). From Lorentz:

$$x'_A = \gamma\left(x_A - vt\right) \qquad x'_B = \gamma\left(x_B - vt\right)$$

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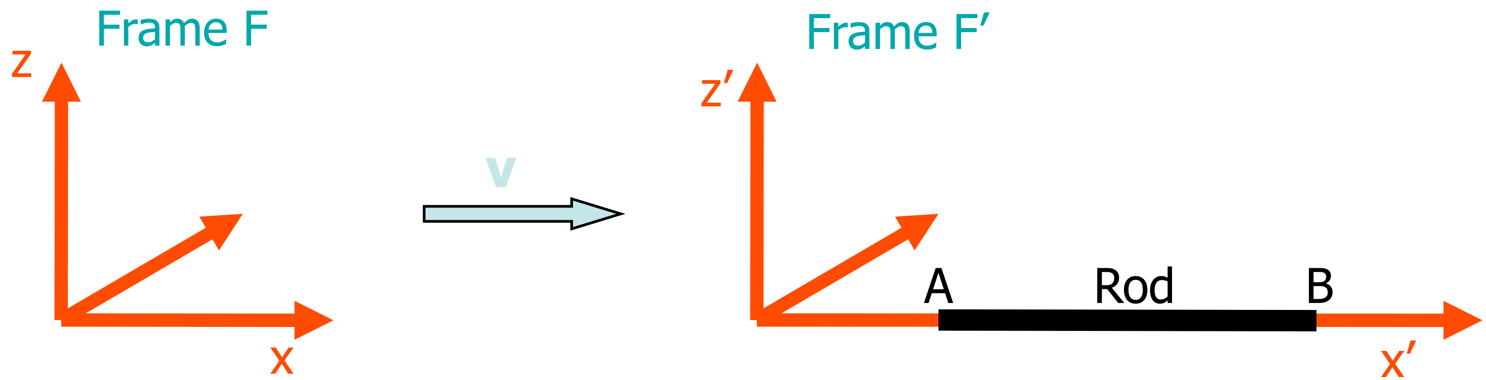

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Moving objects appear contracted in the direction of the motion

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- Clock in frame *F* at point with coordinates (*x*,*y*,*z*) at different times t_A and t_B
- In frame *F*' moving with speed *v*, Lorentz transformation gives

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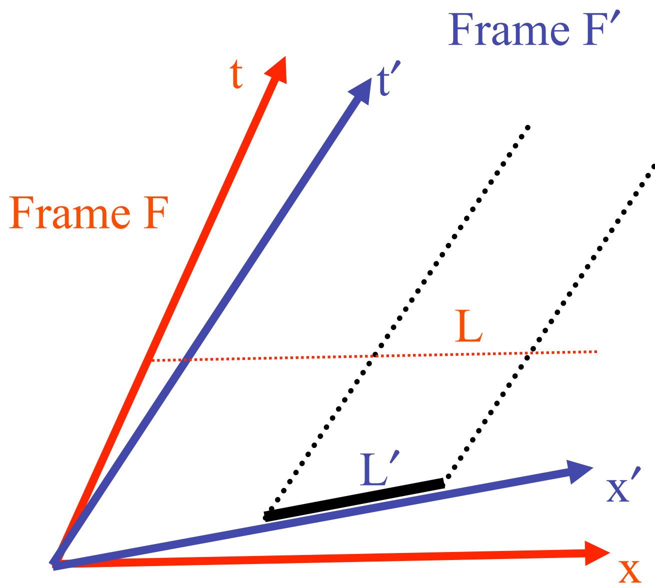
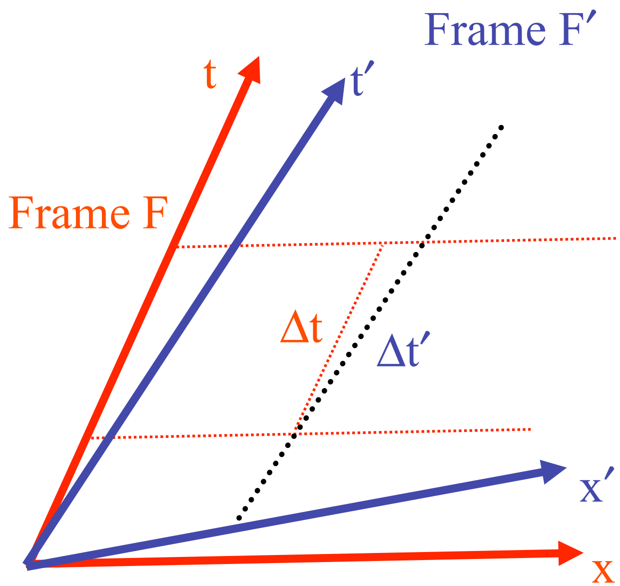
Moving clocks appear to run slow

Schematic Representation of the Lorentz Transformation





Length contraction L<L′

Rod at rest in F′. Measurement in F at fixed time t, along a line parallel to x-axis

Time dilatation: Δt<Δt′

Clock at rest in F. Time difference in F′ from line parallel to x′-axis

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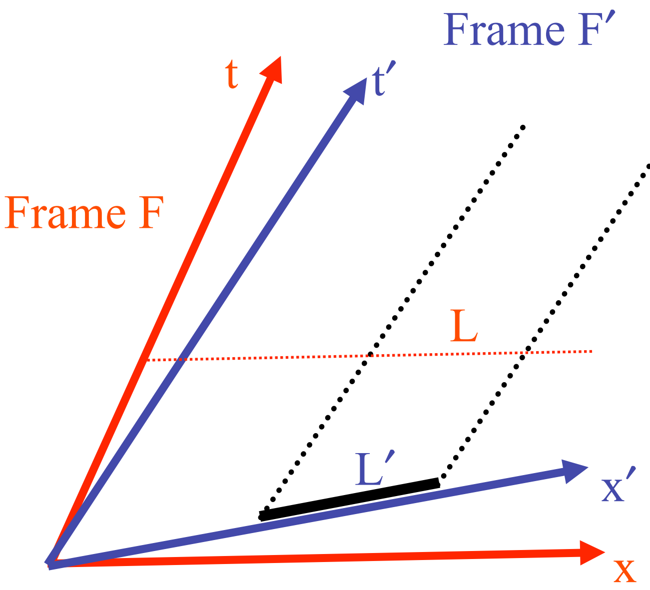
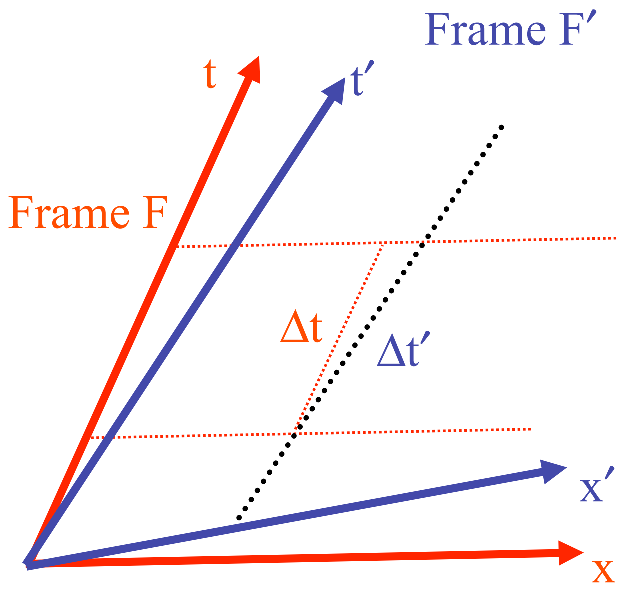
presentation of the Lorentz Transformation





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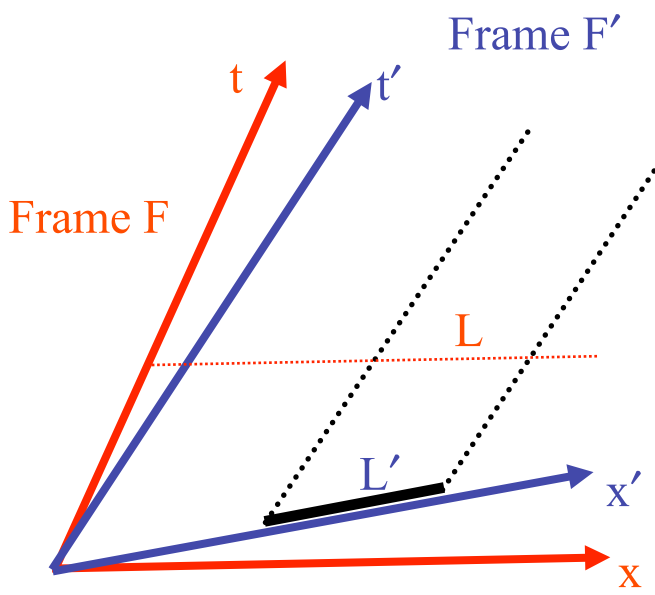

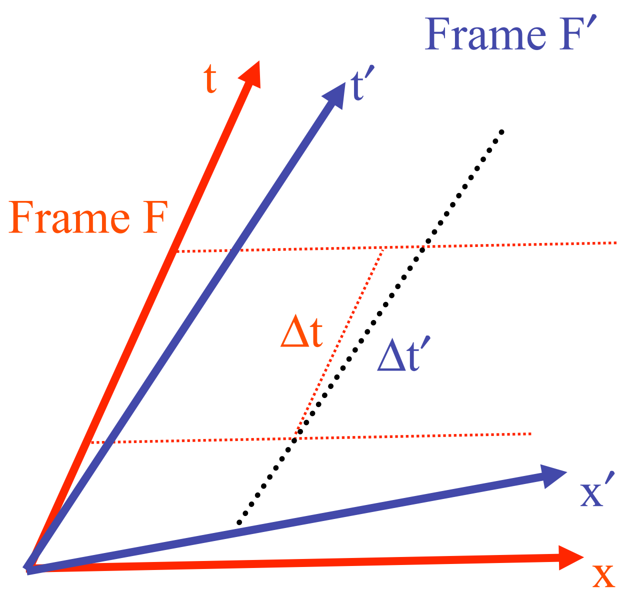

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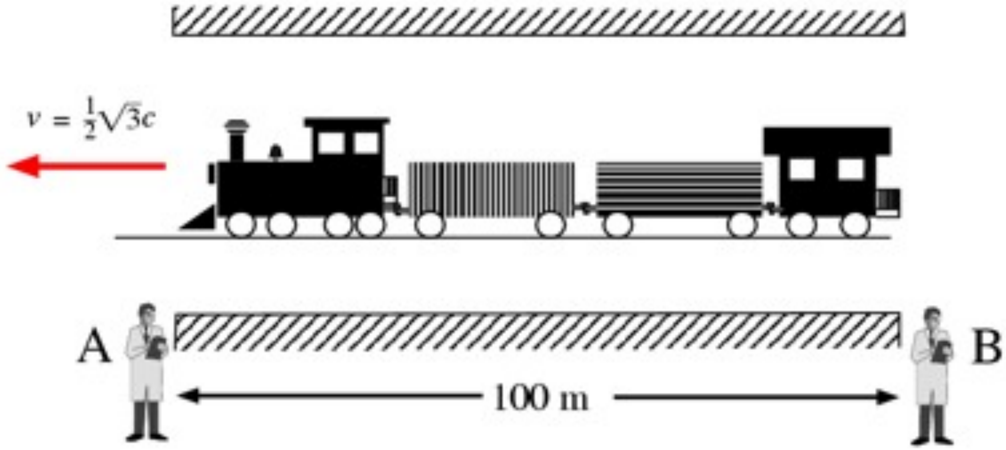
Example: High Speed Train



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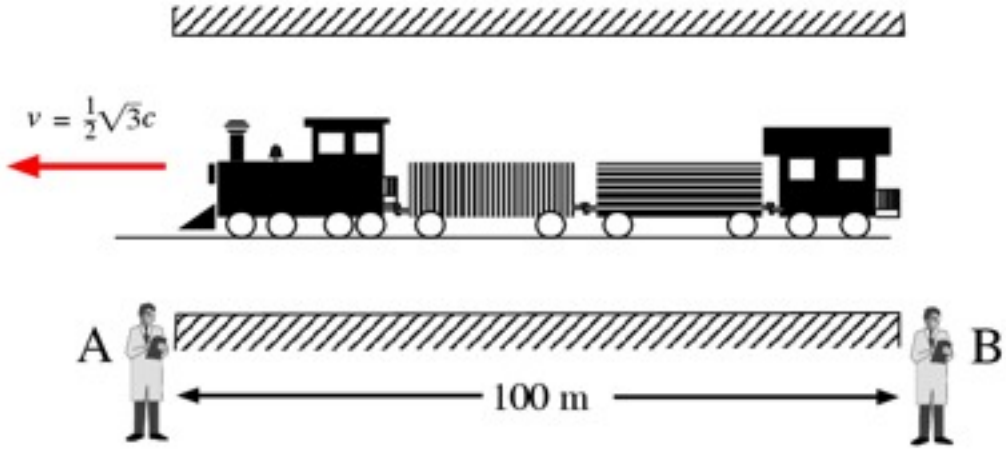

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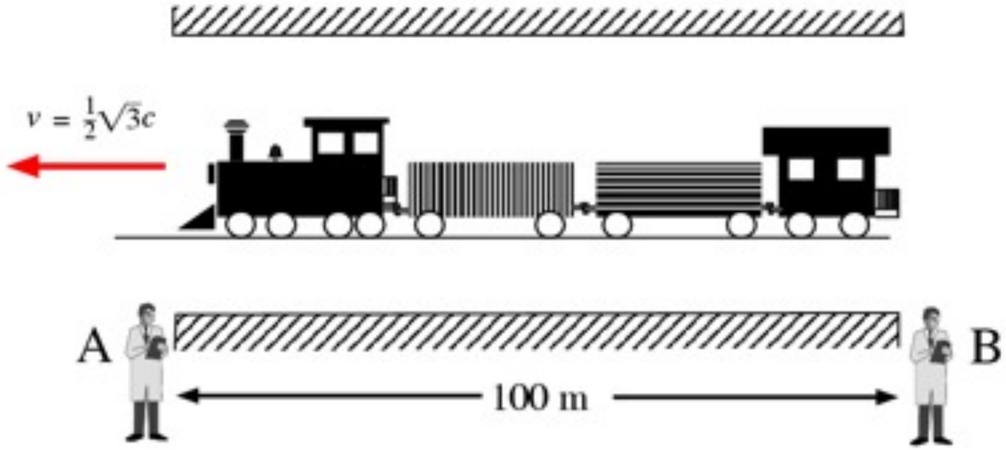

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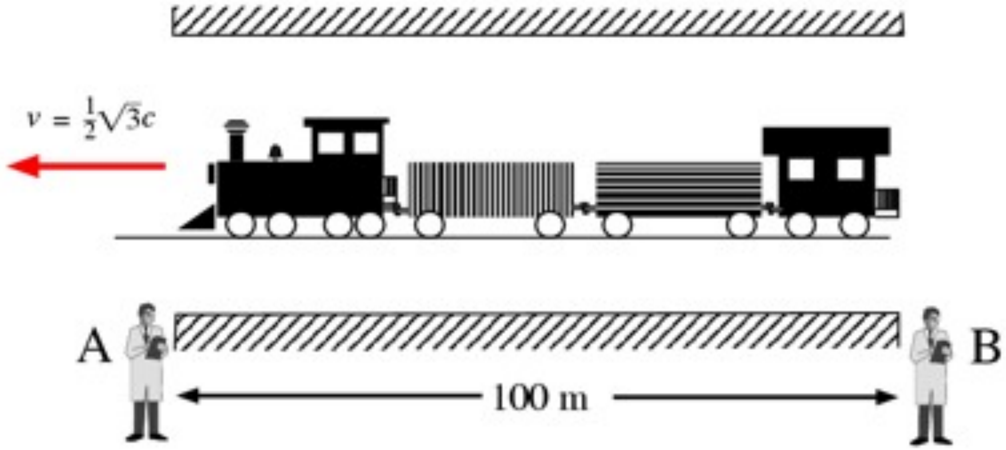

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- But the tunnel is moving relative to the driver and guard on the train and they say the train is 100 m in length but the tunnel has contracted to 50 m

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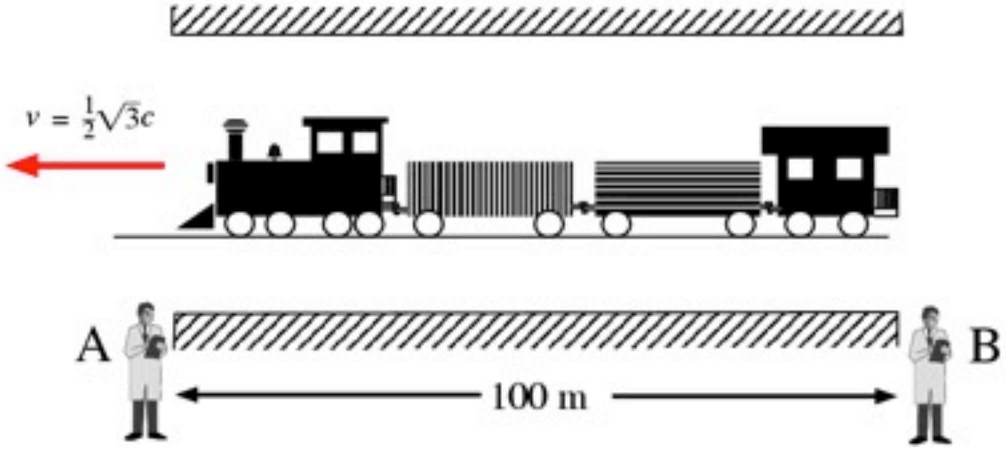
A’s clock (and the driver's clock) reads zero as the driver exits tunnel. What does B’s clock read when the guard goes in?



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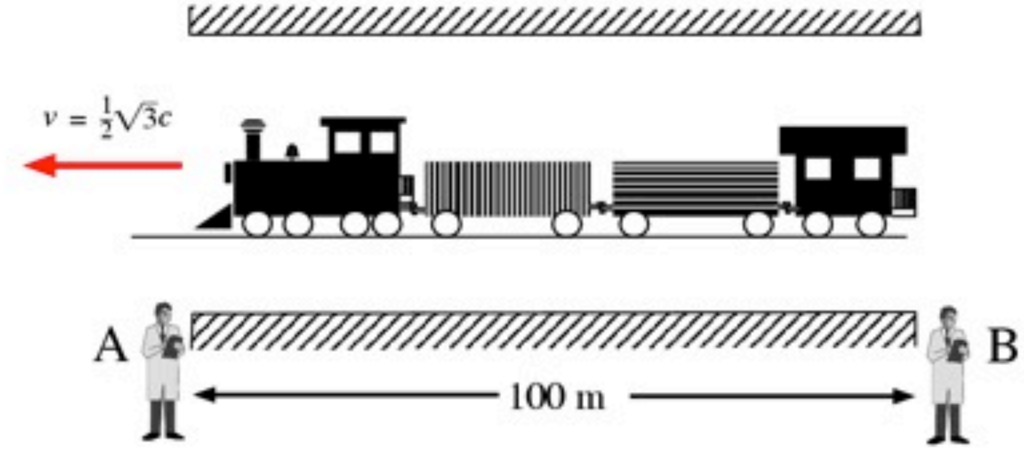

Moving train length 50m, so driver has still 50m to travel before he exits and his clock reads 0. A's clock and B's clock are synchronised. Hence the reading on B's clock is

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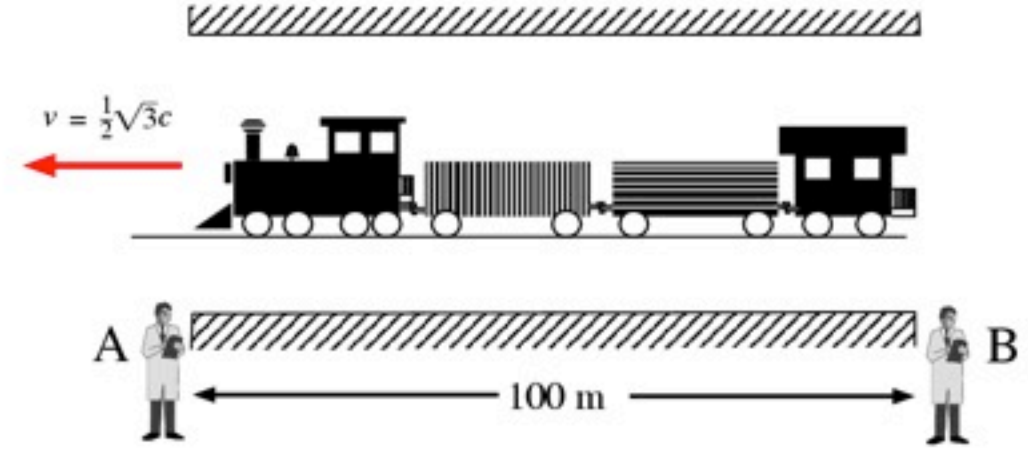

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$$-\frac{50}{v} = -\frac{100}{\sqrt{3}c} \approx -200\,\text{ns}$$

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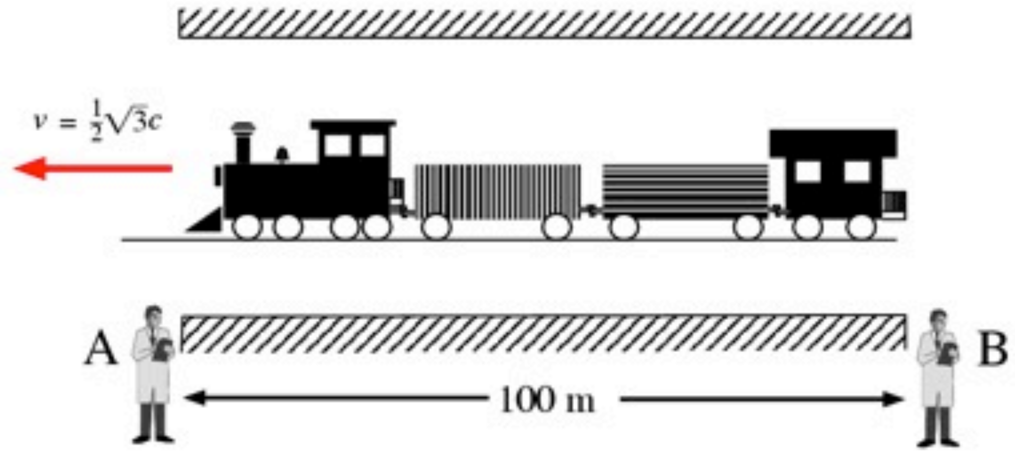
What does the guard's clock read as he goes in?



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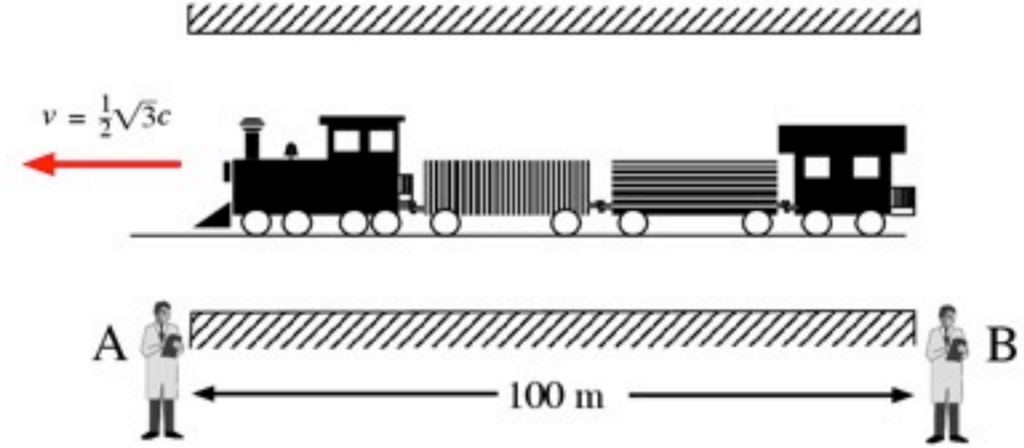

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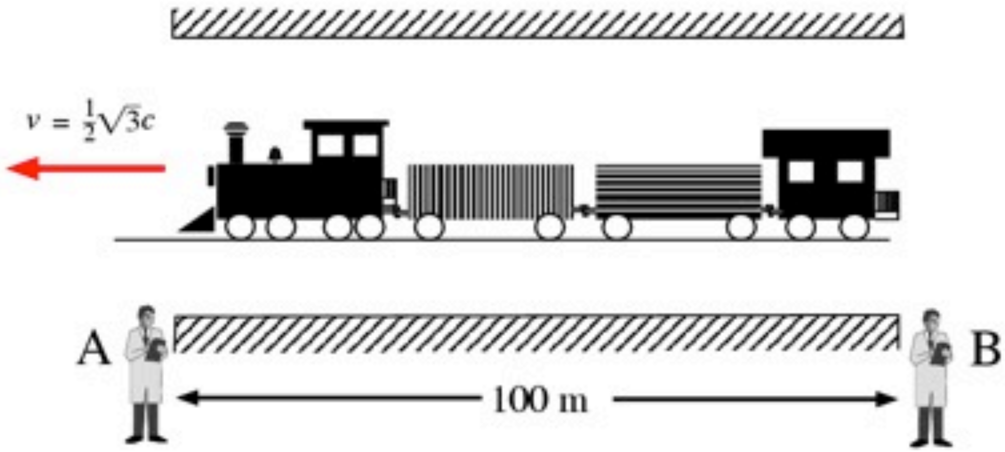

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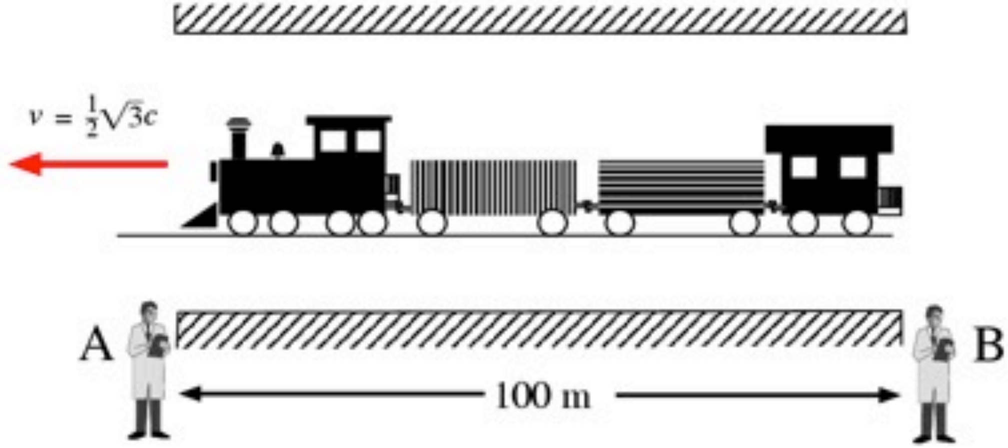

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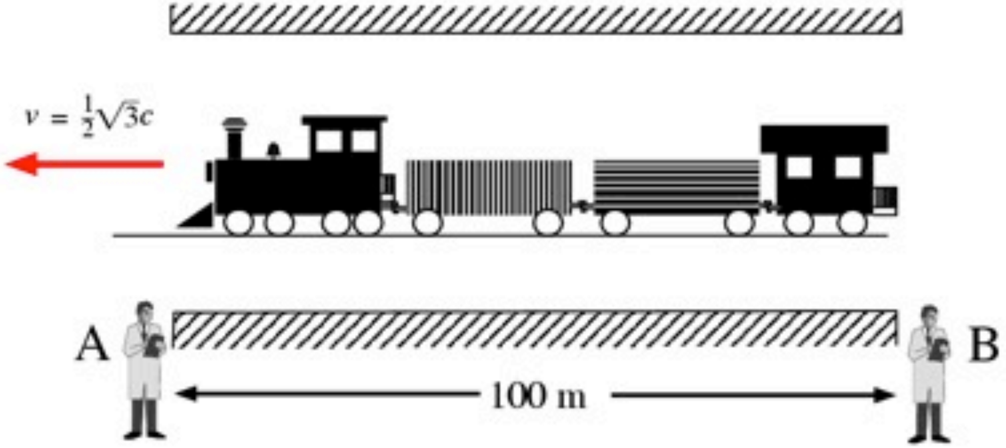

Guard's clock reads 0 when driver's clock reads 0, which is as driver exits the tunnel. To guard and driver, tunnel is 50m, so guard is 50m from the entrance in the train's frame, or 100m in tunnel frame.

So the guard is 100m from the entrance to the tunnel when his clock reads 0.

Repeat within framework of Lorentz transformation

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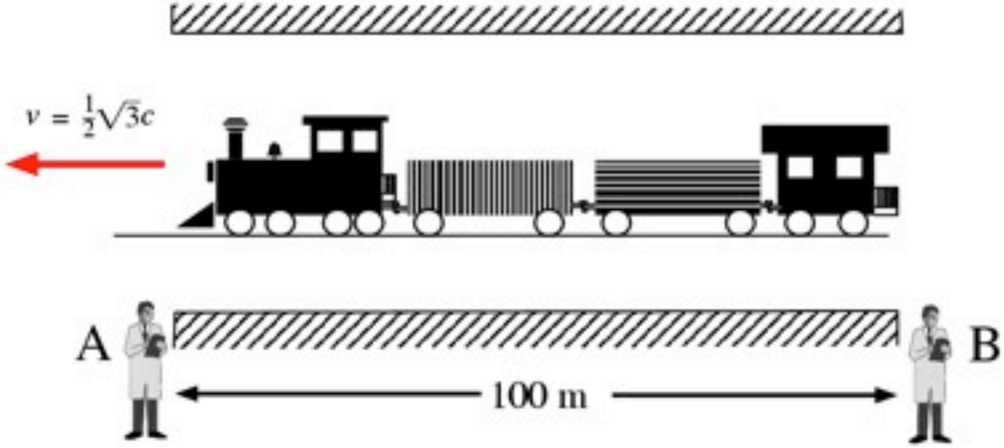

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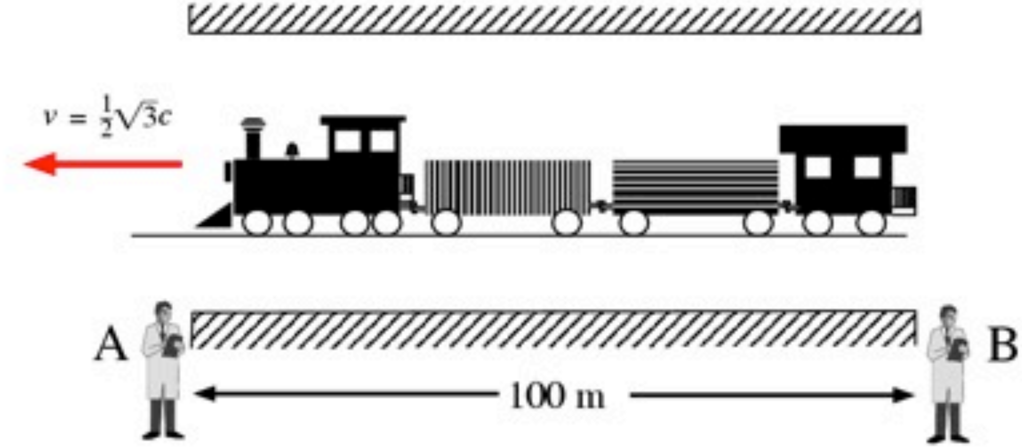

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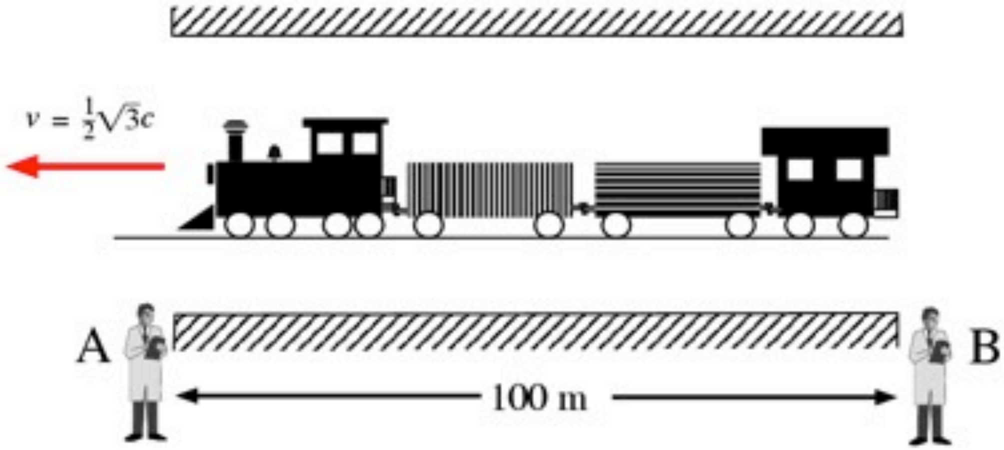

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$$x = \gamma(x' - vt') \quad t = \gamma\left(t' - \frac{vx'}{c^2}\right)$$

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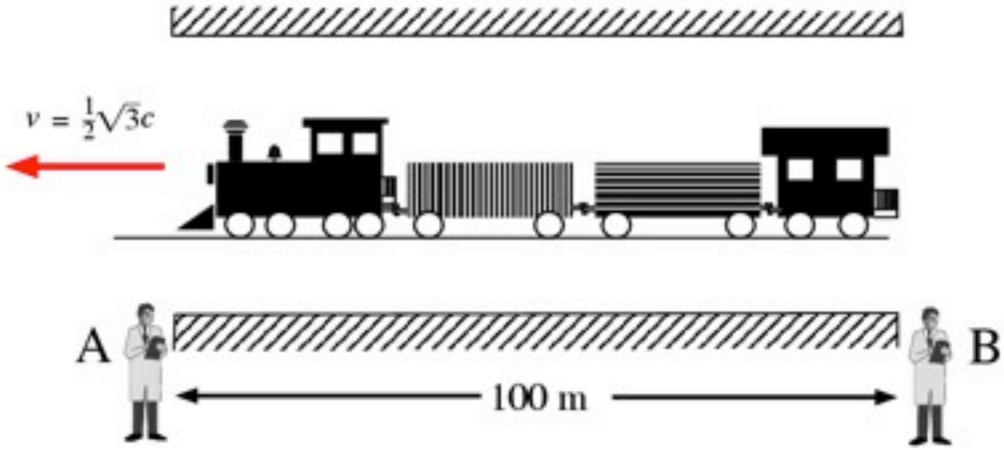
Repeat within framework of
Lorentz transformation

Question 1

A's clock (and the driver's clock) reads zero as the driver exits tunnel. What does B's clock read when the guard goes in?



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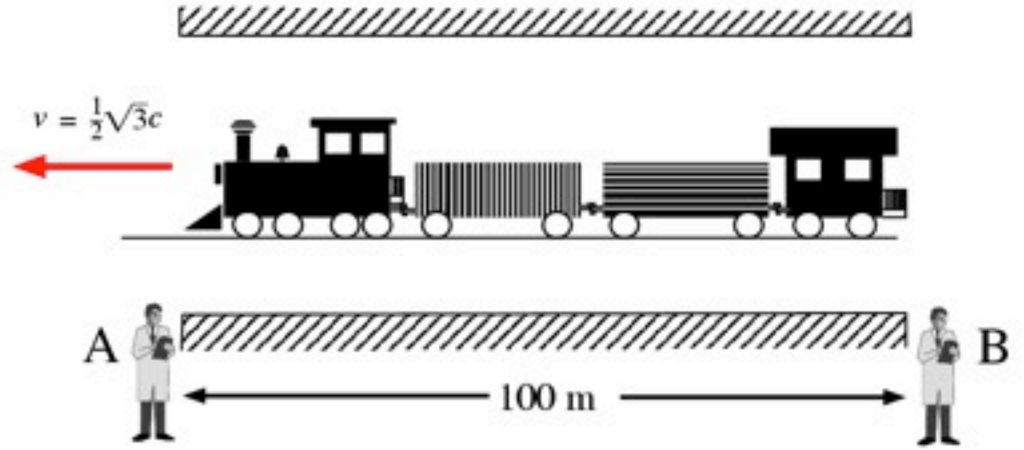
$$x' = \gamma(x + vt) \quad t' = \gamma\left(t + \frac{vx}{c^2}\right)$$

$$x_B = 100, \quad x'_G = 100$$

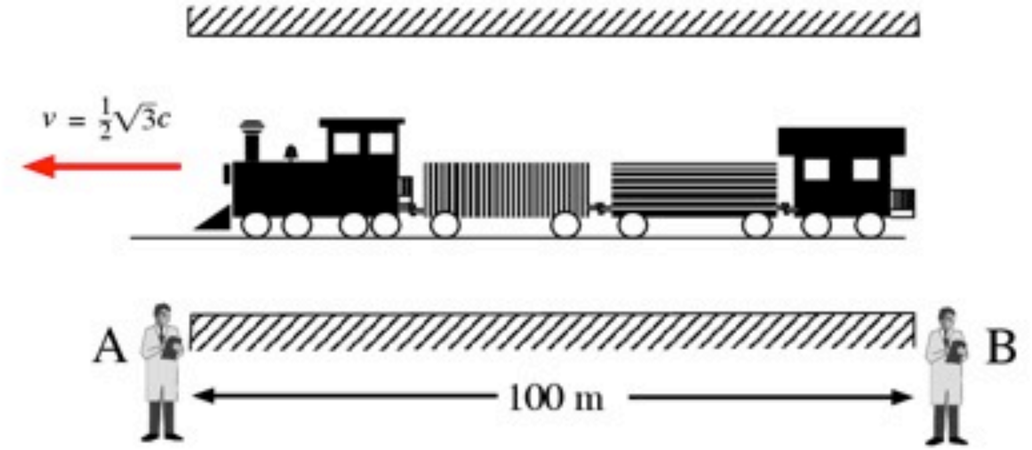
$$\Rightarrow \quad t_B = 100\frac{1-\gamma}{\gamma v} = -\frac{50}{v}$$

Question 2

What does the guard's clock read as he goes in?



Question 2



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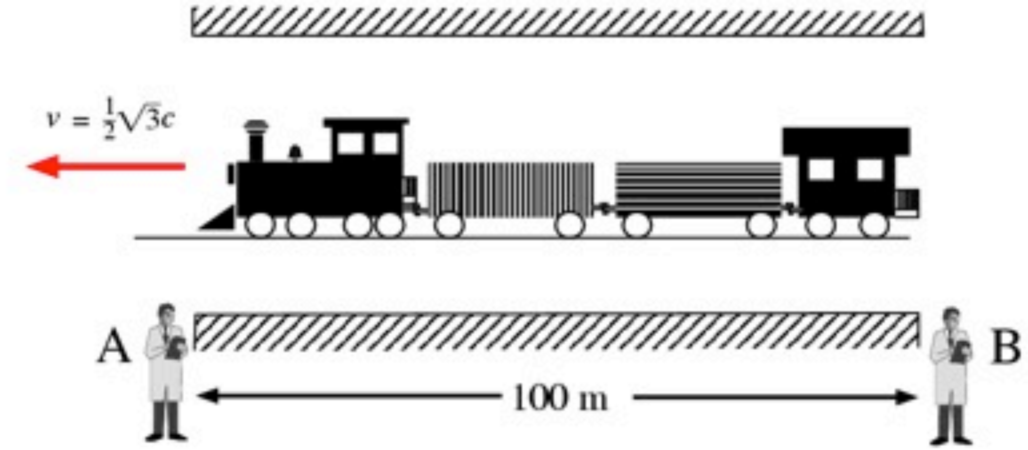

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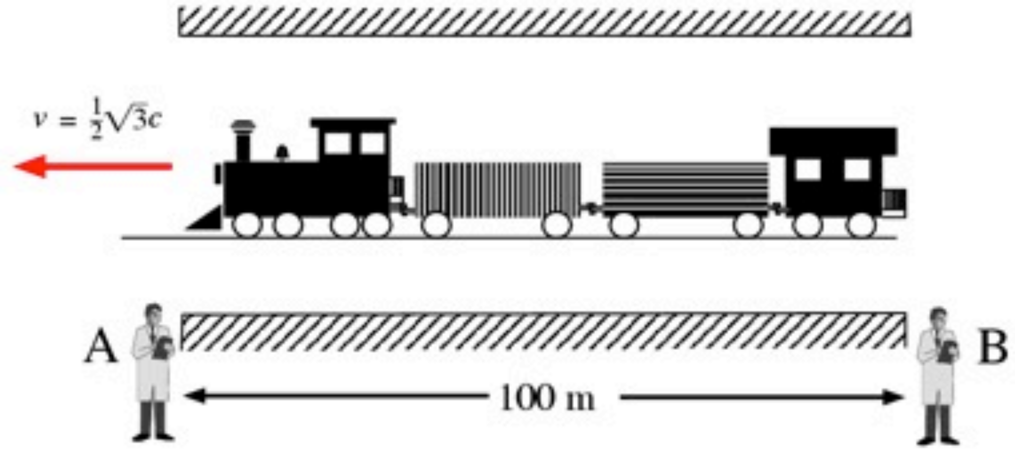

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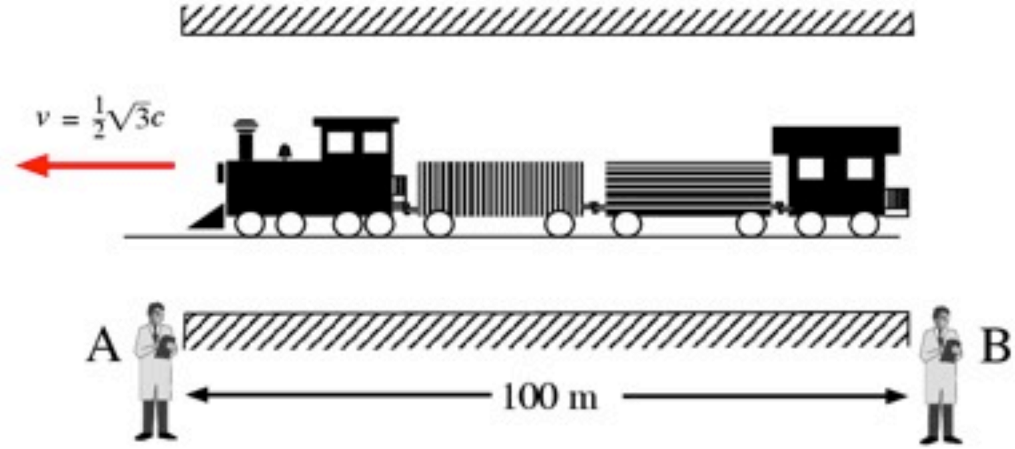
$$
\begin{aligned}
x &= \gamma(x' - vt') & \qquad t &= \gamma\left(t' - \frac{vx'}{c^2}\right) & \qquad x_B &= 100, \quad x'_G = 100 \\
x' &= \gamma(x + vt) & \qquad t' &= \gamma\left(t + \frac{vx}{c^2}\right) \implies & \qquad t'_G &= 100\frac{\gamma - 1}{\gamma v} = \frac{50}{v}
\end{aligned}
$$

Question 3

Where is the guard when his clock reads 0?



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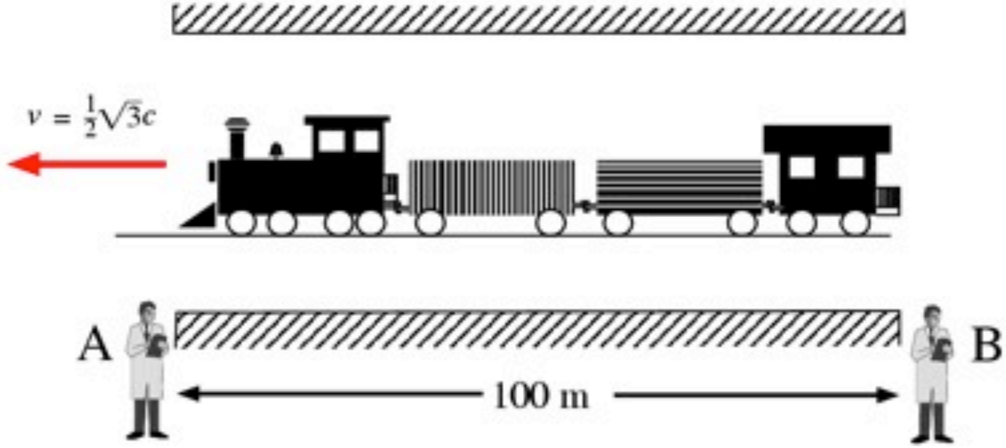

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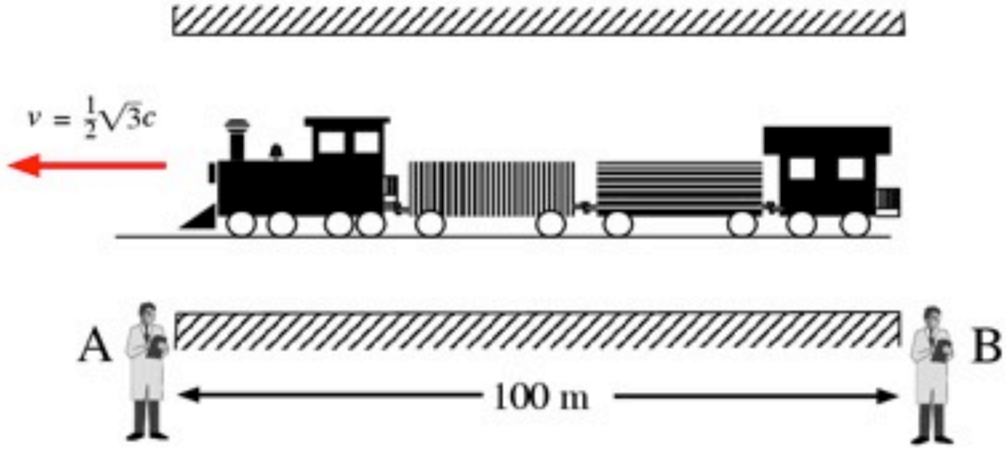

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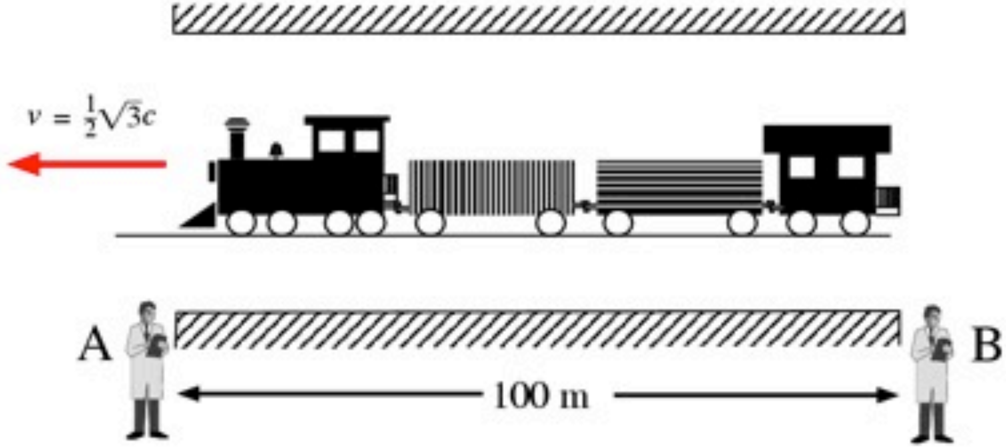

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$$x' = \gamma(x + vt) \qquad t' = \gamma\left(t + \frac{vx}{c^2}\right) \implies x = \gamma x' = 200\,\mathrm{m}$$

Or 100m from the entrance to the tunnel

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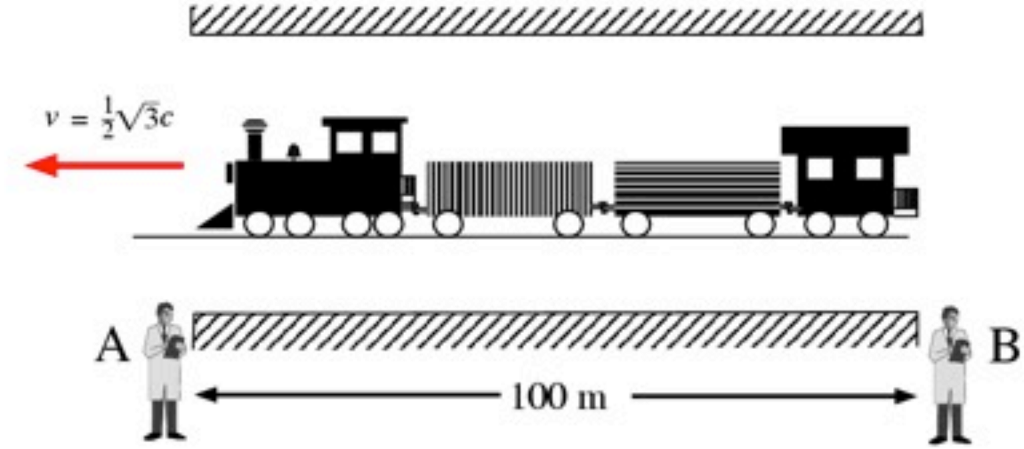
Question 4

Where was the driver when his clock reads the same as the guard's when he enters the tunnel?



Question 4

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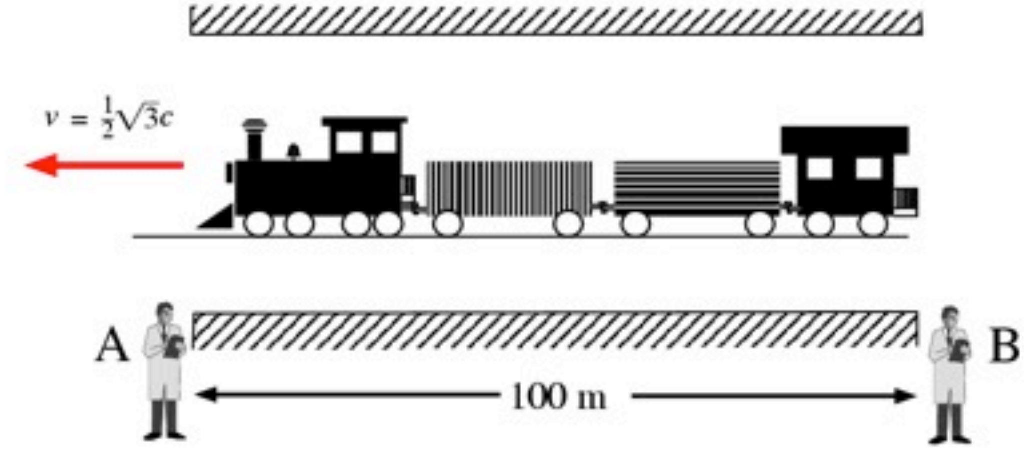

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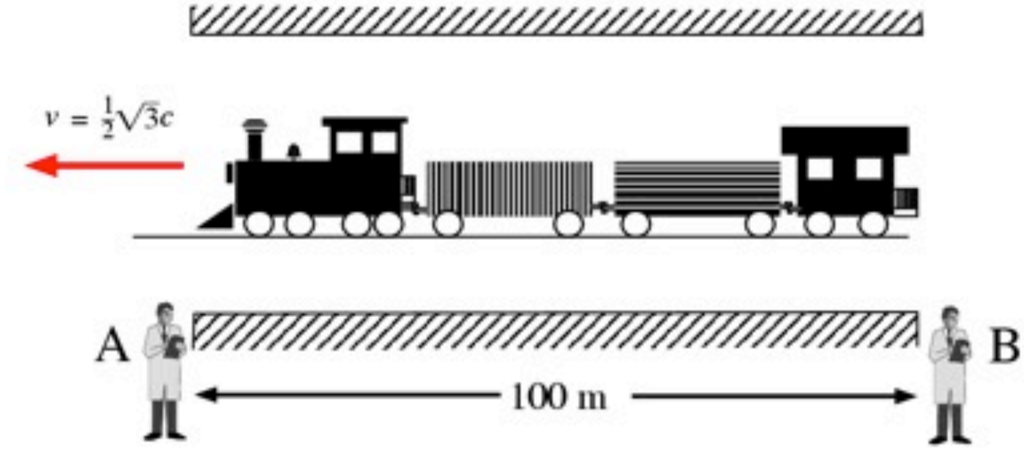

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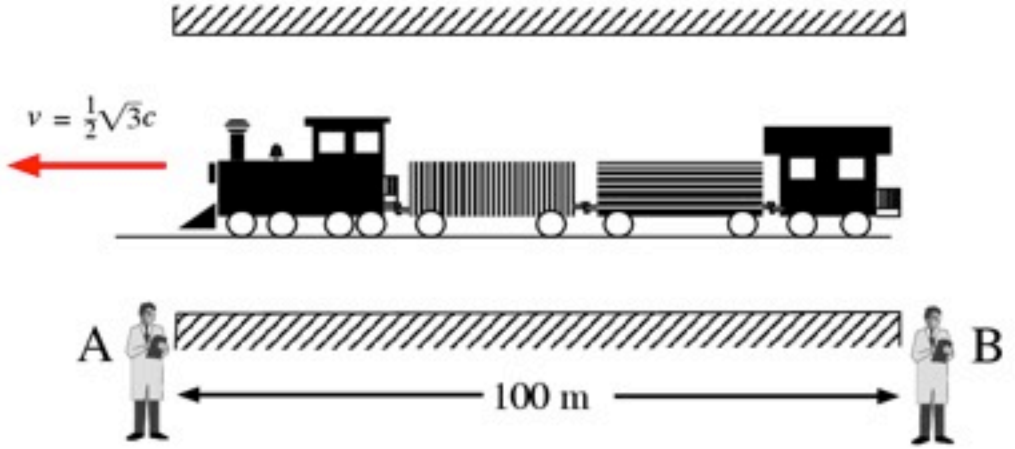

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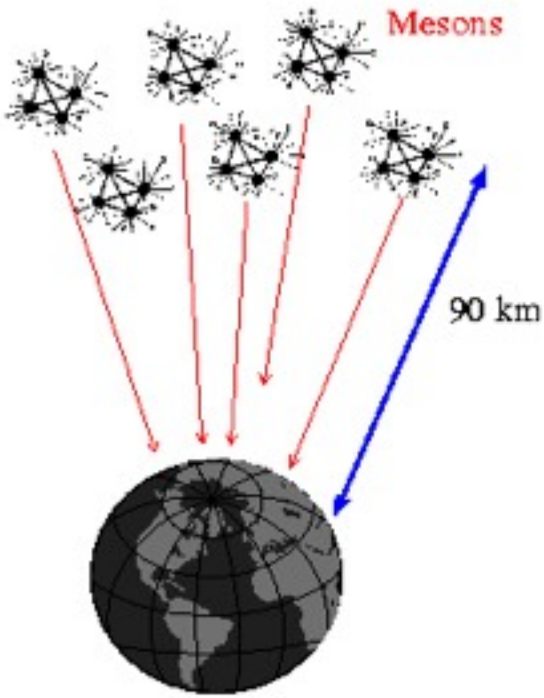
$$x' = \gamma(x + vt) \qquad t' = \gamma\left(t + \frac{vx}{c^2}\right) \qquad \Longrightarrow \qquad x = -\gamma v t'_D = -100$$

or 100 m beyond the tunnel exit

Example: Cosmic Rays



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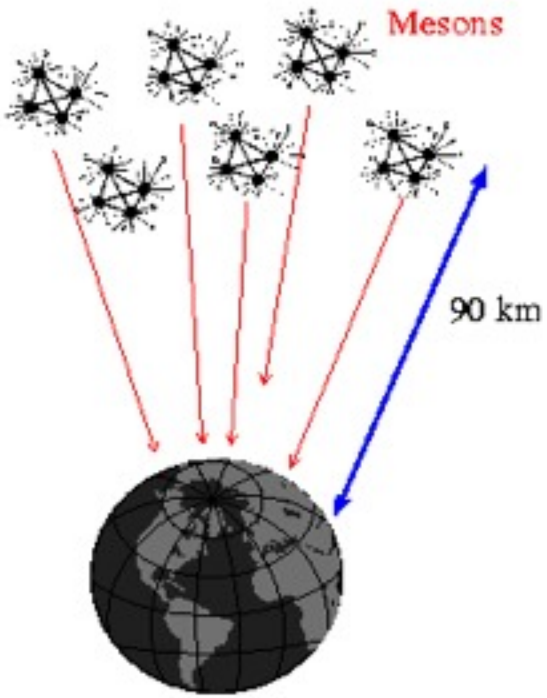

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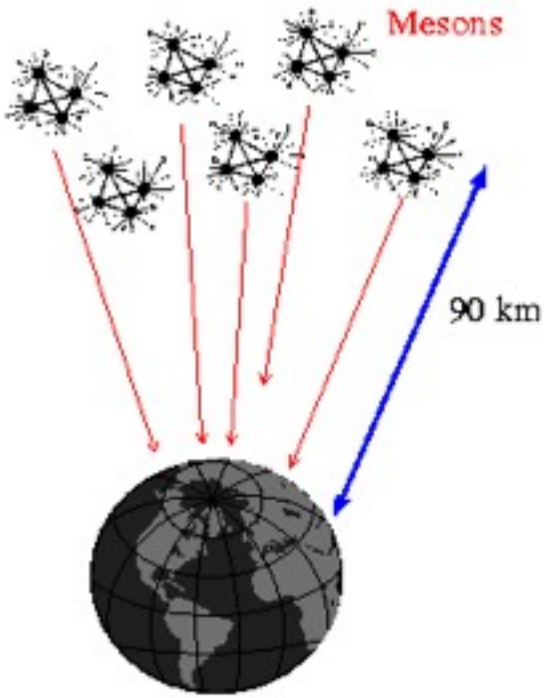

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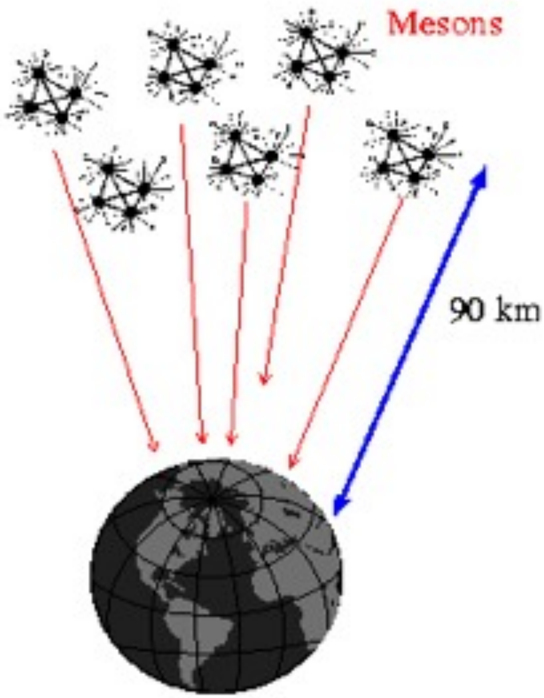

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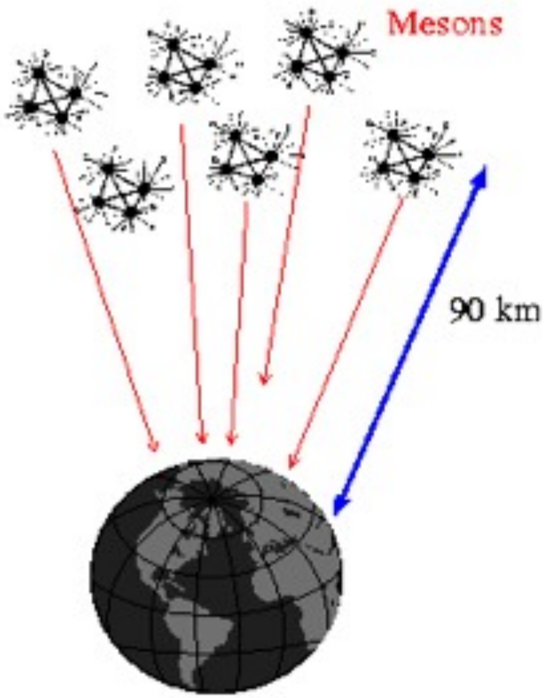

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- Both give

$$\frac{\gamma v}{c} = \frac{90\,km}{c\tau} = 150, \quad v \approx c, \quad \gamma \approx 150$$

Space-time

- An invariant is a quantity that has the same value in all inertial frames.
- Lorentz transformation is based on invariance of

$$c^2t^2 - (x^2 + y^2 + z^2) = (ct)^2 - \vec{x}^2$$

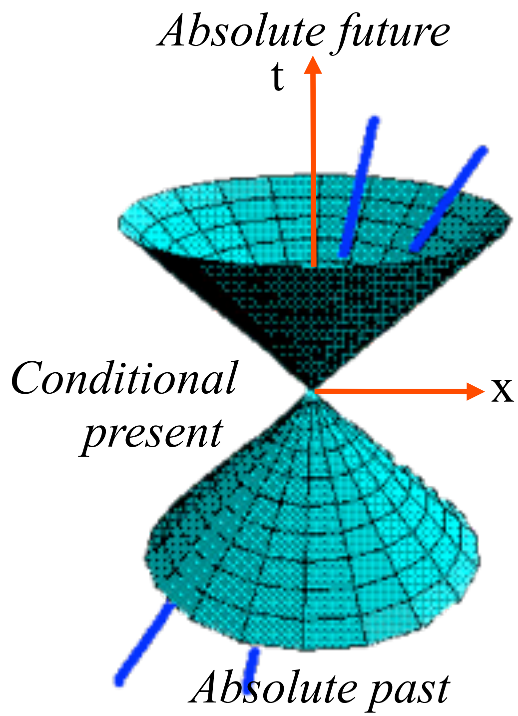
- 4D space with coordinates (t,x,y,z) is called space-time and the point $(t, x, y, z) = (t, \vec{x})$ is called an event.
- Fundamental invariant (preservation of speed of light):

$$\begin{aligned}\Delta s^2 = c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 &= c^2\Delta t^2\left(1 - \frac{\Delta x^2 + \Delta y^2 + \Delta z^2}{c^2\Delta t^2}\right)\\ &= c^2\Delta t^2\left(1 - \frac{v^2}{c^2}\right) = c^2\left(\frac{\Delta t}{\gamma}\right)^2\end{aligned}$$

$$\tau = \int \frac{dt}{\gamma}$$

is called proper time, time in instantaneous rest frame, an invariant. $\Delta s = c\Delta\tau$ is called the separation between two events

Space-time



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4-Vectors

The Lorentz transformation can be written in matrix form as

$$\begin{aligned} t' &= \gamma\left(t - \frac{vx}{c^2}\right) \\ x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \end{aligned} \implies \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\frac{\gamma v}{c} & 0 & 0 \\ -\frac{\gamma v}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

Lorentz matrix *L*

Position 4-vector *X*

An object made up of 4 elements which transforms like X is called a 4-vector

(analogous to the 3-vector of classical mechanics)

$$X' = LX$$

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Invariants

Basic invariant

$$c^2t^2 - x^2 - y^2 - z^2 = (ct, x, y, z)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = X^T g X = X \cdot X$$

Inner product of two 4-vectors $A = (a_0, \vec{a}), \quad B = (b_0, \vec{b})$

$$A \cdot A = A^T g A = a_0 b_0 - a_1 b_1 - a_2 b_2 - a_3 b_3 = a_0 b_0 - \vec{a} \cdot \bar{b}$$

Invariance:

$$A' \cdot A' = (LA)^T g(LA) = A^T (L^T g L) A = A^T g A = A \cdot A$$

Similarly $\quad A' \cdot B' = A \cdot B$

4-Vectors in S.R. Mechanics

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4-Vectors in S.R. Mechanics

- Velocity: $$V = \frac{\mathrm{d}X}{\mathrm{d}\tau} = \gamma\frac{\mathrm{d}X}{\mathrm{d}t} = \gamma\frac{\mathrm{d}}{\mathrm{d}t}(ct, \vec{x}) = \gamma(c, \vec{v})$$

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- Note invariant $$V \cdot V = \gamma^2(c^2 - \vec{v}^2) = \frac{c^2 - \vec{v}^2}{1 - \vec{v}^2/c^2} = c^2$$

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4-Vectors in S.R. Mechanics

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- **Note invariant** $$V \cdot V = \gamma^2(c^2 - \vec{v}^2) = \frac{c^2 - \vec{v}^2}{1 - \vec{v}^2/c^2} = c^2$$
- **Momentum:** $$P = m_0 V = m_0 \gamma(c, \vec{v}) = (mc, \vec{p})$$

$$m = m_0\gamma \quad \text{is the relativistic mass}$$
$$p = m_0\gamma\vec{v} = m\vec{v} \quad \text{is the relativistic 3-momentum}$$

Example of Transformation: Addition of Velocities

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- A particle moves with velocity $\vec{u} = (u_x, u_y, u_z)$ in frame F, so has 4-velocity $V = \gamma_u (c, \vec{u})$

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- Lorentz transformation gives

$$x' = \gamma_v(x + vt) \quad t' = \gamma_v\left(t + \frac{vx}{c^2}\right)$$
$$t \leftrightarrow \gamma_u, \vec{x} \leftrightarrow \gamma_u \vec{u}, t' \leftrightarrow \gamma_w, \vec{x}' \leftrightarrow \gamma_w \vec{w}$$

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$$t \leftrightarrow \gamma_u, \vec{x} \leftrightarrow \gamma_u \vec{u}, t' \leftrightarrow \gamma_w, \vec{x}' \leftrightarrow \gamma_w \vec{w}$$

$$\gamma_w = \gamma_v \left(\gamma_u + \frac{v \gamma_u u_x}{c^2} \right)$$

$$\gamma_w w_x = \gamma_v (\gamma_u u_x + v \gamma_u) \quad \Rightarrow \quad w_x = \frac{u_x + v}{1 + \frac{v u_x}{c^2}} \quad w_y = \frac{u_y}{\gamma_v \left(1 + \frac{v u_x}{c^2} \right)} \quad w_z = \frac{u_z}{\gamma_v \left(1 + \frac{v u_x}{c^2} \right)}$$

$$\gamma_w w_y = \gamma_u u_y$$

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4-Force

From Newton's 2nd Law expect 4-Force given by

$$\begin{aligned} F &= \frac{\mathrm{d}P}{\mathrm{d}\tau} = \gamma\frac{\mathrm{d}P}{\mathrm{d}t} \\ &= \gamma\frac{\mathrm{d}}{\mathrm{d}t}(mc,\vec{p}) = \gamma\left(c\frac{\mathrm{d}m}{\mathrm{d}t},\frac{\mathrm{d}\vec{p}}{\mathrm{d}t}\right) \\ &= \gamma\left(c\frac{\mathrm{d}m}{\mathrm{d}t},\vec{f}\right) \end{aligned}$$

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Note: 3-force equation: $\vec{f} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = m_0 \frac{\mathrm{d}}{\mathrm{d}t}(\gamma \vec{v})$

Einstein's Relation

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- Momentum invariant $P \cdot P = m_0^2 V \cdot V = m_0^2 c^2$

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- Differentiate $P \cdot \frac{\mathrm{d}P}{\mathrm{d}\tau} = 0 \implies V \cdot \frac{\mathrm{d}P}{\mathrm{d}\tau} = 0 \implies V \cdot F = 0$

$$\implies \gamma(c, \vec{v}) \cdot \gamma \left(c \frac{\mathrm{d}m}{\mathrm{d}t}, \vec{f} \right) = 0$$

$$\implies \frac{\mathrm{d}}{\mathrm{d}t}(mc^2) - \vec{v} \cdot \vec{f} = 0$$

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Therefore kinetic energy is

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$E=mc^2$ is total energy

Einstein's Relation

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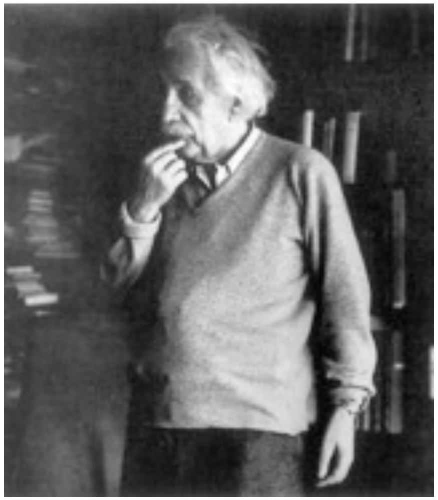
- Differentiate $P \cdot \dfrac{\mathrm{d}P}{\mathrm{d}\tau} = 0 \implies V \cdot \dfrac{\mathrm{d}P}{\mathrm{d}\tau} = 0 \implies V \cdot F = 0$



$$\implies \gamma(c, \vec{v}) \cdot \gamma \left(c\frac{\mathrm{d}m}{\mathrm{d}t}, \vec{f} \right) = 0$$

$$\implies \frac{\mathrm{d}}{\mathrm{d}t}(mc^2) - \vec{v} \cdot \vec{f} = 0$$

$$\begin{aligned} \vec{v} \cdot \vec{f} &= \text{rate at which force does work} \\ &= \text{rate of change of kinetic energy} \end{aligned}$$

Therefore kinetic energy is

$$T = mc^2 + \text{constant} = m_0 c^2 (\gamma - 1)$$

$E=mc^2$ is total energy

Summary of 4-Vectors

Position	X	$=$	$(ct, \vec{x})$
Velocity	V	$=$	$\gamma(c, \vec{v})$
Momentum	P	$=$	$m_0 V = m(c, \vec{v}) = \left(\dfrac{E}{c}, \vec{p}\right)$
Force	F	$=$	$\gamma\left(c\dfrac{\mathrm{d}m}{\mathrm{d}t}, \vec{f}\right) = \gamma\left(\dfrac{1}{c}\dfrac{\mathrm{d}E}{\mathrm{d}t}, \vec{f}\right)$

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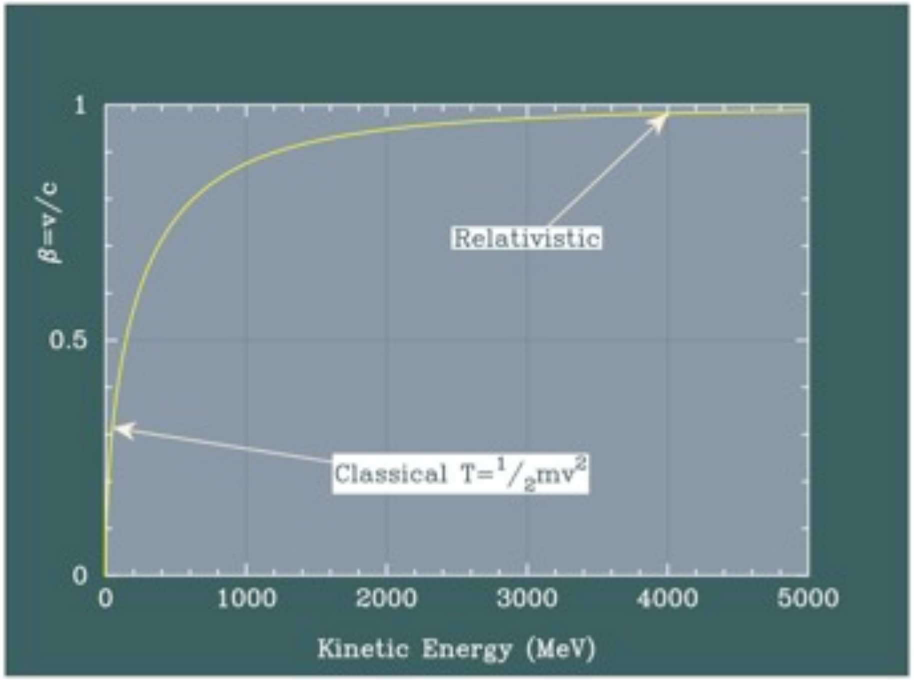
Basic Quantities used in Accelerator Calculations

$$
\begin{array}{llcl}
\text{Relative velocity} & \beta & = & \dfrac{v}{c} \\
\text{Velocity} & v & = & \beta c \\
\text{Momentum} & p & = & mv = m_0\gamma\beta c \\
\text{Kinetic energy} & T & = & (m - m_0)c^2 = m_0c^2(\gamma - 1)
\end{array}
$$

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} = \left(1 - \beta^2\right)^{-\frac{1}{2}}$$

$$\Longrightarrow \quad (\beta\gamma)^2 = \frac{\gamma^2 v^2}{c^2} = \gamma^2 - 1 \quad \Longrightarrow \quad \beta^2 = \frac{v^2}{c^2} = 1 - \frac{1}{\gamma^2}$$

Velocity *v*. Energy



$$T = m_0(\gamma - 1)c^2$$

$$\gamma = 1 + \frac{T}{m_0 c^2}$$

$$\beta = \sqrt{1 - \frac{1}{\gamma^2}}$$

$$p = m_0 \beta \gamma c$$

$$\text{For } v \ll c, \quad \gamma = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} \approx 1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \ldots$$

$$\text{so} \quad T = m_0 c^2(\gamma - 1) \approx \frac{1}{2} m_0 v^2$$

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Energy-Momentum Invariant

$$
\begin{array}{ccccccc}
P \cdot P & = & m_0^2 V \cdot V & = & m_0^2 c^2 & \text{and} & P = \left(\dfrac{E}{c}, \vec{p}\right) \\
\downarrow & & & & \downarrow & & \\
\dfrac{E^2}{c^2} - \vec{p}^2 & & & = & m_0^2 c^2 = \dfrac{1}{c^2} E_0^2 & & \text{where } E_0 \text{ is rest energy} \\
& \Longrightarrow & p^2 c^2 & = & E^2 - E_0^2 & & \\
& & & = & (E - E_0)(E + E_0) & & \\
& & & = & T(T + 2E_0) & &
\end{array}
$$

Energy-Momentum Invariant

$$P \cdot P = m_0^2 V \cdot V = m_0^2 c^2 \quad \text{and} \quad P = \left(\frac{E}{c}, \vec{p} \right)$$

$$\downarrow \qquad\qquad \downarrow$$

$$\frac{E^2}{c^2} - \vec{p}^2 = m_0^2 c^2 = \frac{1}{c^2} E_0^2 \quad \text{where } E_0 \text{ is rest energy}$$

$$\Longrightarrow \quad p^2 c^2 = E^2 - E_0^2$$

$$= (E - E_0)(E + E_0)$$

$$= T(T + 2E_0)$$

Example: ISIS 800 MeV protons
(E_0=938 MeV)

=> pc=1.463 GeV

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Energy-Momentum Invariant

$$P \cdot P = m_0^2 V \cdot V = m_0^2 c^2 \quad \text{and} \quad P = \left(\frac{E}{c}, \vec{p}\right)$$

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$$\Longrightarrow \quad p^2 c^2 = E^2 - E_0^2$$

$$= (E - E_0)(E + E_0)$$

$$= T(T + 2E_0)$$

$$\beta\gamma = \frac{m_0 \beta\gamma c^2}{m_0 c^2} = \frac{pc}{E_0} = 1.56$$

$$\gamma^2 = (\beta\gamma)^2 + 1 \Longrightarrow \gamma = 1.85$$

$$\beta = \frac{\beta\gamma}{\gamma} = 0.84$$

Example: ISIS 800 MeV protons
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Relationships between small variations in parameters ΔE, ΔT, Δp, Δβ, Δγ

$$
\begin{aligned}
(\beta\gamma)^2 &= \gamma^2 - 1 \\
\Longrightarrow \quad \beta\gamma\Delta(\beta\gamma) &= \gamma\Delta\gamma \\
\Longrightarrow \quad \beta\Delta(\beta\gamma) &= \Delta\gamma \qquad (1)
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{\gamma^2} &= 1 - \beta^2 \\
\Longrightarrow \quad \frac{1}{\gamma^3}\Delta\gamma &= \beta\Delta\beta \qquad (2)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\Delta p}{p} &= \frac{\Delta(m_0\beta\gamma c)}{m_0\beta\gamma c} = \frac{\Delta(\beta\gamma)}{\beta\gamma} \\
&= \frac{1}{\beta^2}\frac{\Delta\gamma}{\gamma} = \frac{1}{\beta^2}\frac{\Delta E}{E} \\
&= \gamma^2\frac{\Delta\beta}{\beta} \\
&= \frac{\gamma}{\gamma+1}\frac{\Delta T}{T} \qquad \text{(exercise)}
\end{aligned}
$$

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\frac{\Delta p}{p} &= \frac{\Delta(m_0\beta\gamma c)}{m_0\beta\gamma c} = \frac{\Delta(\beta\gamma)}{\beta\gamma} \\
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&= \gamma^2\frac{\Delta\beta}{\beta} \\
&= \frac{\gamma}{\gamma+1}\frac{\Delta T}{T} \qquad \text{(exercise)}
\end{aligned}
$$

Note: valid to first order only

	$\dfrac{\Delta\beta}{\beta}$	$\dfrac{\Delta p}{p}$	$\dfrac{\Delta T}{T}$	$\dfrac{\Delta E}{E} = \dfrac{\Delta\gamma}{\gamma}$
$\dfrac{\Delta\beta}{\beta} =$	$\dfrac{\Delta\beta}{\beta}$	$\dfrac{1}{\gamma^2}\dfrac{\Delta p}{p}$ $\dfrac{\Delta p}{p} - \dfrac{\Delta\gamma}{\gamma}$	$\dfrac{1}{\gamma(\gamma+1)}\dfrac{\Delta T}{T}$	$\dfrac{1}{\beta^2\gamma^2}\dfrac{\Delta\gamma}{\gamma}$ $\dfrac{1}{\gamma^2-1}\dfrac{\Delta\gamma}{\gamma}$
$\dfrac{\Delta p}{p} =$	$\gamma^2\dfrac{\Delta\beta}{\beta}$	$\dfrac{\Delta p}{p}$	$\dfrac{\gamma}{\gamma+1}\dfrac{\Delta T}{T}$	$\dfrac{1}{\beta^2}\dfrac{\Delta\gamma}{\gamma}$
$\dfrac{\Delta T}{T} =$	$\gamma(\gamma+1)\dfrac{\Delta\beta}{\beta}$	$\left(1+\dfrac{1}{\gamma}\right)\dfrac{\Delta p}{p}$	$\dfrac{\Delta T}{T}$	$\dfrac{\gamma}{\gamma-1}\dfrac{\Delta\gamma}{\gamma}$
$\dfrac{\Delta E}{E} =$	$(\beta\gamma)^2\dfrac{\Delta\beta}{\beta}$	$\beta^2\dfrac{\Delta p}{p}$	$\left(1-\dfrac{1}{\gamma}\right)\dfrac{\Delta T}{T}$	$\dfrac{\Delta\gamma}{\gamma}$
$\dfrac{\Delta\gamma}{\gamma} =$	$(\gamma^2-1)\dfrac{\Delta\beta}{\beta}$	$\dfrac{\Delta p}{p} - \dfrac{\Delta\beta}{\beta}$		

Table 1: Incremental relationships between energy, velocity and momentum.

4-Momentum Conservation

4-Momentum Conservation

- Equivalent expression for 4-momentum

$$P = m_0\gamma(c,\vec{v}) = (mc,\vec{p}) = \left(E/_c,\vec{p}\right)$$

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4-Momentum Conservation

- Equivalent expression for 4-momentum $P = m_0\gamma(c,\vec{v}) = (mc,\vec{p}) = \left(E/_c ,\vec{p}\right)$

- Invariant $m_0^2 c^2 = P \cdot P = \frac{E^2}{c^2} - \vec{p}^2$ $\boxed{E^2/_{c^2} = m_0^2 c^2 + \vec{p}^{\,2}}$

4-Momentum Conservation

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- Classical momentum conservation laws $\rightarrow$ conservation of 4-momentum. Total 3-momentum and total energy are conserved.

$$\sum_{\text{particles, i}} P_i = \text{constant}$$

$$\Rightarrow \sum_{\text{particles, i}} E_i \text{ and } \sum_{\text{particles,i}} \vec{p}_i \text{ constant}$$

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Problem

A body of mass M disintegrates while at rest into two parts of rest masses M_1 and M_2.

Show that the energies of the parts are given by

$$E_1 = c^2 \frac{M^2 + M_1^2 - M_2^2}{2M}, \quad E_2 = c^2 \frac{M^2 - M_1^2 + M_2^2}{2M}$$

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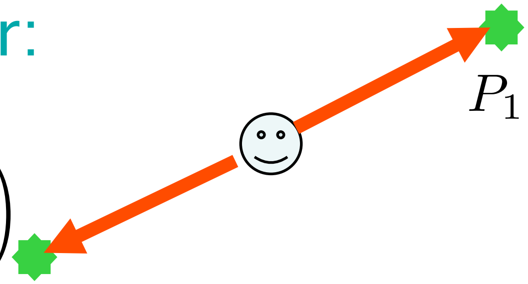
Solution

Before:

$$P = \left(Mc, \vec{0}\right)$$

After:



$$P_1 = \left(\frac{E_1}{c}, \vec{p}\right)$$

$$P_2 = \left(\frac{E_2}{c}, -\vec{p}\right)$$

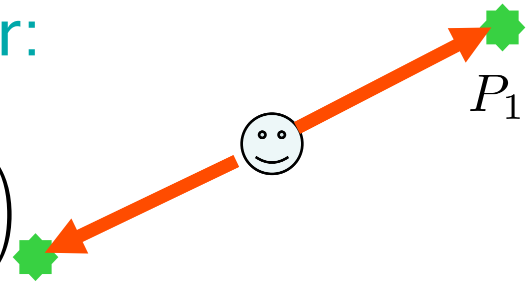
Solution

Before:

$$P = \left(Mc, \vec{0}\right)$$

After:

$$P_2 = \left(\frac{E_2}{c}, -\vec{p}\right)$$



$$P_1 = \left(\frac{E_1}{c}, \vec{p}\right)$$

Conservation of 4-momentum:

Solution

Before:

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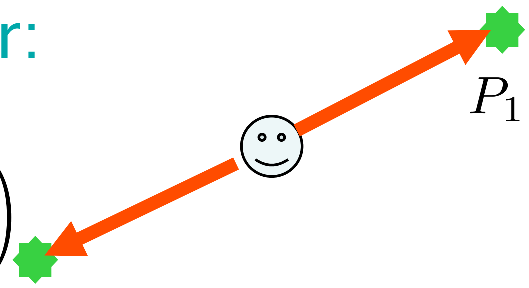
After:



$$P_1 = \left(\frac{E_1}{c}, \vec{p}\right)$$

$$P_2 = \left(\frac{E_2}{c}, -\vec{p}\right)$$

Conservation of 4-momentum:

$$P = P_1 + P_2 \quad \Rightarrow \quad P - P_1 = P_2$$

$$\Rightarrow \quad (P - P_1)\cdot(P - P_1) = P_2 \cdot P_2$$

$$\Rightarrow \quad P\cdot P - 2P\cdot P_1 + P_1\cdot P_1 = P_2 \cdot P_2$$

$$\Rightarrow \quad M^2c^2 - 2ME_1 + M_1^2c^2 = M_2^2c^2$$

$$\Rightarrow \quad E_1 = \frac{M^2 + M_1^2 - M_2^2}{2M}c^2$$

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Example of use of invariants

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Example of use of invariants

- Two particles have equal rest mass m_0.

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Example of use of invariants

- Two particles have equal rest mass m_0.
  - Frame 1: one particle at rest, total energy is E_1.
  - Frame 2: centre of mass frame where velocities are equal and opposite, total energy is E_2.

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Example of use of invariants

- Two particles have equal rest mass m_0.
  - Frame 1: one particle at rest, total energy is E_1.
  - Frame 2: centre of mass frame where velocities are equal and opposite, total energy is E_2.

Problem: Relate E_1 to E_2

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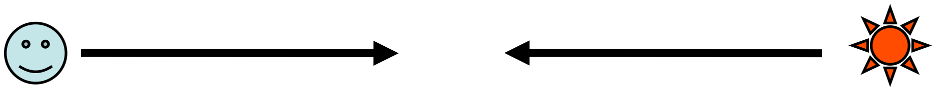
Total energy E_1

(Fixed target experiment)

$$P_1 = \left(\frac{E_1 - m_0 c^2}{c}, \vec{p} \right) \qquad P_2 = \left(m_0 c, \vec{0} \right)$$



Total energy E_2

(Colliding beams expt)

$$P_1 = \left(\frac{E_2}{2c}, \vec{p}' \right) \qquad P_2 = \left(\frac{E_2}{2c}, -\vec{p}' \right)$$

$$\text{Invariant:} \qquad P_2 \cdot (P_1 + P_2)$$

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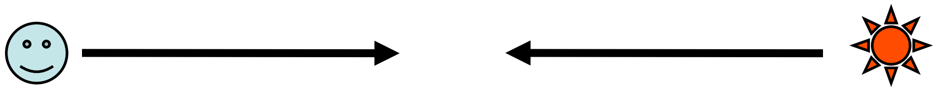
Total energy E_1

(Fixed target experiment)

$$P_1 = \left(\frac{E_1 - m_0 c^2}{c}, \vec{p} \right) \qquad P_2 = \left(m_0 c, \vec{0} \right)$$



Total energy E_2

(Colliding beams expt)

$$P_1 = \left(\frac{E_2}{2c}, \vec{p}' \right) \qquad P_2 = \left(\frac{E_2}{2c}, -\vec{p}' \right)$$

$$\text{Invariant:} \qquad P_2 \cdot (P_1 + P_2)$$

$$m_0 c \times \frac{E_1}{c} - 0 \times p = \frac{E_2}{2c} \times \frac{E_2}{c} + p' \times 0$$

$$\Rightarrow \qquad 2 m_0 c^2 E_1 = E_2^2$$

Collider Problem

Collider Problem

- In an accelerator, a proton p_1 with rest mass m_0 collides with an anti-proton p_2 (with the same rest mass), producing two particles W_1 and W_2 with equal rest mass $M_0=100m_0$

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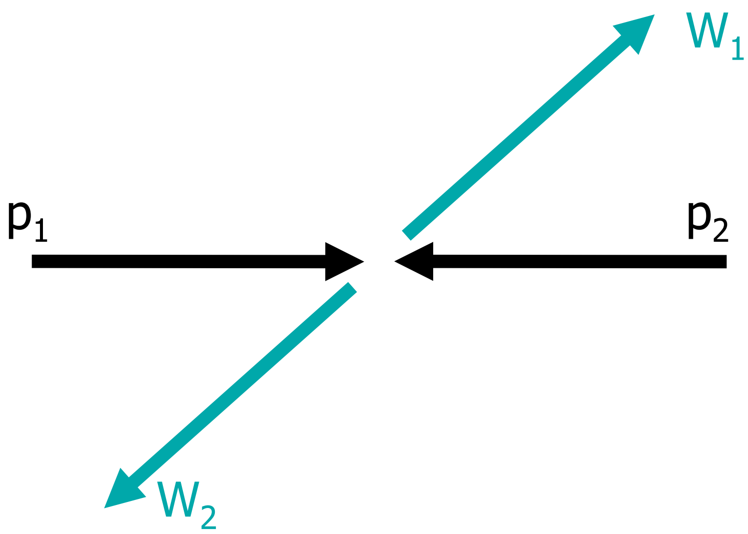
- In an accelerator, a proton p_1 with rest mass m_0 collides with an anti-proton p_2 (with the same rest mass), producing two particles W_1 and W_2 with equal rest mass $M_0=100m_0$
  - Expt 1: p_1 and p_2 have equal and opposite velocities in the lab frame. Find the minimum energy of p_2 in order for W_1 and W_2 to be produced.

Collider Problem

- In an accelerator, a proton p_1 with rest mass m_0 collides with an anti-proton p_2 (with the same rest mass), producing two particles W_1 and W_2 with equal rest mass $M_0=100m_0$
  - Expt 1: p_1 and p_2 have equal and opposite velocities in the lab frame. Find the minimum energy of p_2 in order for W_1 and W_2 to be produced.
  - Expt 2: in the rest frame of p_1, find the minimum energy E' of p_2 in order for W_1 and W_2 to be produced.

Experiment 1

Note: $E^2/c^2 = \vec{p}^{\,2} + m_0^2 c^2 \Rightarrow$ same m_0, same p mean same E.



Total 3-momentum is zero before collision and so is zero afterwards.

4-momenta before collision:

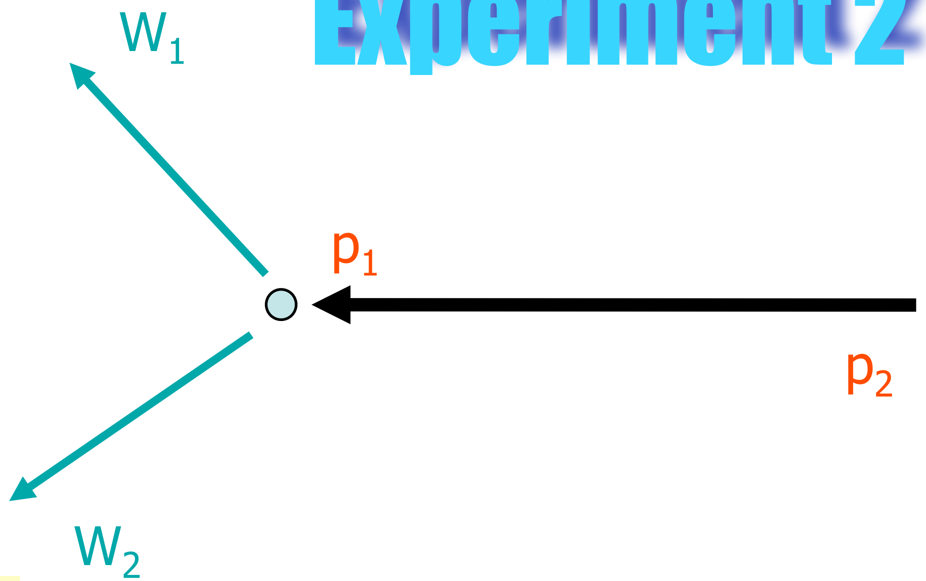
$$P_1 = \left(E/c, \vec{p}\right) \quad P_2 = \left(E/c, -\vec{p}\right)$$

4-momenta after collision:

$$P_1 = \left(E'/c, \vec{q}\right) \quad P_2 = \left(E'/c, -\vec{q}\right)$$

Energy conservation $\Rightarrow$ *E*=*E′* > rest energy = M_0c^2 = 100 m_0c^2

Experiment 2



Before collision:

$$P_1 = \left(m_0 c, \vec{0}\right) \quad P_2 = \left(E'/c, \vec{p}\right)$$

Total energy is

$$E_1 = E' + m_0 c^2$$

Use previous result $2m_0c^2E_1 = E_2^2$ to relate E_1 to total energy E_2 in C.O.M frame

$$2m_0c^2E_1 = E_2^2$$

$$\Rightarrow \quad 2m_0c^2\left(E' + m_0c^2\right) = (2E)^2 > \left(200\, m_0c^2\right)^2$$

$$\Rightarrow \quad E' > \left(2 \times 10^4 - 1\right) m_0c^2 \approx 20{,}000\, m_0c^2$$

4-Acceleration

- 4-Acceleration=rate of change of 4-Velocity

$$A = \frac{\mathrm{d}V}{\mathrm{d}\tau} = \gamma \frac{\mathrm{d}}{\mathrm{d}t}\left(\gamma c, \gamma \vec{v}\right)$$

- Use $\dfrac{1}{\gamma^2} = 1 - \dfrac{\vec{v}\cdot\vec{v}}{c^2} \quad \Longrightarrow \quad \dfrac{1}{\gamma^3}\dfrac{\mathrm{d}\gamma}{\mathrm{d}t} = \dfrac{\vec{v}\cdot\dot{\vec{v}}}{c^2} = \dfrac{\vec{v}\cdot\vec{a}}{c^2}$

$$\boxed{A = \gamma\left(\gamma^3 \frac{\vec{v}\cdot\vec{a}}{c}, \gamma\vec{a} + \gamma^3\left(\frac{\vec{v}\cdot\vec{a}}{c^2}\right)\vec{v}\right)}$$

- In instantaneous rest-frame $A = (0, \vec{a}), \quad A \cdot A = -|\vec{a}|^2$

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Radiation from an accelerating charged particle

- Rate of radiation, *R*, known to be invariant and proportional to $|\vec{a}|^2$ in instantaneous rest frame.
- But in instantaneous rest-frame $A \cdot A = -|\vec{a}|^2$
- Deduce $$R \propto A \cdot A = -\gamma^6 \left(\left(\frac{\vec{v} \cdot \vec{a}}{c} \right)^2 + \frac{1}{\gamma^2} \vec{a}^2 \right)$$
- Rearranged:

$$R = \frac{2e^2}{3c^3} \gamma^6 \left[|\vec{a}|^2 - \frac{(\vec{a} \times \vec{v})^2}{c^2} \right]$$

Relativistic Larmor Formula

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Motion under constant acceleration; world lines

- Introduce *rapidity* ρ defined by

$$\beta = \frac{v}{c} = \tanh\rho \quad \Longrightarrow \quad \gamma = \frac{1}{\sqrt{1-\beta^2}} = \cosh\rho$$

- Then $\quad V = \gamma(c, v) = c(\cosh\rho, \sinh\rho)$

- And $\quad A = \frac{\mathrm{d}V}{\mathrm{d}\tau} = c(\sinh\rho, \cosh\rho)\frac{\mathrm{d}\rho}{\mathrm{d}\tau}$

- So constant acceleration satisfies

$$a^2 = |\vec{a}|^2 = -A\cdot A = c^2\left(\frac{\mathrm{d}\rho}{\mathrm{d}\tau}\right)^2 \quad \Longrightarrow \quad \frac{\mathrm{d}\rho}{\mathrm{d}\tau} = \frac{a}{c}, \quad \text{so } \rho = \frac{a\tau}{c}$$

Particle Paths

$$
\begin{aligned}
c\sinh\rho &= c\sinh\frac{a\tau}{c} = \gamma v = \gamma\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}\tau} \\
\implies\quad x &= x_0 + \frac{c^2}{a}\left(\cosh\frac{a\tau}{c} - 1\right) \\
\cosh\rho &= \cosh\frac{a\tau}{c} = \gamma = \frac{\mathrm{d}t}{\mathrm{d}\tau} \\
\implies\quad t &= \frac{c}{a}\sinh\frac{a\tau}{c}
\end{aligned}
$$

$$
\cosh^2\rho - \sinh^2\rho = 1
$$

$$
\implies \left(x - x_0 + \frac{c^2}{a}\right)^2 - c^2t^2 = \frac{c^4}{a^2}
$$

$$
\begin{aligned}
x &= x_0 + \frac{c^2}{a}\left(1 + \frac{1}{2}\frac{a^2\tau^2}{c^2} + \ldots - 1\right) \\
&\approx x_0 + \frac{1}{2}a\tau^2 \\
t &\approx \frac{c}{a}\times\frac{a\tau}{c} = \tau
\end{aligned}
$$

Particle Paths

$$
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$$

Non-relativistic paths are parabolic

$$
t \approx \frac{c}{a} \times \frac{a\tau}{c} = \tau
$$

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Particle Paths

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\begin{aligned}
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\Longrightarrow \quad t &= \frac{c}{a}\sinh\frac{a\tau}{c}
\end{aligned}
$$

$$
\cosh^2\rho - \sinh^2\rho = 1
$$

$$
\Longrightarrow \quad \left(x - x_0 + \frac{c^2}{a}\right)^2 - c^2t^2 = \frac{c^4}{a^2}
$$

Relativistic paths are hyperbolic

$$
\begin{aligned}
x &= x_0 + \frac{c^2}{a}\left(1 + \frac{1}{2}\frac{a^2\tau^2}{c^2} + \ldots - 1\right) \\
&\approx x_0 + \frac{1}{2}a\tau^2
\end{aligned}
$$

Non-relativistic paths are parabolic

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t \approx \frac{c}{a} \times \frac{a\tau}{c} = \tau
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\begin{aligned}
c\sinh\rho &= c\sinh\frac{a\tau}{c} = \gamma v = \gamma\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}\tau} \\
\Longrightarrow \quad x &= x_0 + \frac{c^2}{a}\left(\cosh\frac{a\tau}{c} - 1\right) \\
\cosh\rho &= \cosh\frac{a\tau}{c} = \gamma = \frac{\mathrm{d}t}{\mathrm{d}\tau} \\
\Longrightarrow \quad t &= \frac{c}{a}\sinh\frac{a\tau}{c}
\end{aligned}
$$

$$
\cosh^2\rho - \sinh^2\rho = 1
$$

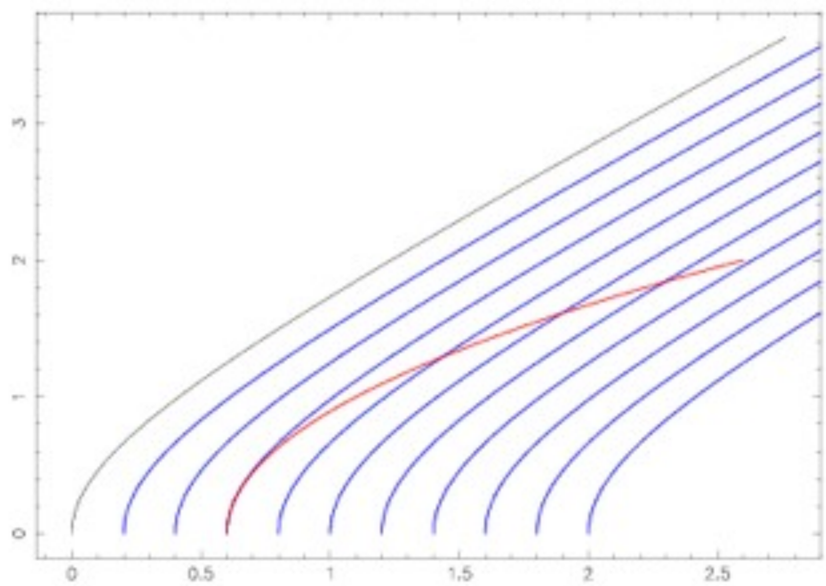
$$
\Longrightarrow \quad \left(x - x_0 + \frac{c^2}{a}\right)^2 - c^2t^2 = \frac{c^4}{a^2}
$$

Relativistic paths are hyperbolic

$$
\begin{aligned}
x &= x_0 + \frac{c^2}{a}\left(1 + \frac{1}{2}\frac{a^2\tau^2}{c^2} + \ldots - 1\right) \\
&\approx x_0 + \frac{1}{2}a\tau^2
\end{aligned}
$$

Non-relativistic paths are parabolic

$$
t \approx \frac{c}{a} \times \frac{a\tau}{c} = \tau
$$



Relativistic Lagrangian and Hamiltonian Formulation

3-force eqn of motion under potential V: $\vec{f} = \frac{d\vec{p}}{dt} \quad \Rightarrow \quad m_0 \frac{d}{dt}\left(\frac{\dot{x}}{\left(1 - v^2/c^2\right)^{1/2}} \right) = -\frac{\partial V}{\partial x}$ etc

Standard Lagrangian formalism: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x} \quad etc \quad \Rightarrow \frac{\partial L}{\partial \dot{x}} = \frac{m_0 \dot{x}}{\left(1 - v^2/c^2\right)^{1/2}}, \quad \frac{\partial L}{\partial x} = -\frac{\partial V}{\partial x}$

Since $v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$, deduce

$$L = -m_0 c^2 \left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}} - V = -\frac{m_0 c^2}{\gamma} - V$$

Relativistic Lagrangian

Hamiltonian $$H = \sum \dot{x}\frac{\partial L}{\partial \dot{x}} - L = \frac{m_0}{\left(1-\frac{v^2}{c^2}\right)^{1/2}}\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right) - L$$

$$= m_0\gamma v^2 + \frac{m_0 c^2}{\gamma} + V = m_0\gamma c^2 + V$$

$$= E + V,$$ total energy

Since $E^2 = \vec{p}^{\,2}c^2 + m_0^2c^4$

$$H = c\left(\vec{p}^{\,2} + m_0^2c^2\right)^{1/2} + V$$

Hamilton's equations of motion $$\dot{x} = \frac{\partial H}{\partial p_x}, \quad \dot{p}_x = -\frac{\partial H}{\partial x}, \quad \text{etc}$$

Science & Technology Facilities Council

Photons and Wave 4-Vectors

Monochromatic plane wave: $\sin(\omega t - \vec{k} \cdot \vec{x})$

$$\vec{k} = \text{wave vector}, \ \left|\vec{k}\right| = \frac{2\pi}{\lambda}; \quad \omega = \text{angular frequency} = 2\pi\nu$$

Phase $\frac{1}{2\pi}\left(\omega t - \vec{k} \cdot \vec{x}\right)$ is the number of wave crests passing an observer, an invariant.

$$\omega t - \vec{k} \cdot \vec{x} = (ct, \vec{x}) \cdot \left(\frac{\omega}{c}, \vec{k}\right)$$

Position 4-vector, X

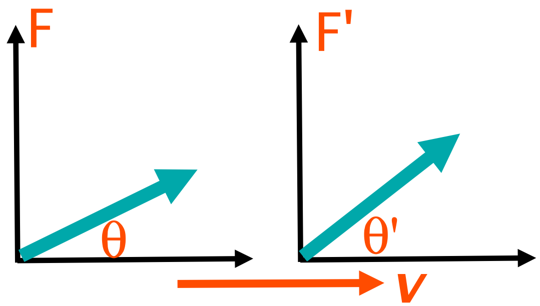
Wave 4-vector, K

Relativistic Doppler Shift

For light rays, phase velocity is $c = \dfrac{\omega}{|\vec{k}|}$

So $K = \dfrac{\omega}{c}(1, \vec{n})$ where $\vec{n} = (\cos\theta, \sin\theta, 0)$ is a unit vector

Lorentz transform $ct \leftrightarrow \dfrac{\omega}{c}, \ \vec{x} \leftrightarrow \dfrac{\omega}{c}\vec{n}$



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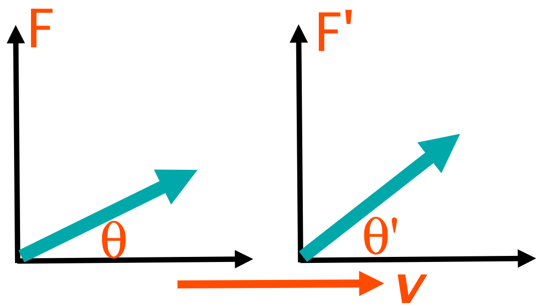

$$
\begin{aligned}
ct' &= \gamma\left(ct - \frac{vx}{c}\right) \\
x' &= \gamma(x - vt) \\
y' &= y
\end{aligned}
$$

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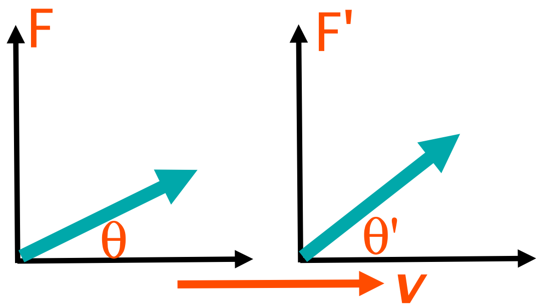

$$
\begin{aligned}
\omega' &= \gamma\left(\omega - \frac{v\omega\cos\theta}{c}\right) \\
\omega' cos\theta' &= \gamma\left(\omega\cos\theta - v\frac{\omega}{c}\right) \\
\omega' \sin\theta' &= \omega\sin\theta
\end{aligned}
$$

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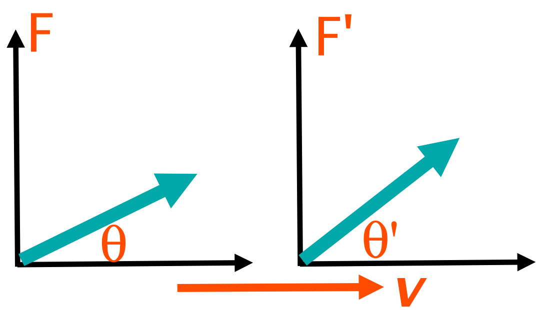

$$\begin{aligned} \omega' &= \gamma\omega\left(1 - \frac{v}{c}\cos\theta\right) \\ \tan\theta' &= \frac{\sin\theta}{\gamma\left(\cos\theta - \frac{v}{c}\right)} \end{aligned}$$

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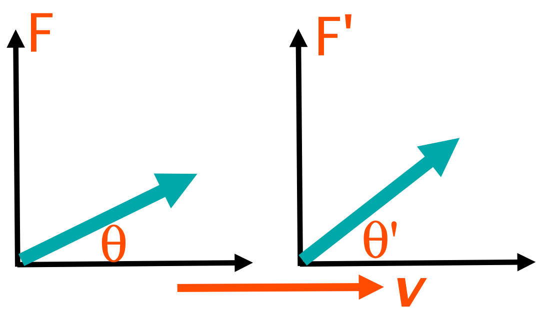

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Note: transverse Doppler effect even when $\theta = \frac{1}{2}\pi$

Motion faster than light

1. Two rods sliding over each other. Speed of intersection point is v/sinα, which can be made greater than c.

2. Explosion of planetary nebula. Observer sees bright spot spreading out. Light from P arrives $t=d\alpha^2/2c$ later.

$$t = \frac{d\alpha^2}{2c} \approx \frac{x}{c}\frac{\alpha}{2} << \frac{x}{c}$$