Introduction to Transverse Beam Dynamics

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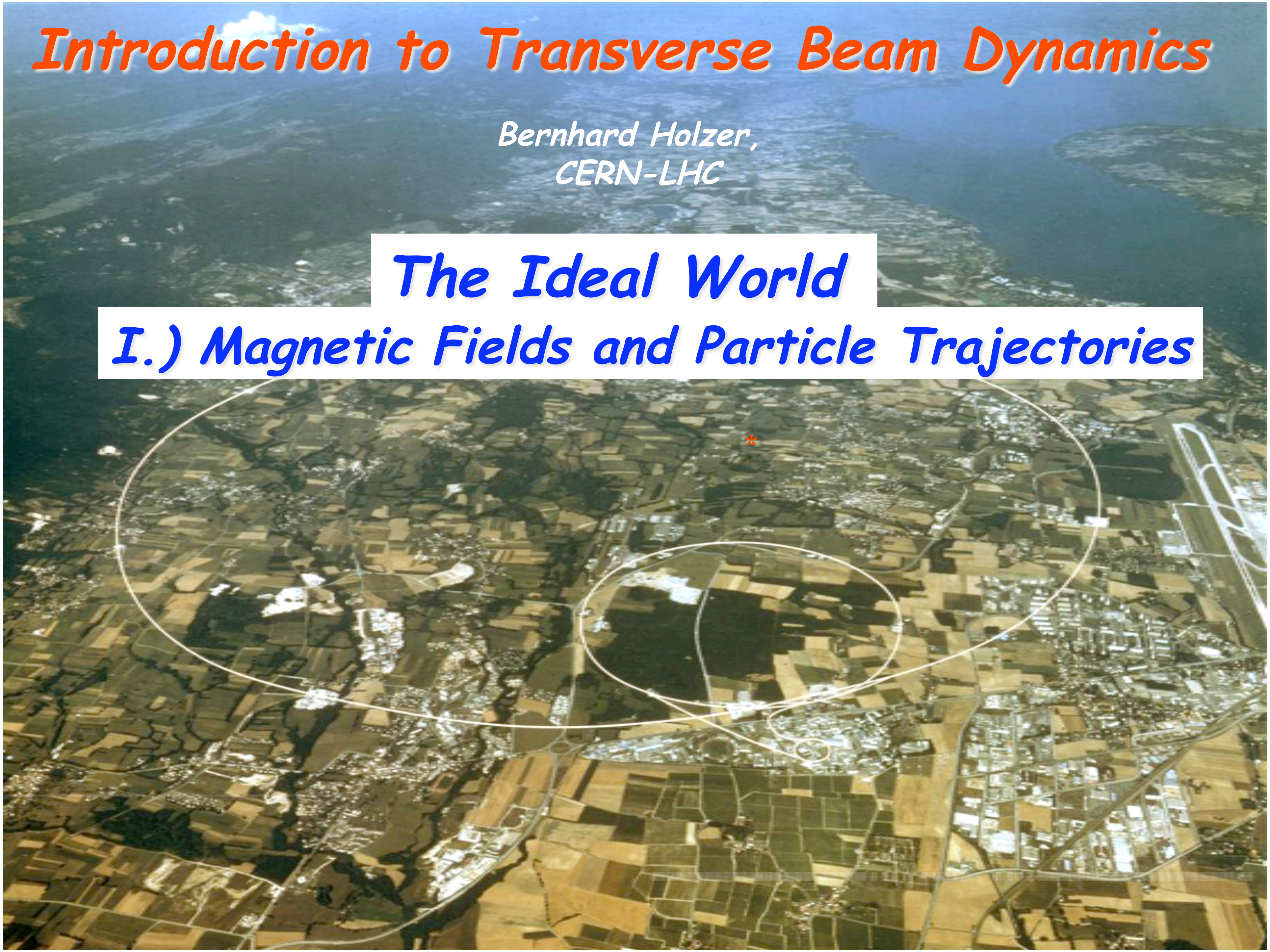
The Ideal World

I.) Magnetic Fields and Particle Trajectories



Luminosity Run of a typical storage ring:

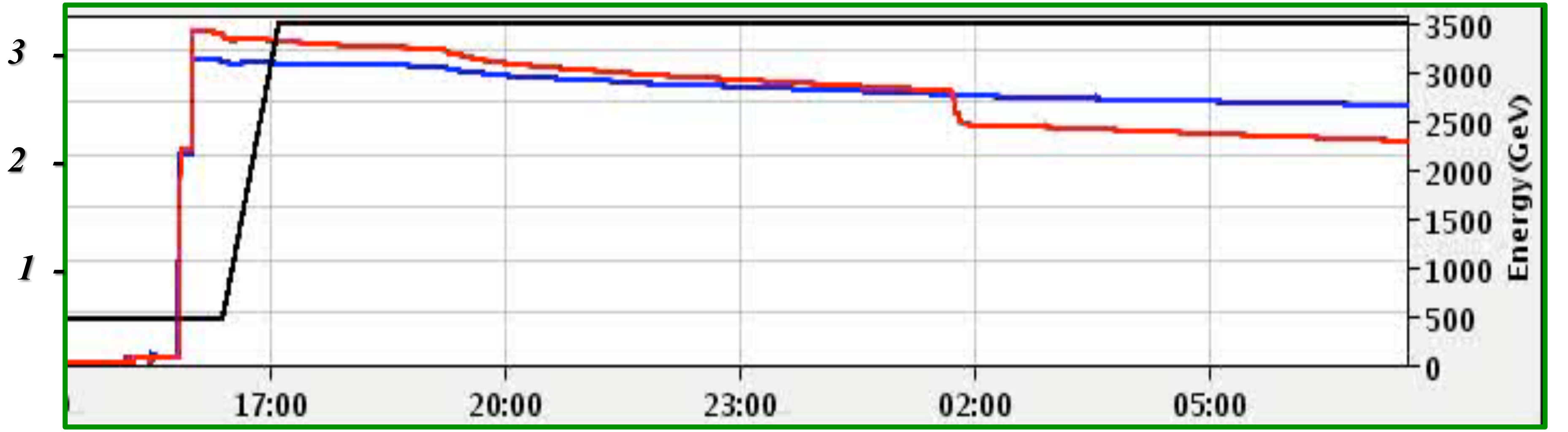
LHC Storage Ring: Protons accelerated and stored for 12 hours

distance of particles travelling at about $v \approx c$

$L = 10^{10}$-10^{11} km

... several times Sun - Pluto and back

intensity (10^{11})



→ *guide the particles on a well defined orbit („design orbit")*

→ *focus the particles to keep each single particle trajectory within the vacuum chamber of the storage ring, i.e. close to the design orbit.*

Transverse Beam Dynamics:

0.) Introduction and Basic Ideas

„ ... in the end and after all it should be a kind of circular machine“

→ need transverse deflecting force

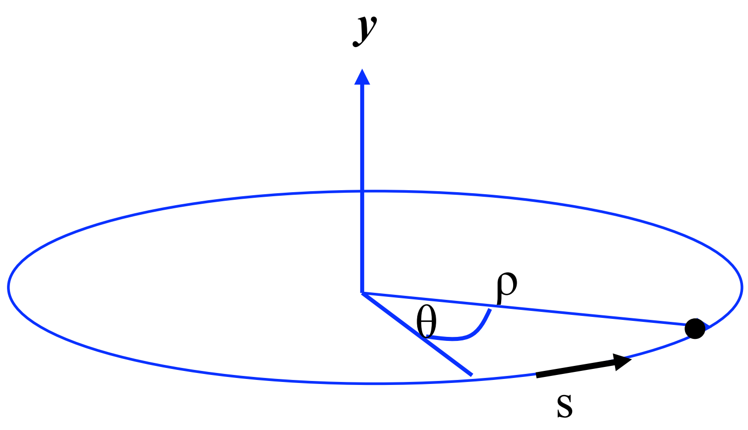
Lorentz force $\qquad \boldsymbol{F} = \boldsymbol{q}\,(\cancel{\vec{E}} + \vec{v} \times \vec{B})$

typical velocity in high energy machines: $\qquad v \approx c \approx 3 * 10^8 \,\frac{m}{s}$

old greek dictum of wisdom:
if you are clever, you use magnetic fields in an accelerator wherever it is possible.

But remember: magn. fields act allways perpendicular to the velocity of the particle
→ only bending forces, → no „beam acceleration“

The ideal circular orbit



circular coordinate system

condition for circular orbit:

Lorentz force $\quad F_L = e\, v\, B$

centrifugal force $\quad F_{centr} = \dfrac{\gamma\, m_0\, v^2}{\rho}$

$$\frac{\gamma\, m_0\, \cancel{v^2}}{\rho} = e\, \cancel{v}\, B$$

$$\boxed{\frac{p}{e} = B\,\rho}$$

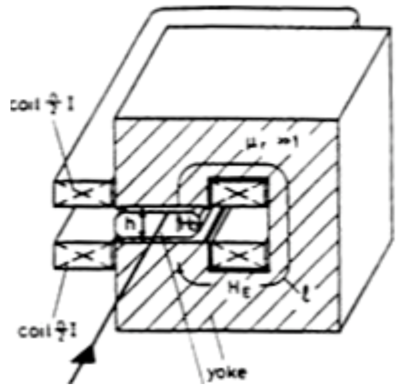
$B\,\rho =$ *"beam rigidity"*

1.) The Magnetic Guide Field

Dipole Magnets:

define the ideal orbit
homogeneous field created
by two flat pole shoes

$$B = \frac{\mu_0 \, n \, I}{h}$$

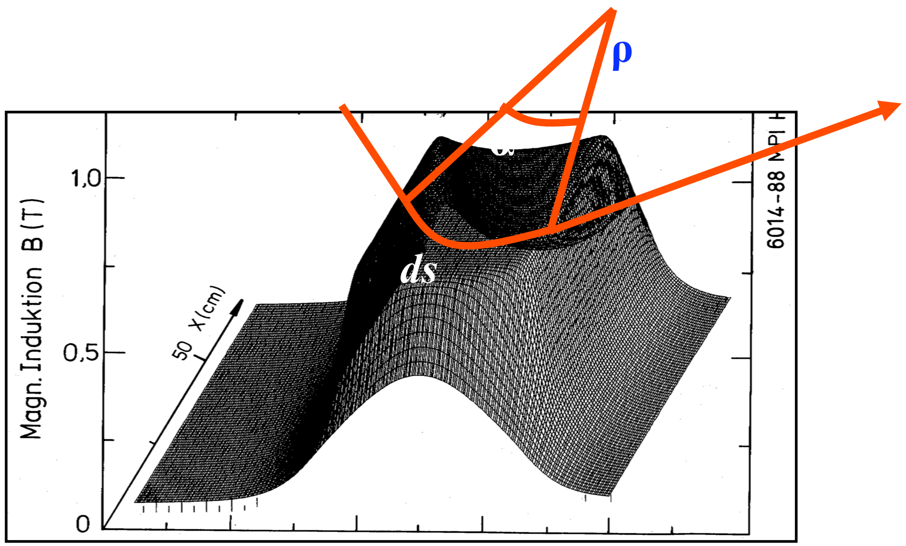


convenient units:

$$\boldsymbol{B} = [\boldsymbol{T}] = \left[\frac{\boldsymbol{Vs}}{\boldsymbol{m}^2}\right] \qquad \boldsymbol{p} = \left[\frac{\boldsymbol{GeV}}{\boldsymbol{c}}\right]$$

Example LHC:

$$\left.\begin{aligned} \boldsymbol{B} &= 8.3\,\boldsymbol{T} \\ \boldsymbol{p} &= 7000\,\frac{\boldsymbol{GeV}}{\boldsymbol{c}} \end{aligned}\right\}$$

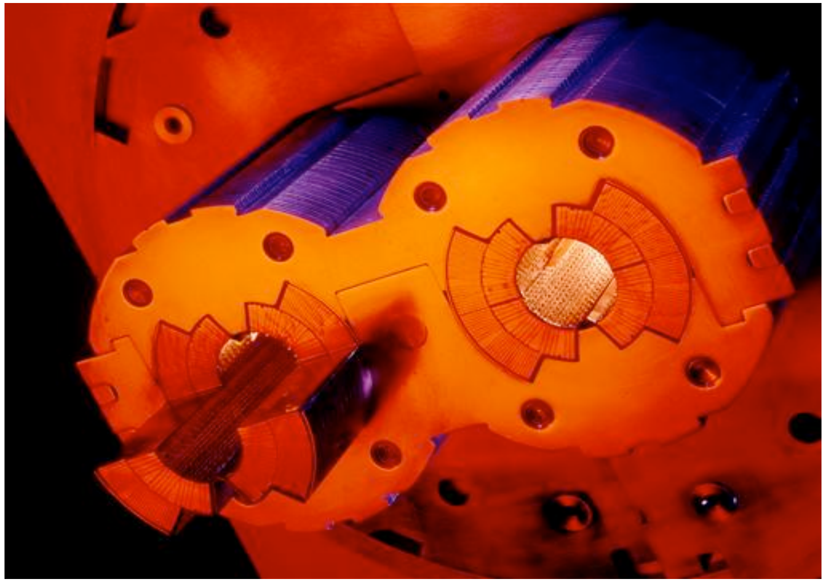


field map of a storage ring dipole magnet

Normalise magnetic field to momentum:

$$\frac{\boldsymbol{p}}{\boldsymbol{e}} = \boldsymbol{B}\,\rho \quad \longrightarrow \quad \frac{1}{\rho} = \frac{\boldsymbol{e}\,\boldsymbol{B}}{\boldsymbol{p}}$$

The Magnetic Guide Field



$$\frac{1}{\rho} = e\,\frac{8.3\ {}^{Vs}\!/_{m^2}}{7000*10^9\ {}^{eV}\!/_{c}} = \frac{8.3\,s\ \ 3*10^8\ {}^{m}\!/_{s}}{7000*10^9\,m^2}$$

$$\frac{1}{\rho} = 0.3\,\frac{8.3}{7000}\ {}^{1}\!/_{m}$$

$\rho = 2.53\ km$ ⟶ $2\pi\rho = 17.6\ km$

$\approx 66\%$

$B \approx 1 \ldots 8\ T$

rule of thumb: $\boxed{\dfrac{1}{\rho} \approx 0.3\,\dfrac{B[T]}{p[GeV/c]}}$

„normalised bending strength"

2.) Quadrupole Magnets:

required: *focusing forces to keep trajectories in vicinity of the ideal orbit*

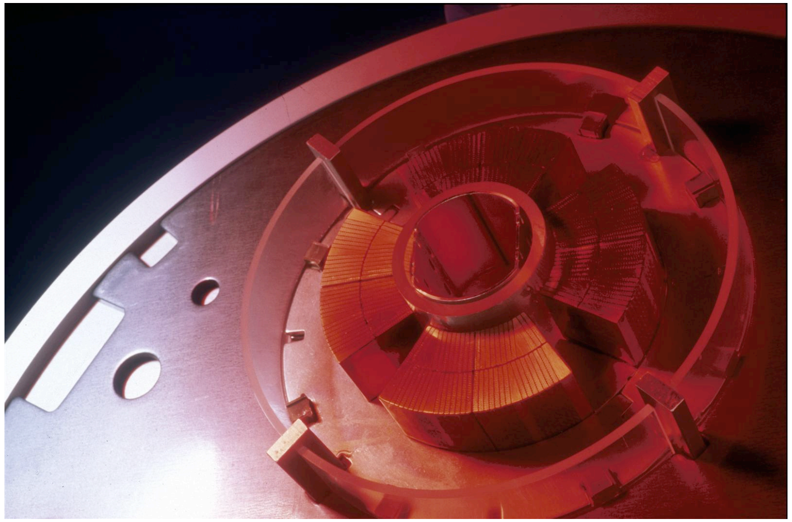
linear increasing Lorentz force

linear increasing magnetic field $B_y = g\,x \qquad B_x = g\,y$

normalised quadrupole field:

gradient of a quadrupole magnet: $$g = \frac{2\mu_0 n I}{r^2}$$

$$\longrightarrow \qquad k = \frac{g}{p/e}$$



LHC main quadrupole magnet

simple rule: $$\boxed{k = 0.3\,\frac{g(T/m)}{p(GeV/c)}}$$

$$g \approx 25 \ldots 220 \;\; T/m$$

what about the vertical plane:
... Maxwell

$$\vec{\nabla} \times \vec{B} = \cancel{\vec{j}} + \cancel{\frac{\partial \vec{E}}{\partial t}} = 0 \qquad \Rightarrow \qquad \frac{\partial B_y}{\partial x} = \frac{\partial B_x}{\partial y}$$

3.) The equation of motion:

Linear approximation:

* *ideal particle* → *design orbit*

* *any other particle* → *coordinates x, y* *small quantities*
$x,y \ll \rho$

→ *magnetic guide field: only linear terms in x & y of B have to be taken into account*

Taylor Expansion of the B field:

$$B_y(x) = B_{y0} + \frac{dB_y}{dx}x + \frac{1}{2!}\frac{d^2B_y}{dx^2}x^2 + \frac{1}{3!}\frac{eg''}{dx^3} + \dots$$

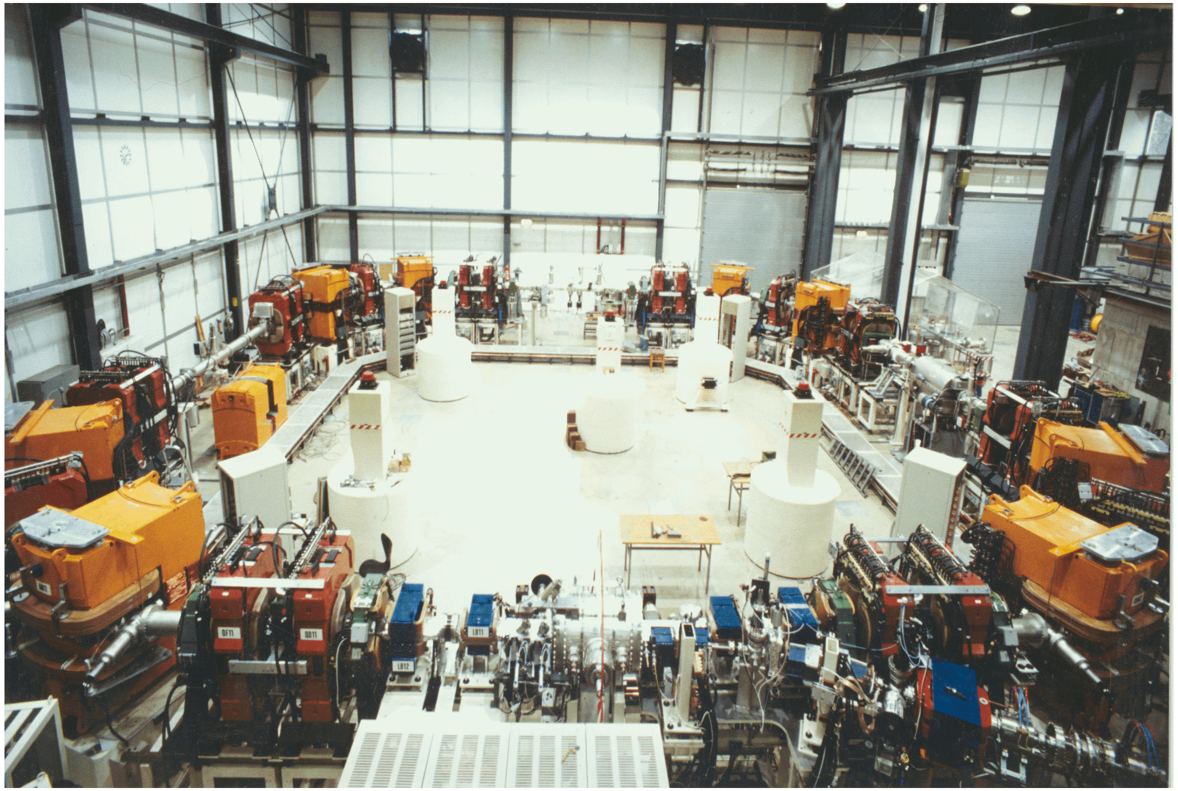
normalise to momentum
$p/e = B\rho$

$$\frac{B(x)}{p/e} = \frac{B_0}{B_0\rho} + \frac{g*x}{p/e} + \frac{1}{2!}\frac{eg'}{p/e} + \frac{1}{3!}\frac{eg''}{p/e} + \dots$$

The Equation of Motion:

$$\frac{B(x)}{p/e} = \frac{1}{\rho} + k\,x + \cancel{\frac{1}{2!}\,m\,x^2} + \cancel{\frac{1}{3!}\,n\,x^3} + \ldots$$

only terms linear in x, y taken into account   *dipole fields*
quadrupole fields



Separate Function Machines:

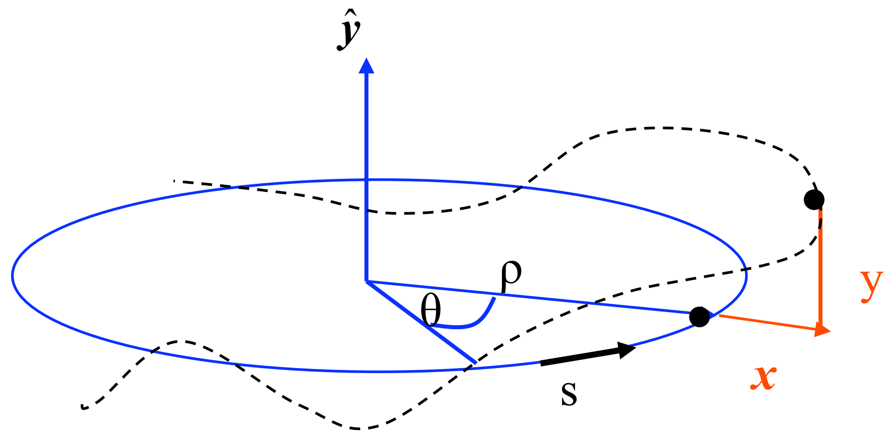
Split the magnets and optimise them according to their job:

bending, focusing etc

Example:
heavy ion storage ring TSR

* *man sieht nur dipole und quads → linear*

Equation of Motion:



Consider local segment of a particle trajectory
... and remember the old days:
(Goldstein page 27)

radial acceleration:

$$a_r = \frac{d^2\rho}{dt^2} - \rho\left(\frac{d\theta}{dt}\right)^2$$

Ideal orbit: $\rho = const, \quad \frac{d\rho}{dt} = 0$

Force: $F = m\rho\left(\frac{d\theta}{dt}\right)^2 = m\rho\omega^2$

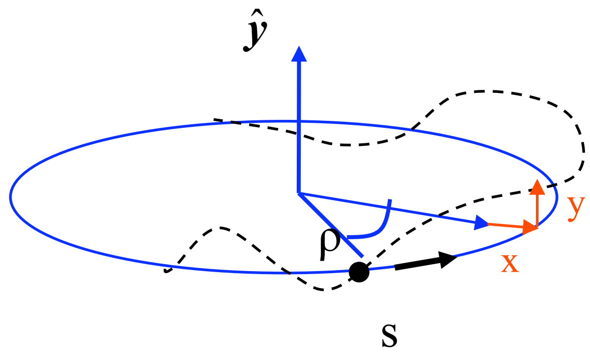
$$F = mv^2 / \rho$$

general trajectory: $\rho \rightarrow \rho + x$

$$\boldsymbol{F = m\frac{d^2}{dt^2}(x+\rho) - \frac{mv^2}{x+\rho} = e\,B_y\,v}$$

$$F = m\frac{d^2}{dt^2}(x+\rho) - \frac{mv^2}{x+\rho} = e\,B_y\,v$$

1 2



1 $$\frac{d^2}{dt^2}(x+\rho) = \frac{d^2}{dt^2}x$$ *... as ρ = const*

2 ***remember:*** $x \approx mm$ *,* $\rho \approx m$ *... →* ***develop for small x***

$$\frac{1}{x+\rho} \approx \frac{1}{\rho}(1-\frac{x}{\rho})$$

Taylor Expansion

$$f(x) = f(x_0) + \frac{(x-x_0)}{1!}f'(x_0) + \frac{(x-x_0)^2}{2!}f''(x_0) + \quad \dots$$

$$m\frac{d^2x}{dt^2} - \frac{mv^2}{\rho}(1-\frac{x}{\rho}) = eB_y v$$

guide field in linear approx.

$$B_y = B_0 + x\frac{\partial B_y}{\partial x} \qquad\qquad m\frac{d^2x}{dt^2} - \frac{mv^2}{\rho}(1-\frac{x}{\rho}) = ev\left\{B_0 + x\frac{\partial B_y}{\partial x}\right\} \qquad \Big| \; : m$$

$$\frac{d^2x}{dt^2} - \frac{v^2}{\rho}(1-\frac{x}{\rho}) = \frac{e\,v\,B_0}{m} + \frac{e\,v\,x\,g}{m}$$

independent variable: $t \rightarrow s$

$$\frac{dx}{dt} = \frac{dx}{ds}\frac{ds}{dt}$$

$$\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{ds}\frac{ds}{dt}\right) = \frac{d}{ds}\left(\underbrace{\frac{dx}{ds}}_{x'}\underbrace{\frac{ds}{dt}}_{v}\right)\frac{ds}{dt}$$

$$\frac{d^2x}{dt^2} = x''\,v^2 + \cancel{\frac{dx}{ds}\frac{dv}{ds}}\,v$$

$$x''v^2 - \frac{v^2}{\rho}(1-\frac{x}{\rho}) = \frac{e\,v\,B_0}{m} + \frac{e\,v\,x\,g}{m} \qquad \Big| \; : v^2$$

$$x'' - \frac{1}{\rho}\left(1 - \frac{x}{\rho}\right) = \frac{e\,B_0}{mv} + \frac{e\,x\,g}{mv}$$

$m\,v = p$

$$x'' - \frac{1}{\rho} + \frac{x}{\rho^2} = \frac{B_0}{p/e} + \frac{x\,g}{p/e}$$

normalize to momentum of particle

$$\frac{B_0}{p/e} = -\frac{1}{\rho}$$

$$\frac{g}{p/e} = k$$

$$x'' - \cancel{\frac{1}{\rho}} + \frac{x}{\rho^2} = -\cancel{\frac{1}{\rho}} + k\,x$$

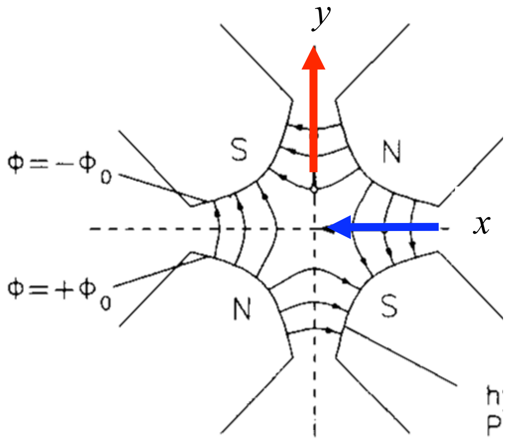
$$\boxed{x'' + x\left(\frac{1}{\rho^2} - k\right) = 0}$$

* *Equation for the vertical motion:*

$\frac{1}{\rho^2} = 0$ *no dipoles … in general …*

$k \leftrightarrow -k$ *quadrupole field changes sign*

$$y'' + k\,y = 0$$



Remarks:

* $$x'' + \left(\frac{1}{\rho^2} - k\right) \cdot x = 0$$

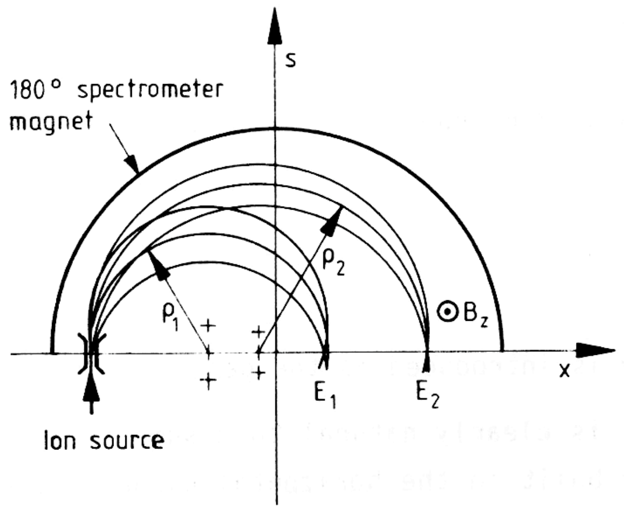
… there seems to be a focusing even without a quadrupole gradient

„weak focusing of dipole magnets“

$$\boldsymbol{k} = 0 \quad \Rightarrow \quad \boldsymbol{x}'' = -\frac{1}{\rho^2}\,\boldsymbol{x}$$

even without quadrupoles there is a retriving force (i.e. focusing) in the bending plane of the dipole magnets

… in large machines it is weak. ***(!)***



Mass spectrometer: particles are separated according to their energy and focused due to the 1/ρ effect of the dipole

* *Hard Edge Model:*

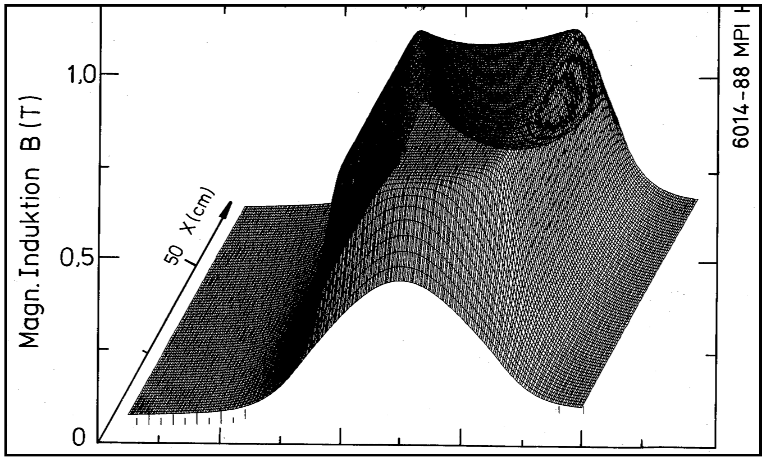
$$x'' + \left\{\frac{1}{\rho^2} - k\right\} x = 0$$

… this equation is not correct !!!

$$x''(s) + \left\{\frac{1}{\rho^2(s)} - k(s)\right\} x(s) = 0$$

bending and focusing fields … are functions of the independent variable „s“

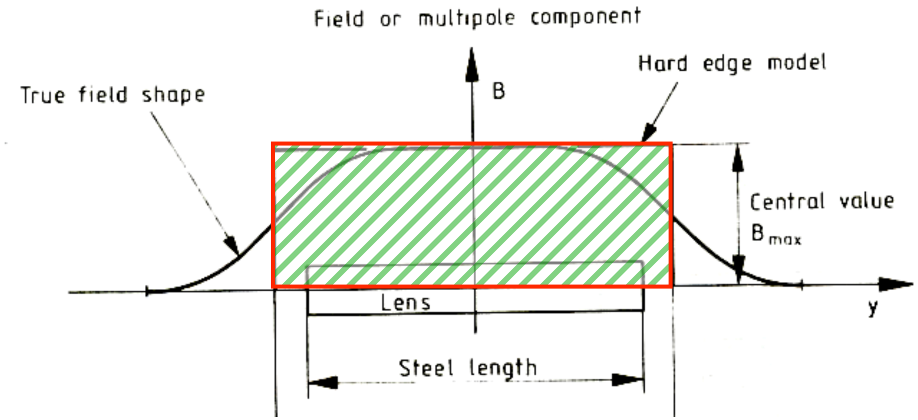
!



Inside a magnet we assume constant focusing properties !

$$\frac{1}{\rho} = const \qquad k = const$$

$$B\, l_{eff} = \int_0^{l_{mag}} B\, ds$$



4.) Solution of Trajectory Equations

Define … hor. plane: $K = 1/\rho^2 - k$

… vert. Plane: $K = k$

$$\boldsymbol{x''} + \boldsymbol{K}\ \boldsymbol{x} = 0$$

Differential Equation of harmonic oscillator … with spring constant K

Ansatz: $x(s) = a_1 \cdot \cos(\omega s) + a_2 \cdot \sin(\omega s)$

general solution: linear combination of two independent solutions

$$x'(s) = -a_1\omega \sin(\omega s) + a_2\omega \cos(\omega s)$$

$$x''(s) = -a_1\omega^2 \cos(\omega s) - a_2\omega^2 \sin(\omega s) = -\omega^2 x(s) \longrightarrow \omega = \sqrt{K}$$

general solution:

$$x(s) = a_1 \cos(\sqrt{K}s) + a_2 \sin(\sqrt{K}s)$$

determine a_1, a_2 by boundary conditions:

$$s = 0 \longrightarrow \begin{cases} x(0) = x_0 \quad , \quad a_1 = x_0 \\ x'(0) = x'_0 \quad , \quad a_2 = \dfrac{x'_0}{\sqrt{K}} \end{cases}$$

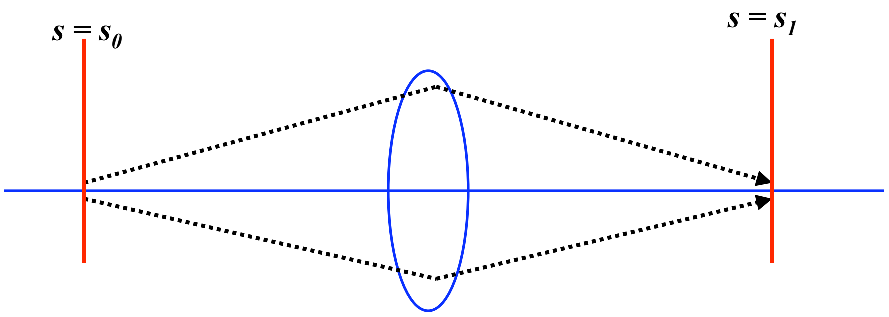
Hor. Focusing Quadrupole $K > 0$:

$$x(s) = x_0 \cdot \cos(\sqrt{|K|}s) + x'_0 \cdot \frac{1}{\sqrt{|K|}} \sin(\sqrt{|K|}s)$$

$$x'(s) = -x_0 \cdot \sqrt{|K|} \cdot \sin(\sqrt{|K|}s) + x'_0 \cdot \cos(\sqrt{|K|}s)$$

For convenience expressed in matrix formalism:

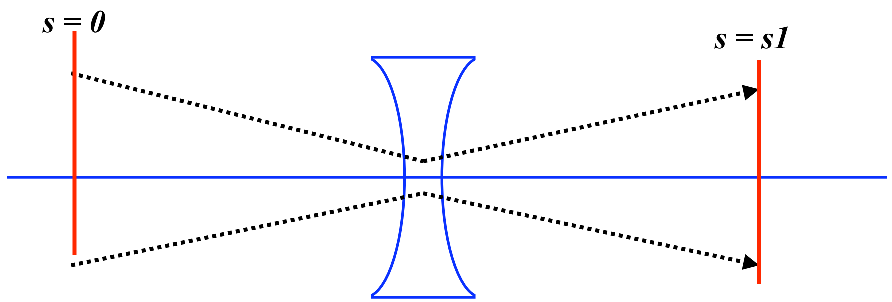
$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s1} = M_{foc} * \begin{pmatrix} x \\ x' \end{pmatrix}_{s0}$$



$$M_{foc} = \begin{pmatrix} \cos(\sqrt{|K|}s) & \dfrac{1}{\sqrt{|K|}} \sin(\sqrt{|K|}s \\ -\sqrt{|K|} \sin(\sqrt{|K|}s) & \cos(\sqrt{|K|}s) \end{pmatrix}_0$$

hor. defocusing quadrupole:

$$x'' - K\,x = 0$$



Remember from school:

$$f(s) = \cosh(s) \quad , \qquad f'(s) = \sinh(s)$$

Ansatz: $$x(s) = a_1 \cdot \cosh(\omega s) + a_2 \cdot \sinh(\omega s)$$

$$M_{def\ oc} = \begin{pmatrix} \cosh \sqrt{|K|}\,l & \frac{1}{\sqrt{|K|}} \sinh \sqrt{|K|}\,l \\ \sqrt{|K|} \sinh \sqrt{|K|}\,l & \cosh \sqrt{|K|}\,l \end{pmatrix}$$

drift space:

K = 0

$$M_{drif\ t} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$$

! with the assumptions made, the motion in the horizontal and vertical planes are independent „ ... the particle motion in x & y is uncoupled“

Thin Lens Approximation:

matrix of a quadrupole lens

$$M = \begin{pmatrix} \cos\sqrt{|k|}l & \frac{1}{\sqrt{|k|}}\sin\sqrt{|k|}l \\ -\sqrt{|k|}\sin\sqrt{|k|}l & \cos\sqrt{|k|}l \end{pmatrix}$$

in many practical cases we have the situation:

$$f = \frac{1}{kl_q} >> l_q$$

... focal length of the lens is much bigger than the length of the magnet

limes: $l_q \rightarrow 0$ ***while keeping*** $k\, l_q = const$

$$M_x = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix} \qquad M_z = \begin{pmatrix} 1 & 0 \\ \frac{-1}{f} & 1 \end{pmatrix}$$

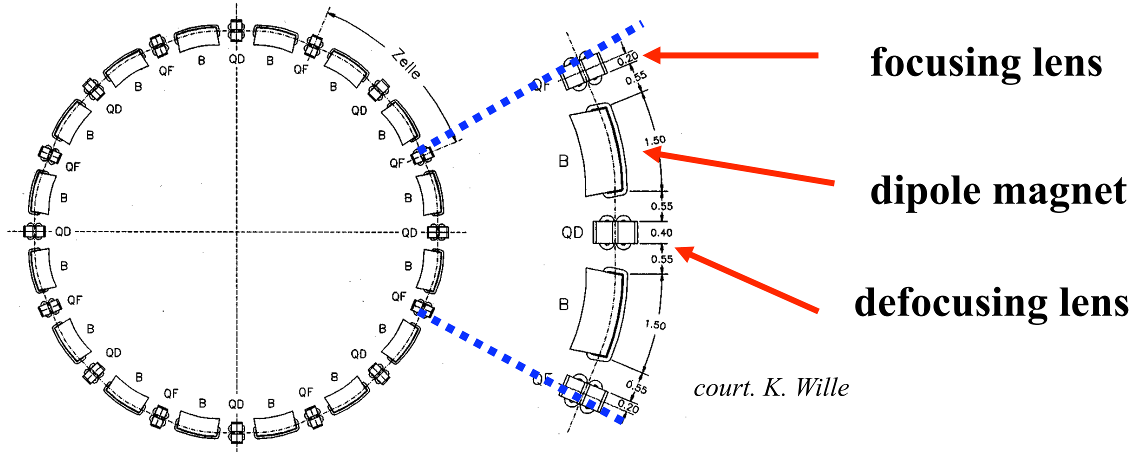
... useful for fast (and in large machines still quite accurate) „back on the envelope calculations“ ... and for the guided studies !

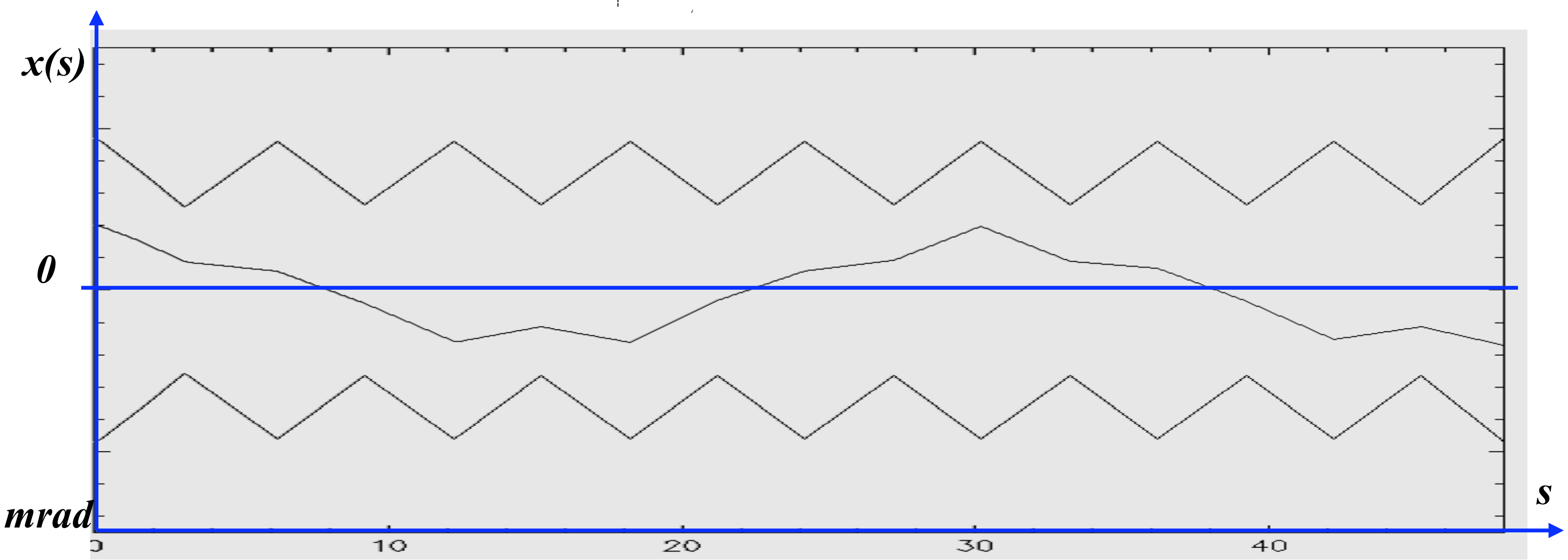
Transformation through a system of lattice elements

combine the single element solutions by multiplication of the matrices

$$M_{total} = M_{QF} * M_D * M_{QD} * M_{Bend} * M_{D*.....}$$

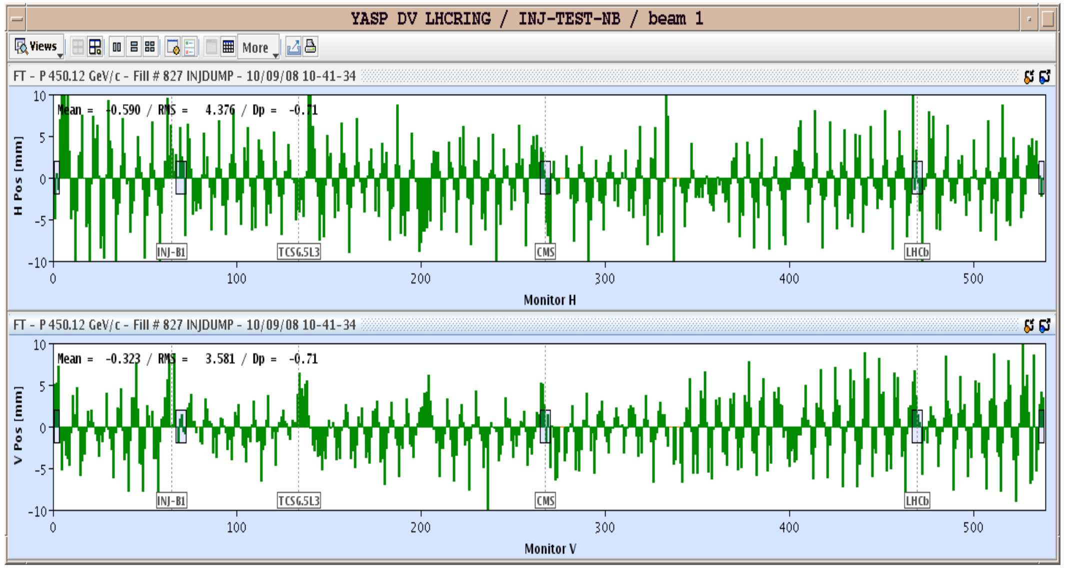
$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s2} = M(s2,s1) * \begin{pmatrix} x \\ x' \end{pmatrix}_{s1}$$





typical values
in a strong
foc. machine:
x ≈ mm, x´ ≤ mrad

5.) Orbit & Tune:



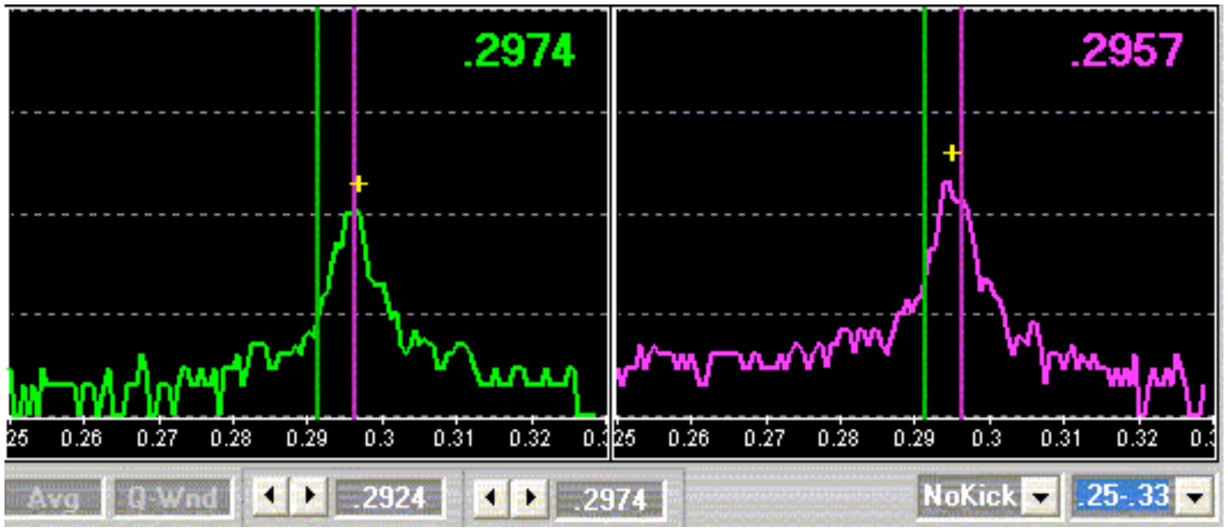
Tune: number of oscillations per turn

64.31
59.32

Relevant for beam stability:
non integer part

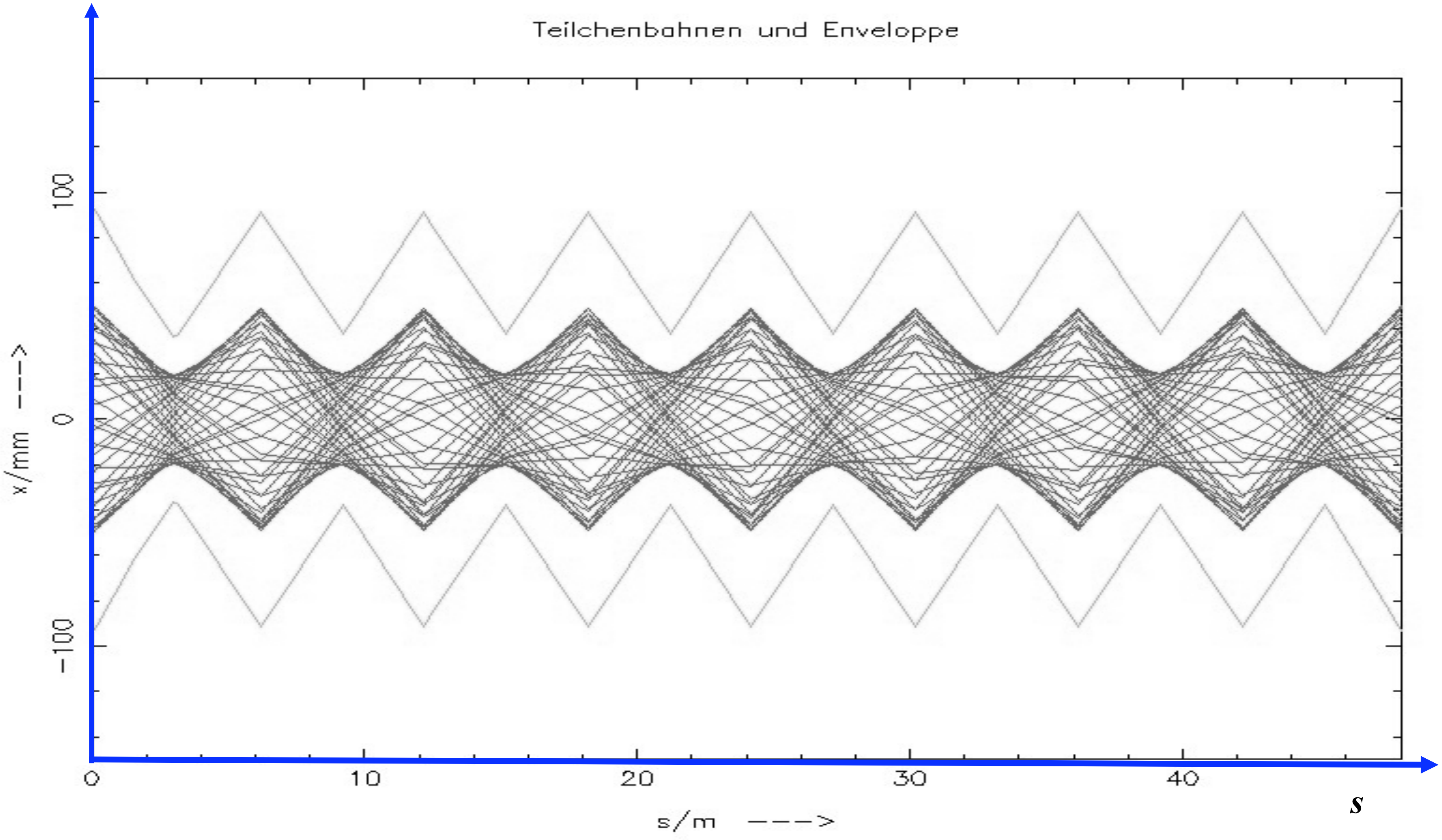
LHC revolution frequency: 11.3 kHz

$$0.31 * 11.3\, kHz = 3.5\, kHz$$



***Question:* what will happen, if the particle performs a second turn ?**

... or a third one or ... 10^{10} turns



Astronomer Hill:

differential equation for motions with periodic focusing properties
„Hill's equation"

Example: particle motion with periodic coefficient

equation of motion: $$x''(s) - k(s)x(s) = 0$$

restoring force $\neq$ ***const,***
k(s) = depending on the position s
k(s+L) = k(s), periodic function

we expect a kind of quasi harmonic oscillation: amplitude & phase will depend on the position s in the ring.

6.) The Beta Function

General solution of Hill´s equation:

(i) $$x(s) = \sqrt{\varepsilon}\,\sqrt{\beta(s)} \cdot \cos(\psi(s) + \phi)$$

ε, Φ = integration constants determined by initial conditions

β(s) periodic function given by focusing properties of the lattice ↔ quadrupoles

$$\beta(s+L) = \beta(s)$$

Inserting (i) into the equation of motion …

$$\psi(s) = \int_0^s \frac{ds}{\beta(s)}$$

Ψ(s) = „phase advance" of the oscillation between point „0" and „s" in the lattice.
For one complete revolution: number of oscillations per turn „Tune"

$$Q_y = \frac{1}{2\pi} \oint \frac{ds}{\beta(s)}$$

7.) Beam Emittance and Phase Space Ellipse

general solution of Hill equation

$$\begin{cases} (1) \quad x(s) = \sqrt{\varepsilon}\,\sqrt{\beta(s)}\,\cos(\psi(s)+\phi) \\ (2) \quad x'(s) = -\dfrac{\sqrt{\varepsilon}}{\sqrt{\beta(s)}}\,\{\alpha(s)\cos(\psi(s)+\phi)+\sin(\psi(s)+\phi)\} \end{cases}$$

from (1) we get

$$\cos(\psi(s)+\phi) = \frac{x(s)}{\sqrt{\varepsilon}\,\sqrt{\beta(s)}}$$

$$\alpha(s) = \frac{-1}{2}\beta'(s)$$

$$\gamma(s) = \frac{1+\alpha(s)^2}{\beta(s)}$$

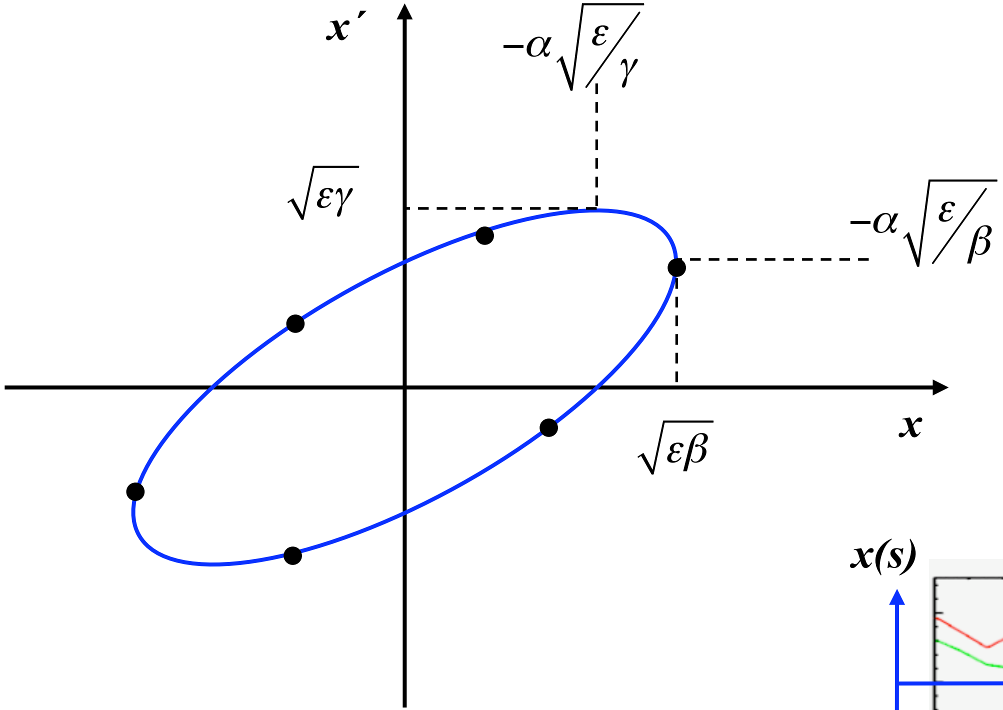
Insert into (2) and solve for ε

$$\varepsilon = \gamma(s)\,x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)\,x'^2(s)$$

**** ε is a constant of the motion … it is independent of „s“***
**** parametric representation of an ellipse in the x x‘ space***
**** shape and orientation of ellipse are given by α, β, γ***

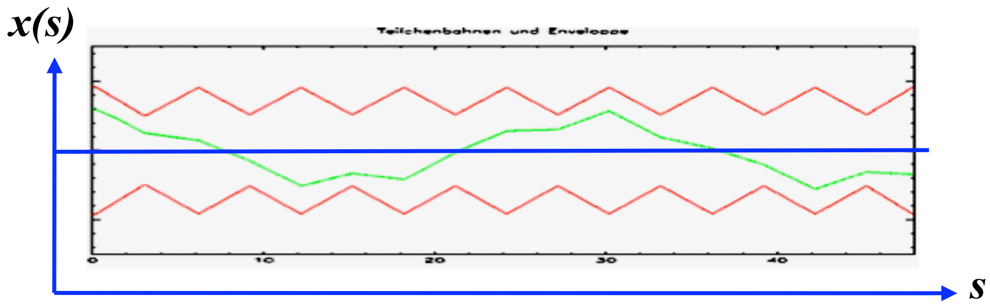
Beam Emittance and Phase Space Ellipse

$$\varepsilon = \gamma(s)\, x^2(s) + 2\alpha(s) x(s) x'(s) + \beta(s)\, x'^2(s)$$



***Liouville:* in reasonable storage rings**
area in phase space is constant.

A = π*ε=const



ε beam emittance = woozilycity of the particle ensemble, intrinsic beam parameter,
cannot be changed by the foc. properties.
Scientifiquely speaking: area covered in transverse x, x´ phase space … and it is constant !!!

Phase Space Ellipse

particel trajectory: $x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}\cos\{\psi(s)+\phi\}$

max. Amplitude: $\hat{x}(s) = \sqrt{\varepsilon\beta}$ $\longrightarrow$ *x´ at that position …?*

… put $\hat{x}(s)$ *into* $\varepsilon = \gamma(s)\,x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)\,x'^2(s)$ *and solve for x´*

$$\varepsilon = \gamma\cdot\varepsilon\beta + 2\alpha\sqrt{\varepsilon\beta}\cdot x' + \beta x'^2$$

$$\longrightarrow \quad \boxed{x' = -\alpha\cdot\sqrt{\varepsilon/\beta}}$$

* *A high β-function means a large beam size and a small beam divergence.* *!*
… et vice versa !!!

* *In the middle of a quadrupole β = maximum,*
α = zero

$$\left.\begin{array}{l}\beta = \text{maximum} \\ \alpha = \text{zero}\end{array}\right\} \quad x' = 0$$

… and the ellipse is flat

Phase Space Ellipse

$$\varepsilon = \gamma(s)\, x^2(s) + 2\alpha(s) x(s) x'(s) + \beta(s)\, x'^2(s)$$

$$\alpha(s) = \frac{-1}{2}\beta'(s)$$

$$\gamma(s) = \frac{1+\alpha(s)^2}{\beta(s)}$$

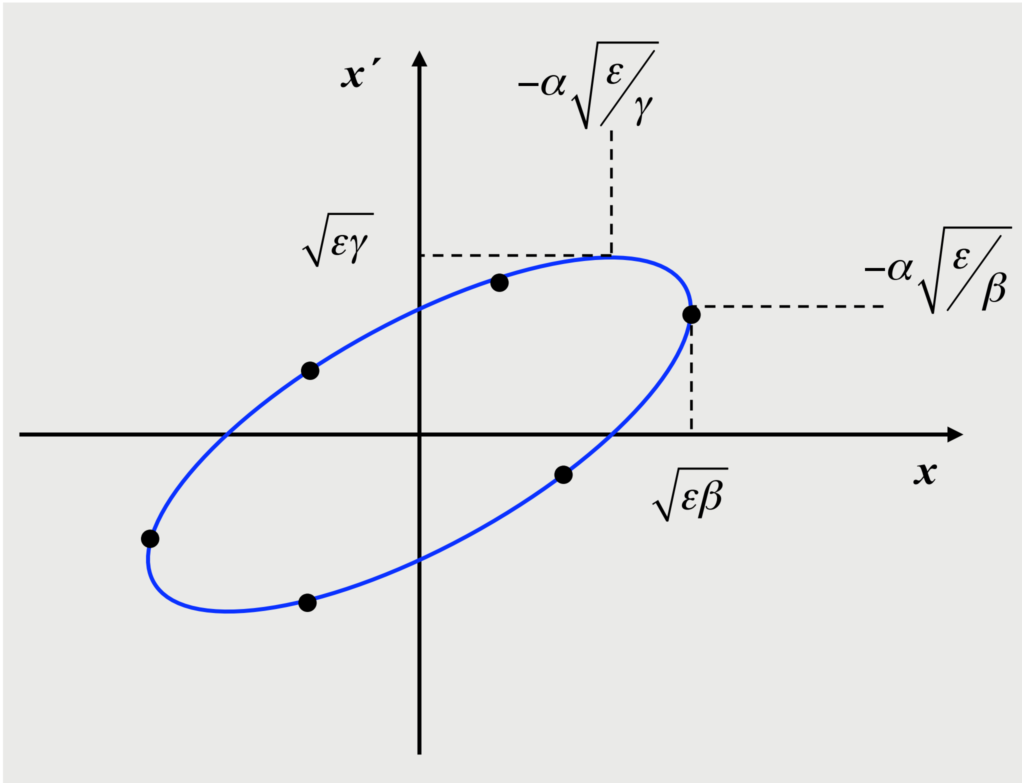
$$\longrightarrow \quad \varepsilon = \frac{x^2}{\beta} + \frac{\alpha^2 x^2}{\beta} + 2\alpha \cdot xx' + \beta \cdot x'^2$$

… solve for x´ $\quad x'_{1,2} = \dfrac{-\alpha \cdot x \pm \sqrt{\varepsilon\beta - x^2}}{\beta}$

… and determine $\hat{x}'$ ***via:*** $\quad \dfrac{dx'}{dx} = 0$

$$\longrightarrow \quad \boxed{\hat{x}' = \sqrt{\varepsilon\gamma}}$$

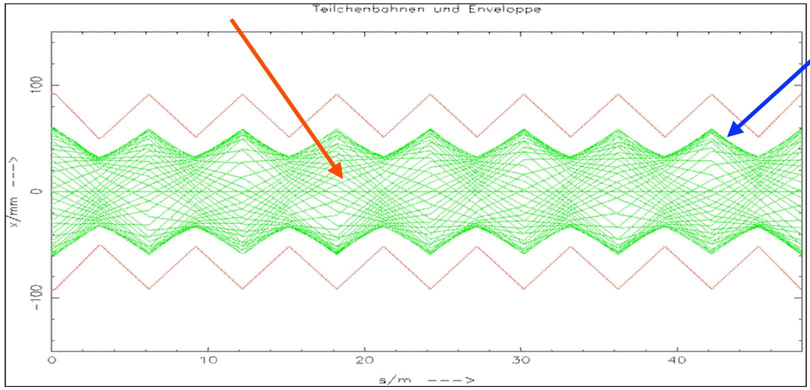
$$\longrightarrow \quad \boxed{\hat{x} = \pm\alpha\sqrt{\varepsilon/\gamma}}$$



shape and orientation of the phase space ellipse depend on the Twiss parameters β α γ

Emittance of the Particle Ensemble:

$$x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)} \cdot \cos(\Psi(s) + \phi) \qquad \hat{x}(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}$$

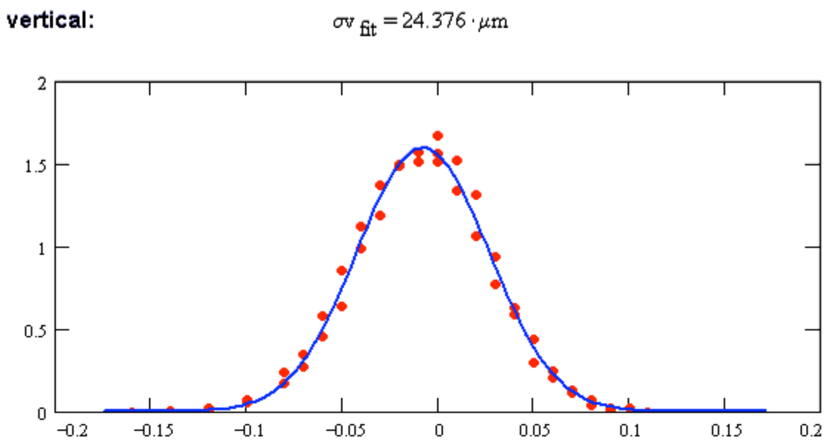


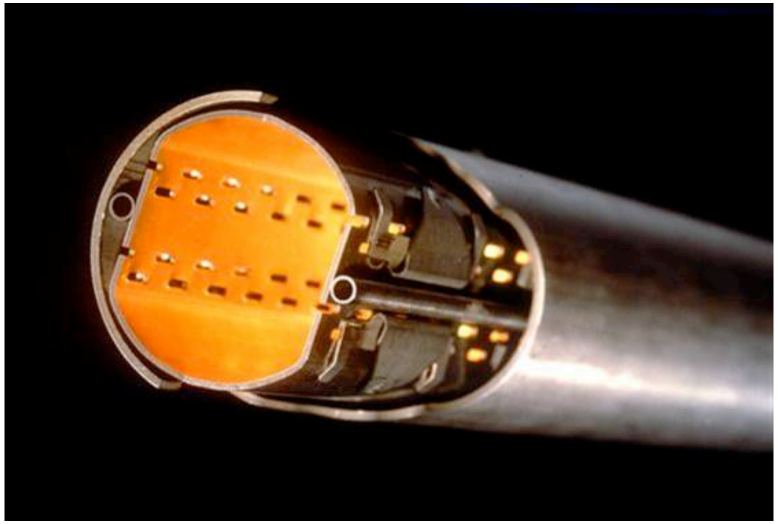
single particle trajectories, N ≈ 10 ^{11} per bunch

Gauß Particle Distribution:

$$\rho(x) = \frac{N \cdot e}{\sqrt{2\pi}\sigma_x} \cdot e^{-\frac{1}{2}\frac{x^2}{\sigma_x^2}}$$

particle at distance 1 σ from centre ↔ 68.3 % of all beam particles





LHC: $\sigma = \sqrt{\varepsilon * \beta} = \sqrt{5 * 10^{-10} m * 180 m} = 0.3\ mm$

aperture requirements: $r_0 = 10 * \sigma$

8.) Transfer Matrix M

… yes we had the topic already

general solution of Hill´s equation

$$\left\{\begin{aligned} x(s) &= \sqrt{\varepsilon}\sqrt{\beta(s)}\cos\{\psi(s)+\phi\} \\ x'(s) &= \frac{-\sqrt{\varepsilon}}{\sqrt{\beta(s)}}\left[\alpha(s)\cos\{\psi(s)+\phi\}+\sin\{\psi(s)+\phi\}\right] \end{aligned}\right.$$

remember the trigonometrical gymnastics: sin(a + b) = … etc

$$x(s) = \sqrt{\varepsilon}\sqrt{\beta_s}\left(\cos\psi_s\cos\phi - \sin\psi_s\sin\phi\right)$$

$$x'(s) = \frac{-\sqrt{\varepsilon}}{\sqrt{\beta_s}}\left[\alpha_s\cos\psi_s\cos\phi - \alpha_s\sin\psi_s\sin\phi + \sin\psi_s\cos\phi + \cos\psi_s\sin\phi\right]$$

starting at point $s(0) = s_0$, where we put $\Psi(0) = 0$

$$\left.\begin{aligned} \cos\phi &= \frac{x_0}{\sqrt{\varepsilon\beta_0}} \quad , \\ \sin\phi &= -\frac{1}{\sqrt{\varepsilon}}\left(x_0'\sqrt{\beta_0} + \frac{\alpha_0 x_0}{\sqrt{\beta_0}}\right) \end{aligned}\right\}$$

inserting above …

$$x(s) = \sqrt{\frac{\beta_s}{\beta_0}}\left\{\cos\psi_s + \alpha_0 \sin\psi_s\right\} x_0 + \left\{\sqrt{\beta_s\beta_0}\,\sin\psi_s\right\} x'_0$$

$$x'(s) = \frac{1}{\sqrt{\beta_s\beta_0}}\left\{(\alpha_0 - \alpha_s)\cos\psi_s - (1+\alpha_0\alpha_s)\sin\psi_s\right\} x_0 + \sqrt{\frac{\beta_0}{\beta_s}}\left\{\cos\psi_s - \alpha_s \sin\psi_s\right\} x'_0$$

which can be expressed ... for convenience ... in matrix form

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M \begin{pmatrix} x \\ x' \end{pmatrix}_0$$

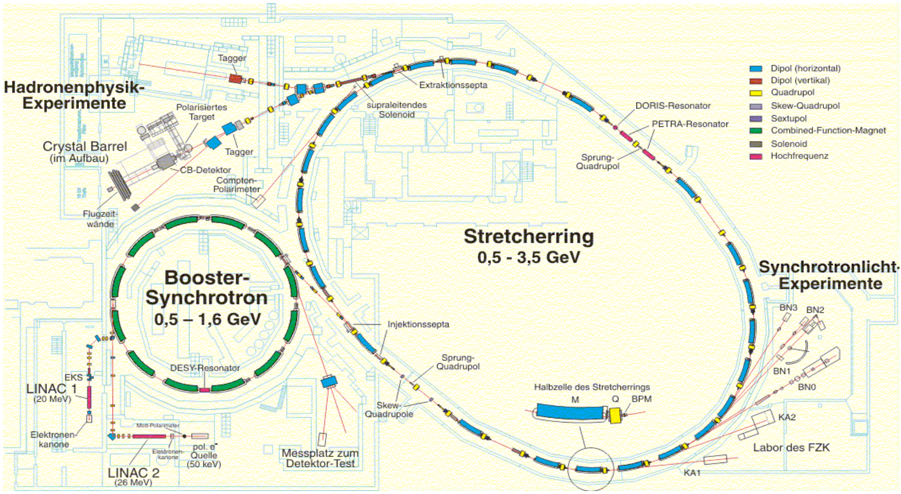
$$M = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}\left(\cos\psi_s + \alpha_0 \sin\psi_s\right) & \sqrt{\beta_s\beta_0}\,\sin\psi_s \\ \frac{(\alpha_0 - \alpha_s)\cos\psi_s - (1+\alpha_0\alpha_s)\sin\psi_s}{\sqrt{\beta_s\beta_0}} & \sqrt{\frac{\beta_0}{\beta s}}\left(\cos\psi_s - \alpha_s \sin\psi_s\right) \end{pmatrix}$$

* ***we can calculate the single particle trajectories between two locations in the ring, if we know the α β γ at these positions.***
* ***and nothing but the α β γ at these positions.***
* ***... !***

*** *Äquivalenz der Matrizen***

9.) Periodic Lattices

$$M = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}\left(\cos\psi_s + \alpha_0 \sin\psi_s\right) & \sqrt{\beta_s\beta_0}\,\sin\psi_s \\ \frac{(\alpha_0 - \alpha_s)\cos\psi_s - (1+\alpha_0\alpha_s)\sin\psi_s}{\sqrt{\beta_s\beta_0}} & \sqrt{\frac{\beta_0}{\beta s}}\left(\cos\psi_s - \alpha_s \sin\psi_s\right) \end{pmatrix}$$



DELTA Electron Storage Ring

„This rather formidable looking matrix simplifies considerably if we consider one complete revolution …"

$$\boldsymbol{M(s)} = \begin{pmatrix} \cos\psi_{turn} + \alpha_s \sin\psi_{turn} & \beta_s \sin\psi_{turn} \\ -\gamma_s \sin\psi_{turn} & \cos\psi_{turn} - \alpha_s \sin\psi_{turn} \end{pmatrix}$$

$$\psi_{turn} = \int_s^{s+L} \frac{ds}{\beta(s)}$$

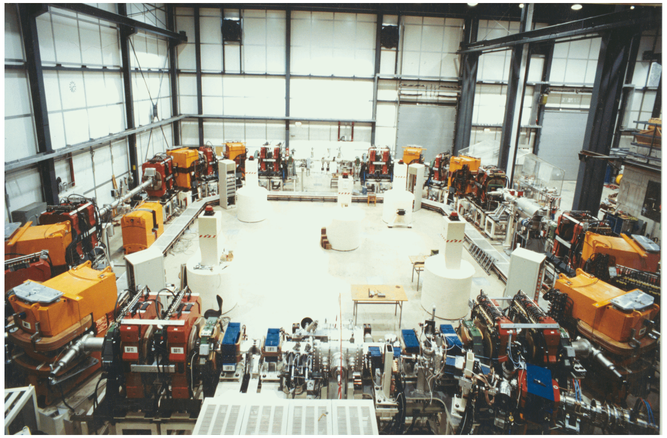
ψ_{turn} = phase advance per period

***Tune:* Phase advance per turn in units of 2π**

$$\boldsymbol{Q} = \frac{1}{2\pi}\oint \frac{\boldsymbol{ds}}{\beta(s)}$$

Stability Criterion:

Question: ***what will happen, if we do not make too many mistakes and your particle performs one complete turn ?***



Matrix for 1 turn:

$$M = \begin{pmatrix} \cos\psi_{turn} + \alpha_s \sin\psi_{turn} & \beta_s \sin\psi_{turn} \\ -\gamma_s \sin\psi_{turn} & \cos\psi_{turn} - \alpha_s \sin\psi_{turn} \end{pmatrix} = \cos\psi \cdot \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\mathbf{1}} + \sin\psi \underbrace{\begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}}_{\mathbf{J}}$$

Matrix for N turns:

$$M^N = \left(1 \cdot \cos\psi + J \cdot \sin\psi\right)^N = 1 \cdot \cos N\psi + J \cdot \sin N\psi$$

The motion for N turns remains bounded, if the elements of M^N remain bounded

$$\psi = real \quad \leftrightarrow \quad |\cos\psi| \le 1 \quad \leftrightarrow \quad Tr(M) \le 2$$

stability criterion …. proof for the disbelieving collegues !!

Matrix for 1 turn:

$$M=\begin{pmatrix}\cos\psi_{turn}+\alpha_s\sin\psi_{turn} & \beta_s\sin\psi_{turn}\\ -\gamma_s\sin\psi_{turn} & \cos\psi_{turn}-\alpha_s\sin\psi_{turn}\end{pmatrix} \;=\; \cos\psi\cdot\underbrace{\begin{pmatrix}1&0\\0&1\end{pmatrix}}_{\boldsymbol{I}} + \sin\psi\underbrace{\begin{pmatrix}\alpha&\beta\\-\gamma&-\alpha\end{pmatrix}}_{\boldsymbol{J}}$$

Matrix for 2 turns:

$$\boldsymbol{M}^2=(\boldsymbol{I}\cos\psi_1+\boldsymbol{J}\sin\psi_1)(\boldsymbol{I}\cos\psi_2+\boldsymbol{J}\sin\psi_2)$$

$$=\boldsymbol{I}^2\cos\psi_1\cos\psi_2+\boldsymbol{IJ}\cos\psi_1\sin\psi_2+\boldsymbol{JI}\sin\psi_1\cos\psi_2+\boldsymbol{J}^2\sin\psi_1\sin\psi_2$$

now …

$$\boldsymbol{I}^2=\boldsymbol{I}$$

$$\left.\begin{aligned}\boldsymbol{I}\,\boldsymbol{J}&=\begin{pmatrix}1&0\\0&1\end{pmatrix}*\begin{pmatrix}\alpha&\beta\\-\gamma&-\alpha\end{pmatrix}=\begin{pmatrix}\alpha&\beta\\-\gamma&-\alpha\end{pmatrix}\\ \boldsymbol{J}\,\boldsymbol{I}&=\begin{pmatrix}\alpha&\beta\\-\gamma&-\alpha\end{pmatrix}*\begin{pmatrix}1&0\\0&1\end{pmatrix}=\begin{pmatrix}\alpha&\beta\\-\gamma&-\alpha\end{pmatrix}\end{aligned}\right\}\quad \boldsymbol{I}\,\boldsymbol{J}=\boldsymbol{J}\,\boldsymbol{I}$$

$$\boldsymbol{J}^2=\begin{pmatrix}\alpha&\beta\\-\gamma&-\alpha\end{pmatrix}*\begin{pmatrix}\alpha&\beta\\-\gamma&-\alpha\end{pmatrix}=\begin{pmatrix}\alpha^2-\gamma\beta&\alpha\beta-\beta\alpha\\-\gamma\alpha+\alpha\gamma&\alpha^2-\gamma\beta\end{pmatrix}=\begin{pmatrix}-1&0\\0&-1\end{pmatrix}=-\boldsymbol{I}$$

$$\boldsymbol{M}^2=\boldsymbol{I}\cos(\psi_1+\psi_2)+\boldsymbol{J}\sin(\psi_1+\psi_2)$$

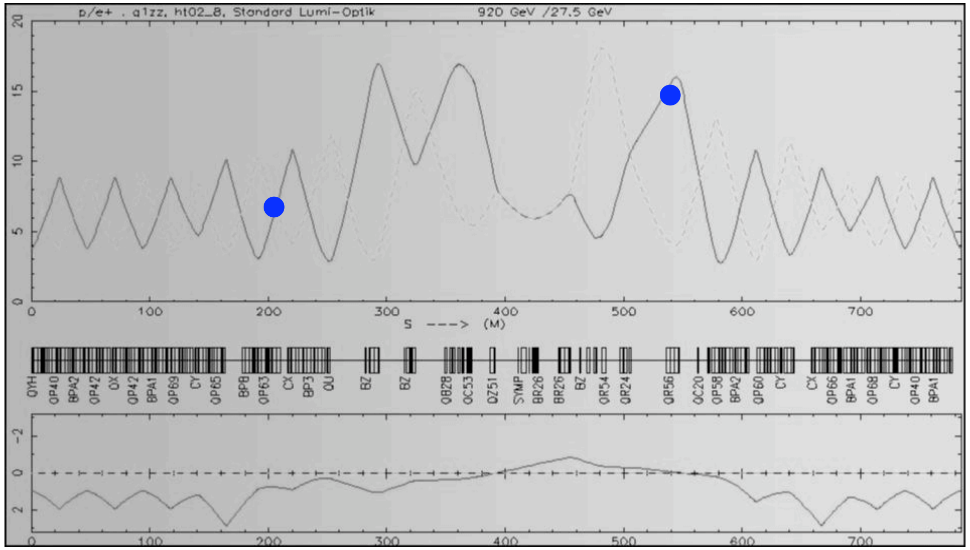
$$\boxed{\boldsymbol{M}^2=\boldsymbol{I}\cos(2\psi)+\boldsymbol{J}\sin(2\psi)}$$

10.) Transformation of α, β, γ

consider two positions in the storage ring: s_0 , s

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M * \begin{pmatrix} x \\ x' \end{pmatrix}_{s0}$$

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$



Betafunction in a Storage Ring

since ε = const (Liouville):

$$\varepsilon = \beta_s x'^2 + 2\alpha_s x x' + \gamma_s x^2$$

$$\varepsilon = \beta_0 x_0'^2 + 2\alpha_0 x_0 x_0' + \gamma_0 x_0{}^2$$

... remember W = CS´-SC´ = 1

$$\left.\begin{aligned} \begin{pmatrix} x \\ x' \end{pmatrix}_0 &= M^{-1} * \begin{pmatrix} x \\ x' \end{pmatrix}_s \\ M^{-1} &= \begin{pmatrix} m_{22} & -m_{12} \\ -m_{21} & m_{11} \end{pmatrix} \end{aligned}\right\} \rightarrow \quad \begin{aligned} x_0 &= m_{22}x - m_{12}x' \\ x_0' &= -m_{21}x + m_{11}x' \end{aligned}$$

... inserting into ε

$$\varepsilon = \beta_0 (m_{11}x' - m_{21}x)^2 + 2\alpha_0 (m_{22}x - m_{12}x')(m_{11}x' - m_{21}x) + \gamma_0 (m_{22}x - m_{12}x')^2$$

sort via x, x´and compare the coefficients to get ....

The Twiss parameters α, β, γ can be transformed through the lattice via the matrix elements defined above.

$$\beta(s) = m_{11}^2 \beta_0 - 2m_{11}m_{12}\alpha_0 + m_{12}^2 \gamma_0$$

$$\alpha(s) = -m_{11}m_{21}\beta_0 + (m_{12}m_{21} + m_{11}m_{22})\alpha_0 - m_{12}m_{22}\gamma_0$$

$$\gamma(s) = m_{21}^2 \beta_0 - 2m_{21}m_{22}\alpha_0 + m_{22}^2 \gamma_0$$

in matrix notation:

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_{s2} = \begin{pmatrix} m_{11}^2 & -2m_{11}m_{12} & m_{12}^2 \\ -m_{11}m_{21} & m_{12}m_{21} + m_{22}m_{11} & -m_{12}m_{22} \\ m_{12}^2 & -2m_{22}m_{21} & m_{22}^2 \end{pmatrix} * \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_{s1}$$

!

1.) this expression is important

2.) given the twiss parameters α, β, γ at any point in the lattice we can transform them and calculate their values at any other point in the ring.

3.) the transfer matrix is given by the focusing properties of the lattice elements, the elements of M are just those that we used to calculate single particle trajectories.

4.) go back to point 1.)

II.) Acceleration and Momentum Spread

The „ not so ideal world "

Remember:

Beam Emittance and Phase Space Ellipse:

equation of motion: $x''(s) - k(s)\, x(s) = 0$

general solution of Hills equation: $x(s) = \sqrt{\varepsilon}\,\sqrt{\beta(s)}\,\cos(\psi(s) + \varphi)$

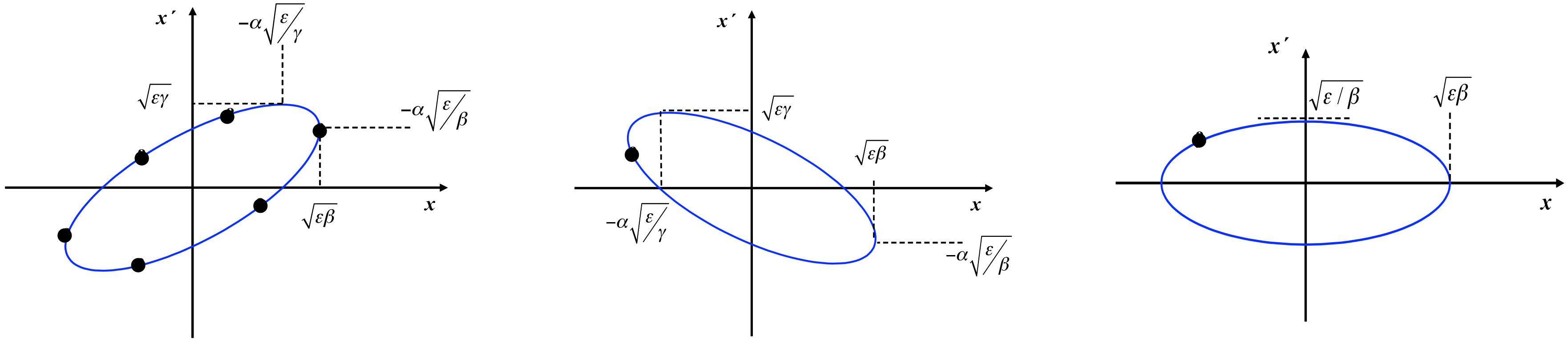
beam size: $\sigma = \sqrt{\varepsilon\beta} \approx \text{"}mm\text{"}$

$$\varepsilon = \gamma(s)\, x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)\, x'^2(s)$$

$$\alpha(s) = \frac{-1}{2}\beta'(s)$$

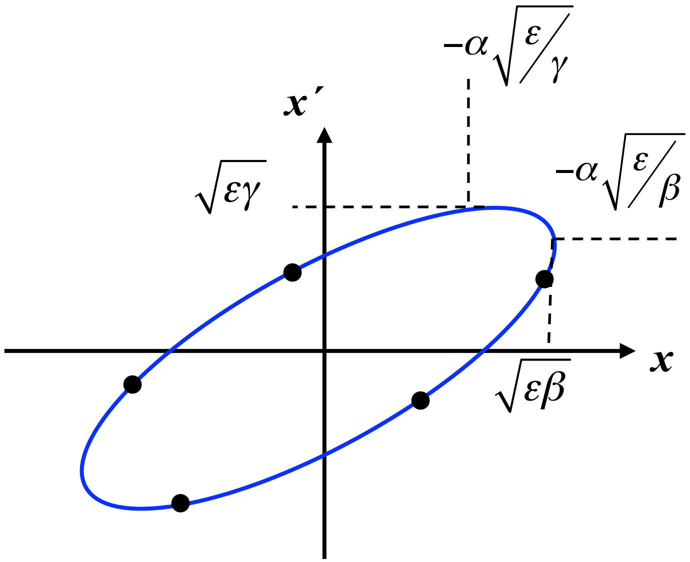
$$\gamma(s) = \frac{1 + \alpha(s)^2}{\beta(s)}$$

* *ε is a constant of the motion … it is independent of „s"*
* *parametric representation of an ellipse in the x x' space*
* *shape and orientation of ellipse are given by α, β, γ*



11.) Liouville during Acceleration

$$\varepsilon = \gamma(s)\, x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)\, x'^2(s)$$



***Beam Emittance** corresponds to the area covered in the x, x´ Phase Space Ellipse*

***Liouville:** Area in phase space is constant.*

But so sorry ... $\varepsilon \neq$ const !

Classical Mechanics:

phase space = diagram of the two canonical variables

position & momentum

x $\qquad$ p_x

$$p_j = \frac{\partial L}{\partial \dot{q}_j} \quad ; \quad L = T - V = kin.\,Energy - pot.\,Energy$$

According to Hamiltonian mechanics:
phase space diagram relates the variables q and p

$$q = position = x$$
$$p = momentum = \gamma m v = m c \gamma \beta_x$$

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \quad ; \quad \beta_x = \frac{\dot{x}}{c}$$

Liouvilles Theorem: $\int p\,dq = const$

for convenience (i.e. because we are lazy bones) we use in accelerator theory:

$$x' = \frac{dx}{ds} = \frac{dx}{dt}\frac{dt}{ds} = \frac{\beta_x}{\beta}$$

where $\beta_x = v_x / c$

$$\int p\,dq = mc \int \gamma \beta_x dx$$

$$\int p\,dq = mc\gamma\beta \underbrace{\int x' dx}_{\varepsilon}$$

$$\boxed{\Rightarrow \quad \varepsilon = \int x' dx \propto \frac{1}{\beta\gamma}}$$

the beam emittance shrinks during acceleration $\varepsilon \sim 1/\gamma$

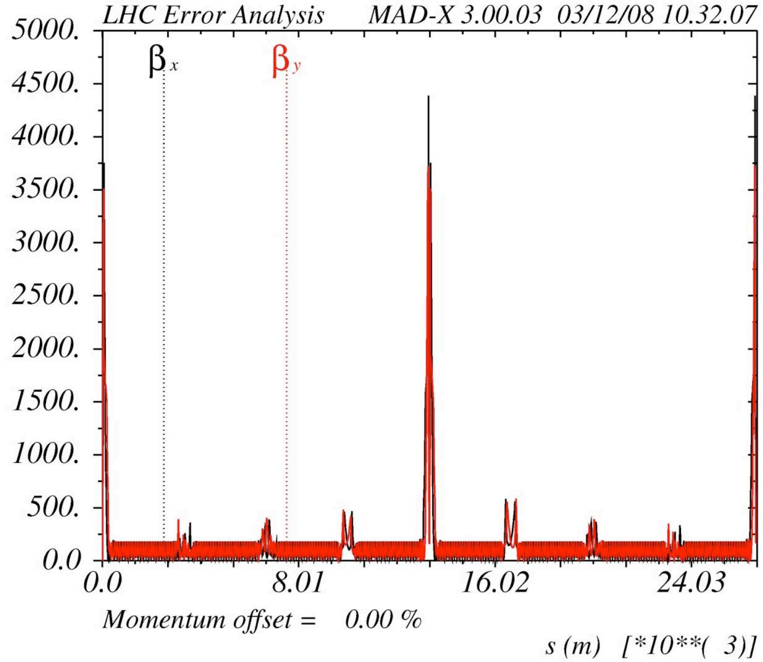
Nota bene:

1.) A proton machine … or an electron linac … needs the highest aperture at injection energy !!! as soon as we start to accelerate the beam size shrinks as $\gamma^{-1/2}$ in both planes.

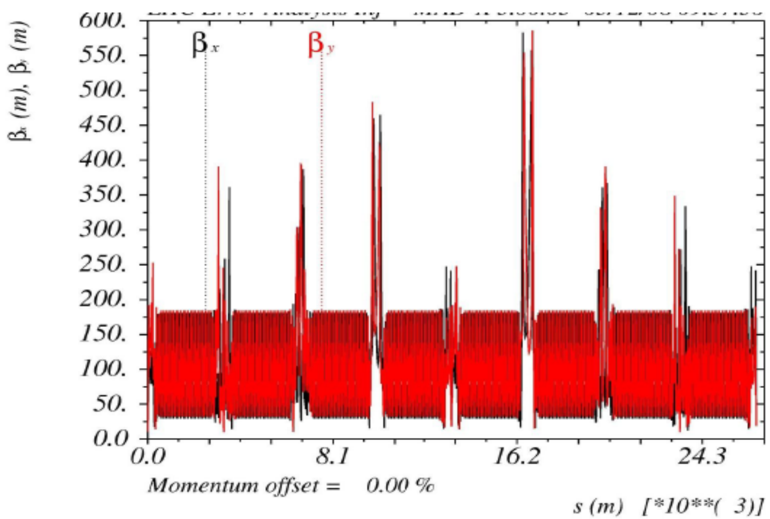
$$\sigma = \sqrt{\varepsilon\beta}$$

2.) At lowest energy the machine will have the major aperture problems,
→ here we have to minimise $\hat{\beta}$

3.) we need different beam optics adopted to the energy:
A Mini Beta concept will only be adequate at flat top.



LHC mini beta optics at 7000 GeV

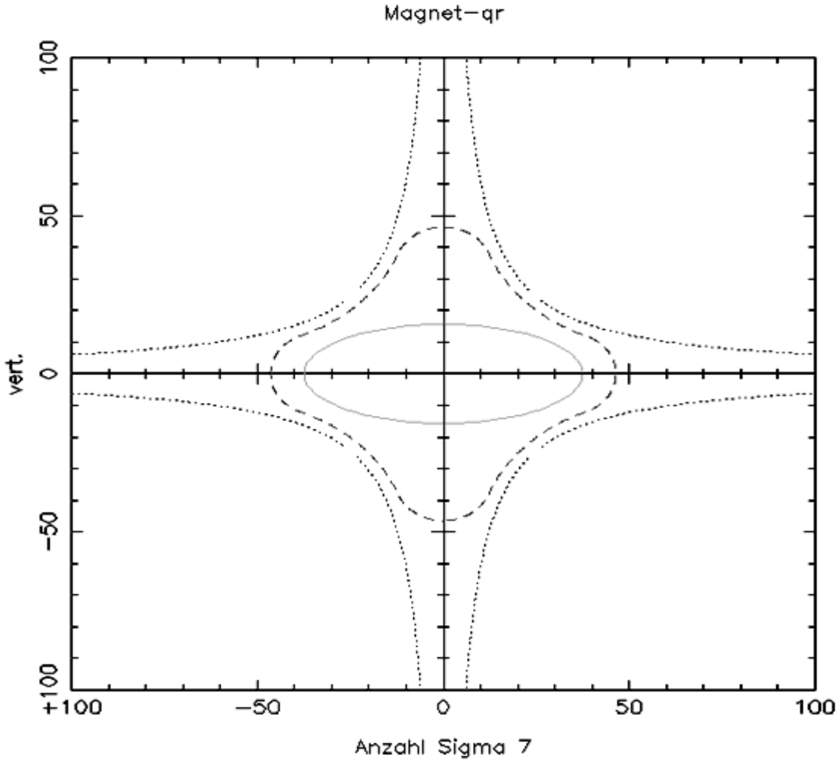


LHC injection optics at 450 GeV

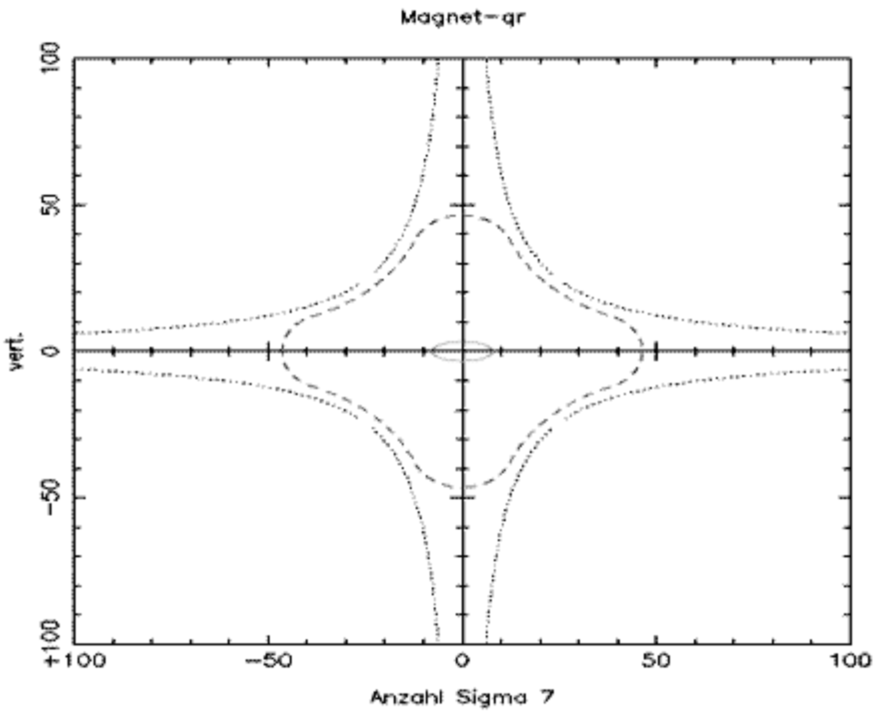
Example: HERA proton ring

injection energy: 40 GeV $\gamma = 43$
flat top energy: 920 GeV $\gamma = 980$

emittance ε *(40GeV)* $= 1.2 * 10^{-7}$
ε *(920GeV)* $= 5.1 * 10^{-9}$



7 σ beam envelope at E = 40 GeV



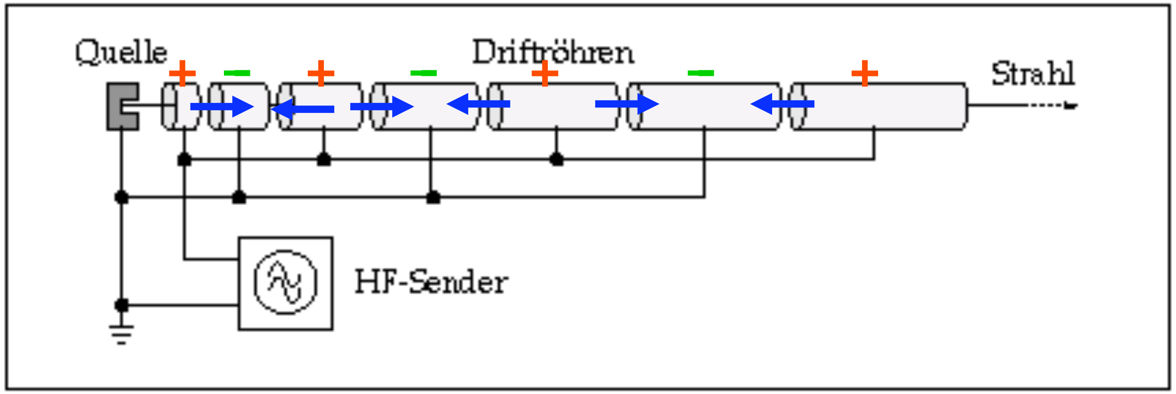
… and at E = 920 GeV

12.) The „ $\Delta p / p \neq 0$" Problem

Linear Accelerator

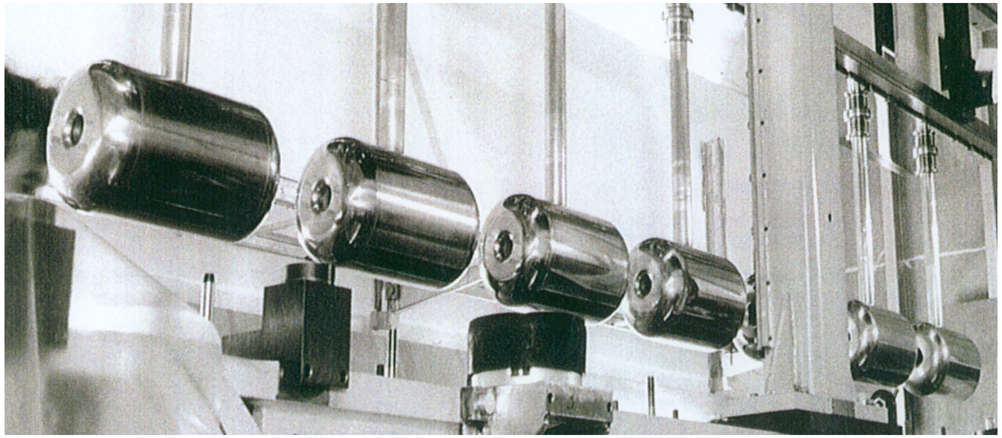
Energy Gain per „Gap":

$$W = q\, U_0 \sin \omega_{RF} t$$

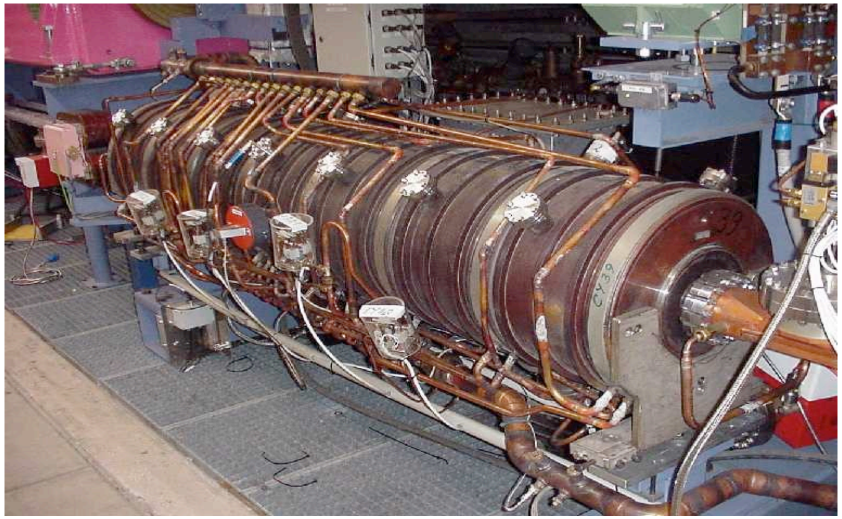


1928, Wideroe

drift tube structure at a proton linac



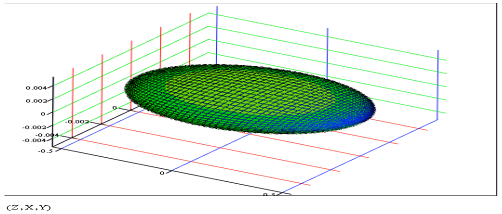
500 MHz cavities in an electron storage ring



* *RF Acceleration:* *multiple application of the same acceleration voltage;*
brillant idea to gain higher energies
... but changing acceleration voltage

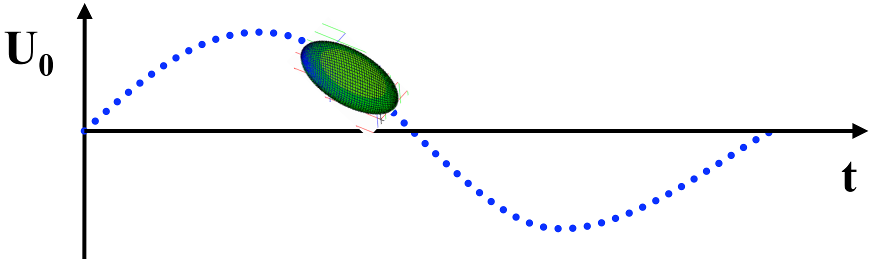
Problem: panta rhei !!!

(Heraklit: 540-480 v. Chr.)



Bunch length of Electrons ≈ 1cm

Example: HERA RF:



$\lambda = 60\ cm$

$$\left.\begin{array}{l} \nu = 500\, MHz \\ c = \lambda\, \nu \end{array}\right\} \quad \lambda = 60\ cm$$

$$\sin(90^o) = 1$$

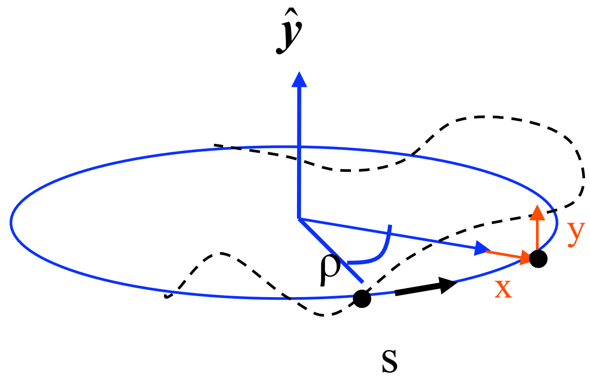
$$\sin(84^o) = 0.994$$

$$\frac{\Delta U}{U} = 6.0\ 10^{-3}$$

typical momentum spread of an electron bunch:

$$\frac{\Delta p}{p} \approx 1.0\ 10^{-3}$$

13.) Dispersion: trajectories for $\Delta p / p \neq 0$



***Force** acting on the particle*

$$F = m\frac{d^2}{dt^2}(x+\rho) - \frac{mv^2}{x+\rho} \;=\; e\,B_y\,v$$

remember: $x \approx mm$, $\rho \approx m$ … → develop for small x

$$m\frac{d^2x}{dt^2} - \frac{mv^2}{\rho}\left(1-\frac{x}{\rho}\right) = eB_y v$$

consider only linear fields, and change independent variable: $t \rightarrow s$

$$B_y = B_0 + x\frac{\partial B_y}{\partial x}$$

$$x'' - \frac{1}{\rho}\left(1-\frac{x}{\rho}\right) = \frac{e\;B_0}{mv} + \frac{e\;x\,g}{mv}$$

$p = p_0 + \Delta p$

… but now take a small momentum error into account !!!

Dispersion:

develop for small momentum error

$$\Delta p << p_0 \Rightarrow \frac{1}{p_0 + \Delta p} \approx \frac{1}{p_0} - \frac{\Delta p}{p_0^2}$$

$$x'' - \frac{1}{\rho} + \frac{x}{\rho^2} \approx \underbrace{\frac{e\ B_0}{p_0}}_{-\frac{1}{\rho}} - \frac{\Delta p}{p_0^2} eB_0 + \underbrace{\frac{xeg}{p_0}}_{k*x} - \underbrace{xeg\frac{\Delta p}{p_0^2}}_{\approx 0}$$

$$x'' + \frac{x}{\rho^2} \approx \frac{\Delta p}{p_0} * \underbrace{\frac{(-eB_0)}{p_0}}_{\frac{1}{\rho}} + k * x = \frac{\Delta p}{p_0} * \frac{1}{\rho} + k * x$$

$$x'' + \frac{x}{\rho^2} - kx = \frac{\Delta p}{p_0}\frac{1}{\rho} \quad \longrightarrow \quad \boxed{x'' + x(\frac{1}{\rho^2} - k) = \frac{\Delta p}{p_0}\frac{1}{\rho}}$$

***Momentum spread** of the beam adds a term on the r.h.s. of the equation of motion.*

→ ***inhomogeneous differential equation.***

Dispersion:

$$x'' + x\left(\frac{1}{\rho^2} - k\right) = \frac{\Delta p}{p} \cdot \frac{1}{\rho}$$

general solution:

$$x(s) = x_h(s) + x_i(s)$$

$$\begin{cases} x_h''(s) + K(s) \cdot x_h(s) = 0 \\ x_i''(s) + K(s) \cdot x_i(s) = \dfrac{1}{\rho} \cdot \dfrac{\Delta p}{p} \end{cases}$$

Normalise with respect to Δp/p:

$$D(s) = \frac{x_i(s)}{\Delta p / p}$$

Dispersion function D(s)

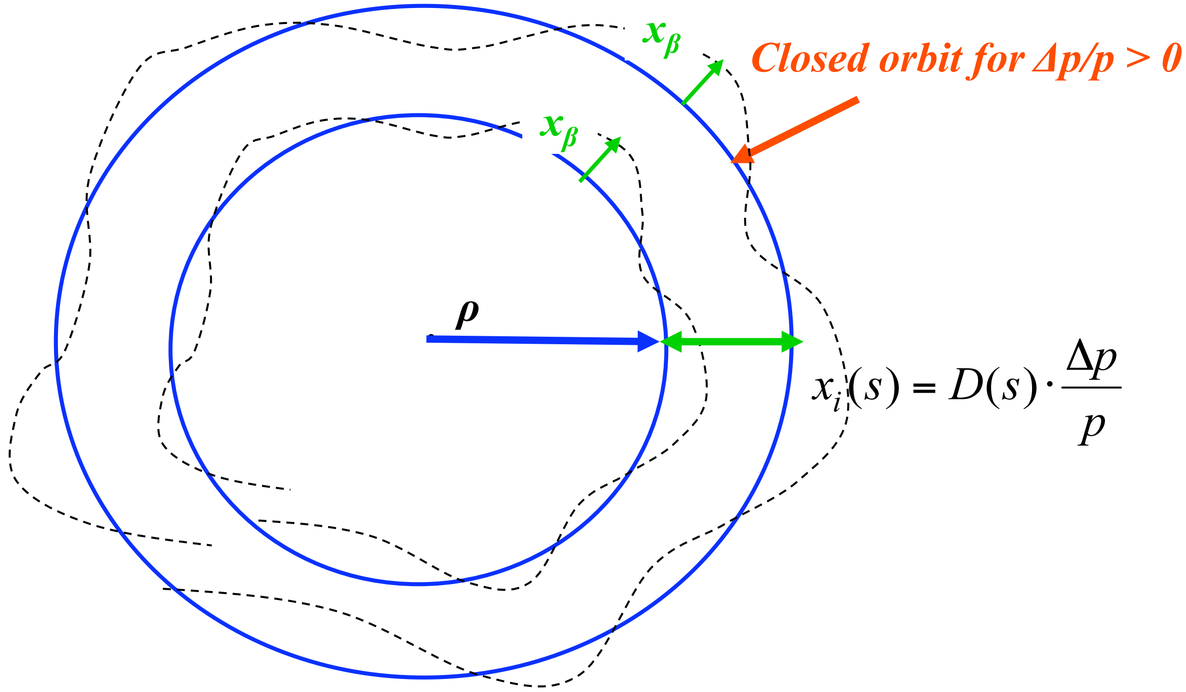
**** is that special orbit, an ideal particle would have for Δp/p = 1***

**** the orbit of any particle is the sum of the well known x_β and the dispersion***

**** as D(s) is just another orbit it will be subject to the focusing properties of the lattice***

Dispersion

Example: homogeneous dipole field



Matrix formalism:

e.g. matrix for a quadrupole lens:

$$M_{foc} = \begin{pmatrix} \cos(\sqrt{|K|}s & \frac{1}{\sqrt{|K|}}\sin(\sqrt{|K|}s \\ -\sqrt{|K|}\sin(\sqrt{|K|}s & \cos(\sqrt{|K|}s \end{pmatrix} = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix}$$

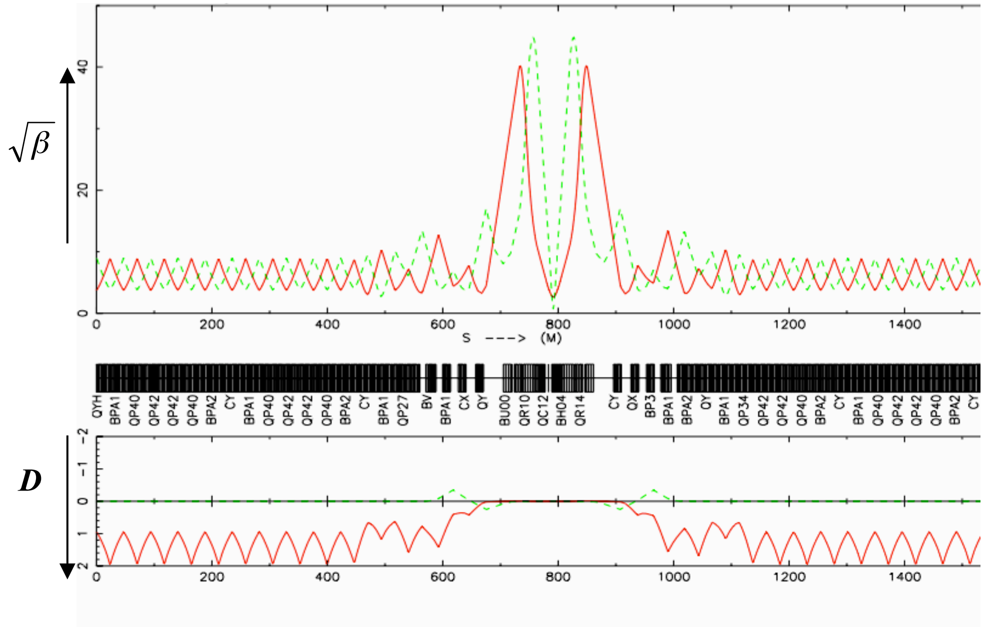
$$x(s) = x_\beta(s) + D(s) \cdot \frac{\Delta p}{p}$$

$$x(s) = C(s) \cdot x_0 + S(s) \cdot x_0' + D(s) \cdot \frac{\Delta p}{p}$$

$$\boxed{\begin{pmatrix} x \\ x' \end{pmatrix}_s = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_0 + \frac{\Delta p}{p} \begin{pmatrix} D \\ D' \end{pmatrix}}$$

or expressed as 3x3 matrix

$$\begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_s = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_0$$



Example HERA

$$\left.\begin{aligned} x_\beta &= 1 \ldots 2\ mm \\ D(s) &\approx 1 \ldots 2\ m \\ \Delta p/p &\approx 1 \cdot 10^{-3} \end{aligned}\right\}$$

Amplitude of Orbit oscillation

contribution due to Dispersion ≈ beam size

→ Dispersion must vanish at the collision point

!

Calculate D, D´

$$D(s) = S(s) \int_{s0}^{s1} \frac{1}{\rho} C(\tilde{s}) d\tilde{s} - C(s) \int_{s0}^{s1} \frac{1}{\rho} S(\tilde{s}) d\tilde{s}$$

(proof: see appendix)

Example: Drift

$$M_{Drift} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$$

$$D(s) = S(s)\underbrace{\int_{s0}^{s1} \frac{1}{\rho} C(\tilde{s})\, d\tilde{s}}_{=0} - C(s)\underbrace{\int_{s0}^{s1} \frac{1}{\rho} S(\tilde{s})\, d\tilde{s}}_{=0}$$

Example: Dipole

$$M_{foc} = \begin{pmatrix} \cos(\sqrt{|K|}s) & \frac{1}{\sqrt{|K|}}\sin(\sqrt{|K|}s \\ -\sqrt{|K|}\sin(\sqrt{|K|}s) & \cos(\sqrt{|K|}s) \end{pmatrix}_0$$

$$\boldsymbol{K} = \frac{1}{\rho^2} \cancel{- k}$$

$$\boldsymbol{s} = \boldsymbol{l_B}$$

$$M_{Dipole} = \begin{pmatrix} \cos\frac{l}{\rho} & \rho\sin\frac{l}{\rho} \\ -\frac{1}{\rho}\sin\frac{l}{\rho} & \cos\frac{l}{\rho} \end{pmatrix} \quad \rightarrow$$

$$D(s) = \rho \cdot (1 - \cos\frac{l}{\rho})$$

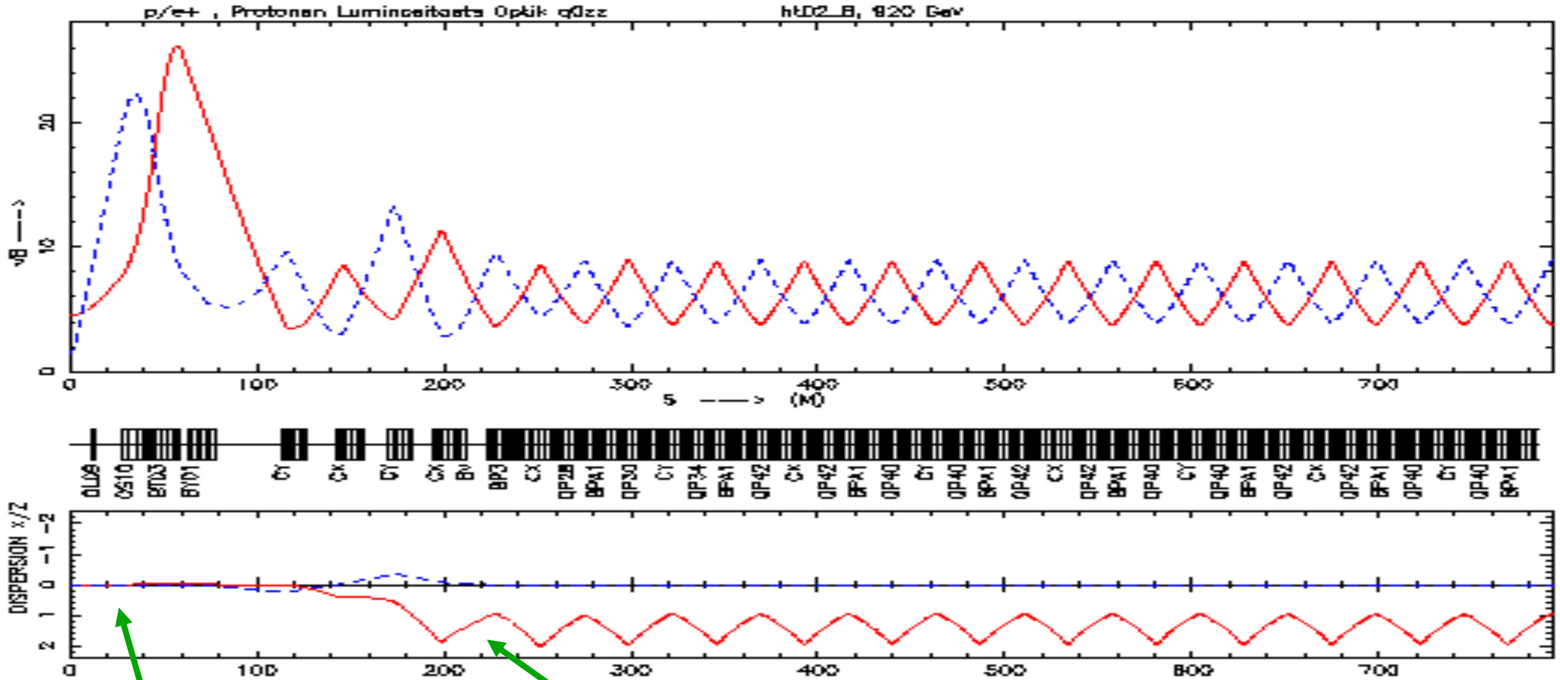
$$D'(s) = \sin\frac{l}{\rho}$$

Example: Dispersion, calculated by an optics code for a real machine

$$x_D = D(s)\,\frac{\Delta p}{p}$$

* D(s) is created by the dipole magnets

… and afterwards focused by the quadrupole fields

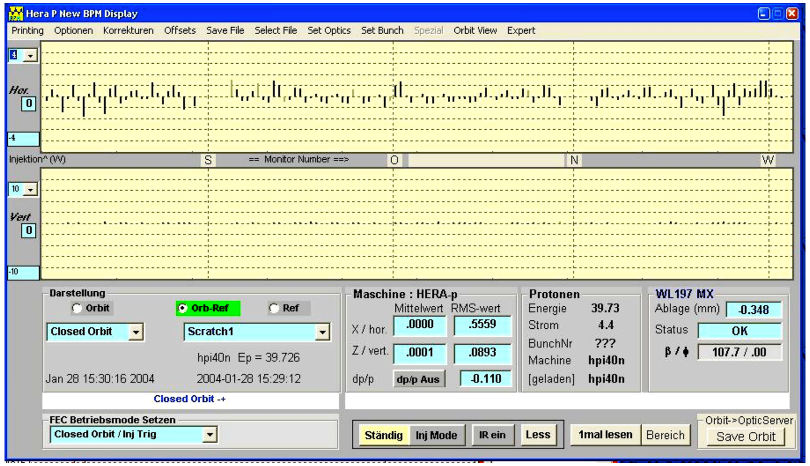


$\longrightarrow$ s

Mini Beta Section,
→ no dipoles !!!

$D(s) \approx 1 \ldots 2$ m

Dispersion is visible



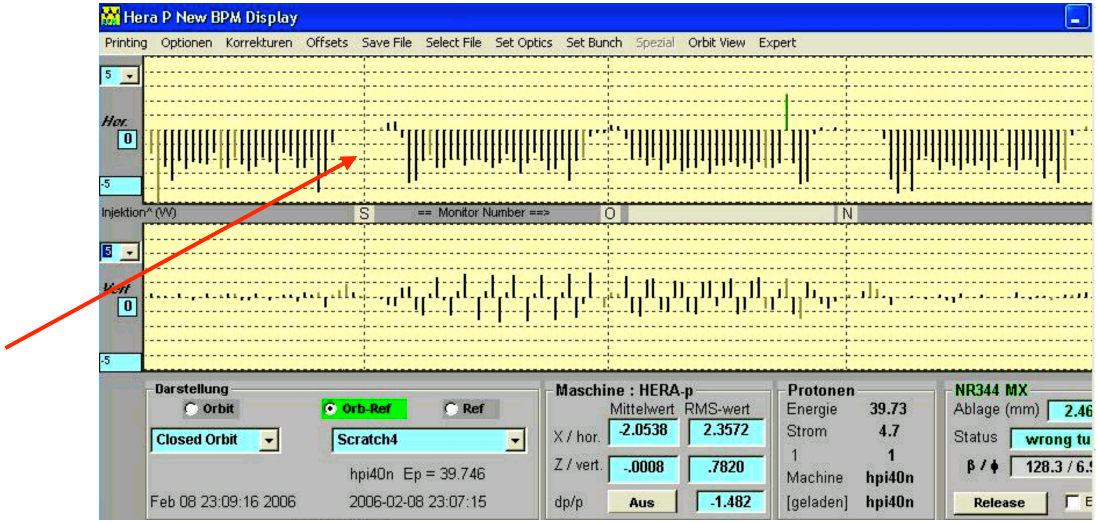
HERA Standard Orbit

HERA Dispersion Orbit

dedicated energy change of the stored beam

→ closed orbit is moved to a dispersions trajectory

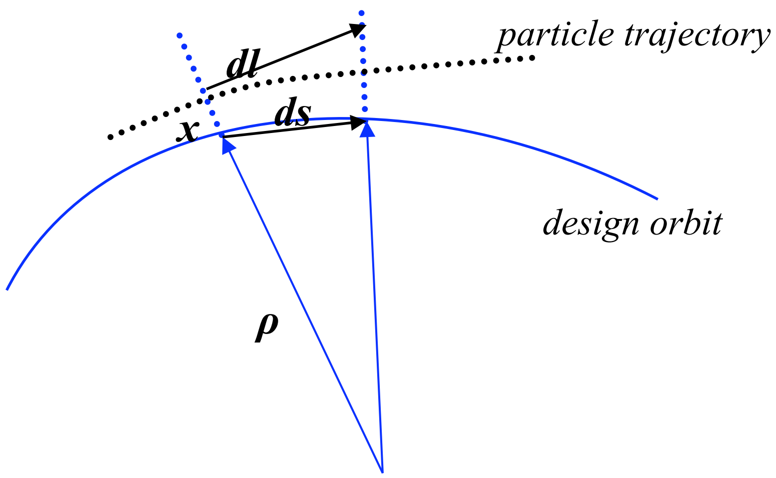
$$x_D = D(s) * \frac{\Delta p}{p}$$



Attention: at the Interaction Points we require D=D′= 0

14.) Momentum Compaction Factor: α_p

particle with a displacement x to the design orbit
→ path length dl ...



$$\frac{dl}{ds} = \frac{\rho + x}{\rho}$$

$$\rightarrow \quad dl = \left(1 + \frac{x}{\rho(s)}\right) ds$$

circumference of an off-energy closed orbit

$$l_{\Delta E} = \oint dl = \oint \left(1 + \frac{x_{\Delta E}}{\rho(s)}\right) ds$$

remember:

$$x_{\Delta E}(s) = D(s) \frac{\Delta p}{p}$$

$$\delta l_{\Delta E} = \frac{\Delta p}{p} \oint \left(\frac{D(s)}{\rho(s)}\right) ds$$

**** The lengthening of the orbit for off-momentum particles is given by the dispersion function and the bending radius.***

Definition:

$$\frac{\delta l_\varepsilon}{L} = \alpha_p \frac{\Delta p}{p}$$

$$\rightarrow \quad \alpha_p = \frac{1}{L}\oint\left(\frac{D(s)}{\rho(s)}\right)ds$$

For first estimates assume: $\quad \frac{1}{\rho} = const.$

$$\int_{dipoles} D(s)\,ds \approx l_{\Sigma(dipoles)} \cdot \langle D \rangle_{dipole}$$

$$\alpha_p = \frac{1}{L}\, l_{\Sigma(dipoles)} \cdot \langle D \rangle \frac{1}{\rho} = \frac{1}{L}\, 2\pi\rho \cdot \langle D \rangle \frac{1}{\rho} \quad \rightarrow \quad \alpha_p \approx \frac{2\pi}{L}\langle D \rangle \approx \frac{\langle D \rangle}{R}$$

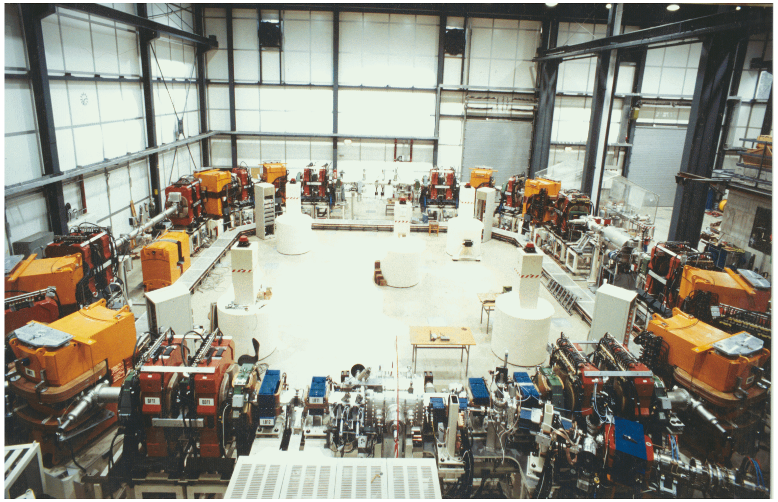
Assume: $\quad v \approx c$

$$\rightarrow \quad \frac{\delta T}{T} = \frac{\delta l_\varepsilon}{L} = \alpha_p \frac{\Delta p}{p}$$

α_p combines via the dispersion function the momentum spread with the longitudinal motion of the particle.

15.) Gradient Errors

Matrix in Twiss Form



Transfer Matrix from point „0“ in the lattice to point „s“:

$$M(s) = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}(\cos\psi_s + \alpha_0 \sin\psi_s) & \sqrt{\beta_s \beta_0}\sin\psi_s \\ \frac{(\alpha_0 - \alpha_s)\cos(\psi_s - (1+\alpha_0\alpha_s)\sin\psi_s}{\sqrt{\beta_s\beta_0}} & \sqrt{\frac{\beta_0}{\beta_s}}(\cos(\psi_s - \alpha_0 \sin\psi_s) \end{pmatrix}$$

For one complete turn the Twiss parameters have to obey periodic bundary conditions:

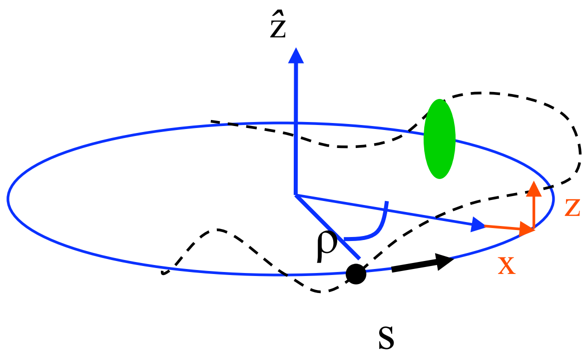
$$\beta(s+L) = \beta(s)$$
$$\alpha(s+L) = \alpha(s)$$
$$\gamma(s+L) = \gamma(s)$$

$$M(s) = \begin{pmatrix} \cos\psi_{turn} + \alpha_s \sin\psi_{turn} & \beta_s \sin\psi_{turn} \\ -\gamma_s \sin\psi_s & \cos\psi_{turn} - \alpha_s \sin\psi_{turn} \end{pmatrix}$$

Quadrupole Error in the Lattice

optic perturbation described by thin lens quadrupole

$$M_{dist} = M_{\Delta k} \cdot M_0 = \underbrace{\begin{pmatrix} 1 & 0 \\ \Delta kds & 1 \end{pmatrix}}_{\textit{quad error}} \cdot \underbrace{\begin{pmatrix} \cos\psi_{turn} + \alpha \sin\psi_{turn} & \beta \sin\psi_{turn} \\ -\gamma \sin\psi_{turn} & \cos\psi_{turn} - \alpha \sin\psi_{turn} \end{pmatrix}}_{\textit{ideal storage ring}}$$



$$M_{dist} = \begin{pmatrix} \cos\psi_0 + \alpha \sin\psi_0 & \beta \sin\psi_0 \\ \Delta kds(\cos\psi_0 + \alpha \sin\psi_0) - \gamma \sin\psi_0 & \Delta kds\,\beta \sin\psi_0 + \cos\psi_0 - \alpha \sin\psi_0 \end{pmatrix}$$

rule for getting the tune

$$Trace(M) = 2\cos\psi = 2\cos\psi_0 + \Delta kds\beta \sin\psi_0$$

Quadrupole error → Tune Shift

$$\boxed{\psi = \psi_0 + \Delta\psi} \longrightarrow \cos(\psi_0 + \Delta\psi) = \cos\psi_0 + \frac{\Delta \boldsymbol{kds}\, \beta \sin\psi_0}{2}$$

remember the old fashioned trigonometric stuff and assume that the error is small !!!

$$\cos\psi_0 \underbrace{\cos\Delta\psi}_{\approx 1} - \sin\psi_0 \underbrace{\sin\Delta\psi}_{\approx \Delta\psi} = \cos\psi_0 + \frac{\boldsymbol{kds}\,\beta \sin\psi_0}{2}$$

$$\Delta\psi = \frac{\boldsymbol{kds}\,\beta}{2}$$

and referring to Q instead of ψ:

$$\psi = 2\pi \boldsymbol{Q}$$

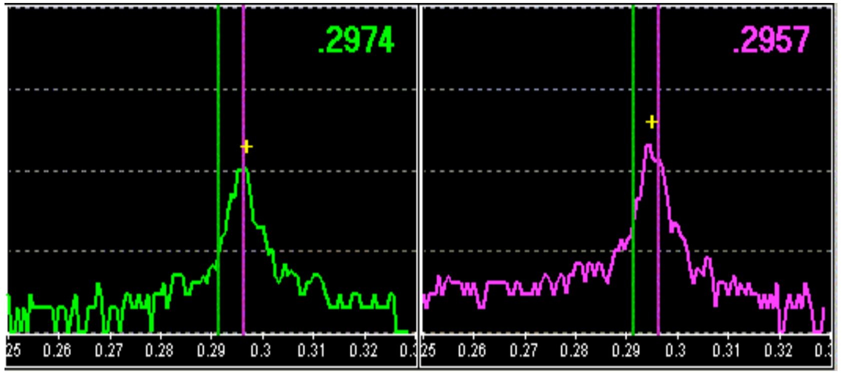
$$\boxed{\Delta \boldsymbol{Q} = \int_{s0}^{s0+l} \frac{\Delta \boldsymbol{k}(\boldsymbol{s})\beta(\boldsymbol{s})\boldsymbol{ds}}{4\pi}}$$

! the tune shift is proportional to the β-function at the quadrupole

!! field quality, power supply tolerances etc are much tighter at places where β is large

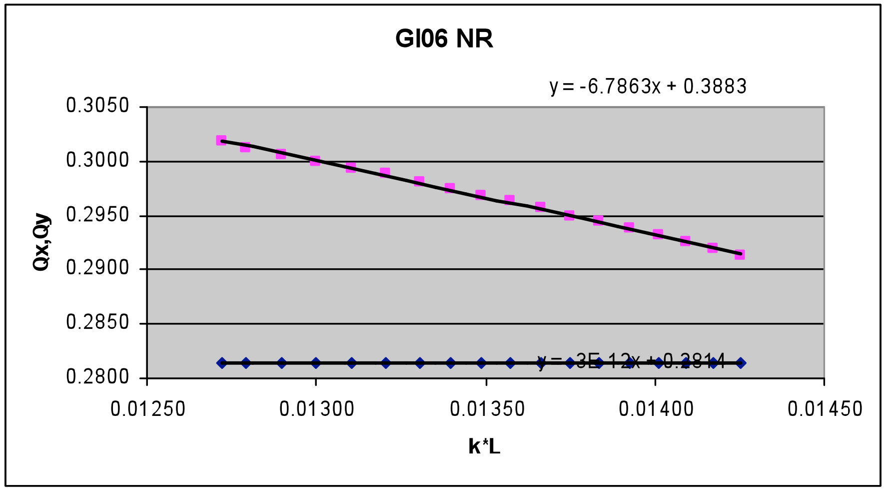
!!! mini beta quads: β ≈ 1900 m
arc quads: β ≈ 80 m

!!!! β is a measure for the sensitivity of the beam *a quadrupol error leads to a shift of the tune:*



$$\Delta Q = \int_{s0}^{s0+l} \frac{\Delta k \beta(s)}{4\pi} ds \approx \frac{\Delta k l_{quad} \overline{\beta}}{4\pi}$$

Example: measurement of β in a storage ring:
tune spectrum

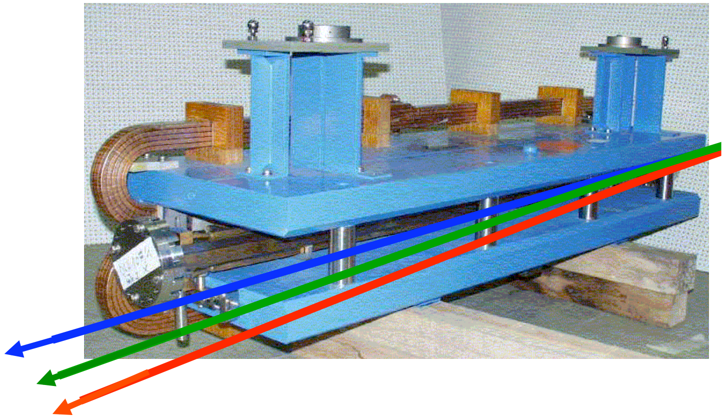


16.) Chromaticity:
A Quadrupole Error for $\Delta p/p \neq 0$

Influence of external fields on the beam: *prop. to magn. field & prop. zu 1/p*

dipole magnet

$$\alpha = \frac{\int B\, dl}{p/e}$$



$$x_D(s) = D(s)\,\frac{\Delta p}{p}$$

focusing lens

$$k = \frac{g}{p/e}$$

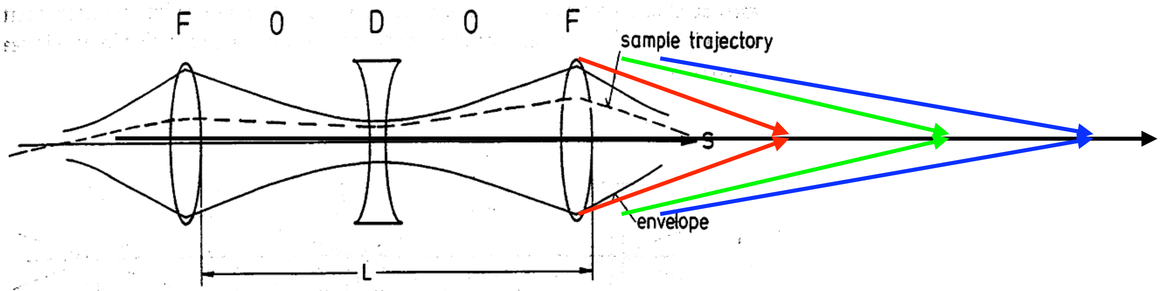


Figure 29: FODO cell

particle having ...

to high energy

to low energy

ideal energy

Chromaticity: Q'

$$k = \frac{g}{p/e} \qquad\qquad p = p_0 + \Delta p$$

in case of a momentum spread:

$$\boldsymbol{k = \frac{eg}{p_0 + \Delta p} \approx \frac{e}{p_0}(1 - \frac{\Delta p}{p_0})\, g = k_0 + \Delta k}$$

$$\Delta k = -\frac{\Delta p}{p_0} k_0$$

… which acts like a quadrupole error in the machine and leads to a tune spread:

$$\Delta Q = -\frac{1}{4\pi} \frac{\Delta p}{p_0} k_0 \beta(s) ds$$

definition of chromaticity:

$$\boxed{\Delta Q = Q' \frac{\Delta p}{p} \quad ; \quad Q' = -\frac{1}{4\pi} \oint k(s) \beta(s) ds}$$

… what is wrong about Chromaticity:

Problem: chromaticity is generated by the lattice itself !!

Q' is a number indicating the size of the tune spot in the working diagram,
Q' is always created if the beam is focussed
→ *it is determined by the focusing strength k of all quadrupoles*

$$\boxed{Q' = -\frac{1}{4\pi}\oint k(s)\beta(s)ds}$$

k = quadrupole strength
β = betafunction indicates the beam size … and even more the sensitivity of the beam to external fields

Example: LHC

$Q' = 250$
$\Delta p/p = +/- 0.2 * 10^{-3}$
$\Delta Q = 0.256 \ldots 0.36$

→ *Some particles get very close to resonances and are lost*

in other words: the tune is not a point it is a pancake

Correction of Q':

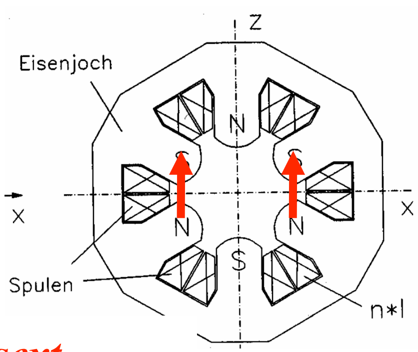
1.) sort the particles acording to their momentum $x_D(s) = D(s)\frac{\Delta p}{p}$

2.) apply a magnetic field that rises quadratically with x (sextupole field)

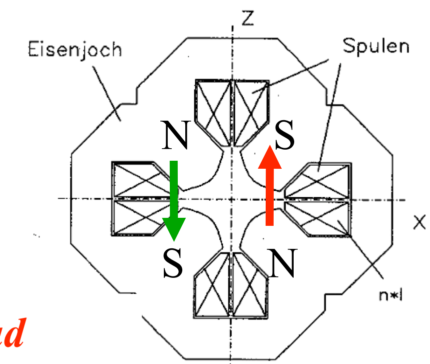
$$\left.\begin{aligned} B_x &= \tilde{g}xz \\ B_z &= \frac{1}{2}\tilde{g}(x^2 - z^2) \end{aligned}\right\} \qquad \frac{\partial B_x}{\partial z} = \frac{\partial B_z}{\partial x} = \tilde{g}x$$

linear rising „gradient“:

Sextupole Magnets:



sext geometry



quad geometry

normalised quadrupole strength:

$$k_{sext} = \frac{\tilde{g}x}{p/e} = m_{sext.}x$$

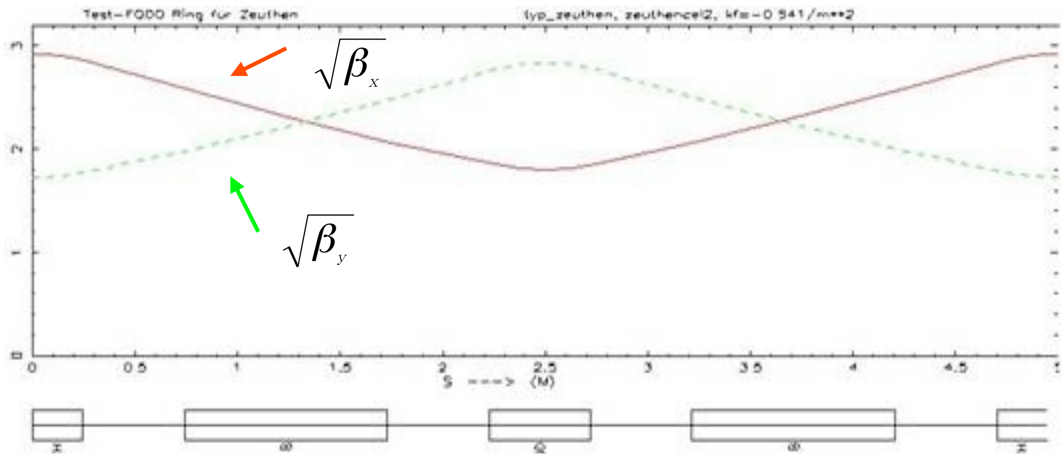
$$k_{sext} = m_{sext.}D\frac{\Delta p}{p}$$

corrected chromaticity:

$$\boldsymbol{Q' = -\frac{1}{4\pi}\oint\{K(s) - mD(s)\}\beta(s)ds}$$

Chromaticity in the FoDo Lattice

$$\boldsymbol{Q'} = \frac{-1}{4\pi} * \oint \boldsymbol{k(s)}\beta(\boldsymbol{s})\,\boldsymbol{ds}$$



β-Function in a FoDo

$$\hat{\beta} = \frac{(1+\sin\frac{\psi_{cell}}{2})L}{\sin\psi_{cell}} \qquad \breve{\beta} = \frac{(1-\sin\frac{\psi_{cell}}{2})L}{\sin\psi_{cell}}$$

$$\boldsymbol{Q'} = \frac{-1}{4\pi}\boldsymbol{N} * \frac{\hat{\beta}-\breve{\beta}}{\boldsymbol{f_Q}}$$

$$Q' = \frac{-1}{4\pi}N * \frac{1}{f_Q} * \left\{ \frac{L(1+\sin\frac{\psi_{cell}}{2}) - L(1-\sin\frac{\psi_{cell}}{2})}{\sin\mu} \right\}$$

using some TLC transformations ... ξ can be expressed in a very simple form:

$$Q' = \frac{-1}{4\pi} N * \frac{1}{f_Q} * \frac{2L \sin\frac{\psi_{cell}}{2}}{\sin\psi_{cell}}$$

$$Q' = \frac{-1}{4\pi} N * \frac{1}{f_Q} * \frac{L \sin\frac{\psi_{cell}}{2}}{\sin\frac{\psi_{cell}}{2} \cos\frac{\psi_{cell}}{2}}$$

remember ...

$$\sin x = 2 \sin\frac{x}{2} \cos\frac{x}{2}$$

$$Q'_{cell} = \frac{-1}{4\pi f_Q} * \frac{L \tan\frac{\psi_{cell}}{2}}{\sin\frac{\psi_{cell}}{2}}$$

putting ...

$$\sin\frac{\psi_{cell}}{2} = \frac{L}{4 f_Q}$$

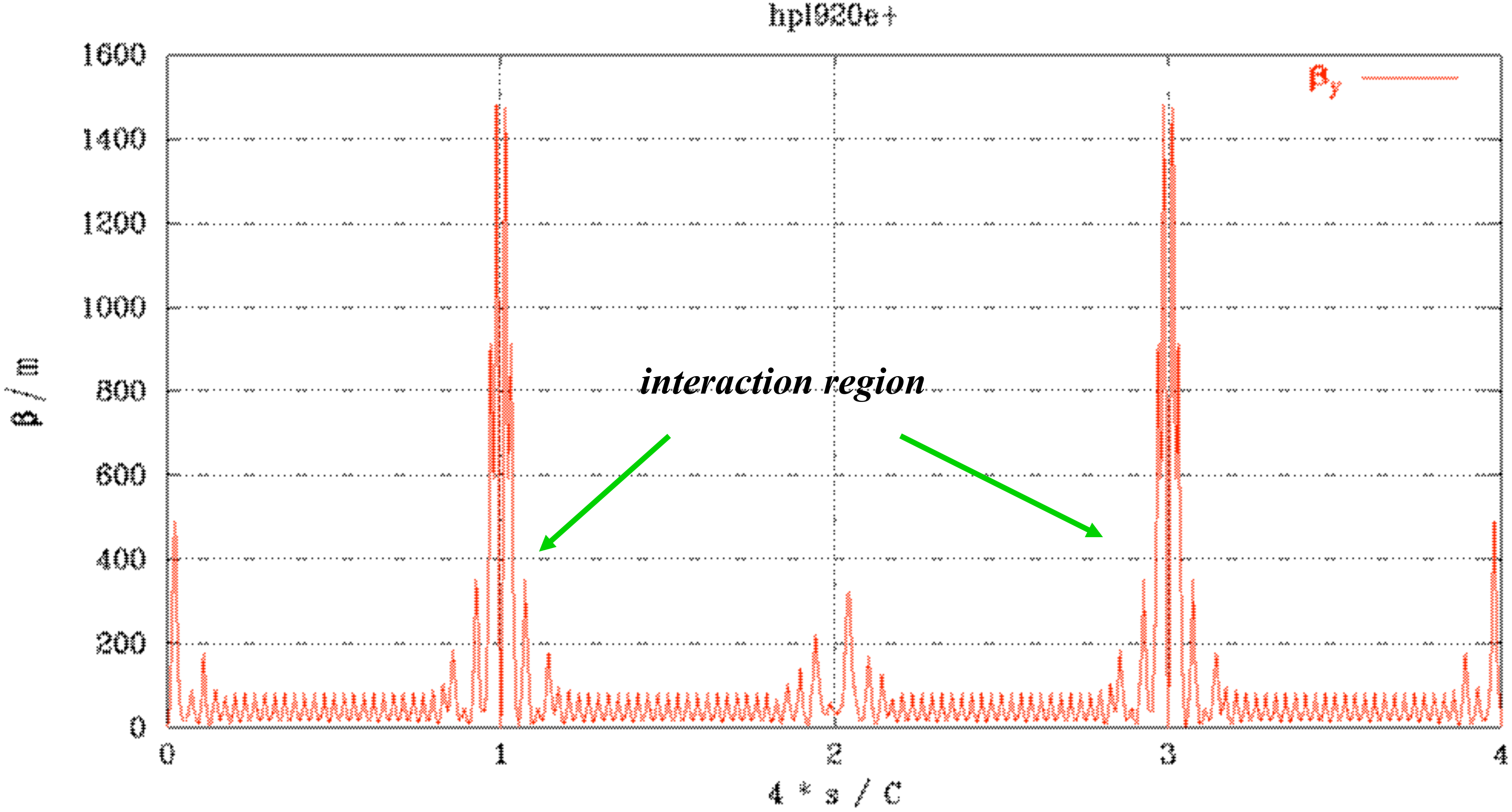
$$\boxed{Q'_{cell} = \frac{-1}{\pi} * \tan\frac{\psi_{cell}}{2}}$$

contribution of one FoDo Cell to the chromaticity of the ring:

Chromaticity

$$Q' = -\frac{1}{4\pi}\oint K(s)\beta(s)ds$$

question: main contribution to ξ in a lattice … ?



17.) Résumé:

beam rigidity: $B \cdot \rho = p/q$

bending strength of a dipole: $\frac{1}{\rho}\left[m^{-1}\right] = \frac{0.2998 \cdot B_0(T)}{p(GeV/c)}$

focusing strength of a quadrupole: $k\left[m^{-2}\right] = \frac{0.2998 \cdot g}{p(GeV/c)}$

focal length of a quadrupole: $f = \frac{1}{k \cdot l_q}$

equation of motion: $x'' + Kx = \frac{1}{\rho}\frac{\Delta p}{p}$

matrix of a foc. quadrupole: $x_{s2} = M \cdot x_{s1}$

$$M = \begin{pmatrix} \cos\sqrt{|K|}l & \frac{1}{\sqrt{|K|}}\sin\sqrt{|K|}l \\ -\sqrt{|K|}\sin\sqrt{|K|}l & \cos\sqrt{|K|}l \end{pmatrix}, \qquad M = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}$$

Resume´:

beam emittance: $$\varepsilon \propto \frac{1}{\beta\gamma}$$

beta function in a drift: $$\beta(s) = \beta_0 - 2\alpha_0 s + \gamma_0 s^2$$

… and for α = 0 $$\beta(s) = \beta_0 + \frac{s^2}{\beta_0}$$

particle trajectory for Δp/p ≠ 0
inhomogenious equation: $$\boldsymbol{x}'' + \boldsymbol{x}\left(\frac{1}{\rho^2} - \boldsymbol{k}\right) = \frac{\Delta \boldsymbol{p}}{\boldsymbol{p}_0}\frac{1}{\rho}$$

… and its solution: $$x(s) = x_\beta(s) + D(s)\cdot\frac{\Delta p}{p}$$

momentum compaction: $$\frac{\delta\, l_\varepsilon}{L} = \alpha_{cp}\frac{\Delta p}{p} \qquad \alpha_{cp} \approx \frac{2\pi}{L}\langle D\rangle \approx \frac{\langle D\rangle}{R}$$

quadrupole error: $$\Delta Q = \int_{s0}^{s0+l} \frac{\Delta K(s)\beta(s)ds}{4\pi}$$

chromaticity: $$Q' = -\frac{1}{4\pi}\oint K(s)\beta(s)ds$$

18.) Bibliography

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