Linac Driven Free Electron Lasers (II)

Massimo.Ferrario@lnf.infn.it

CERN CAS THE CERN ACCELERATOR SCHOOL

SASE FEL Electron Beam Requirements: High Brightness B_n

$$\lambda_r^{MIN} \propto \sigma_\delta \sqrt{\frac{\left(1+K^2/2\right)}{\gamma B_n K^2}}$$

minimum radiation wavelength

energy spread

undulator parameter

$$B_n = \frac{2I}{\varepsilon_n^2}$$

$$L_g \propto \frac{\gamma^{3/2}}{K\sqrt{B_n\left(1+K^2/2\right)}}$$

gain length

R. Saldin et al. in *Conceptual Design of a 500 GeV e+e- Linear Collider with Integrated X-ray Laser Facility*, DESY-1997-048

Short Wavelength SASE FEL Electron Beam Requirement: High Brightness B_n > 10^{15} A/m^2

$$\uparrow \quad B_n \approx \frac{2I}{\varepsilon_n^2}$$

Bunch compressors

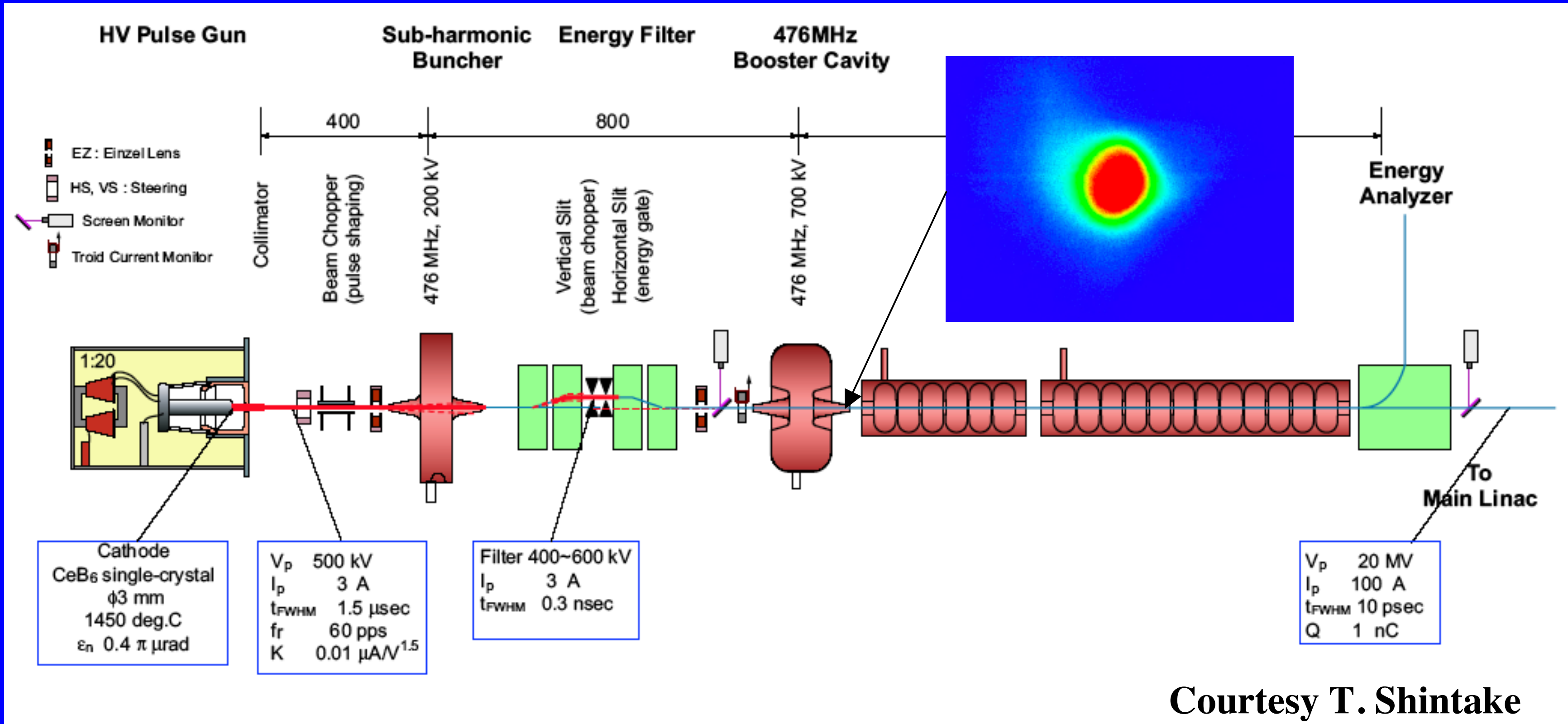
(RF & magnetic)

Laser Pulse shaping

Emittance compensation

Cathode emittance

500 kV pulsed thermionic gun for SCSS

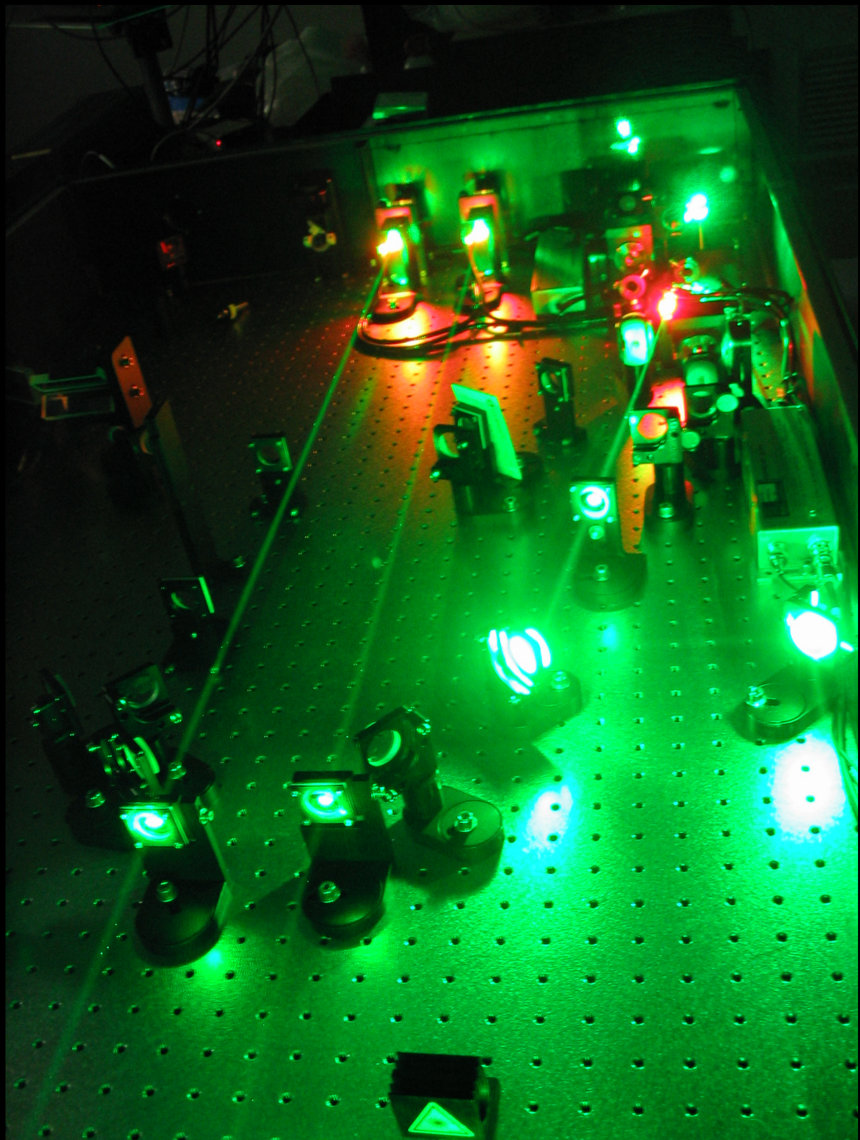


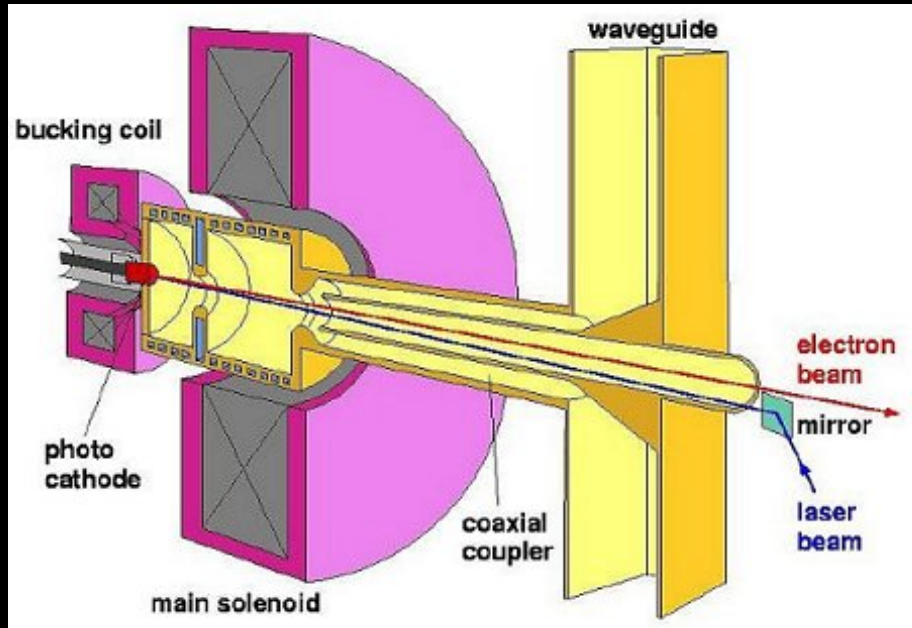
Stable operation with uniform beam quality

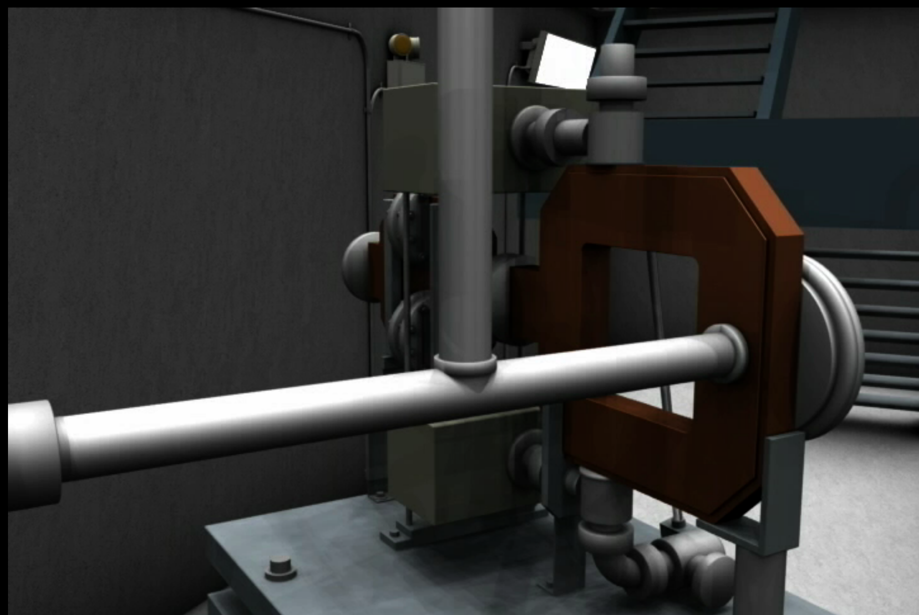
Low thermal emittance single crystal CeB_6 (Cerium Hexaborite)

Low accelerating gradient ==> Low charge density
(10 MV/m) ==> Free from dark current

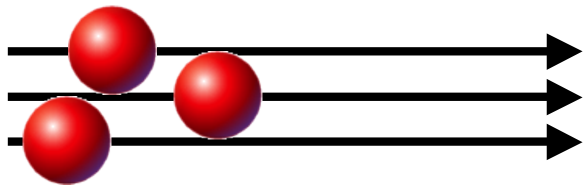
RF photoinjectors

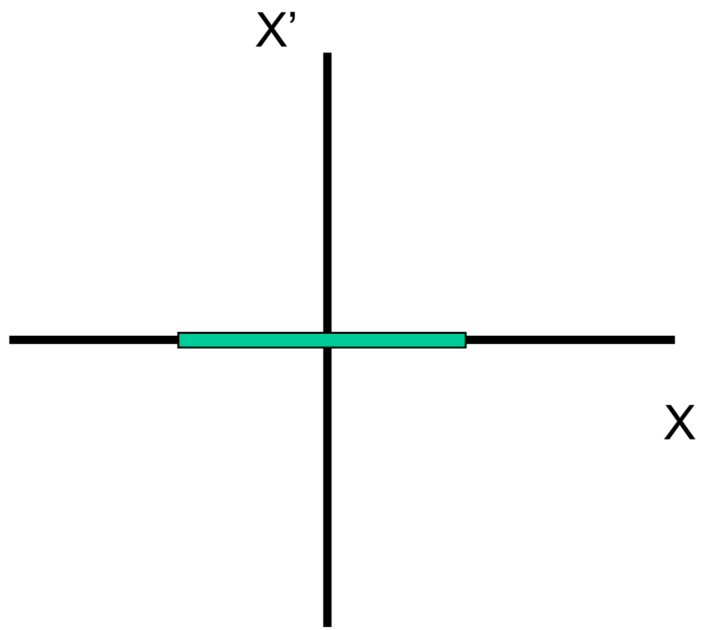




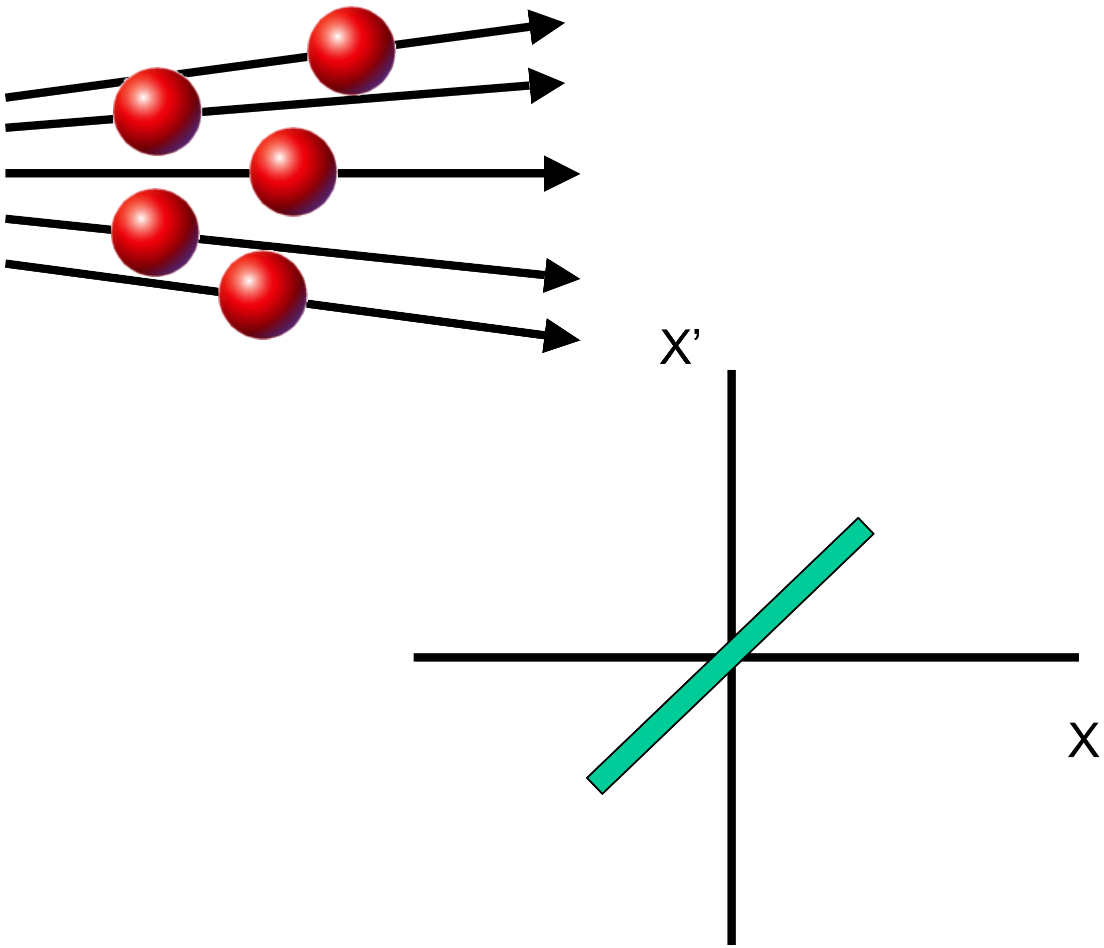


Phase space of a parallel laminar beam

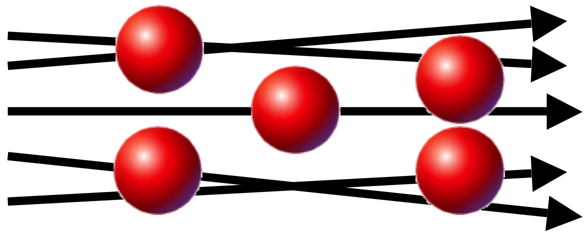


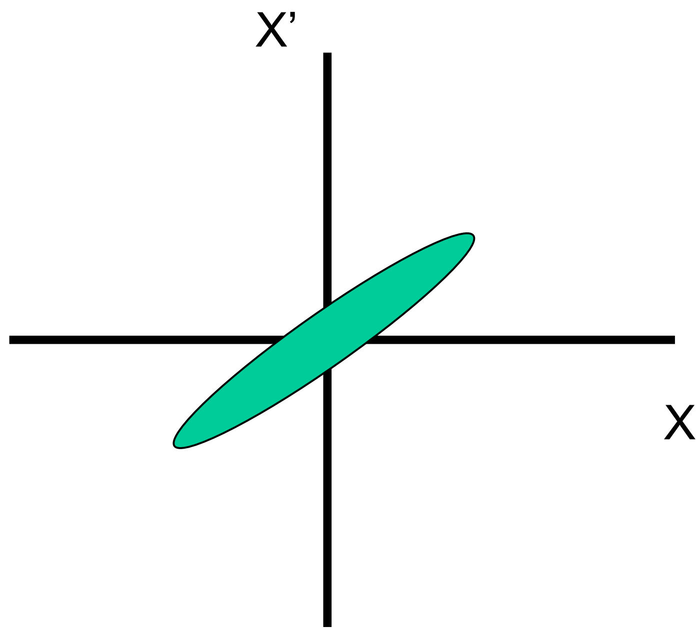


Phase space laminar beam



Phase space of non laminar beam



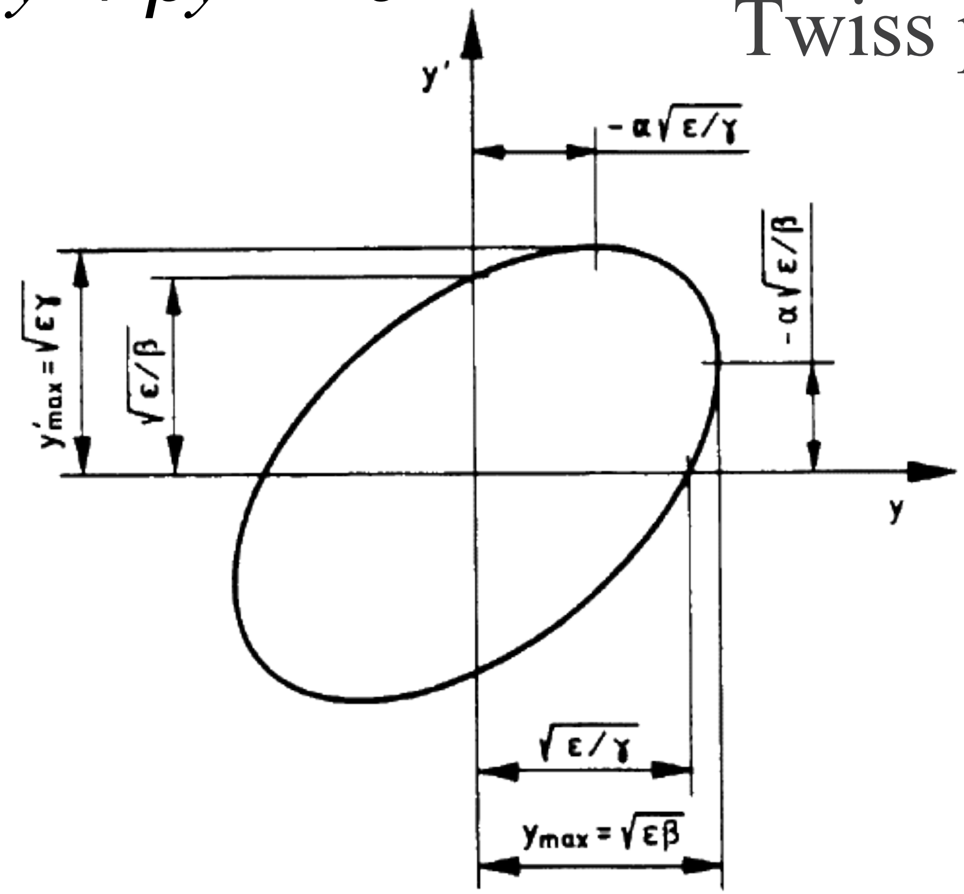


Ellipse equation

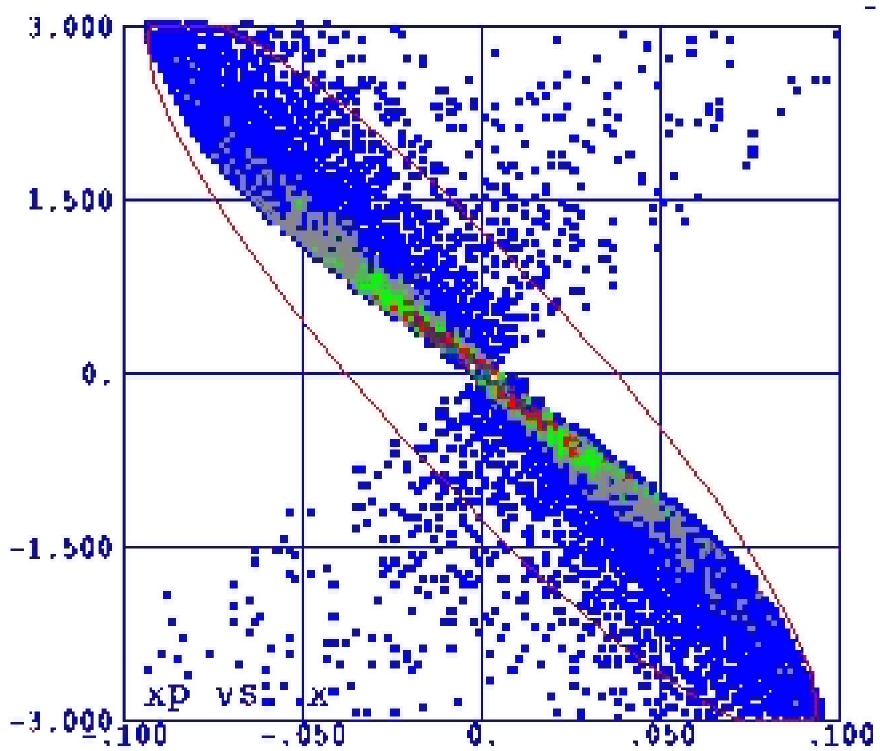
$$\gamma y^2 + 2\alpha y y' + \beta y'^2 = \varepsilon$$

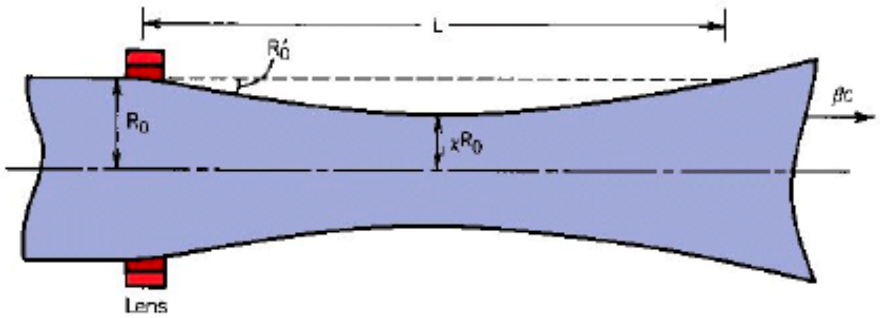
Twiss parameters

$$\beta\gamma - \alpha^2 = 1$$

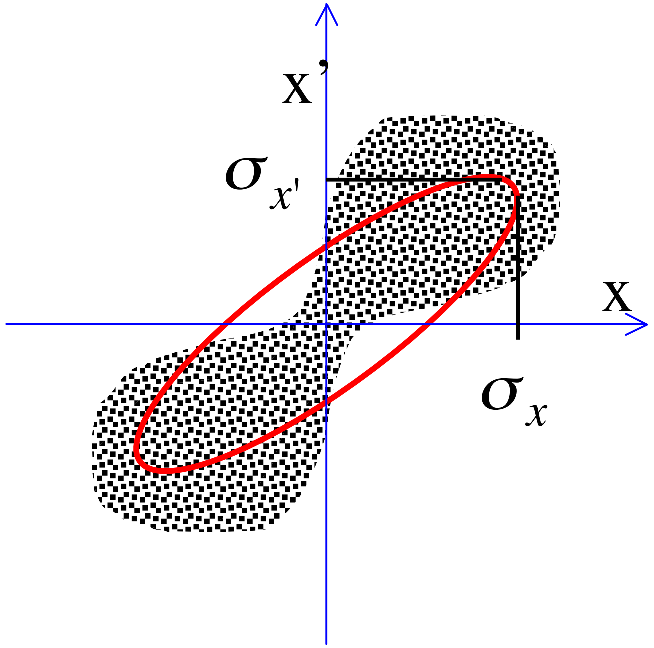


Phase space evolution at injector exit





rms Envelope Equations and rms Emittance



rms beam envelope:

$$\sigma_x^2 = \langle x^2 \rangle = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x^2 f(x,x')\,dx\,dx'$$

Define rms emittance:

$$\boxed{\gamma x^2 + 2\alpha x x' + \beta x'^2 = \varepsilon_{rms}}$$

such that:

$$\sigma_x = \sqrt{\langle x^2 \rangle} = \sqrt{\beta \varepsilon_{rms}}$$

$$\sigma_{x'} = \sqrt{\langle x'^2 \rangle} = \sqrt{\gamma \varepsilon_{rms}}$$

Since: $\alpha = -\dfrac{\beta'}{2}$

it follows: $\alpha = -\dfrac{1}{2\varepsilon_{rms}} \dfrac{d}{dz}\langle x^2 \rangle = -\dfrac{\langle x x' \rangle}{\varepsilon_{rms}} = -\dfrac{\sigma_{xx'}}{\varepsilon_{rms}}$

$$\boxed{\begin{aligned}\sigma_x &= \sqrt{\langle x^2 \rangle} = \sqrt{\beta \varepsilon_{rms}} \\ \sigma'_x &= \sqrt{\langle x'^2 \rangle} = \sqrt{\gamma \varepsilon_{rms}} \\ \sigma_{xx'} &= \langle x x' \rangle = \alpha \varepsilon_{rms}\end{aligned}}$$

It holds also the relation: $\gamma\beta - \alpha^2 = 1$

Substituting α, β, γ we get $\frac{\sigma_{x'}^2}{\varepsilon_{rms}} \frac{\sigma_x^2}{\varepsilon_{rms}} - \left(\frac{\sigma_{xx'}}{\varepsilon_{rms}} \right)^2 = 1$

We end up with the definition of rms emittance in terms of the second moments of the distribution:

$$\boxed{\varepsilon_{rms} = \sqrt{\sigma_x^2 \sigma_{x'}^2 - \sigma_{xx'}^{\ 2}} = \sqrt{\left(\langle x^2 \rangle \langle x'^2 \rangle - \langle x x' \rangle^2 \right)}}$$

$$\varepsilon_n = \langle \beta\gamma \rangle \varepsilon_{rms}$$

Envelope Equation without Acceleration

Now take the derivatives:

$$\frac{d\sigma_x}{dz} = \frac{d}{dz}\sqrt{\left\langle x^2 \right\rangle} = \frac{1}{2\sigma_x}\frac{d}{dz}\left\langle x^2 \right\rangle = \frac{1}{2\sigma_x}2\left\langle xx' \right\rangle = \frac{\sigma_{xx'}}{\sigma_x}$$

$$\frac{d^2\sigma_x}{dz^2} = \frac{d}{dz}\frac{\sigma_{xx'}}{\sigma_x} = \frac{1}{\sigma_x}\frac{d\sigma_{xx'}}{dz} - \frac{\sigma_{xx'}^2}{\sigma_x^3} = \frac{1}{\sigma_x}\left(\left\langle x'^2 \right\rangle - \left\langle xx'' \right\rangle\right) - \frac{\sigma_{xx'}^2}{\sigma_x^3} = \frac{\sigma_{x'}^2 + \left\langle xx'' \right\rangle}{\sigma_x} - \frac{\sigma_{xx'}^2}{\sigma_x^3}$$

And simplify:

$$\boxed{\sigma_x'' = \frac{\sigma_x^2\sigma_{x'}^2 - \sigma_{xx'}^2}{\sigma_x^3} - \frac{\left\langle xx'' \right\rangle}{\sigma_x} = \frac{\varepsilon_{rms}^2}{\sigma_x^3} - \frac{\left\langle xx'' \right\rangle}{\sigma_x}}$$

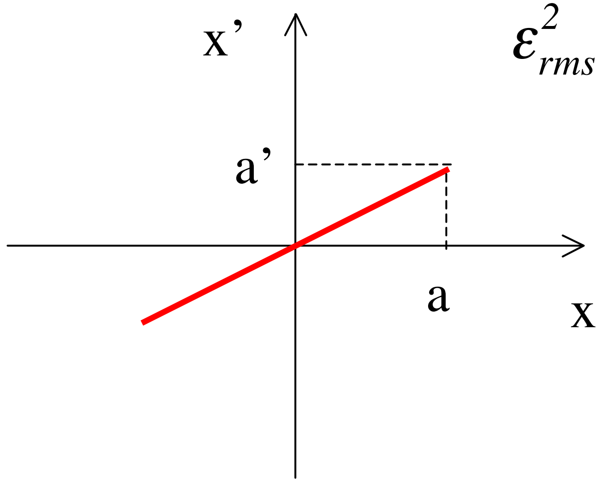
Assuming that each particle is subject only to a linear focusing force, without acceleration: $x'' + k_x^2(z)x = 0$

take the average over the entire particle ensemble $\left\langle xx'' \right\rangle = -k^2\left\langle x^2 \right\rangle$

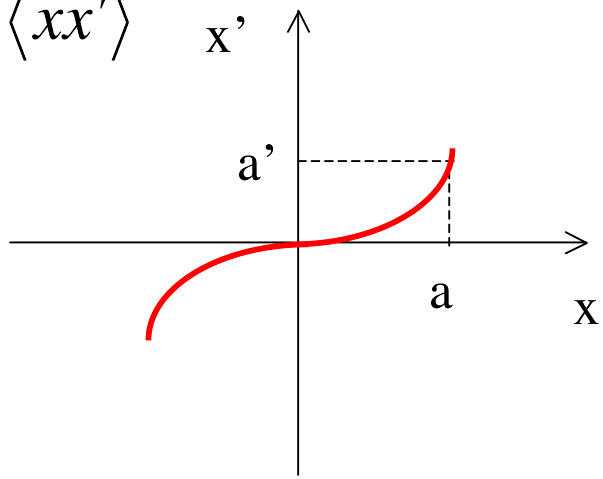
$$\boxed{\sigma_x'' + k_x^2\sigma_x = \frac{\varepsilon_{rms}^2}{\sigma_x^3}}$$

We obtain the rms envelope equation in which the rms emittance enters as defocusing pressure like term

What does rms emittance tell us about phase space distributions under linear or non-linear forces acting on the beam?



$$\varepsilon_{rms}^2 = \langle x^2 \rangle \langle x'^2 \rangle - \langle x x' \rangle^2$$



Assuming a generic x, x' correlation of the type: $x' = C x^n$

$$\varepsilon_{rms}^2 = C^2 \left(\langle x^2 \rangle \langle x^{2n} \rangle - \langle x^{n+1} \rangle^2 \right)$$

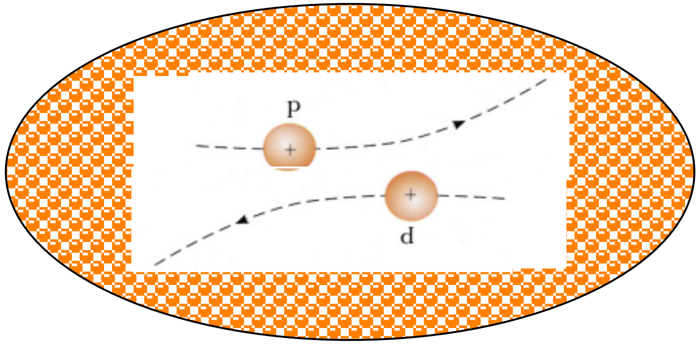
When $n = 1$ ==> $\varepsilon_{rms} = 0$

When $n \neq 1$ ==> $\varepsilon_{rms} \neq 0$

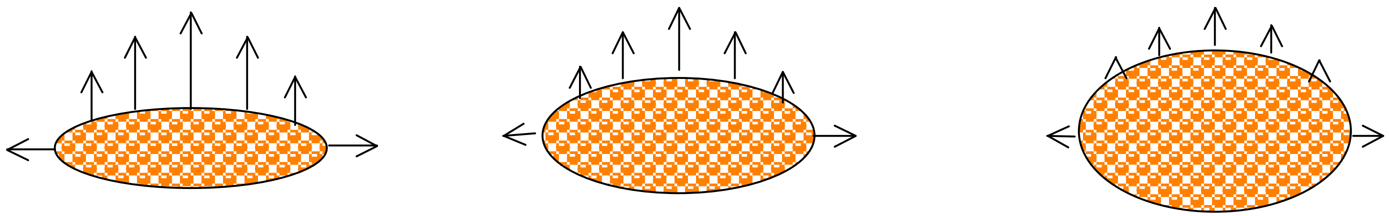
Space Charge: What does it mean?

The net effect of the **Coulomb** interactions in a multi-particle system can be classified into two regimes:

1) **Collisional Regime** ==> dominated by **binary collisions** caused by close particle encounters ==> **Single Particle Effects**

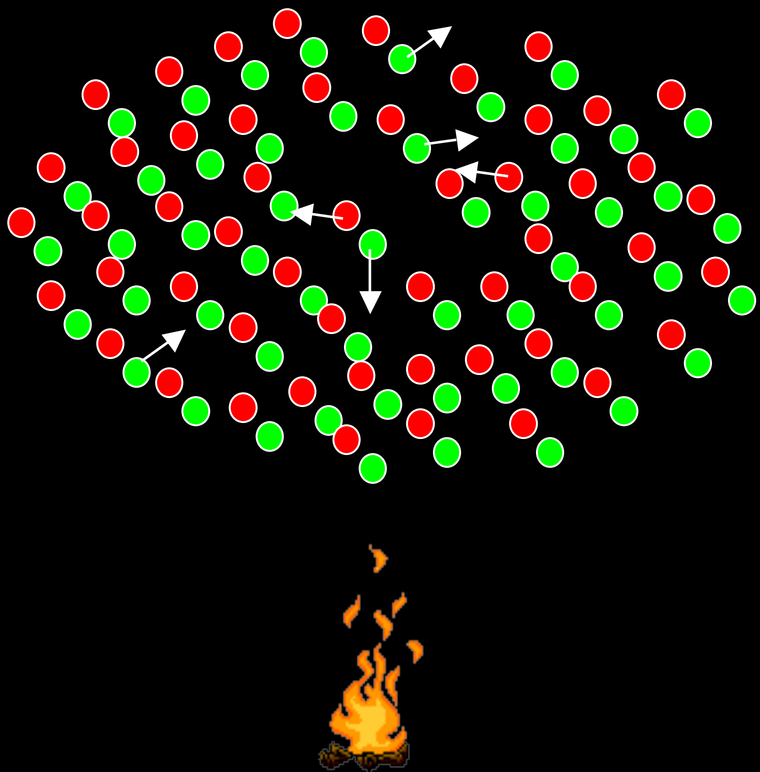


2) **Space Charge Regime** ==> dominated by the **self field** produced by the particle distribution, which varies appreciably only over large distances compare to the average separation of the particles ==> **Collective Effects, Single Component Cold Plasma**



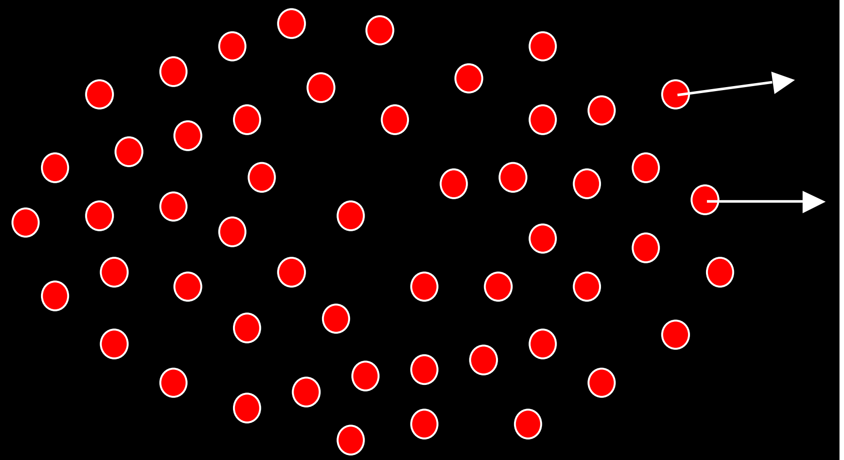
Neutral Plasma

- Oscillations
- Instabilities
- EM Wave propagation



Single Component Cold Relativistic Plasma

Magnetic focusing

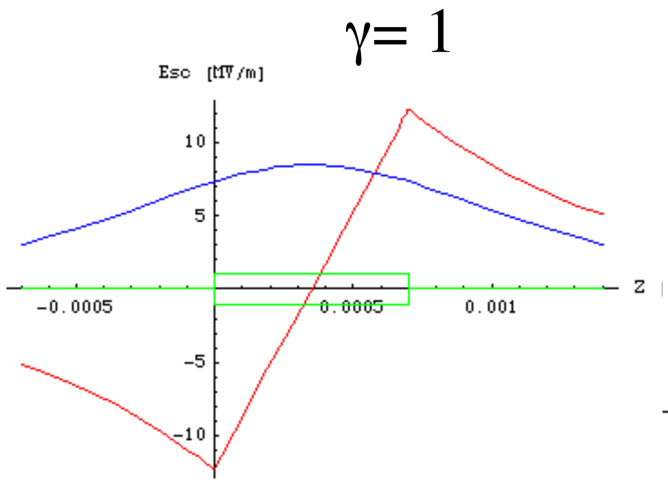


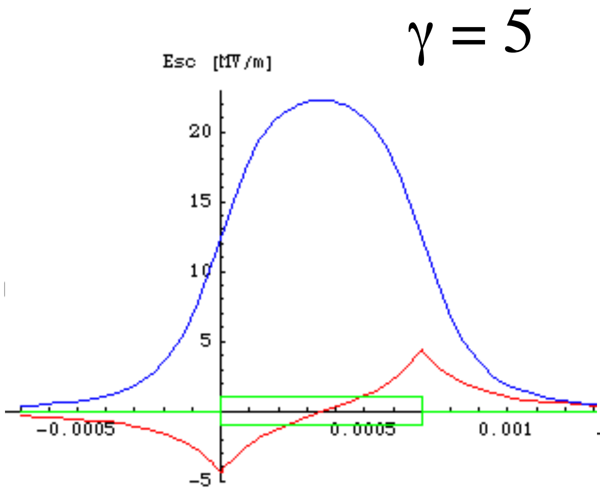
Magnetic focusing

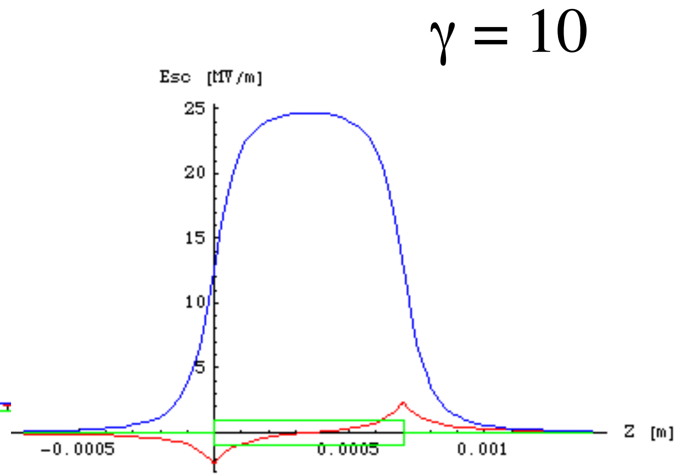
Longitudinal and Transverse Space charge Fields In a uniform charged cilindrical bunch

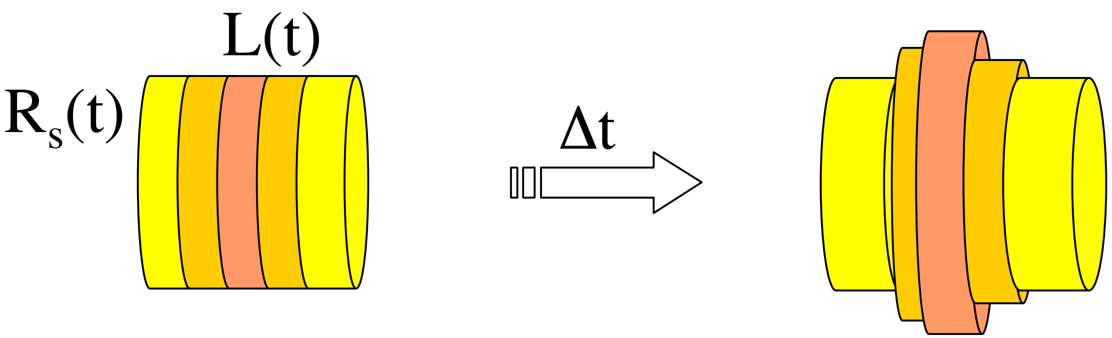
$$\boxed{E_z(0,s,\gamma) = \frac{I}{2\pi\gamma\varepsilon_0 R^2 \beta c} h(s,\gamma)}$$

$$\boxed{E_r(r,s,\gamma) = \frac{Ir}{2\pi\varepsilon_0 R^2 \beta c} g(s,\gamma)}$$









$$\boxed{B_\vartheta = \frac{\beta}{c} E_r}$$

Lorentz Force

$$\boxed{E_r(r,s,\gamma) = \frac{Ir}{2\pi\varepsilon_0 R^2 \beta c} g(s,\gamma)}$$

$$\boxed{F_r = e\left(E_r - \beta c B_\vartheta\right) = e\left(1-\beta^2\right)E_r = \frac{eE_r}{\gamma^2}}$$

is a **linear** function of the transverse coordinate

$$\boxed{\frac{dp_r}{dt} = F_r = \frac{eE_r}{\gamma^2} = \frac{eIr}{2\pi\gamma^2\varepsilon_0 R^2 \beta c} g(s,\gamma)}$$

The attractive magnetic force , which becomes significant at high velocities, tends to compensate for the repulsive electric force. Therefore space charge defocusing is primarily a non-relativistic effect.

$$\boxed{F_x = \frac{eIx}{2\pi\gamma^2\varepsilon_0 \sigma_x^2 \beta c} g(s,\gamma)}$$

Envelope Equation with Space Charge

Single particle transverse motion:

$$\frac{dp_x}{dt} = F_x \qquad p_x = p_o x' = \beta\gamma m_o c x'$$

$$\frac{d}{dt}\left(p_o x'\right) = \beta c \frac{d}{dz}\left(p_o x'\right) = F_x$$

$$x'' = \frac{F_x}{\beta c p_o}$$

$$\boxed{x'' = \frac{k_{sc}\left(s,\gamma\right)}{\sigma_x^2} x}$$

Space Charge de-focusing force

Generalized perveance

$$k_{sc}\left(s,\gamma\right) = \frac{2I}{I_A\left(\beta\gamma\right)^3} g\left(s,\gamma\right)$$

$$I_A = \frac{4\pi\varepsilon_o m_o c^3}{e} = 17kA$$

Now we can calculate the term $\langle xx'' \rangle$ that enters in the envelope equation

$$\langle xx'' \rangle = \frac{k_{sc}}{\sigma_x^2} \langle x^2 \rangle = k_{sc} \qquad \boxed{\sigma_x'' = \frac{\varepsilon_{rms}^2}{\sigma_x^3} - \frac{\langle xx'' \rangle}{\sigma_x}}$$

Including all the other terms the envelope equation reads:

Space Charge De-focusing Force

$$\boxed{\sigma_x'' + k^2 \sigma_x = \frac{\varepsilon_n^2}{(\beta\gamma)^2 \sigma_x^3} + \frac{k_{sc}}{\sigma_x}}$$

Emittance Pressure

External Focusing Forces

Laminarity Parameter: $\boxed{\rho = \frac{(\beta\gamma)^2 k_{sc} \sigma_x^2}{\varepsilon_n^2}}$

The beam undergoes two regimes along the accelerator

$$\sigma_x'' + k^2\sigma_x = \cancel{\frac{\varepsilon_n^2}{(\beta\gamma)^2\sigma_x^3}} + \frac{k_{sc}}{\sigma_x}$$

ρ>>1 Laminar Beam

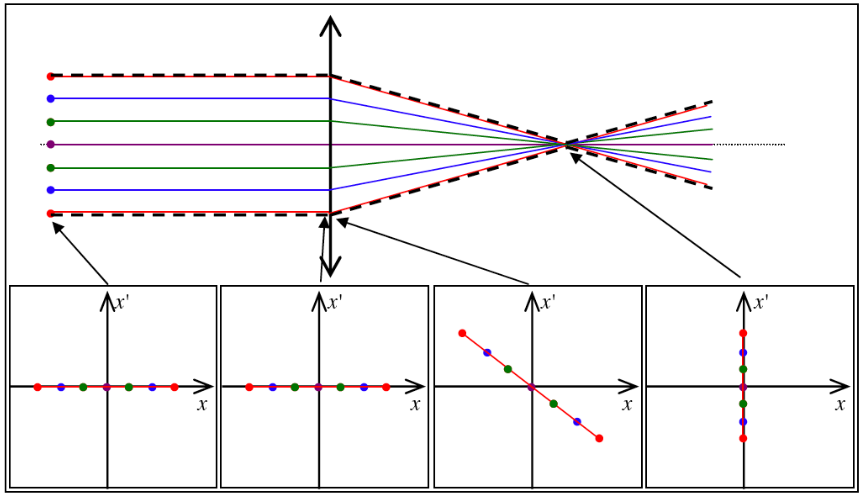


Fig. 10: Particle trajectories in laminar beam

$$\sigma_x'' + k^2\sigma_x = \frac{\varepsilon_n^2}{(\beta\gamma)^2\sigma_x^3} + \cancel{\frac{k_{sc}}{\sigma_x}}$$

ρ<<1 Thermal Beam

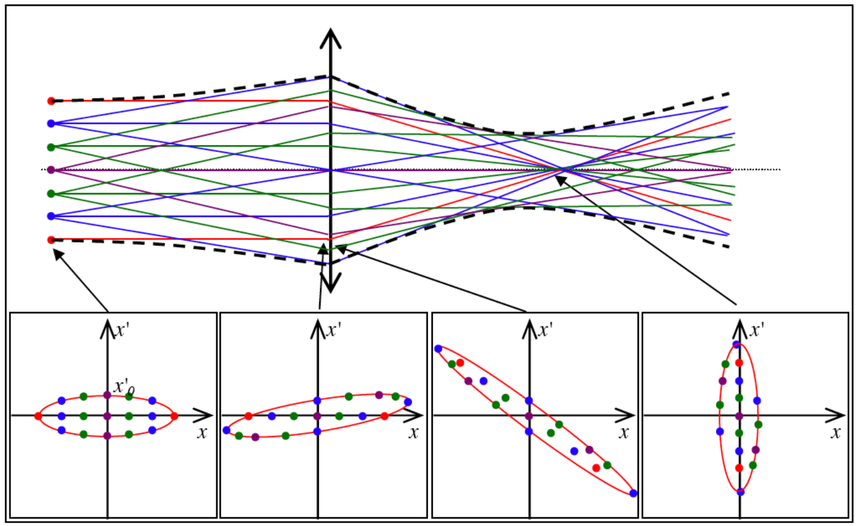


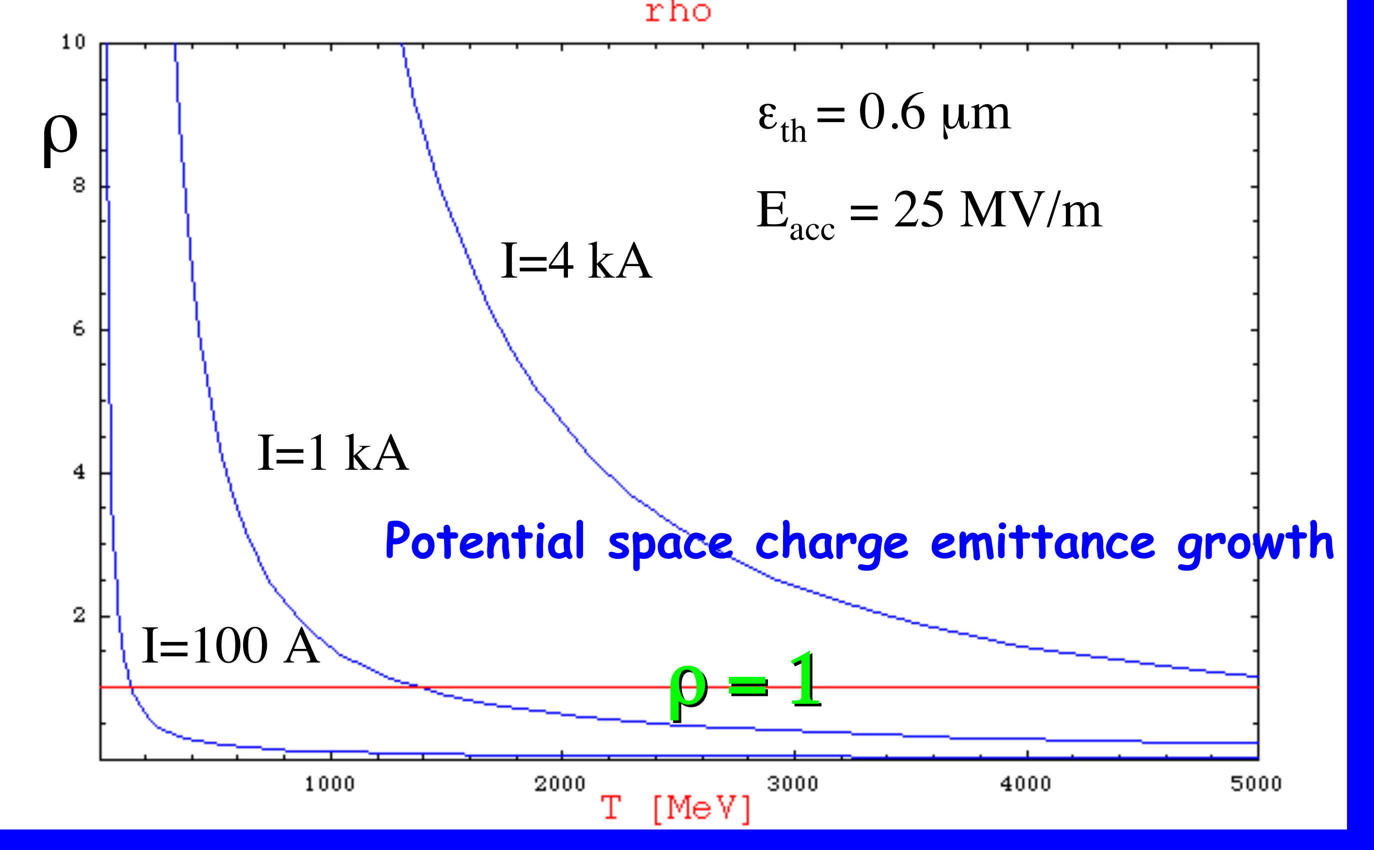
Fig. 11: Particle trajectories in non-zero emittance beam

Laminarity parameter

$$\rho = \frac{2I\sigma^2}{\gamma I_A \varepsilon_n^2} \equiv \frac{2I\sigma_q^2}{\gamma I_A \varepsilon_n^2} = \frac{4I^2}{\gamma'^2 I_A^2 \varepsilon_n^2 \gamma^2}$$

Transition Energy (ρ=1)

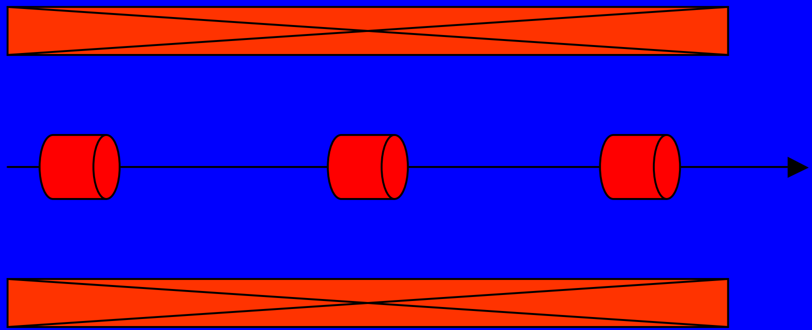
$$\gamma_{tr} = \frac{2I}{\gamma' I_A \varepsilon_n}$$



Space Charge induced emittance oscillations in a laminar beam

Simple Case: Transport in a Long Solenoid

$$k_s = \frac{qB}{2mc\beta\gamma}$$



$$\sigma'' + k_s^2\sigma = \frac{k_{sc}(s,\gamma)}{\sigma}$$

$\sigma = \sigma_{eq}$ ==> Equilibrium solution ? ==> $$\boxed{\sigma_{eq}(s,\gamma) = \frac{\sqrt{k_{sc}(s,\gamma)}}{k_s}}$$

Small perturbations around the equilibrium solution

$$\boxed{\sigma'' + k_s^2 \sigma = \frac{k_{sc}(s,\gamma)}{\sigma}}$$

$$\boxed{\sigma(\zeta) = \sigma_{eq}(s) + \delta\sigma(s)}$$

$$\delta\sigma'' + k_s^2 \left(\sigma_{eq} + \delta\sigma\right) = \frac{k_{sc}(s,\gamma)}{\sigma_{eq}}\left(1 - \frac{\delta\sigma}{\sigma_{eq}}\right)$$

$$\boxed{\sigma_{eq}(\zeta) = \frac{\sqrt{k_{sc}(s,\gamma)}}{k_s}}$$

$$\boxed{\delta\sigma''(s) + 2k_s^2 \delta\sigma(s) = 0}$$

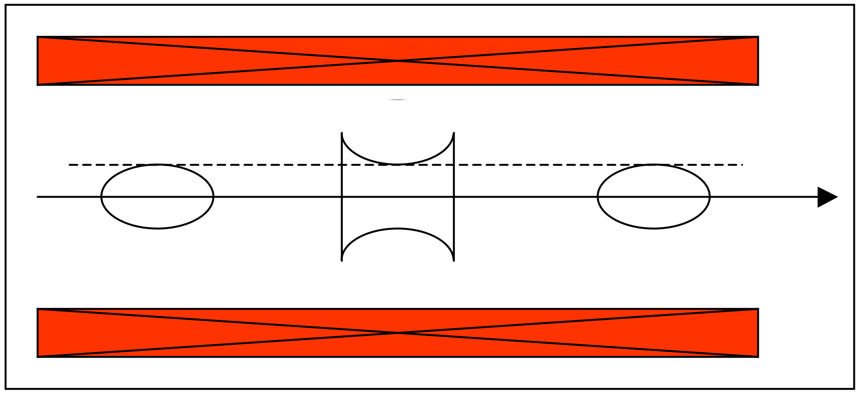
$$\boxed{\delta\sigma'' + 2k_s^2\delta\sigma = 0}$$

$$\sigma = \sigma_{eq} + \delta\sigma$$

Perturbed trajectories oscillate around the equilibrium with the same frequency but with different amplitudes

$$\sigma(s) = \sigma_{eq}(s) + \left(\sigma(s) - \sigma_{eq}(s)\right)cos\left(\sqrt{2}k_s z\right)$$

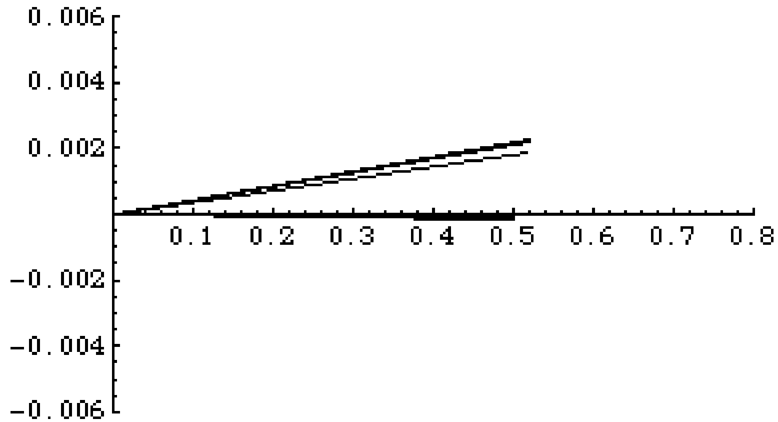
$$\sigma'(s) = -\sqrt{2}k\left(\sigma(s) - \sigma_{eq}(s)\right)sin\left(\sqrt{2}k_s z\right)$$

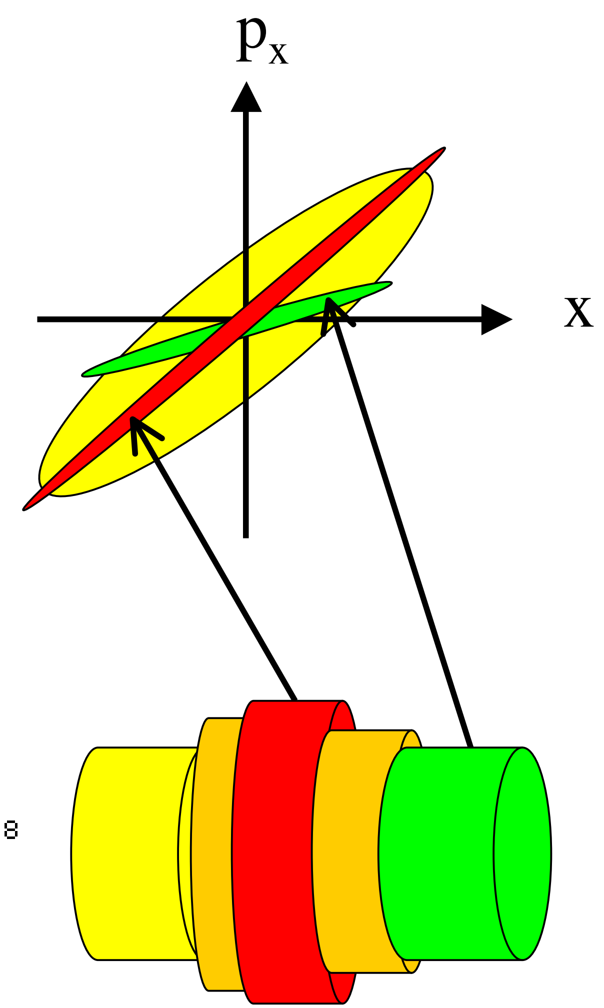


Plasma frequency

Emittance Oscillations are driven by space charge differential defocusing in core and tails of the beam

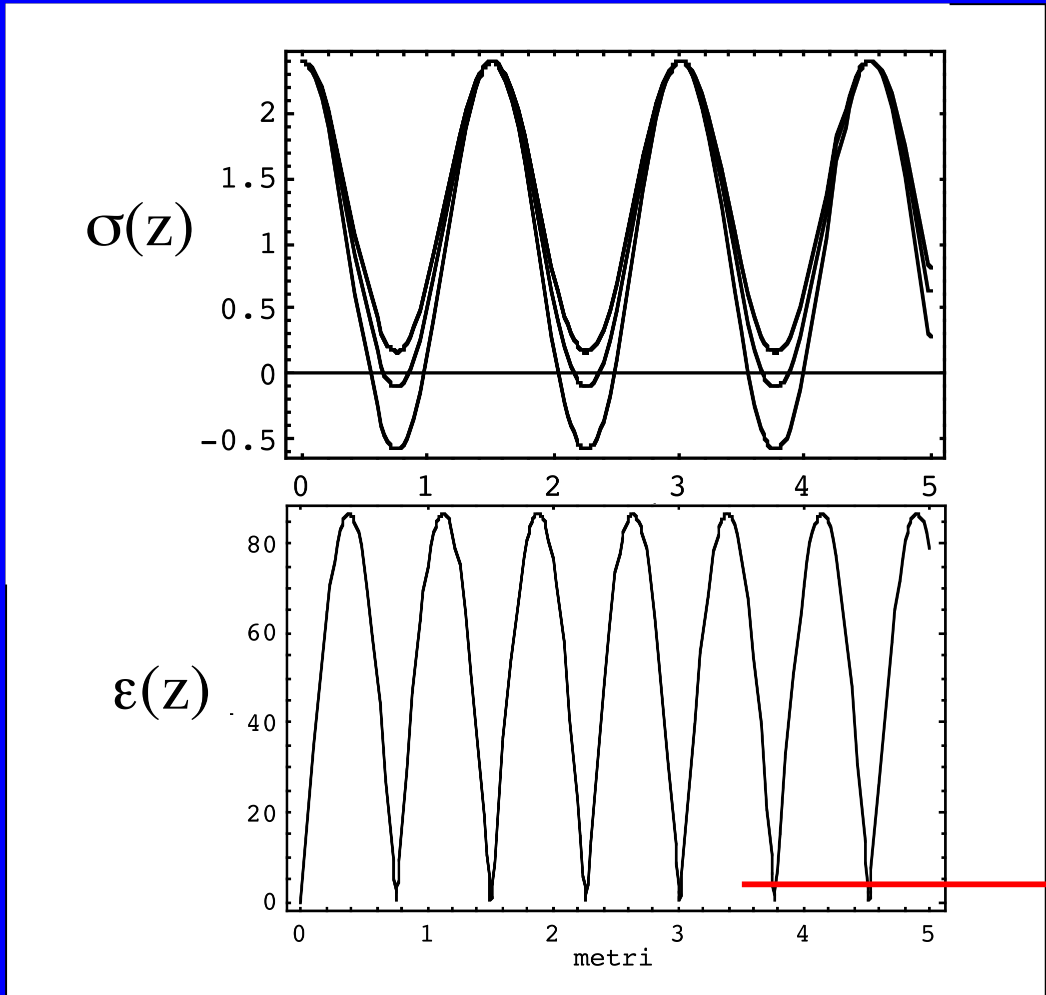
Projected Phase Space





Slice Phase Spaces

Envelope oscillations drive Emittance oscillations

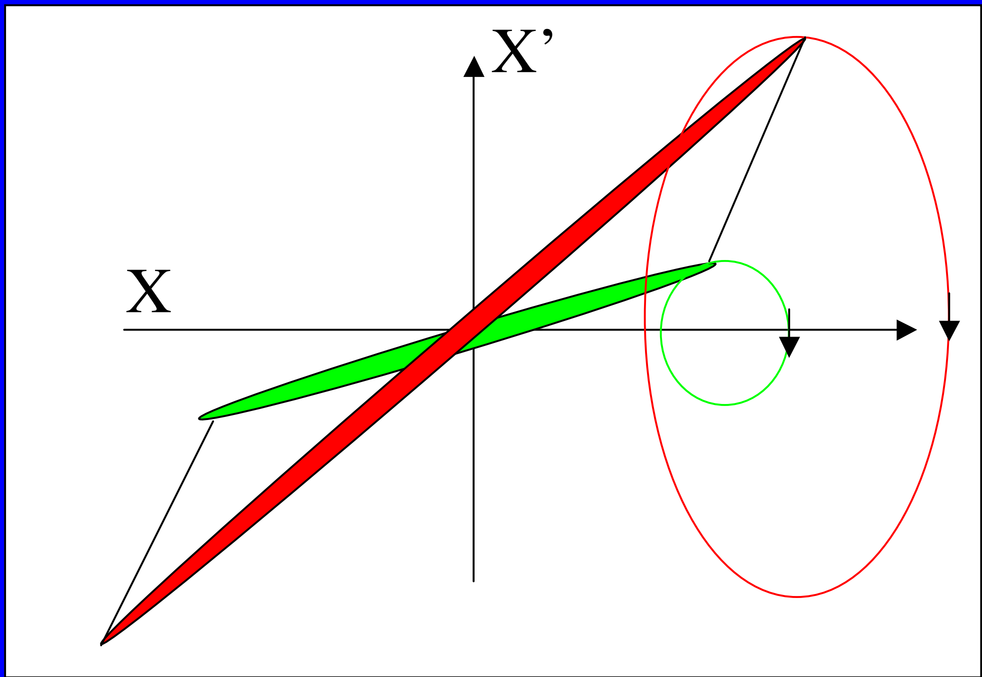


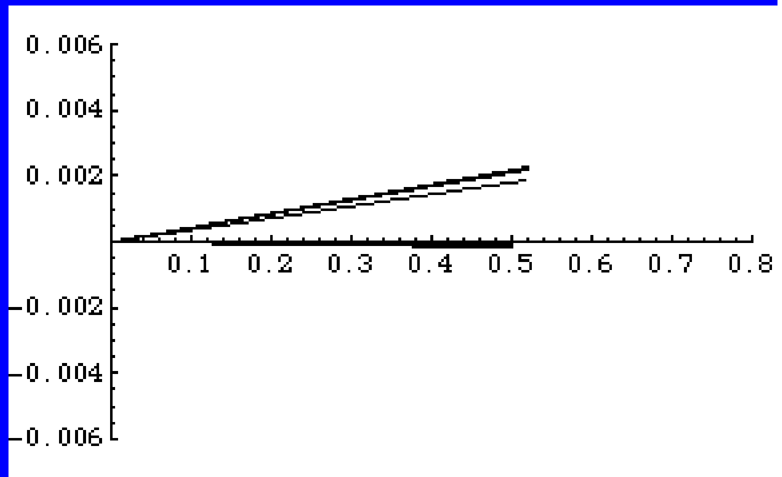
$$\frac{\delta\gamma}{\gamma} = 0$$

$$\sigma' = 0$$

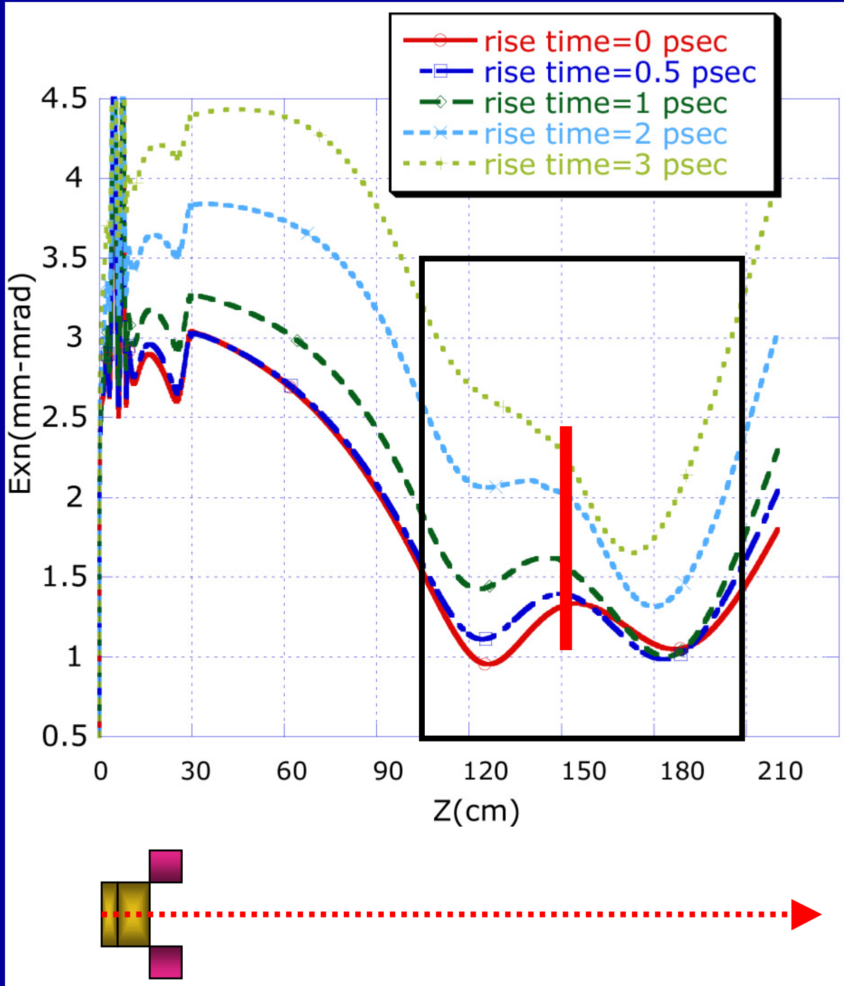
$$\varepsilon_{rms} = \sqrt{\sigma_x^2 \sigma_{x'}^2 - \sigma_{xx'}{}^2} = \sqrt{\left(\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2\right)} \approx \left|\sin\left(\sqrt{2} k_s z\right)\right|$$

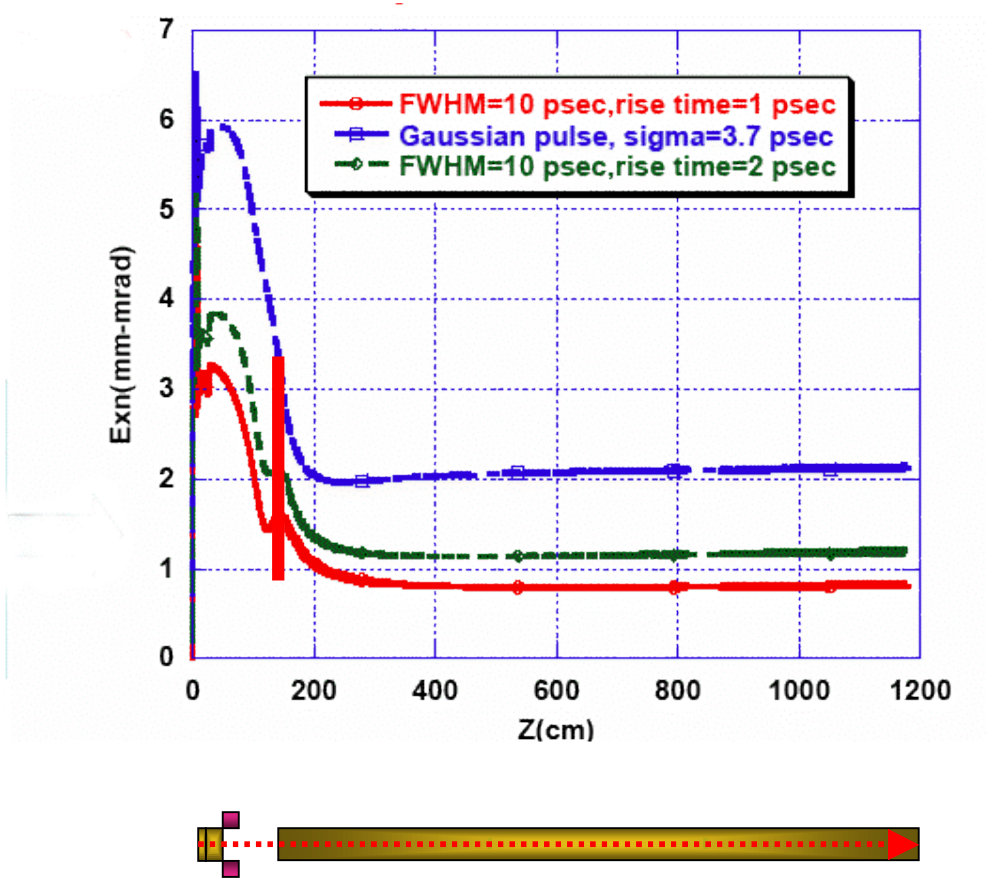
Perturbed trajectories oscillate around the equilibrium with the same frequency but with different amplitudes





Emittance evolution for different pulse shapes





Optimum injection in to the linac with:

$$\sigma' = 0$$

$$\gamma' = \frac{eE_{acc}}{mc^2} = \frac{2}{\sigma}\sqrt{\frac{I}{2\gamma I_A}}$$

Short Wavelength SASE FEL Electron Beam Requirement: High Brightness $B_n > 10^{15}$ A/m^2

$$\uparrow B_n \approx \frac{2I \uparrow}{\varepsilon_n^2 \downarrow}$$

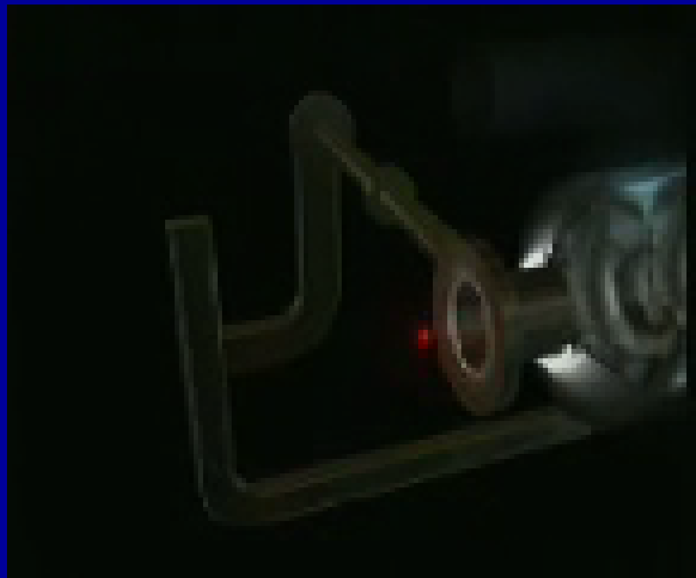
Bunch compressors
(RF & magnetic)

Laser Pulse shaping
Emittance compensation
Cathode emittance

The paradox of relativistic bunch compression

Low energy electron bunch injected in a linac:

$$\gamma \approx 1$$
$$L_b = 3mm = L'_b$$
$$I = 100\,A$$



Length contraction?

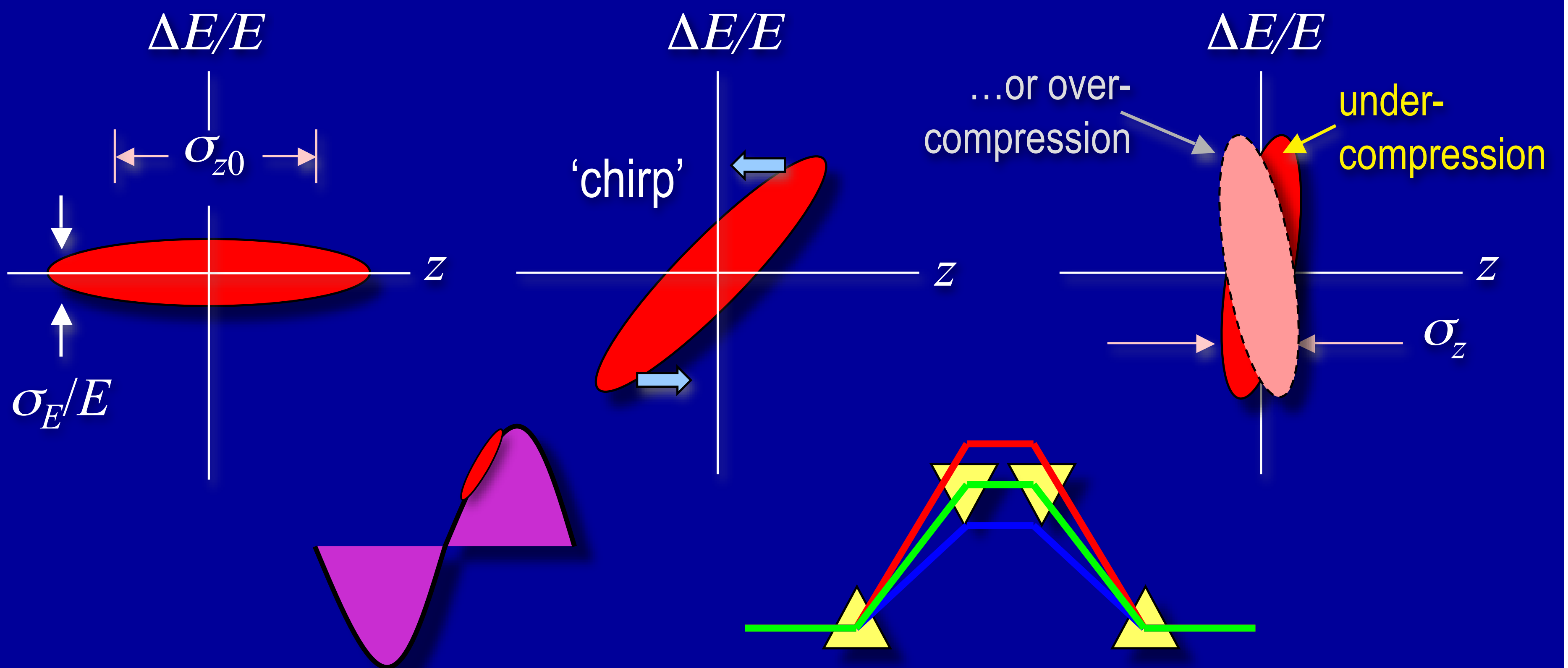
$$\cancel{\gamma = 1000}$$
$$\cancel{L_b = \frac{L'_b}{\gamma} = 3\,\mu m}$$
$$\cancel{I = 100\,kA}$$

Why do we need a bunch compressor?

$$L''_b = \gamma L_b = 30m$$
$$L_b = \frac{L''_b}{\gamma} = 3mm$$
$$I = 100\,A$$

Magnetic compressor (Chicane)

Magnetic compressor (Chicane)



$$V = V_0 \sin(\omega\tau)$$

$$\Delta z = R_{56} \Delta E/E$$

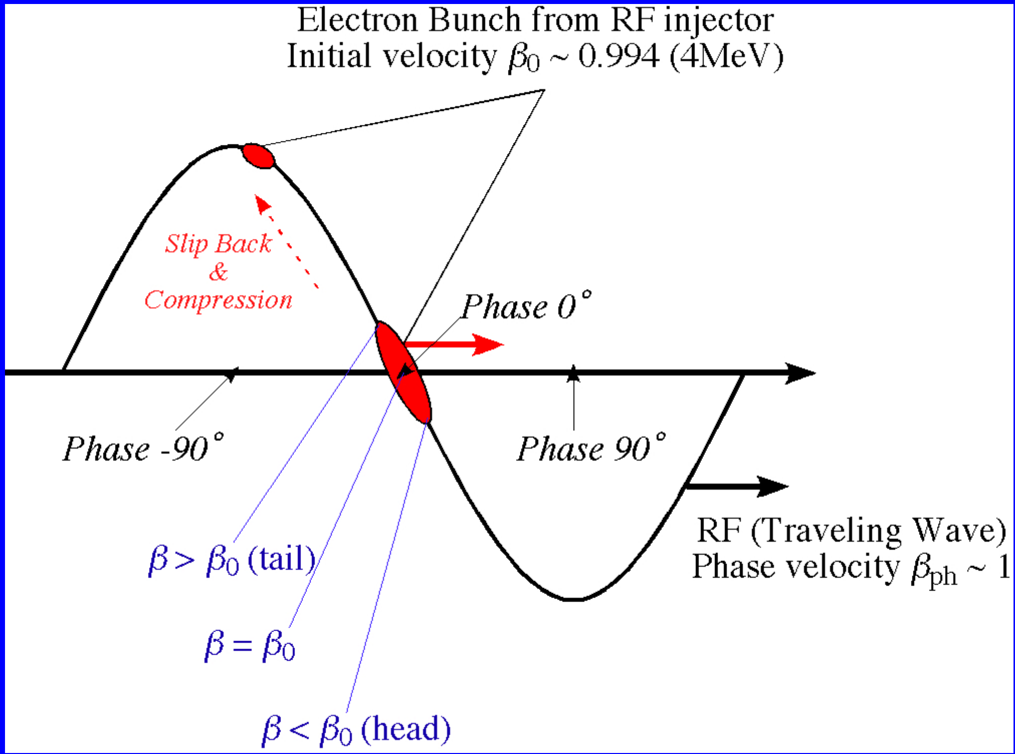
RF Accelerating Voltage

Path Length-Energy Dependent Beamline

Courtesy P. Emma

Velocity bunching concept (RF Compressor)

If the beam injected in a long accelerating structure at the crossing field phase and it is slightly slower than the phase velocity of the RF wave , it will slip back to phases where the field is accelerating, but at the same time it will be chirped and compressed.



The key point is that compression and acceleration take place at the same time within the same linac section, actually the first section following the gun, that typically accelerates the beam, under these conditions, from a few MeV (> 4) up to 25-35 MeV.

Longitudinal beam dynamics

$$\frac{d}{dz}\left(\gamma m_o c^2\right) = eE\,sin\left(kz - \omega t + \varphi_o\right)$$

$$\frac{d\gamma}{dz} = \frac{eE}{m_o c^2} sin\left(kz - \omega t + \varphi_o\right)$$

$$\frac{d\gamma}{dz} = \alpha k\,sin\left(\varphi(z,t)\right) \qquad \alpha = \frac{eE}{km_o c^2}$$

$$\frac{d\varphi}{dz} = \frac{d}{dz}\left(kz - \omega t + \varphi_o\right) = \left(k - \omega\frac{dt}{dz}\right) = \left(k - \frac{\omega}{\beta c}\right)$$

$$= k\left(1 - \frac{1}{\beta}\right) = k\left(1 - \frac{\gamma}{\sqrt{\gamma^2 - 1}}\right)$$

$$\begin{cases} \dfrac{d\gamma}{dz} = \alpha k\,sin(\varphi) \\ \dfrac{d\varphi}{dz} = k\left(1 - \dfrac{\gamma}{\sqrt{\gamma^2 - 1}}\right) \end{cases}$$

$$\begin{cases} \dfrac{d\gamma}{dz} \xrightarrow{\gamma \to \infty} \alpha k\,sin(\varphi_\infty) \\ \dfrac{d\varphi}{dz} \xrightarrow{\gamma \to \infty} 0 \end{cases}$$

Such a system is solved using the variable separation technique to yield a constant of the motion (total energy):

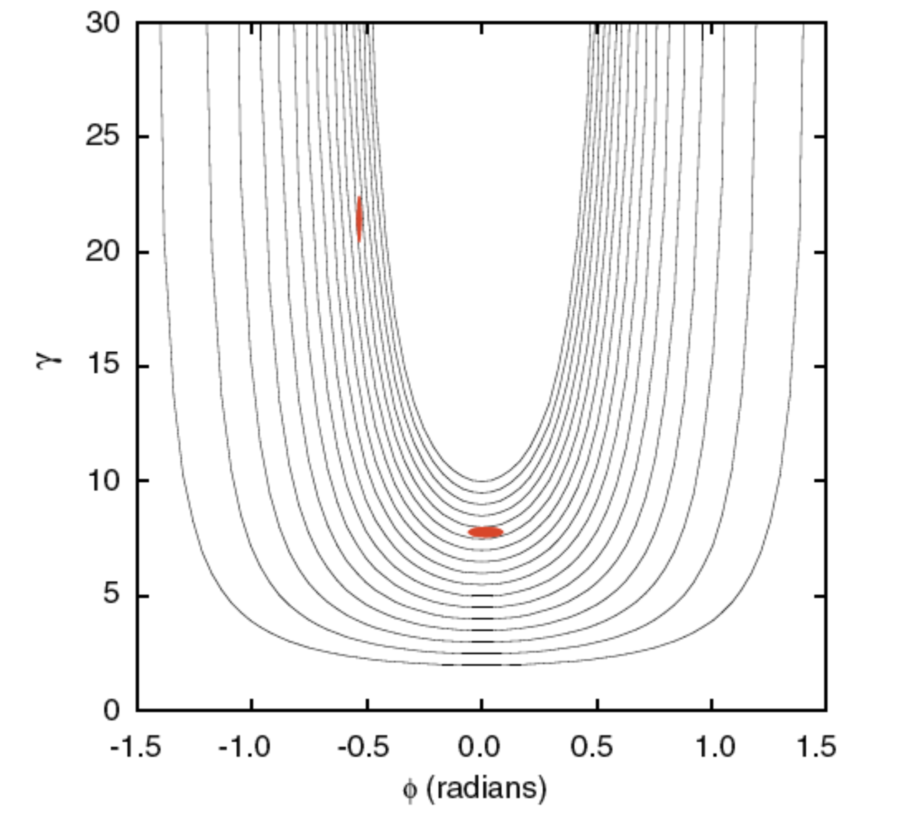
$$H = \gamma - \sqrt{\gamma^2 - 1} - \alpha \cos(\phi)$$

$$\frac{1}{C} = \frac{\Delta\varphi_\infty}{\Delta\varphi_o} = \frac{\sin\varphi_o}{\sin\varphi_\infty} + \frac{1}{2\alpha\gamma_o^2 \sin\varphi_\infty}\frac{\Delta\gamma_o}{\Delta\varphi_o}$$

$$H = \gamma - \sqrt{\gamma^2 - 1} - \alpha\cos(\phi)$$

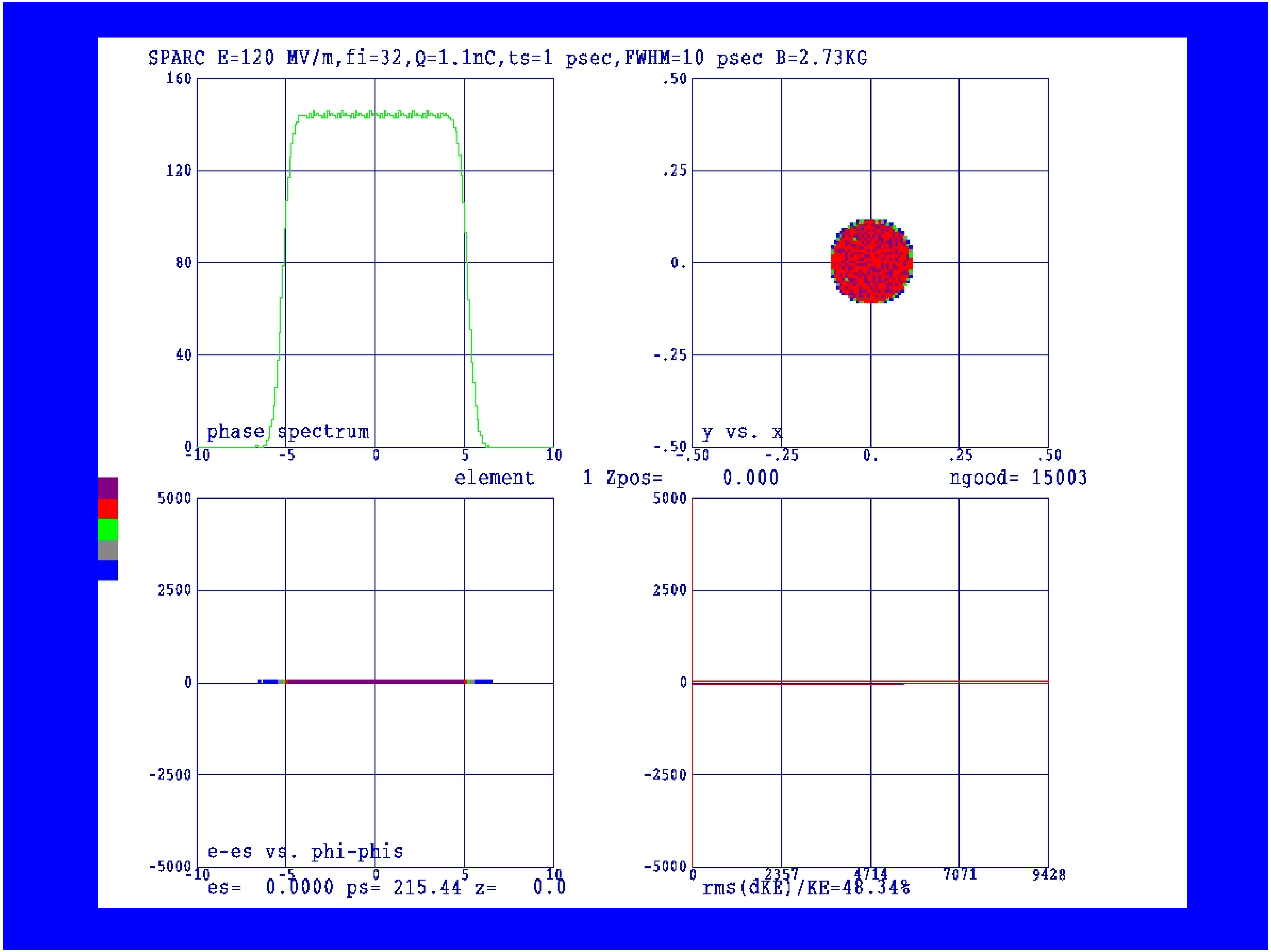
$$-\alpha\cos\phi_\infty = \gamma_0 - \sqrt{\gamma_0^2 - 1} - \alpha\cos\phi_0$$

$$\phi_\infty \cong \cos^{-1}\left[\cos\phi_0 - \frac{1}{2\alpha\gamma_0}\right]$$

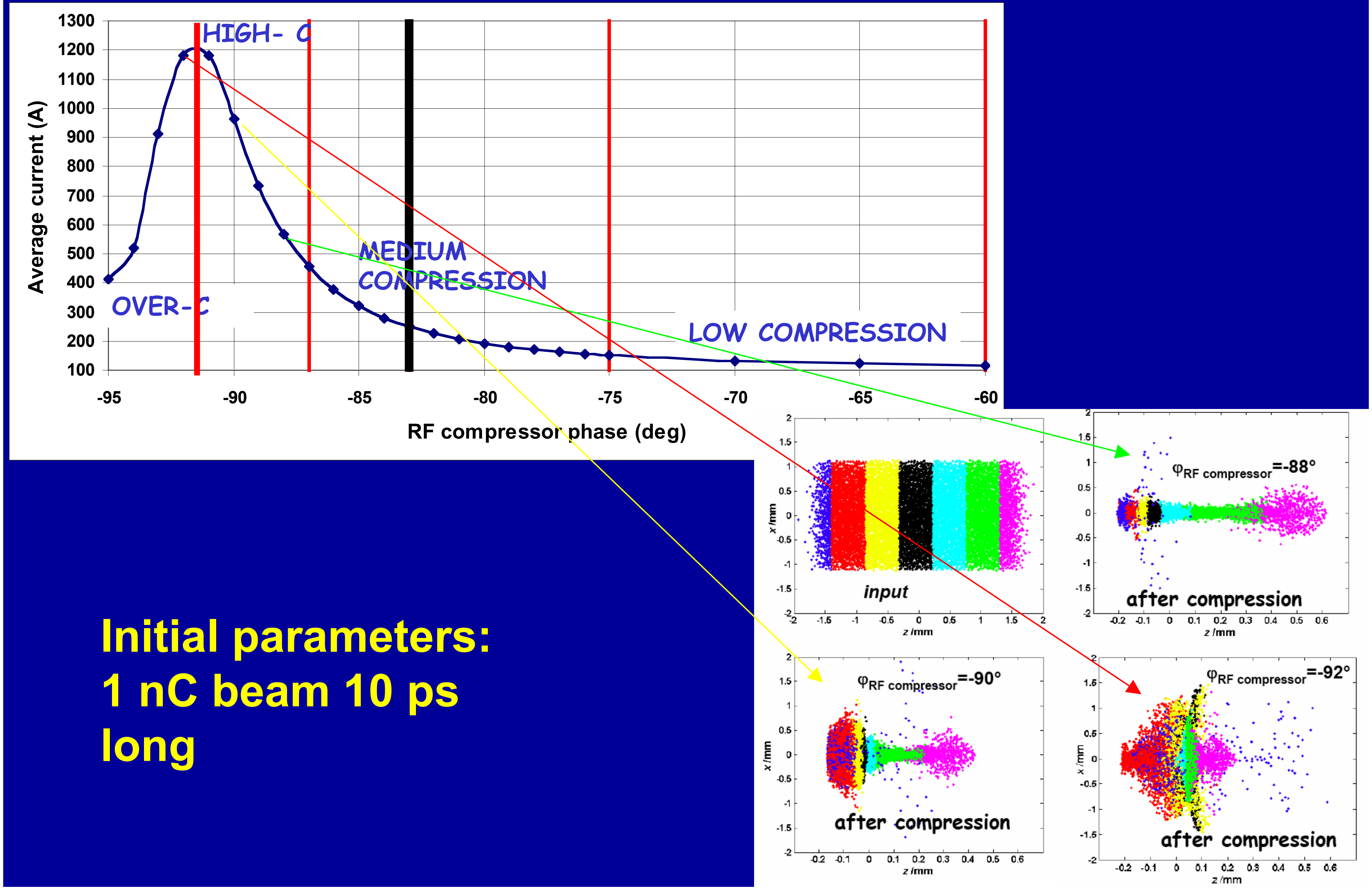


$$\alpha\sin\varphi_\infty\Delta\varphi_\infty = \alpha\sin\varphi_o\Delta\varphi_o + \frac{1}{2\gamma_o^2}\Delta\gamma_o$$

$$\Delta\phi_\infty = \frac{\sin\phi_0}{\sin\phi_\infty}\Delta\phi_0 + \frac{1}{2\alpha\gamma_0^2\sin\phi_\infty}\Delta\gamma_0$$

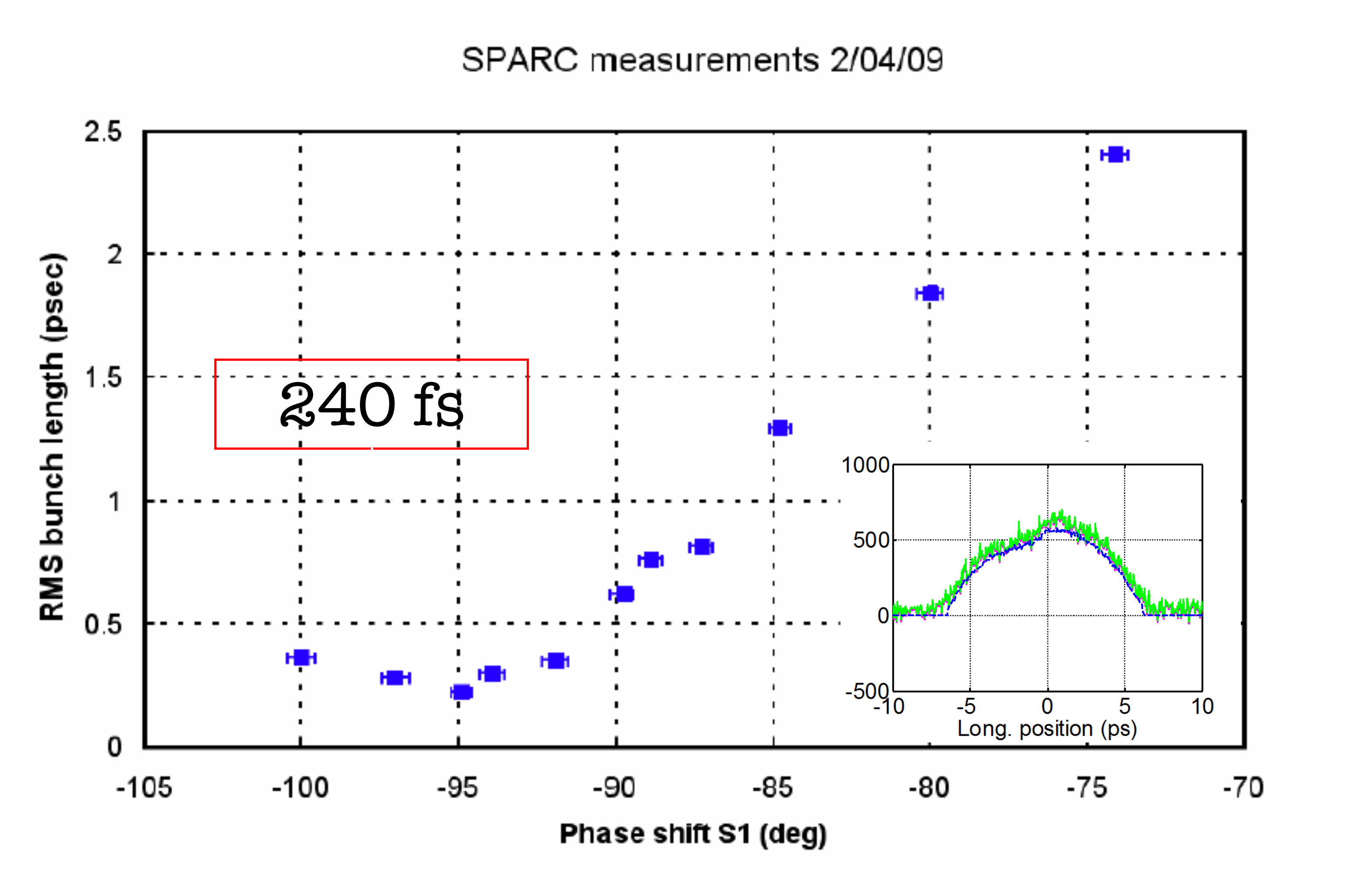

SPARC E=120 MV/m,fi=32,Q=1.1nC,ts=1 psec,FWHM=10 psec B=2.73KG
phase spectrum
y vs. x
element 1 Zpos= 0.000 ngood= 15003
e-es vs. phi-phis
es= 0.0000 ps= 215.44 z= 0.0
rms(dKE)/KE=48.34%

Peak current vs RF compressor phase

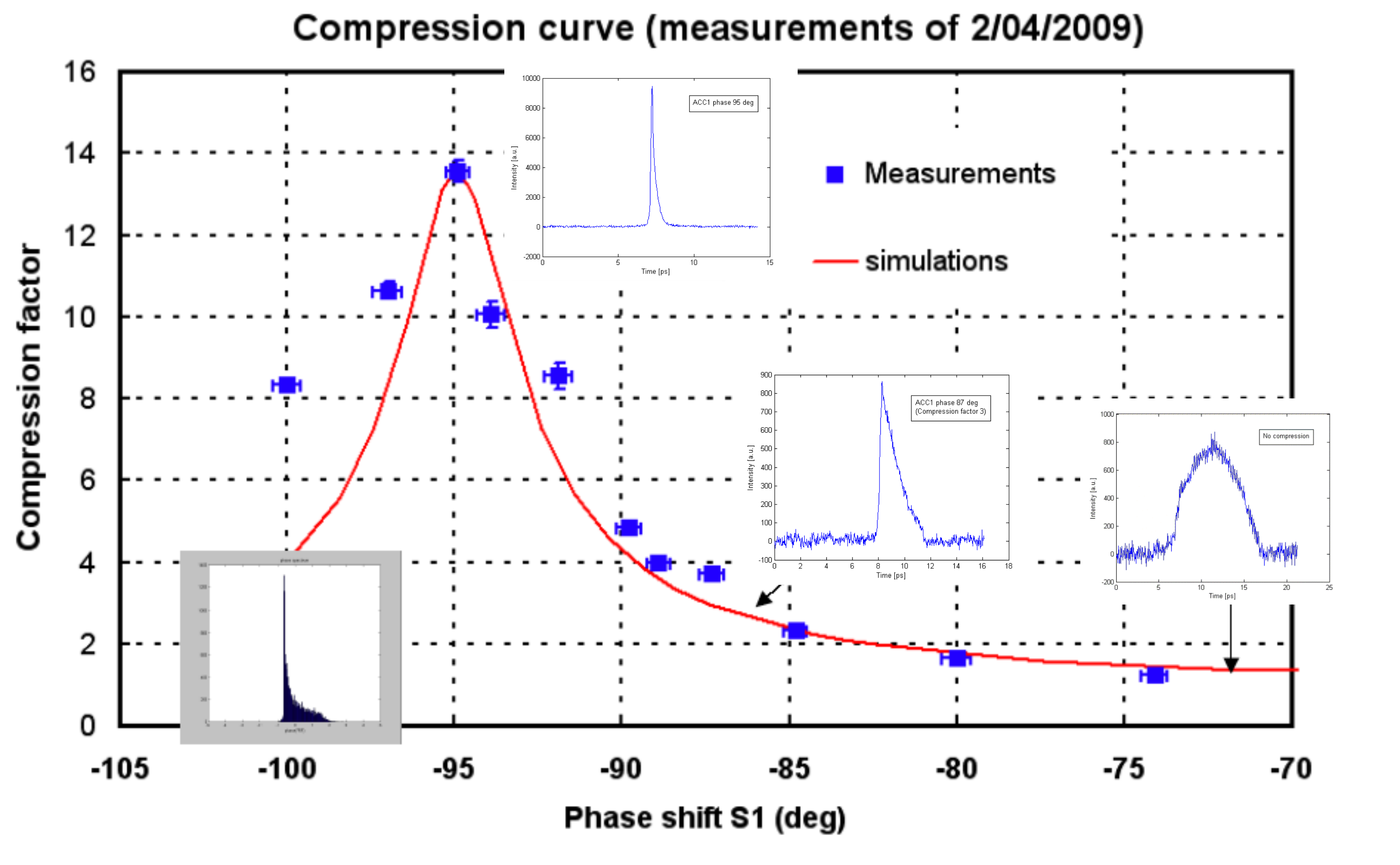


Initial parameters:
1 nC beam 10 ps long

Pulse length versus Velocity Bunching phase



C-factor versus injection phase



THE END

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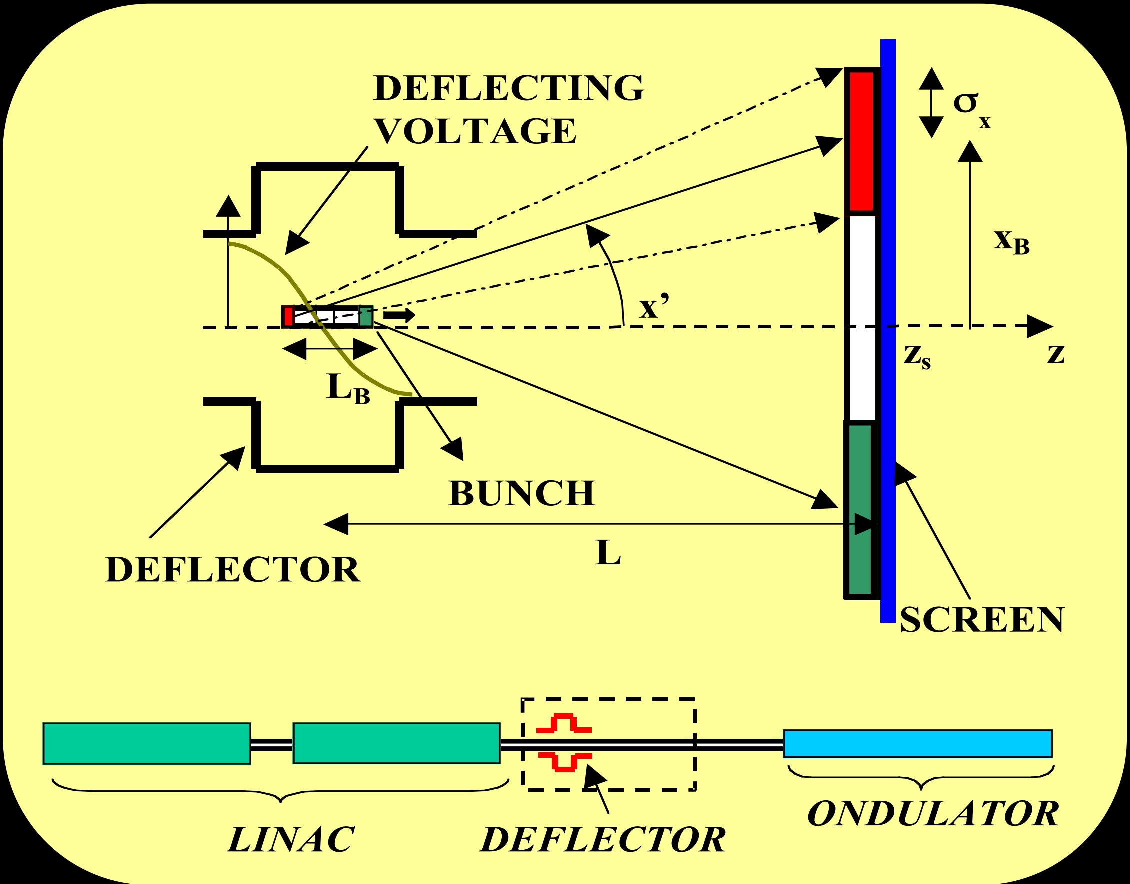
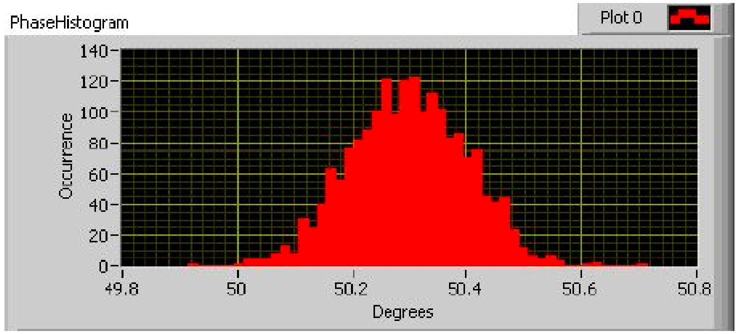


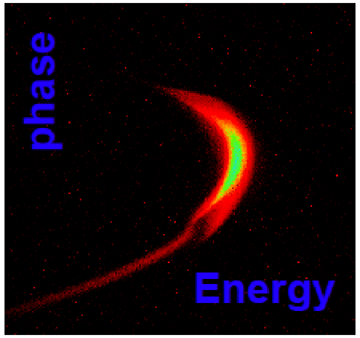
$$x_B = \frac{\pi \ f_{RF} L L_B V_\perp}{cE/e}$$

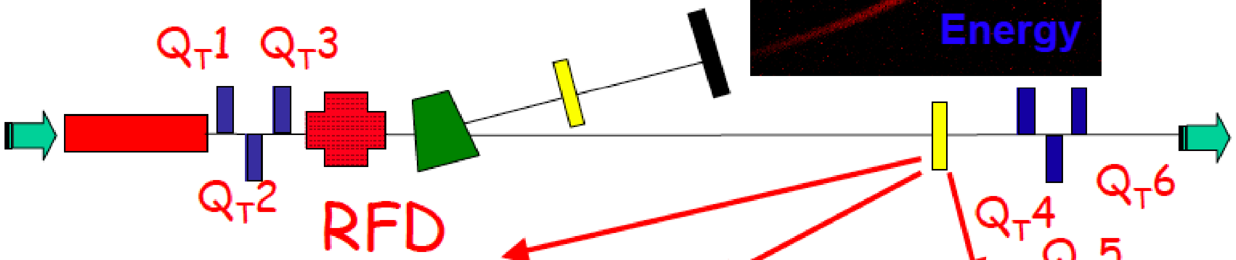
$$V_\perp = \frac{\sigma_x cE/e}{\pi f_{RF} L L_{res}}$$

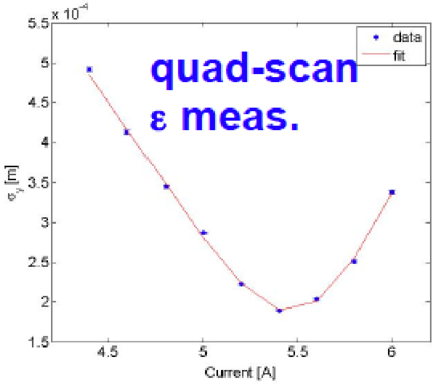
Diagnostic Section

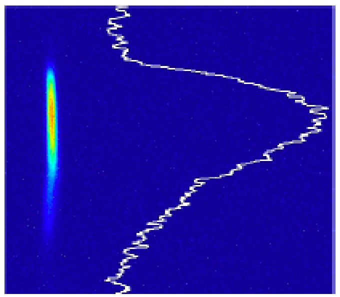


Rf deflector phase jitter ~100 fs

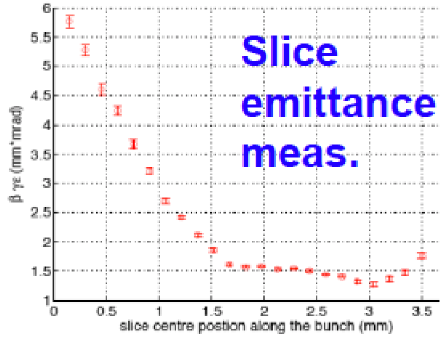








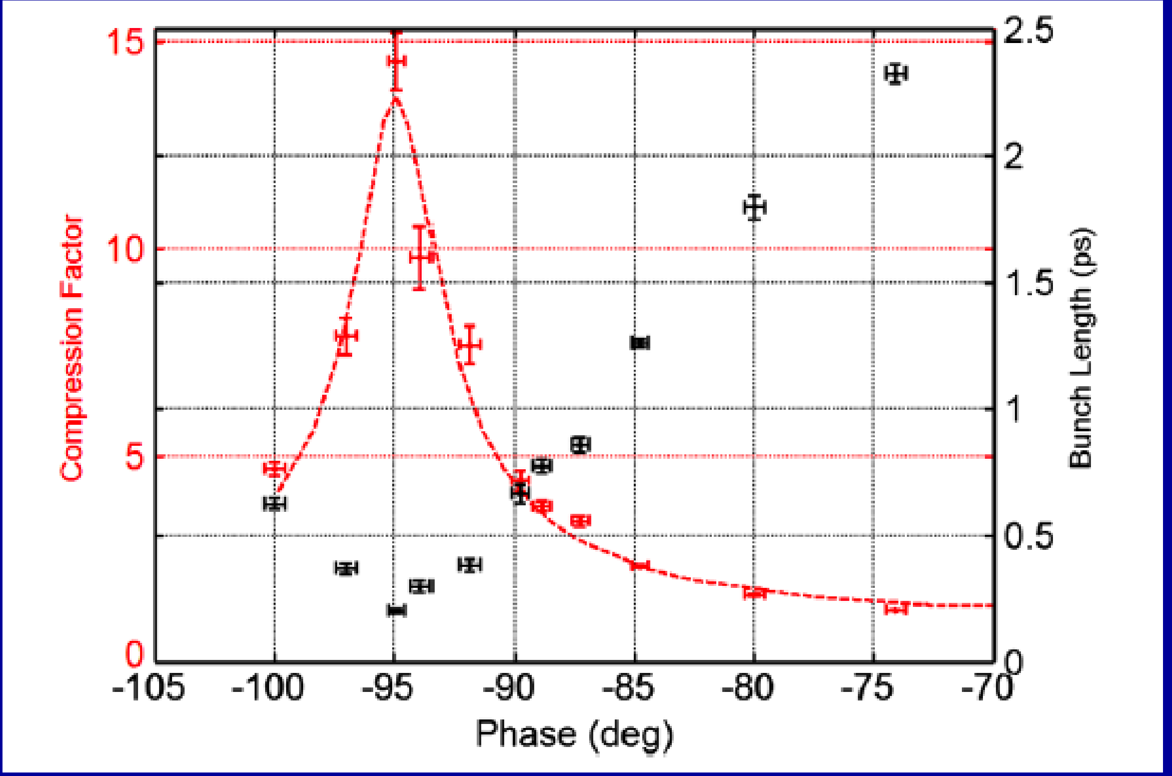
bunch length meas.



PRL 104, 054801 (2010) PHYSICAL REVIEW LETTERS week ending 5 FEBRUARY 2010

Experimental Demonstration of Emittance Compensation with Velocity Bunching

M. Ferrario,[1] D. Alesini,[1] A. Bacci,[3] M. Bellaveglia,[1] R. Boni,[1] M. Boscolo,[1] M. Castellano,[1] E. Chiadroni,[1] A. Cianchi,[2] L. Cultrera,[1] G. Di Pirro,[1] L. Ficcadenti,[1] D. Filippetto,[1] V. Fusco,[1] A. Gallo,[1] G. Gatti,[1] L. Giannessi,[4] M. Labat,[4] B. Marchetti,[2] C. Marrelli,[1] M. Migliorati,[1] A. Mostacci,[1] E. Pace,[1] L. Palumbo,[1] M. Quattromini,[4] C. Ronsivalle,[4] A. R. Rossi,[3] J. Rosenzweig,[5] L. Serafini,[3] M. Serluca,[6] B. Spataro,[1] C. Vaccarezza,[1] and C. Vicario[1]



Envelope Equation with Longitudinal Acceleration

$$\boxed{\begin{aligned} &p_o = \gamma_o m_o \beta_o c \\ &p_x << p_o \\ &p = p_o + p'z \\ &p' = (\beta\gamma)' m_o c \end{aligned}}$$

$$\frac{dp_x}{dt} = 0$$

$$\frac{d}{dt}(px') = \beta c \frac{d}{dz}(px') = 0$$

$$x'' + \frac{p'}{p} x' = 0$$

$$\boxed{x'' + \frac{(\beta\gamma)'}{\beta\gamma} x' = 0}$$

$$\langle xx'' \rangle = \frac{(\beta\gamma)'}{\beta\gamma} \langle xx' \rangle$$

Space Charge De-focusing Force

$$\boxed{\sigma_x'' + \frac{(\beta\gamma)'}{\beta\gamma}\sigma_x' + k^2\sigma_x = \frac{\varepsilon_n^2}{(\beta\gamma)^2\sigma_x^3} + \frac{k_{sc}}{\sigma_x}}$$

Adiabatic Damping

Emittance Pressure

Other External Focusing Forces

$$\boxed{\varepsilon_n = \beta\gamma\varepsilon_{rms}}$$

Beam subject to strong acceleration

$$\boxed{\sigma_x'' + \frac{(\beta\gamma)'}{\beta\gamma}\sigma_x' + k_{RF}^2\sigma_x = \frac{\varepsilon_n^2}{(\beta\gamma)^2\sigma_x^3} + \frac{k_{sc}^o}{(\beta\gamma)^3\sigma_x}}$$

We must include also the RF focusing force $k_{RF} = \frac{1}{4}\left(\frac{\gamma'}{\gamma}\right)$

$$k_{sc}^o = \frac{2I}{I_A}g(s,\gamma)$$

$$\sigma_x'' + \frac{(\beta\gamma)'}{\beta\gamma}\sigma_x' + k_{RF}^2\sigma_x = \frac{\varepsilon_n^2}{(\beta\gamma)^2\sigma_x^3} + \frac{k_{sc}^o}{(\beta\gamma)^3\sigma_x}$$

$\gamma = 1 + \alpha z$ ==> $\gamma'' = 0$

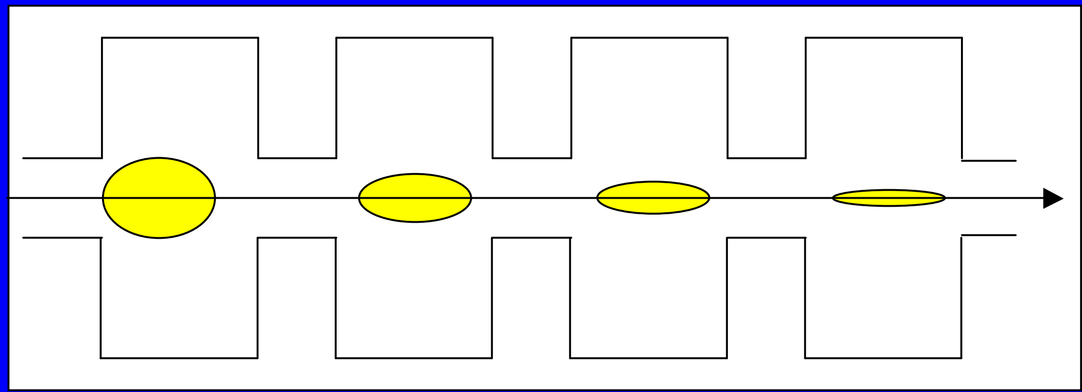
Looking for an "equilibrium" solution $\sigma_{inv} = \sigma_o \gamma^n$
==> all terms must have the same dependence on γ

Laminar beam $\rho >> 1 \Rightarrow n = -\frac{1}{2}$ $\sigma_q = \frac{\sigma_o}{\sqrt{\gamma}}$

Thermal beam $\rho << 1 \Rightarrow n = 0$ $\sigma_\varepsilon = \sigma_o$

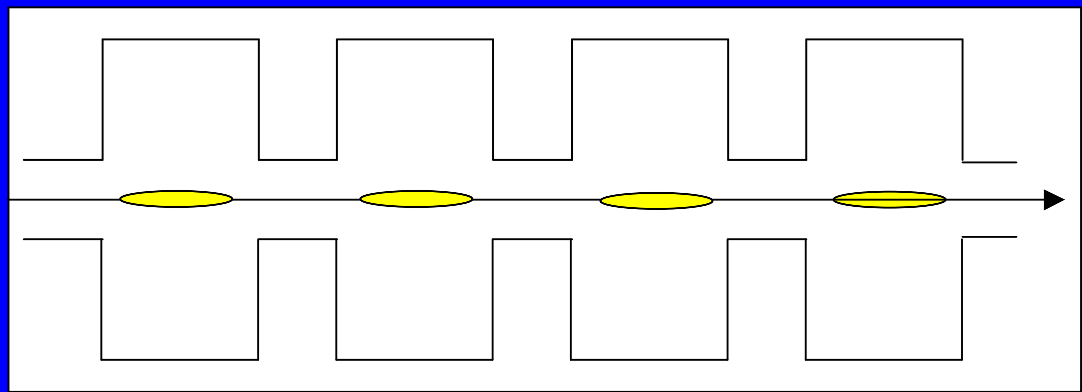
Space charge dominated beam (Laminar)

$$\sigma_q = \frac{1}{\gamma'}\sqrt{\frac{2I}{I_A \gamma}}$$

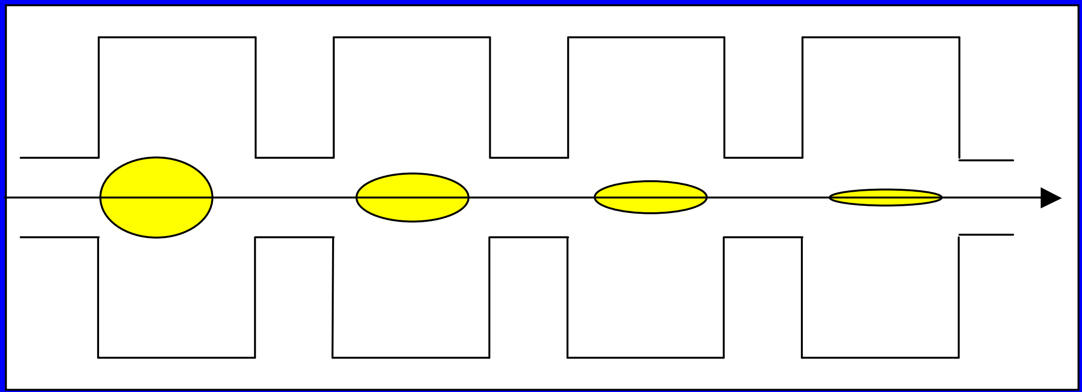


Emittance dominated beam (Thermal)

$$\sigma_\varepsilon = \sqrt{\frac{2\varepsilon_n}{\gamma'}}$$



$$\sigma_q = \frac{1}{\gamma'}\sqrt{\frac{2I}{I_A \gamma}}$$

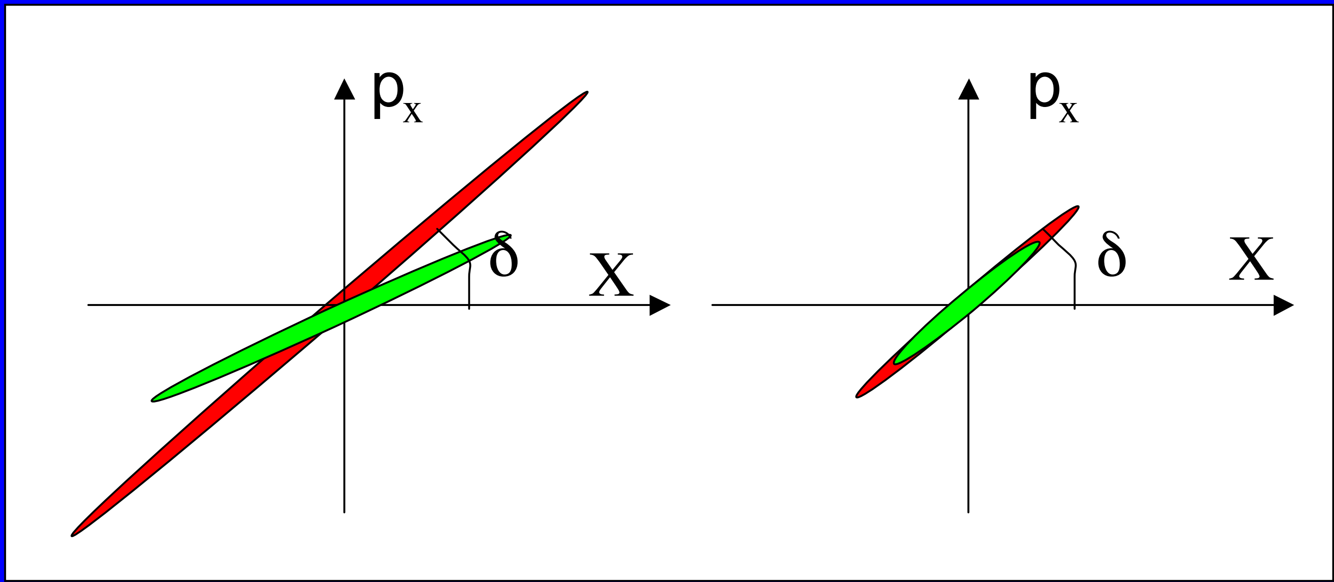


This solution represents a **beam equilibrium mode** that turns out to be the transport mode for achieving minimum emittance at the end of the **emittance correction process**

An important property of the laminar beam

$$\sigma_q = \frac{1}{\gamma'}\sqrt{\frac{2I}{I_A\gamma}} \qquad \sigma_q' = -\sqrt{\frac{2I}{I_A\gamma^3}}$$

Constant phase space angle: $\delta = \frac{\gamma\sigma_q'}{\sigma_q} = -\frac{\gamma'}{2}$

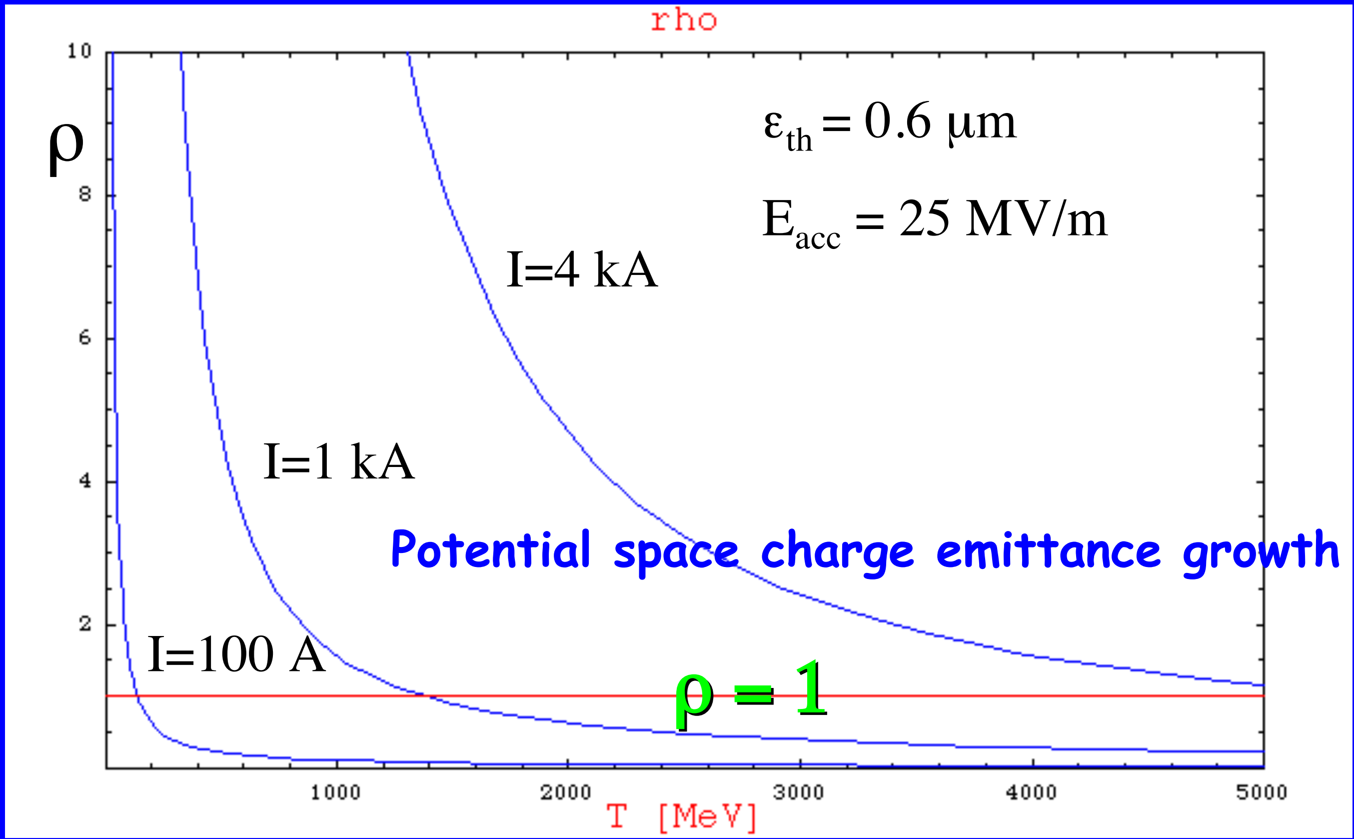


Laminarity parameter

$$\rho = \frac{2I\sigma^2}{\gamma I_A \varepsilon_n^2} \equiv \frac{2I\sigma_q^2}{\gamma I_A \varepsilon_n^2} = \frac{4I^2}{\gamma'^2 I_A^2 \varepsilon_n^2 \gamma^2}$$

Transition Energy (ρ=1)

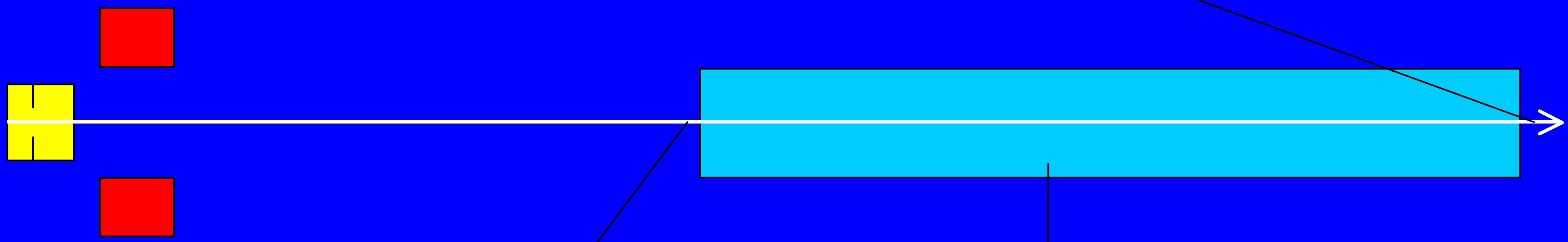
$$\gamma_{tr} = \frac{2I}{\gamma' I_A \varepsilon_n}$$



Matching Conditions with a TW Linac

150 MeV

$$\gamma_{tr} = \frac{2I}{\gamma' I_A \varepsilon_n}$$



$$\sigma' = 0$$

$$\sigma_q = \frac{1}{\gamma'}\sqrt{\frac{2I}{I_A \gamma}}$$

25 MV/m

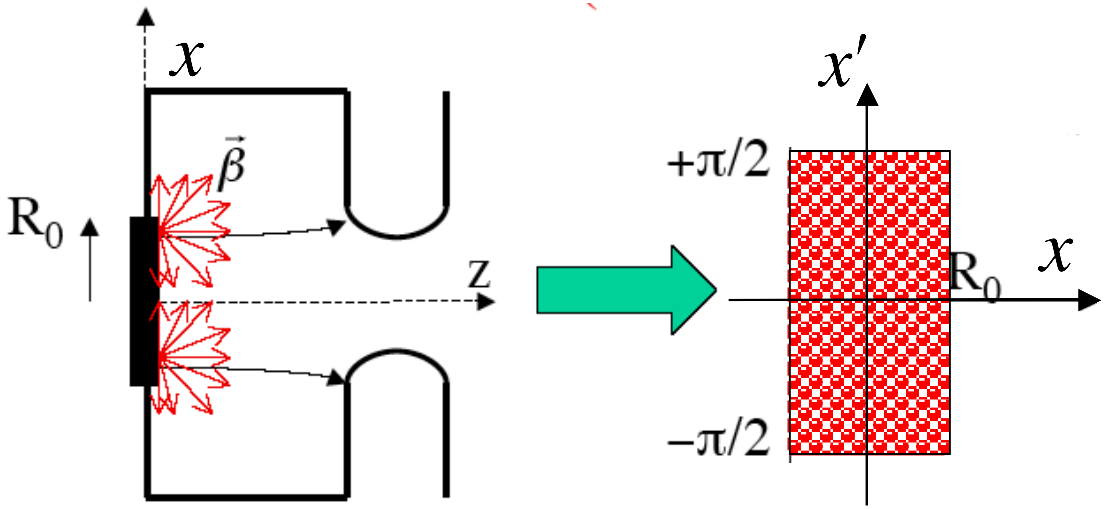
Emittance Compensation in a Photoinjector: Controlled Damping of Plasma Oscillations

- ε_n oscillations are driven by Space Charge
- propagation close to the laminar solution allows control of ε_n oscillation "phase"
- ε_n sensitive to SC up to the transition energy

Thermal emittance

$$\boxed{\varepsilon_{rms} = \langle \beta\gamma \rangle \sqrt{\sigma_x^2 \sigma_{x'}^2 - \sigma_{xx'}{}^2}}$$

Emittance evaluation close to the cathode sourface:



$$\sigma_{xx'} = 0$$

$$\sigma_{x'} = \frac{\pi}{4}$$

$$\langle \beta\gamma \rangle = \langle \beta \rangle = \sqrt{\frac{2E_{kin}}{m_o c^2}}$$

$$\boxed{\varepsilon_{rms}^{th} = \langle \beta \rangle \sigma_x \sigma_{x'} = \sigma_x \frac{\pi \langle \beta \rangle}{4}}$$