
CERN Accelerator School

**Advanced Accelerator Physics Course**

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# Low Emittance Machines

## Part 2: Equilibrium Emittance and Storage Ring Lattice Design

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# Lecture 1 summary

In Lecture 1, we:

- discussed the effect of synchrotron radiation on the (linear) motion of particles in storage rings;
- derived expressions for the damping times of the vertical, horizontal, and longitudinal emittances;
- discussed the effects of quantum excitation, and derived expressions for the equilibrium horizontal and longitudinal emittances in an electron storage ring in terms of the lattice functions and beam energy.

The natural emittance is:

$$\varepsilon_0 = C_q \gamma^2 \frac{I_5}{j_x I_2}, \qquad C_q = 3.832 \times 10^{-13} \text{ m}. \tag{1}$$

The natural energy spread and bunch length are given by:

$$\sigma_\delta^2 = C_q \gamma^2 \frac{I_3}{j_z I_2}, \qquad \sigma_z = \frac{\alpha_p c}{\omega_s} \sigma_\delta. \tag{2}$$

The momentum compaction factor is:

$$\alpha_p = \frac{I_1}{C_0}. \tag{3}$$

The synchrotron frequency and synchronous phase are given by:

$$\omega_s^2 = -\frac{eV_{RF}}{E_0} \frac{\omega_{RF}}{T_0} \alpha_p \cos\phi_s, \qquad \sin\phi_s = \frac{U_0}{eV_{RF}}. \tag{4}$$

The synchrotron radiation integrals are:

$$I_1 = \oint \frac{\eta_x}{\rho}\, ds, \tag{5}$$

$$I_2 = \oint \frac{1}{\rho^2}\, ds, \tag{6}$$

$$I_3 = \oint \frac{1}{|\rho|^3}\, ds, \tag{7}$$

$$I_4 = \oint \frac{\eta_x}{\rho}\left(\frac{1}{\rho^2} + 2k_1\right) ds, \qquad k_1 = \frac{e}{P_0}\frac{\partial B_y}{\partial x}, \tag{8}$$

$$I_5 = \oint \frac{\mathcal{H}_x}{|\rho|^3}\, ds, \qquad \mathcal{H}_x = \gamma_x \eta_x^2 + 2\alpha_x \eta_x \eta_{px} + \beta_x \eta_{px}^2. \tag{9}$$

In this lecture, we shall:

- derive expressions for the natural emittance in four types of lattices:
  - FODO;
  - double-bend achromat (DBA);
  - multi-bend achromats, including the triple-bend achromat (TBA);
  - theoretical minimum emittance (TME).

- consider how the emittance of an achromat may be reduced by "detuning" from the zero-dispersion conditions.

In Lecture 1, we showed that the natural emittance in a storage ring is given by:

$$\varepsilon_0 = C_q \gamma^2 \frac{I_5}{j_x I_2}, \tag{10}$$

where $C_q$ is a physical constant, $\gamma$ is the relativistic factor, $j_x$ is the horizontal damping partition number, and $I_5$ and $I_2$ are synchrotron radiation integrals.

Note that $j_x$, $I_5$ and $I_2$ are all functions of the lattice, and are independent of the beam energy.

In most storage rings, if the bends have no quadrupole component, the damping partition number $j_x \approx 1$.

In this case we just need to evaluate the two synchrotron radiation integrals:

$$I_2 = \oint \frac{1}{\rho^2}\, ds, \qquad I_5 = \oint \frac{\mathcal{H}_x}{|\rho|^3}\, ds. \tag{11}$$

If we know the strength and length of all the dipoles in the lattice, it is straightforward to calculate $I_2$.

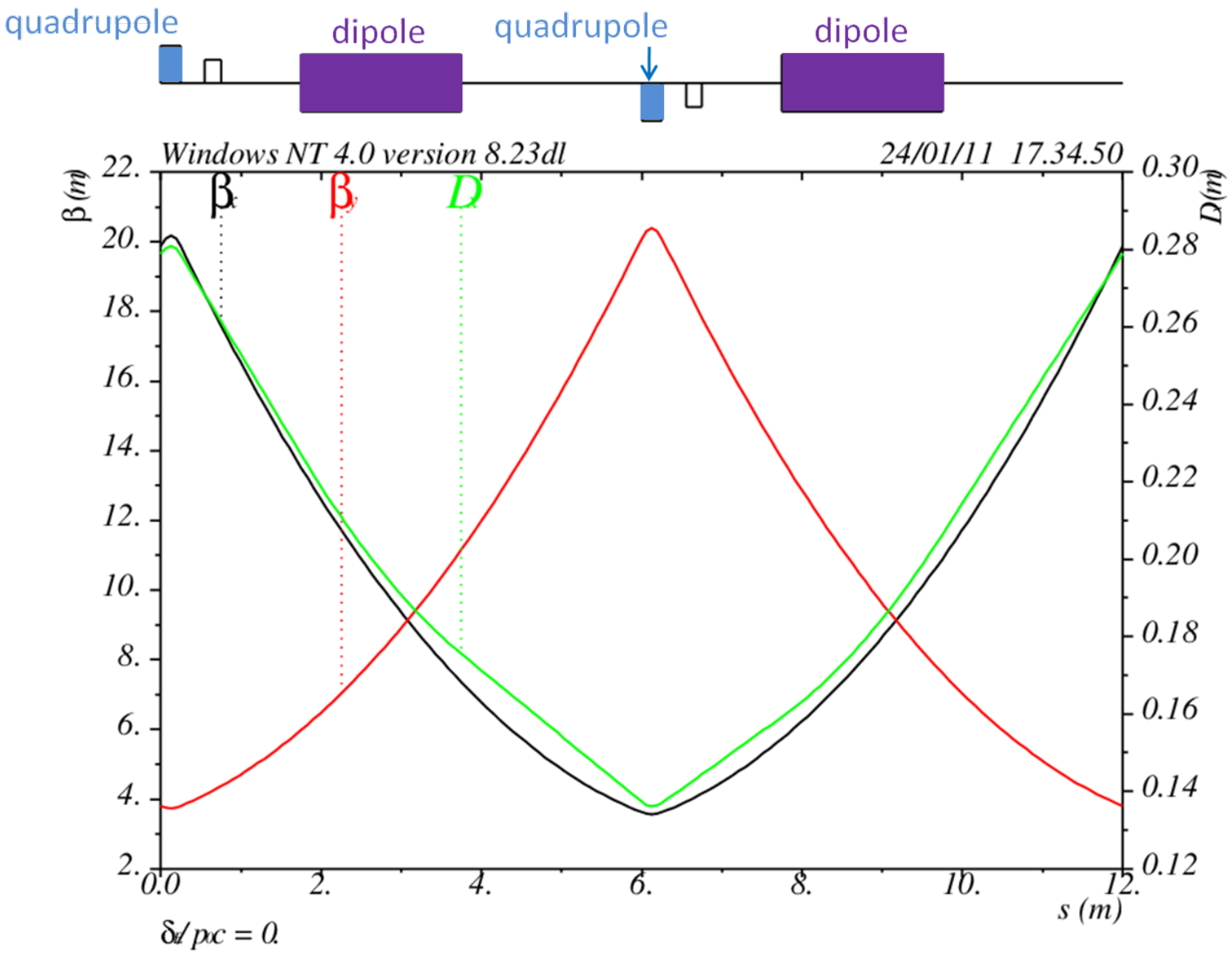
For example, if all the bends are identical, then in a complete ring (total bending angle $= 2\pi$):

$$I_2 = \oint \frac{1}{\rho^2} ds = \oint \frac{B}{(B\rho)} \frac{ds}{\rho} = \frac{2\pi B}{(B\rho)} \approx 2\pi \frac{cB}{E/e}, \tag{12}$$

where $E$ is the beam energy.

Evaluating $I_5$ is more complicated: it depends on the lattice functions...

# Case 1: natural emittance in a FODO lattice



Let us consider the case of a simple FODO lattice. To simplify the system, we use the following approximations:

- the quadrupoles are represented as thin lenses;
- the space between the quadrupoles is completely filled by the dipoles.

With these approximations, the lattice functions (Twiss parameters and dispersion) are completely determined by the following parameters:

- the focal length $f$ of a quadrupole;
- the bending radius $\rho$ of a dipole;
- the length $L$ of a dipole.

In terms of $f$, $\rho$ and $L$, the horizontal beta function at the horizontally-focusing quadrupole is given by:

$$\beta_x = \frac{4f\rho\sin\theta(2f\cos\theta + \rho\sin\theta)}{\sqrt{16f^4 - [\rho^2 - (4f^2+\rho^2)\cos 2\theta]^2}}, \tag{13}$$

where $\theta = L/\rho$ is the bending angle of a single dipole.

The dispersion at a horizontally-focusing quadrupole is given by:

$$\eta_x = \frac{2f\rho(2f + \rho\tan\frac{\theta}{2})}{4f^2+\rho^2}. \tag{14}$$

By symmetry, at the centre of a quadrupole, $\alpha_x = \eta_{px} = 0$.

We also know how to evolve the lattice functions through the lattice, using the transfer matrices, $M$.

For the Twiss parameters:

$$A(s_1) = M \cdot A(s_0) \cdot M^{\mathsf{T}}, \tag{15}$$

where $M = M(s_1; s_0)$ is the transfer matrix from $s_0$ to $s_1$, and:

$$A = \begin{pmatrix} \beta_x & -\alpha_x \\ -\alpha_x & \gamma_x \end{pmatrix}. \tag{16}$$

The dispersion can be evolved (over a distance $\Delta s$, with constant bending radius $\rho$) using:

$$\begin{pmatrix} \eta_x \\ \eta_{px} \end{pmatrix}_{s_1} = M \cdot \begin{pmatrix} \eta_x \\ \eta_{px} \end{pmatrix}_{s_0} + \begin{pmatrix} \rho(1 - \cos\frac{\Delta s}{\rho}) \\ \sin\frac{\Delta s}{\rho} \end{pmatrix}. \tag{17}$$

For a thin quadrupole, the transfer matrix is: $M = \begin{pmatrix} 1 & 0 \\ -1/f & 0 \end{pmatrix}$.

For a dipole, the transfer matrix is: $M = \begin{pmatrix} \cos\frac{s}{\rho} & \rho\sin\frac{s}{\rho} \\ -\frac{1}{\rho}\sin\frac{s}{\rho} & \cos\frac{s}{\rho} \end{pmatrix}$.

We now have all the information we need to find an expression for $I_5$ in the FODO cell.

However, the algebra is rather formidable. The result is most easily expressed as a power series in the dipole bending angle, $\theta$:

$$\frac{I_5}{I_2} = \left(4 + \frac{\rho^2}{f^2}\right)^{-\frac{3}{2}} \left[8 - \frac{\rho^2}{2f^2}\theta^2 + O(\theta^4)\right]. \qquad (18)$$

For small $\theta$, the expression for $I_5/I_2$ can be written:

$$\frac{I_5}{I_2} \approx \left(1 - \frac{\rho^2}{16f^2}\theta^2\right)\left(1 + \frac{\rho^2}{4f^2}\right)^{-\frac{3}{2}} = \left(1 - \frac{L^2}{16f^2}\right)\left(1 + \frac{\rho^2}{4f^2}\right)^{-\frac{3}{2}}. \tag{19}$$

This can be further simplified if $\rho \gg 2f$ (often the case):

$$\frac{I_5}{I_2} \approx \left(1 - \frac{L^2}{16f^2}\right)\frac{8f^3}{\rho^3}, \tag{20}$$

and still further simplified if $4f \gg L$ (less often the case):

$$\frac{I_5}{I_2} \approx \frac{8f^3}{\rho^3}. \tag{21}$$

Making the approximation $j_x \approx 1$ (since there is no quadrupole component in the dipole), and writing $\rho = L/\theta$, we have:

$$\varepsilon_0 \approx C_q \gamma^2 \left(\frac{2f}{L}\right)^3 \theta^3. \tag{22}$$

Notice how the emittance scales with the beam and lattice parameters:

- The emittance is proportional to the square of the energy.
- The emittance is proportional to the cube of the bending angle. Increasing the number of cells in a complete circular lattice reduces the bending angle of each dipole, and reduces the emittance.
- The emittance is proportional to the cube of the quadrupole focal length: stronger quads means lower emittance.
- The emittance is inversely proportional to the cube of the cell (or dipole) length.

The phase advance in a FODO cell is given by:

$$\cos\mu_x = 1 - \frac{L^2}{2f^2}. \tag{23}$$

This means that a stable lattice must have:

$$\frac{f}{L} \geq \frac{1}{2}. \tag{24}$$

In the limiting case, $\mu_x = 180^\circ$, and $f$ has the minimum value $f = L/2$. Using the approximation (22):

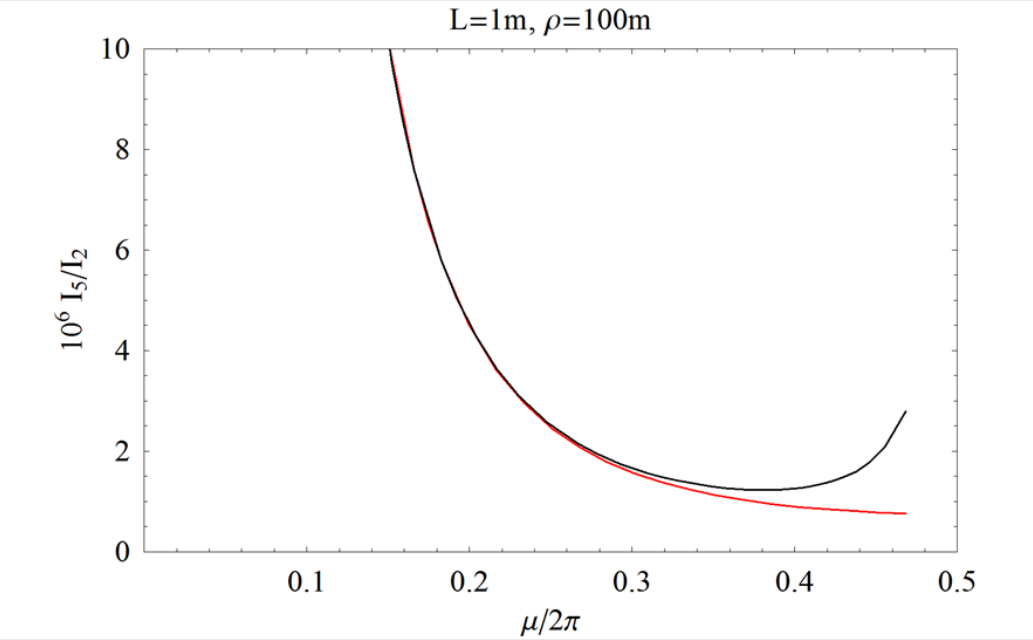
$$\varepsilon_0 \approx C_q \gamma^2 \left(\frac{2f}{L}\right)^3 \theta^3,$$

the *minimum emittance in a FODO lattice* is expected to be:

$$\varepsilon_0 \approx C_q \gamma^2 \theta^3. \tag{25}$$

However, as we increase the focusing strength, the approximations we used to obtain the simple expression for $\varepsilon_0$ start to break down...

Plotting the exact formula for $I_5/I_2$ as a function of the phase advance, we find that there is a minimum in the natural emittance, at $\mu \approx 137^\circ$.




Black line:
exact formula.

Red line:
approximation,

$$\frac{I_5}{I_2} \approx \left(1 - \frac{L^2}{16f^2}\right)\frac{8f^3}{\rho^3}.$$

It turns out that the minimum value of the natural emittance in a FODO lattice is given by:

$$\varepsilon_{0,\mathrm{FODO,min}} \approx 1.2 C_q \gamma^2 \theta^3. \qquad (26)$$

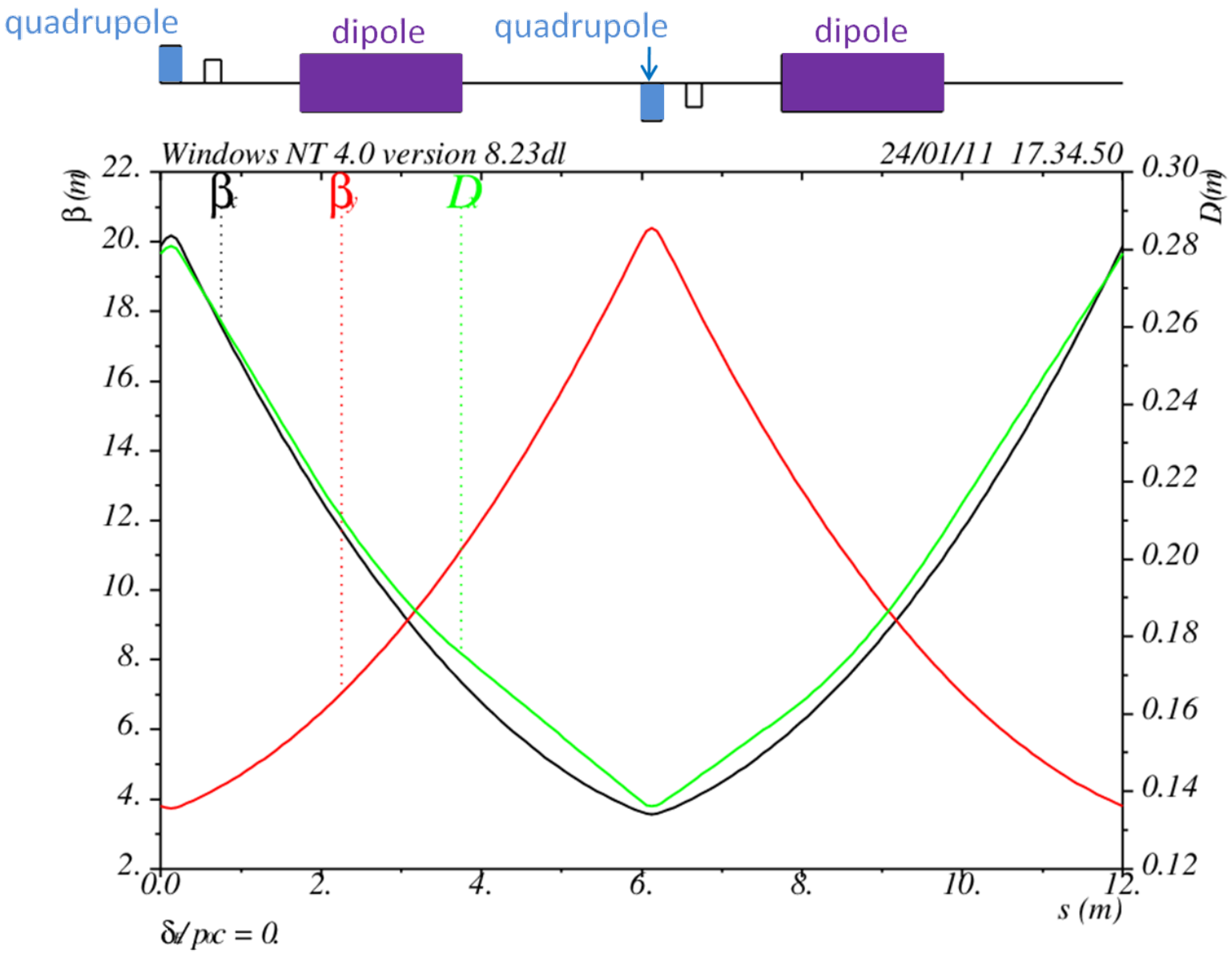
Using Eq. (22), we estimate that a storage ring constructed from 16 FODO cells (32 dipoles) with 90° phase advance per cell ($f = L/\sqrt{2}$), and storing beam at 2 GeV would have a natural emittance of around 125 nm.

Many modern applications (including synchrotron light sources) demand emittances smaller by one or two orders of magnitude.

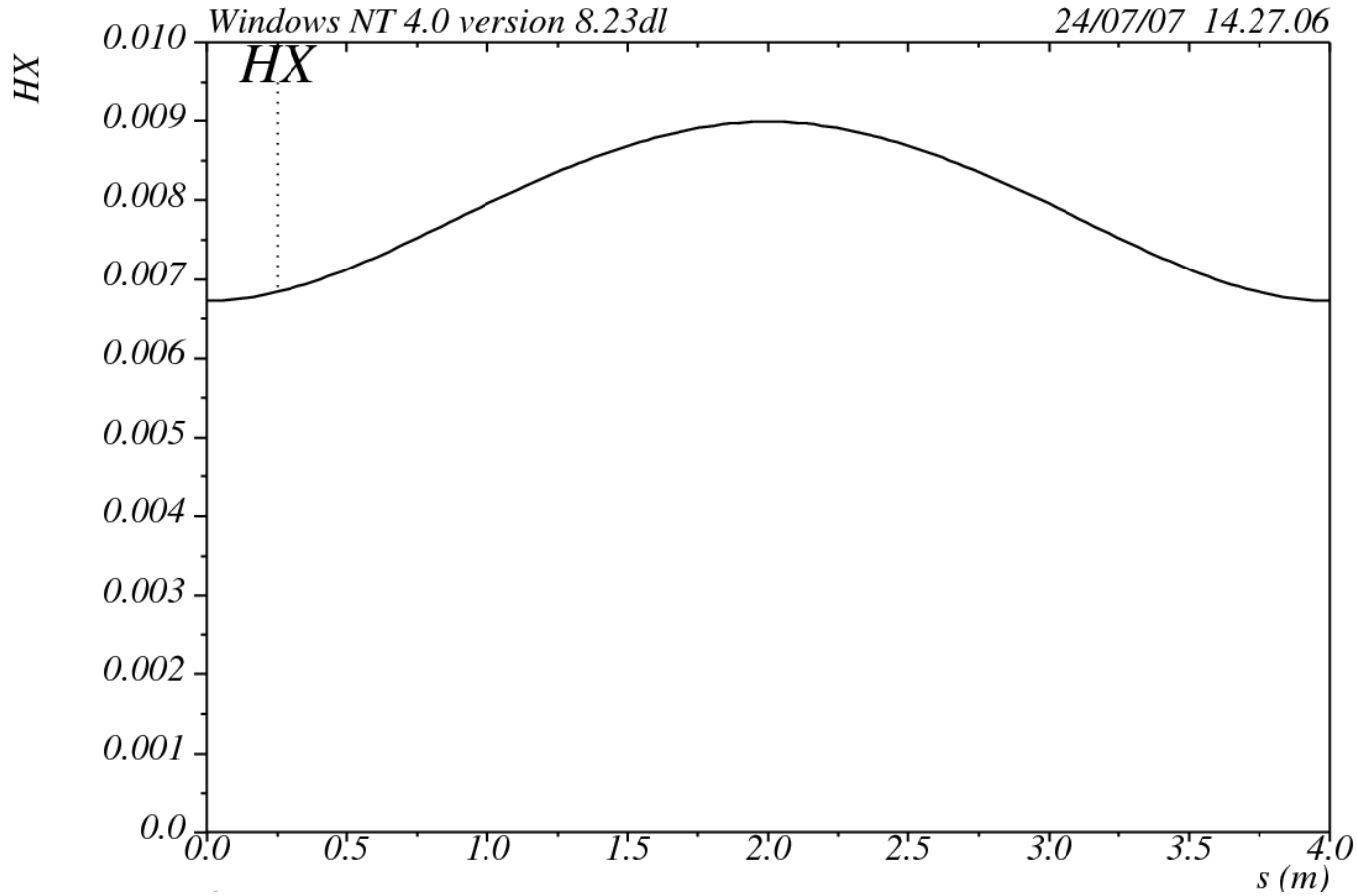
How can we design a lattice with a smaller natural emittance?

Looking at curly-H in a FODO cell provides a clue...

# Case 1: natural emittance in a FODO lattice

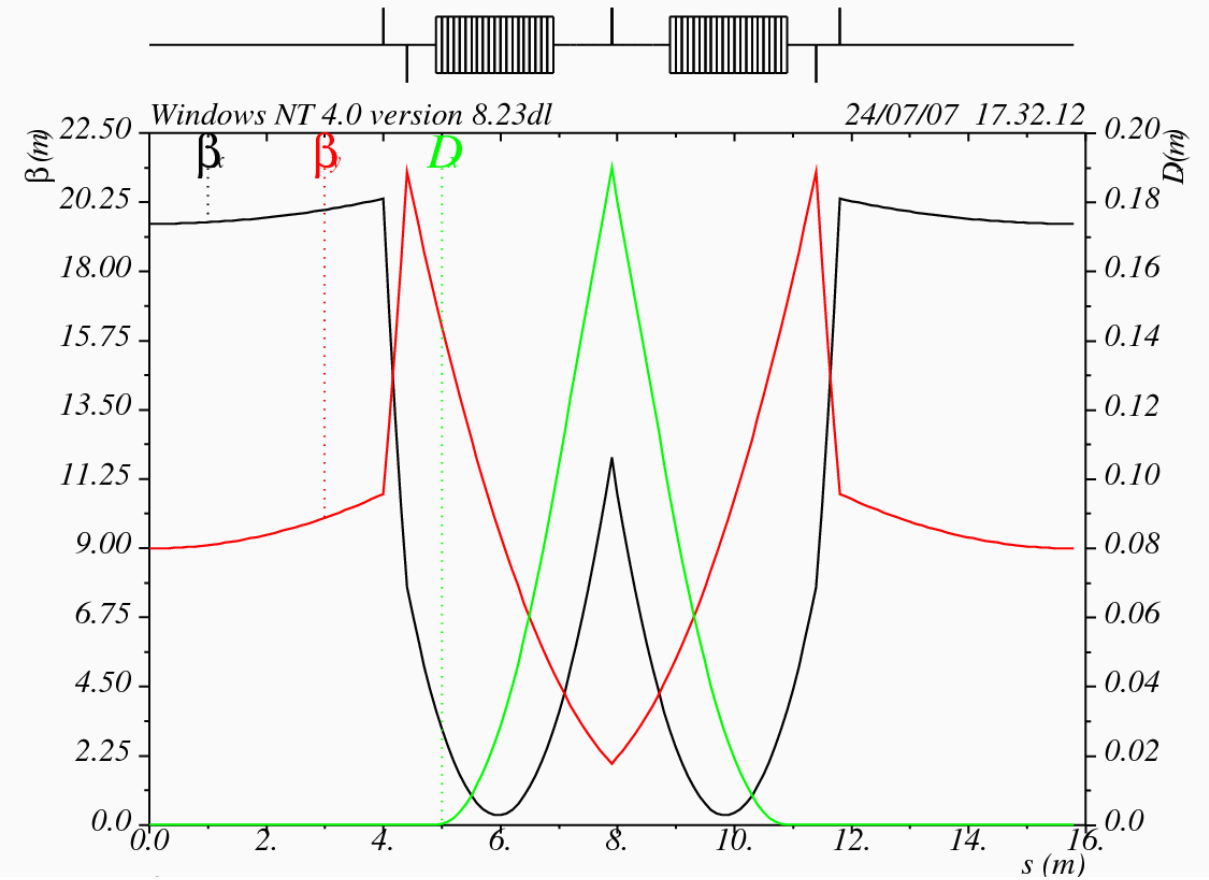


The curly-H function remains at a relatively constant value throughout the lattice:



As a first attempt at reducing the natural emittance, we can try reducing the curly-H function in the dipoles, by designing a lattice that has zero dispersion at either end of a dipole pair.

The result is a double bend achromat (DBA) cell:



To calculate the natural emittance in a DBA, let us begin by considering the conditions for zero dispersion at the start and the exit of the cell.

Assume that the dispersion is zero at the start of the cell.

Place a quadrupole midway between the dipoles, to reverse the gradient of the dispersion.

By symmetry, the dispersion at the exit of the cell will be zero.

In the thin lens approximation, this condition can be written:

$$\begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \cdot \begin{pmatrix} \eta_x \\ \eta_{px} \end{pmatrix} = \begin{pmatrix} \eta_x \\ \eta_{px} - \frac{\eta_x}{f} \end{pmatrix} = \begin{pmatrix} \eta_x \\ -\eta_{px} \end{pmatrix}. \qquad (27)$$

Hence the central quadrupole must have focal length:

$$f = \frac{\eta_x}{2\eta_{px}}. \tag{28}$$

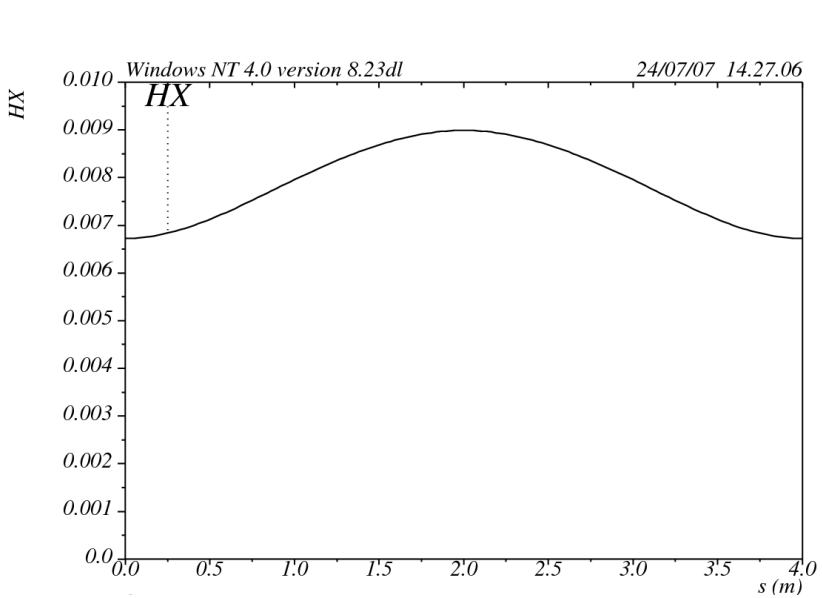
The actual value of the dispersion (and its gradient) is determined by the dipole bending angle $\theta$, the bending radius $\rho$, and the drift length $L$:

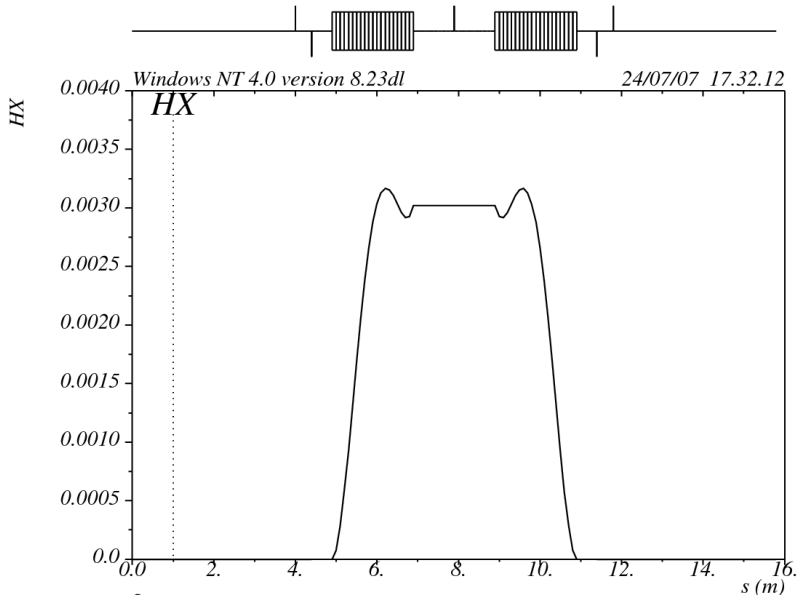
$$\eta_x = \rho(1 - \cos\theta) + L\sin\theta, \qquad \eta_{px} = \sin\theta. \tag{29}$$

Is this style of lattice likely to have a lower natural emittance than a FODO lattice?

We can get some idea by looking at the curly-H function...

# Case 2: natural emittance in a DBA lattice




The curly-H function is much smaller in the DBA lattice (right) than in the FODO lattice (left).

Note that we use the same dipoles (bending angle and length) in both cases.

Let us calculate the minimum natural emittance of a DBA lattice, for given bending radius $\rho$ and bending angle $\theta$ in the dipoles.

To do this, we need to calculate the minimum value of:

$$I_5 = \int \frac{\mathcal{H}_x}{\rho^3}\, ds \tag{30}$$

in one dipole, subject to the constraints:

$$\eta_{x,0} = \eta_{px,0} = 0, \tag{31}$$

where $\eta_{x,0}$ and $\eta_{px,0}$ are the dispersion and gradient of the dispersion at the entrance of a dipole.

We know how the dispersion and the Twiss parameters evolve through the dipole, so we can calculate $I_5$ for one dipole, for given initial values of the Twiss parameters $\alpha_{x,0}$ and $\beta_{x,0}$.

Then, we simply have to minimise the value of $I_5$ with respect to $\alpha_{x,0}$ and $\beta_{x,0}$.

Again, the algebra is rather formidable, and the full expression for $I_5$ is not especially enlightening.

Therefore, we just quote the significant results...

We find that, for given $\rho$ and $\theta$ and with the constraints:

$$\eta_{x,0} = \eta_{px,0} = 0, \tag{32}$$

the minimum value of $I_5$ is given by:

$$I_{5,\mathrm{min}} = \frac{1}{4\sqrt{15}}\frac{\theta^4}{\rho} + O(\theta^6). \tag{33}$$

This minimum occurs for values of the Twiss parameters at the entrance to the dipole given by:

$$\beta_{x,0} = \sqrt{\frac{12}{5}}L + O(\theta^3), \qquad \alpha_{x,0} = \sqrt{15} + O(\theta^2), \tag{34}$$

where $L = \rho\theta$ is the length of a dipole.

Since we know that $I_2$ in a single dipole is given by:

$$I_2 = \int \frac{1}{\rho^2}\, ds = \frac{\theta}{\rho}, \tag{35}$$

we can now write down an expression for the minimum emittance in a DBA lattice:

$$\varepsilon_{0,\mathrm{DBA,min}} = C_q \gamma^2 \frac{I_{5,\mathrm{min}}}{j_x I_2} \approx \frac{1}{4\sqrt{15}} C_q \gamma^2 \theta^3. \tag{36}$$

The approximation is valid for small $\theta$. Note that we have again assumed that, since there is no quadrupole component in the dipole, $j_x \approx 1$.

Compare the above expression with that for the minimum emittance in a FODO lattice:

$$\varepsilon_{0,\mathrm{FODO,min}} \approx C_q \gamma^2 \theta^3. \tag{37}$$

We see that in both cases (FODO and DBA), the emittance scales with the square of the beam energy, and with the cube of the bending angle.

However, the emittance in a DBA lattice is smaller than that in a FODO lattice (for given energy and dipole bending angle) by a factor $4\sqrt{15} \approx 15.5$.

This is a significant improvement... *but can we do even better?*

For a DBA lattice, we imposed the constraints:

$$\eta_{x,0} = \eta_{px,0} = 0. \tag{38}$$

To get a lower emittance, we can consider relaxing these constraints.

To derive the conditions for a "theoretical minimum emittance" (TME) lattice, we write down an expression for:

$$I_5 = \int \frac{\mathcal{H}_x}{\rho}\, ds, \tag{39}$$

with *arbitrary* dispersion $\eta_{x,0}$, $\eta_{px,0}$ and Twiss parameters $\alpha_{x,0}$ and $\beta_{x,0}$ in a dipole with given bending radius $\rho$ and angle $\theta$.

Then, we minimise $I_5$ with respect to $\eta_{x,0}$, $\eta_{px,0}$, $\alpha_{x,0}$ and $\beta_{x,0}$...

The result is:

$$\varepsilon_{0,\mathrm{TME,min}} \approx \frac{1}{12\sqrt{15}} C_q \gamma^2 \theta^3. \tag{40}$$

The minimum emittance is obtained with dispersion at the entrance to the dipole given by:

$$\eta_{x,0} = \frac{1}{6} L\theta + O(\theta^3), \qquad \eta_{px,0} = -\frac{\theta}{2} + O(\theta^3), \tag{41}$$

and with Twiss functions at the entrance:

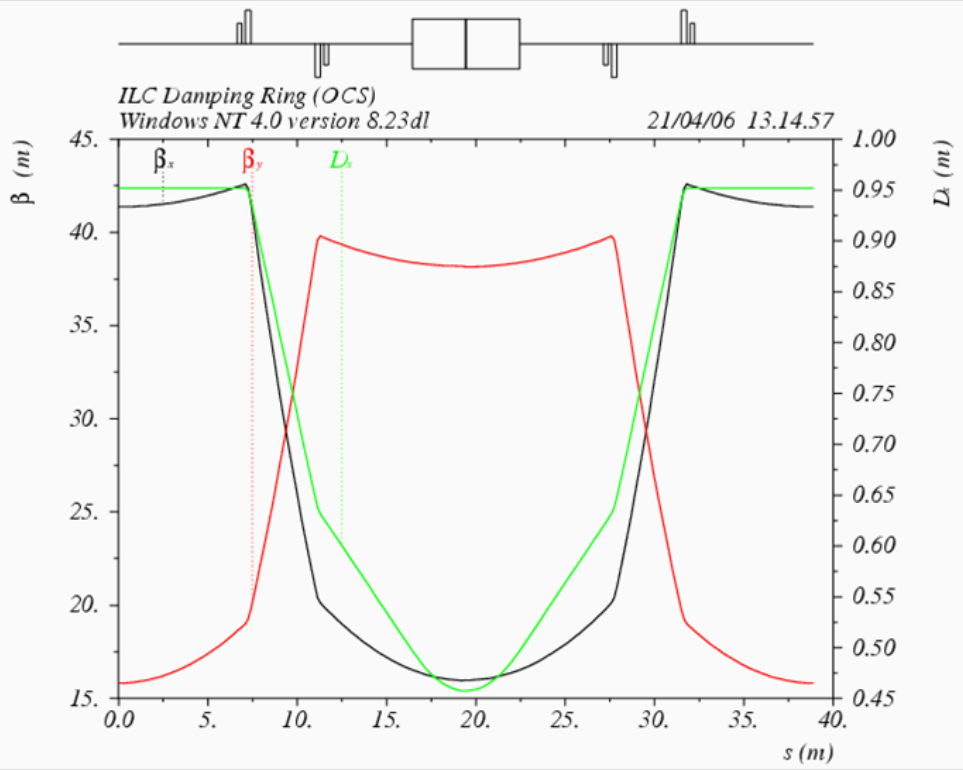
$$\beta_{x,0} = \frac{8}{\sqrt{15}} L + O(\theta^2), \qquad \alpha_{x,0} = \sqrt{15} + O(\theta^2). \tag{42}$$

The dispersion and beta function reach minimum values in the centre of the dipole:

$$\eta_{x,\mathrm{min}} = \rho \left( 1 - 2\frac{\sin\frac{\theta}{2}}{\theta} \right) = \frac{L\theta}{24} + O(\theta^4), \qquad \beta_{x,\mathrm{min}} = \frac{L}{2\sqrt{15}} + O(\theta^2). \tag{43}$$

# Case 3: natural emittance in a TME lattice

By symmetry, we can consider a single TME cell to contain a single dipole, rather than a pair of dipoles as was necessary for the FODO and DBA cells.




Outside the dipole, the dispersion is relatively large. This is not ideal for a light source, since insertion devices at locations with large dispersion will blow up the emittance.

*Note that the cell shown here does not achieve the exact conditions for a TME lattice: a more complicated design would be needed for this.*

## Summary: natural emittance in FODO, DBA and TME lattices

| Lattice style | Minimum emittance | Conditions/comments |
|---|---|---|
| 90° FODO | $\varepsilon_0 \approx 2\sqrt{2}C_q\gamma^2\theta^3$ | $\frac{f}{L} = \frac{1}{\sqrt{2}}$ |
| 137° FODO | $\varepsilon_0 \approx 1.2C_q\gamma^2\theta^3$ | minimum emittance FODO |
| DBA | $\varepsilon_0 \approx \frac{1}{4\sqrt{15}}C_q\gamma^2\theta^3$ | $\eta_{x,0} = \eta_{px,0} = 0$ <br> $\beta_{x,0} \approx \sqrt{12/5}L \quad \alpha_{x,0} \approx \sqrt{15}$ |
| TME | $\varepsilon_0 \approx \frac{1}{12\sqrt{15}}C_q\gamma^2\theta^3$ | $\eta_{x,\mathrm{min}} \approx \frac{L\theta}{24} \quad \beta_{x,\mathrm{min}} \approx \frac{L}{2\sqrt{15}}$ |

The results we have derived have been for "ideal" lattices that perfectly achieve the stated conditions in each case.

In practice, lattices rarely, if ever, achieve the ideal conditions. In particular, the beta function in an achromat is usually not optimal for low emittance; and it is difficult to tune the dispersion for the ideal TME conditions.

The main reasons for this are:

- Beam dynamics issues generally impose a number of strong constraints on the design.

- Optimizing the lattice functions while respecting all the various constraints can require complex configurations of quadrupoles.

A particularly challenging constraint on design of a low-emittance lattice is the dynamic aperture.

Storage rings require a large dynamic aperture in order to achieve good injection efficiency and good beam lifetime.

However, low emittance lattices generally need low dispersion and beta functions, and hence require strong quadrupoles. As a result, the chromaticity can be large, and requires strong sextupoles for its correction.

Strong sextupoles lead to strongly nonlinear motion, and limit the dynamic aperture (the trajectories of particles at large betatron amplitudes or large energy deviations become unstable).

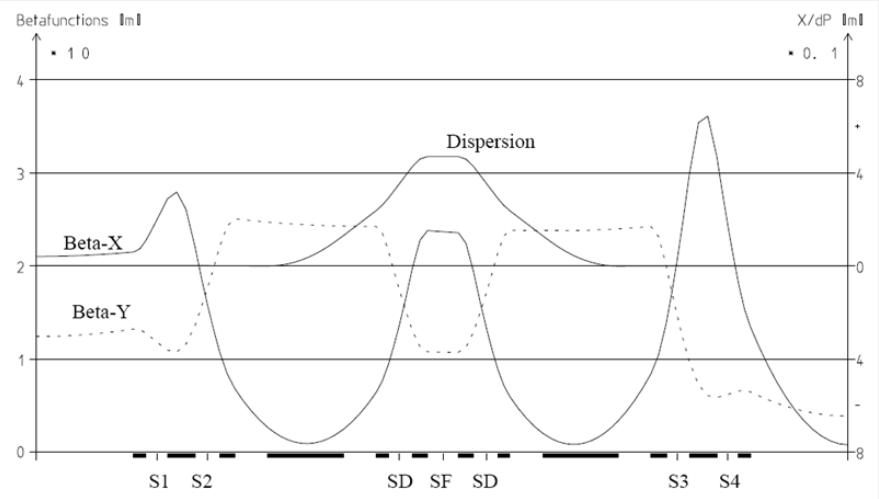
We have derived the main results for this lecture.

However, there are many other options besides FODO, DBA and TME for the lattice "style".

In the remainder of this lecture, we will discuss:

- the use of the DBA lattice in third-generation synchrotron light sources;
- detuning a DBA lattice to reduce the emittance;
- the use of multi-bend achromats.

Lattices composed of DBA cells have been a popular choice for third generation synchrotron light sources, e.g. the ESRF.




The DBA structure provides a lower natural emittance than a FODO lattice with the same number of dipoles.

The long, dispersion-free straight sections provide ideal locations for insertion devices such as undulators and wigglers.

## "Detuning" a DBA

If an insertion device, such as an undulator or wiggler, is incorporated in a storage ring at a location with large dispersion, then the dipole fields in the device can make a significant contribution to the quantum excitation ($I_5$).

As a result, the insertion device can lead to an increase in the natural emittance of the storage ring.

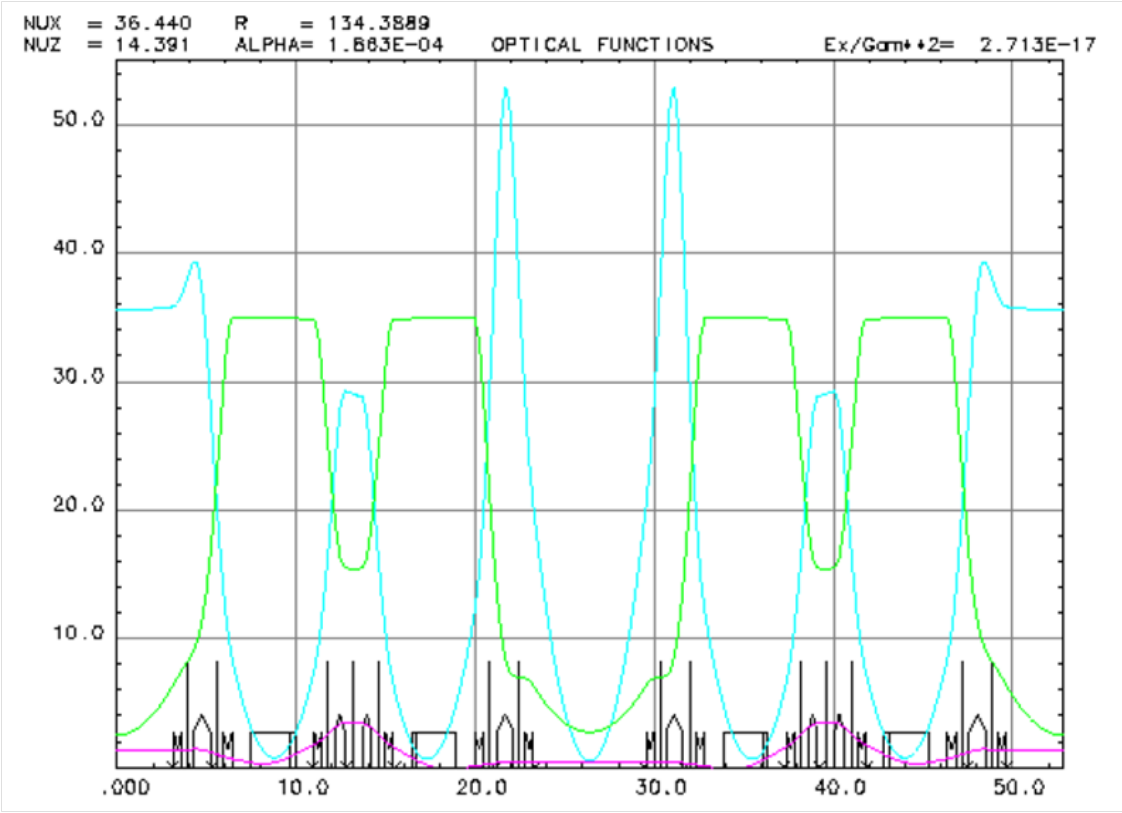
By using a DBA lattice, we provide dispersion-free straights in which we can locate undulators and wigglers without blowing up the natural emittance.

However, there is some tolerance. In many cases, it is possible to "detune" the lattice from the strict DBA conditions, thereby allowing some reduction in natural emittance at the cost of some dispersion in the straights.

The insertion devices will then contribute to the quantum excitation; but depending on the lattice and the insertion devices, there may still be a net benefit.

Some light sources that were originally designed with zero-dispersion straights take advantage of tuning flexibility to operate with non-zero dispersion in the straights.

This provides a lower natural emittance, and better output for users. For example, the ESRF:



In principle, it is possible to combine the DBA and TME lattices by having an arc cell consisting of more than two dipoles.

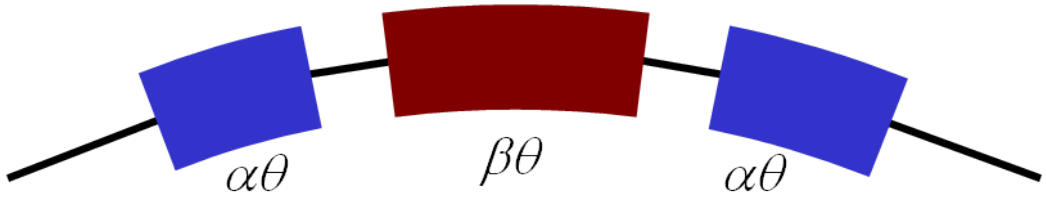
The dipoles at either end of the cell have zero dispersion (and gradient of the dispersion) at their outside faces, thus satisfying the achromat condition.

The lattice is tuned so that in the "central" dipoles, the Twiss parameters and dispersion satisfy the TME conditions.

Since the lattice functions are different in the central dipoles compared to the end dipoles, we have additional degrees of freedom we can use to minimise the quantum excitation.

Therefore, it is possible to have cases where the end dipoles and central dipoles differ in: the bend angle (i.e. length of dipole), and/or the bend radius (i.e. strength of dipole).

For simplicity, let us consider the case where the dipoles all have the same bending radius (i.e. they all have the same field strength), but they vary in length.




Assuming each arc cell has a fixed number, $M$, of dipoles, and $\theta = 2\pi/MN_{\text{cells}}$, the bending angles satisfy:

$$2\alpha + (M-2)\beta = M. \tag{44}$$

Since the synchrotron radiation integrals are additive, for an $M$-bend achromat, we can write:

$$I_{5,\text{cell}} \approx \frac{2}{4\sqrt{15}}\frac{(\alpha\theta)^4}{\rho} + \frac{(M-2)}{12\sqrt{15}}\frac{(\beta\theta)^4}{\rho} = \frac{6\alpha^4 + (M-2)\beta^4}{12\sqrt{15}}\frac{\theta^4}{\rho}, \tag{45}$$

$$I_{2,\text{cell}} \approx 2\frac{\alpha\theta}{\rho} + (M-2)\frac{\beta\theta}{\rho} = [2\alpha + (M-2)\beta]\,\frac{\theta}{\rho}. \tag{46}$$

Hence, in an $M$-bend achromat:

$$\frac{I_{5,\text{cell}}}{I_{2,\text{cell}}} \approx \frac{1}{12\sqrt{15}} \left[ \frac{6\alpha^4 + (M-2)\beta^4}{2\alpha + (M-2)\beta} \right] \theta^3. \tag{47}$$

Minimising the ratio $I_5/I_2$ with respect to $\alpha$ gives:

$$\frac{\alpha}{\beta} = \frac{1}{\sqrt[3]{3}}, \qquad \frac{6\alpha^4 + (M-2)\beta^4}{2\alpha + (M-2)\beta} \approx \frac{M+1}{M-1}. \tag{48}$$

The central bending magnets should be longer than the outer bending magnets by a factor $\sqrt[3]{3}$.

Then, the minimum natural emittance in an $M$-bend achromat is given by:

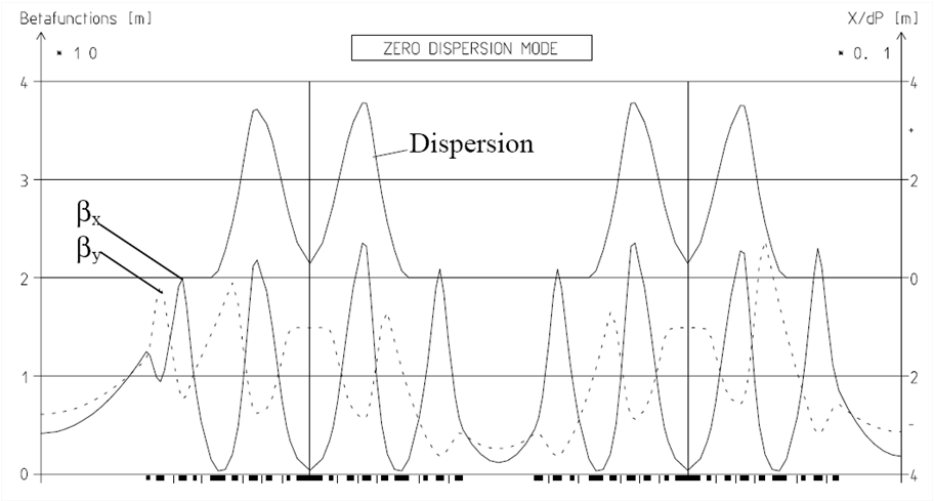
$$\varepsilon_0 \approx C_q \gamma^2 \frac{1}{12\sqrt{15}} \left( \frac{M+1}{M-1} \right) \theta^3, \qquad 2 < M < \infty. \tag{49}$$

Note that $\theta$ is the *average* bending angle per dipole.

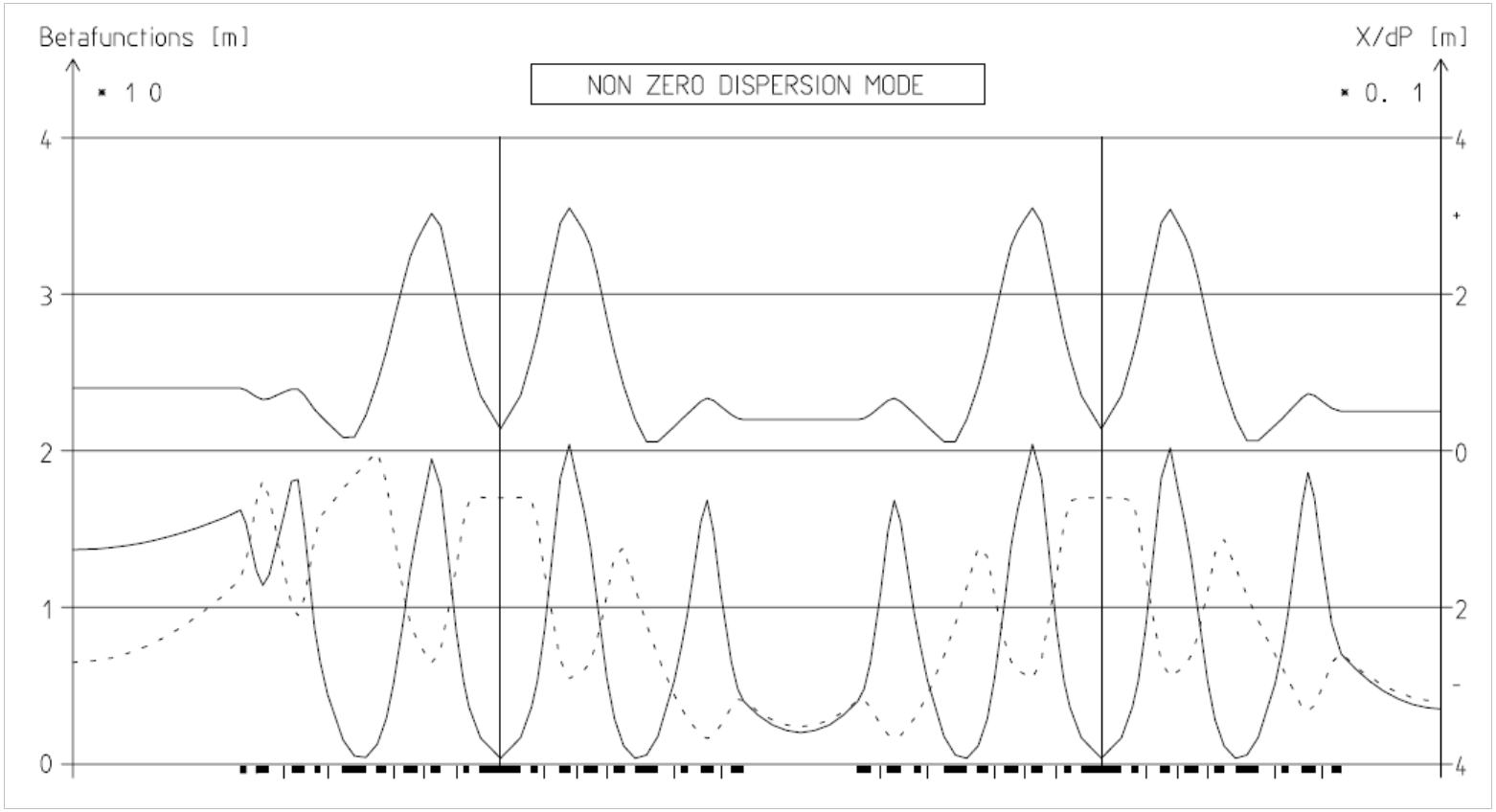
# Example of a triple-bend achromat: the Swiss Light Source

The storage ring in the Swiss Light Source consists of 12 TBA cells. The circumference is 288 m, and the beam energy is 2.4 GeV.

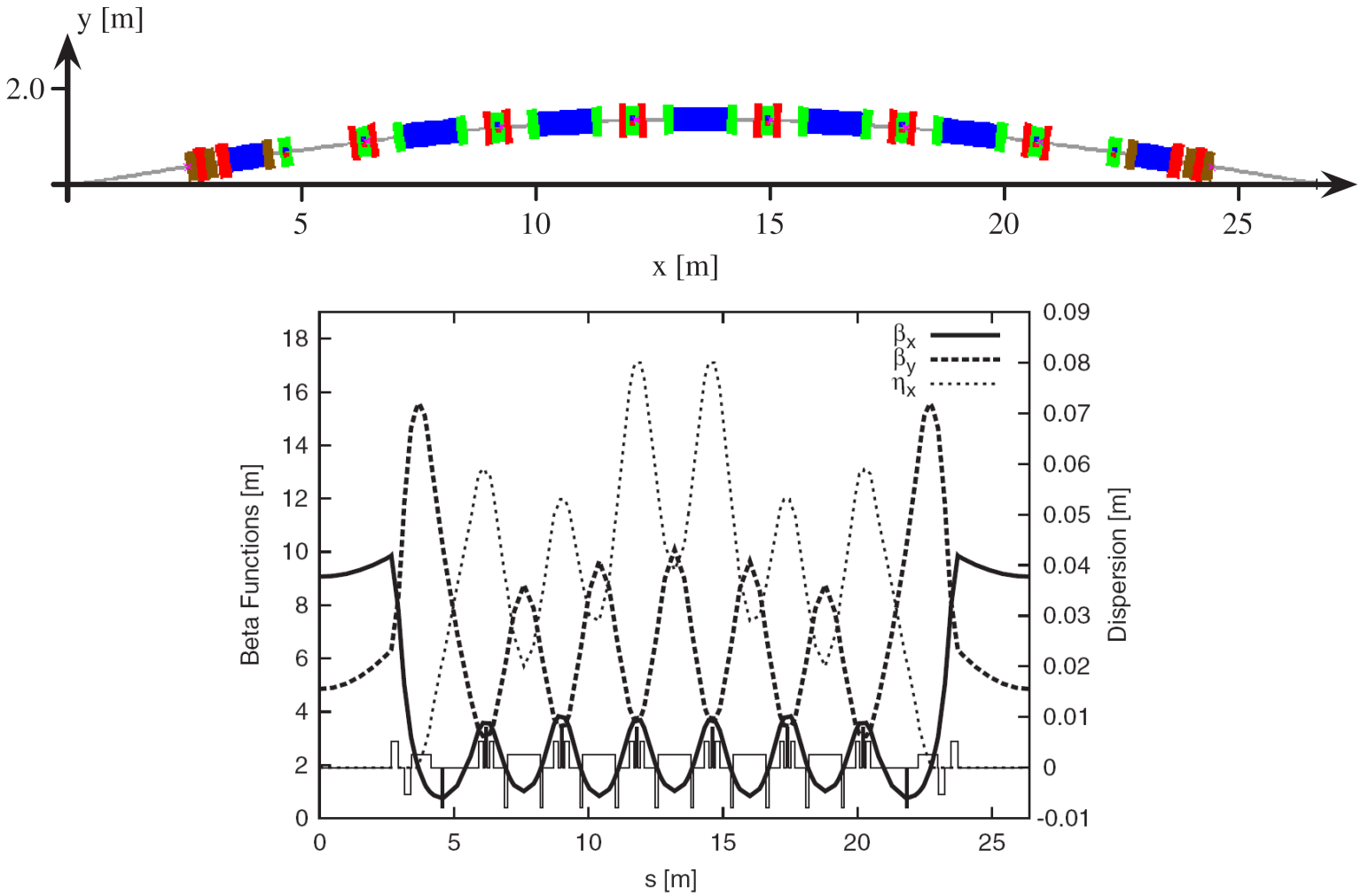
In the "zero-dispersion" mode, the natural emittance is 4.8 nm-rad.




Detuning the achromat to allow dispersion in the straights reduces the natural emittance from 4.8 nm-rad to 3.9 nm-rad (a reduction of about 20%).



# A 7-bend achromat: MAX IV




Note: vertical focusing provided by gradient in the bending magnets.

S.C. Leeman et al, "Beam dynamics and expected performance of Sweden's new storage-ring light source: MAX IV," PRST-AB 12, 120701 (2009).

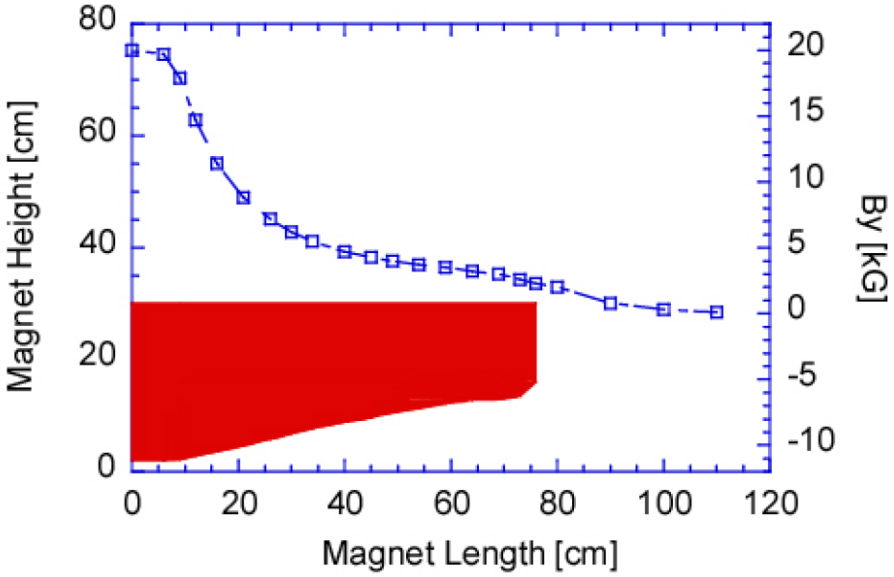
# A 7-bend achromat: MAX IV

| | |
|---|---|
| Beam energy | 3 GeV |
| Circumference | 528 m |
| Number of cells | 20 |
| Horizontal emittance (no IDs) | 0.326 nm |
| Horizontal emittance (with IDs) | 0.263 nm |

# Final thought: variational (longitudinal gradient) bends

In principle, we can relax the constraint that the field strength in a dipole is constant along the length of the dipole.

Allowing a longitudinal variation in the strength provides another degree of freedom in reducing the emittance. We expect an optimised design to have the strongest field at the centre of the dipole, where the dispersion can be minimised.




| | Uniform Bend | Variational Bend | |
|---|---|---|---|
| Max B (T) | 1.2 | 2.0 | 5.10 |
| Length (m) | 0.96 | 1.5 | 1.5 |
| Bend angle | $2\pi/36$ | $2\pi/36$ | $2\pi/36$ |
| I2 | 3.173e-2 | 3.139e-2 | 6.887e-2 |
| I5/I2 | 1.142e-4 | 2.945e-5 | 1.5793e-5 |
| $\beta_{x0}$ (m) | 0.124 | 0.0803 | 0.0222 |
| $\eta_{x0}$ (m) | 6.98e-3 | 1.40e-3 | 3.62e-4 |

J. Guo and T. Raubenheimer, Proceedings of EPAC'02, Paris, France.

The natural emittance in a storage ring is determined by the balance between the radiation damping (given by $I_2$) and the quantum excitation (given by $I_5$).

The quantum excitation depends on the lattice functions. Different "styles" of lattice can be used, depending on the emittance specification for the storage ring.

In general, for small bending angle $\theta$ the natural emittance can be written as:

$$\varepsilon_0 \approx F C_q \gamma^2 \theta^3, \tag{50}$$

where $\theta$ is the bending angle of a single dipole, and the numerical factor $F$ is determined by the lattice style...

# Summary (2)

$$\varepsilon_0 \approx F C_q \gamma^2 \theta^3$$

| Lattice style | $F$ |
|---|---|
| 90° FODO | $2\sqrt{2}$ |
| 137° FODO | 1.2 |
| Double-bend achromat (DBA) | $\frac{1}{4\sqrt{15}}$ |
| Multi-bend achromat | $\frac{1}{12\sqrt{15}}\left(\frac{M+1}{M-1}\right)$ |
| TME | $\frac{1}{12\sqrt{15}}$ |

Achromats have been popular choices for storage ring lattices in third-generation synchrotron light sources for two reasons:

- they provide lower natural emittance than FODO lattices;
- they provide zero-dispersion locations appropriate for insertion devices (wigglers and undulators).

Light sources have been built using double-bend achromats (e.g. ESRF, APS, SPring-8, DIAMOND, SOLEIL) and triple-bend achromats (e.g. ALS, SLS).

# Summary (4)

Increasing the number of bends in a single cell of an achromat ("multiple-bend achromats") reduces the emittance, since the lattice functions in the "central" bends can be tuned to conditions for minimum emittance.

"Detuning" an achromat to allow some dispersion in the straights provides the possibility of further reduction in natural emittance, by moving towards the conditions for a theoretical minimum emittance (TME) lattice.