Physics of Landau Damping

An introduction (to a mysterious topic)

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http://cern.ch/Werner.Herr/CAS2013/lectures/Trondheim_landau.pdf

Landau damping - the mystery

- **First publication in 1946**
- **Applied to longitudinal oscillations of an electron plasma**
  - **Was not believed for $\approx$ 20 years (but worked in simulations and experiment)**
  - **Still plenty of papers every year ($\approx$ 6000 in 2012) (and many attempts to teach it ...)**
  - **Many applications: plasma physics, accelerators**
  - **Physical interpretation often unclear**
  - **Many mathematical subtleties ...**

Landau damping - the mystery

- **Chronology:**

Landau	**(1946)**	**Plasma physics**
Bohm, Gross	**(1949)**	**Plasma physics**
van Kampen	**(1955)**	**Mathematical foundation**
Sessler et al.	**(1959)**	**Accelerators**
...	**...**	**...**
Mouhot, Villani	**(2010)**	**Non-linear Landau Damping**
		(Very detailed maths)

Landau damping - why confusing ?

- **In a plasma:**
  - **Landau damping damps collective oscillations**
  - **Leads to exponentially decaying oscillations**

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  - **Landau damping does not damp anything !!**
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    "Landau damping" is confused with decoherence
  - **Landau damping stabilizes the beam, i.e.**

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Landau damping - why confusing ?

- **In a plasma:**
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  - **Leads to exponentially decaying oscillations**
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Landau damping - why confusing ?

- **The non-trivial part:**
  - **In a beam (any plasma) particles interact via Coulomb forces (binary collisions)**
  - **For Landau damping: particles "interact" with the beam (collective modes)**
- **Must distinguish:**
  - **Binary interactions (collisions) of particles**
  - **Interactions of particles with a collective mode**

Landau damping - why confusing ?

- **In accelerators different mechanisms have been associated with "Landau damping", most popular:**
  - **"Resonance damping"**
  - **"Phase mixing"**
- **Often confused with "decoherence"**
  - **Landau damping <u>does not</u> lead to emittance growth**
  - **Decoherence <u>does</u> !**
- **Different treatment (and results !) for**
  - **Bunch and unbunched beams**
  - **Transverse and longitudinal motion**

Landau damping - the menu

- **Sketch Landau's treatment for plasmas**
- **Mechanisms of stabilization - physical origin**
- **Conditions for stabilization - beam transfer function and stability diagrams**
- **Collective motion, physics and description**
- **Example: how it is used, limits, problems ...**
- **Do not go through formal mathematics (found in many places, or discussed in the bar), rather <u>intuitive</u> approach to touch the concepts, give hints ..**

Why an intuitive approach ?

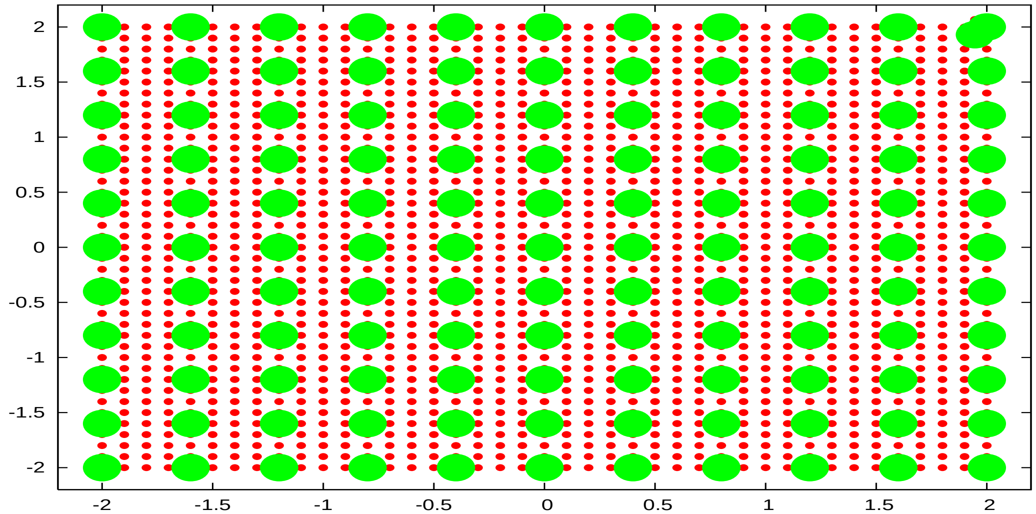
A lot of attention is often paid to interpretation of subtle (mathematical and philosophical) problems:

- **Singularities**
- **Reversibility versus Irreversibility**
- **Linearity versus Non-linearity**

The truth is:

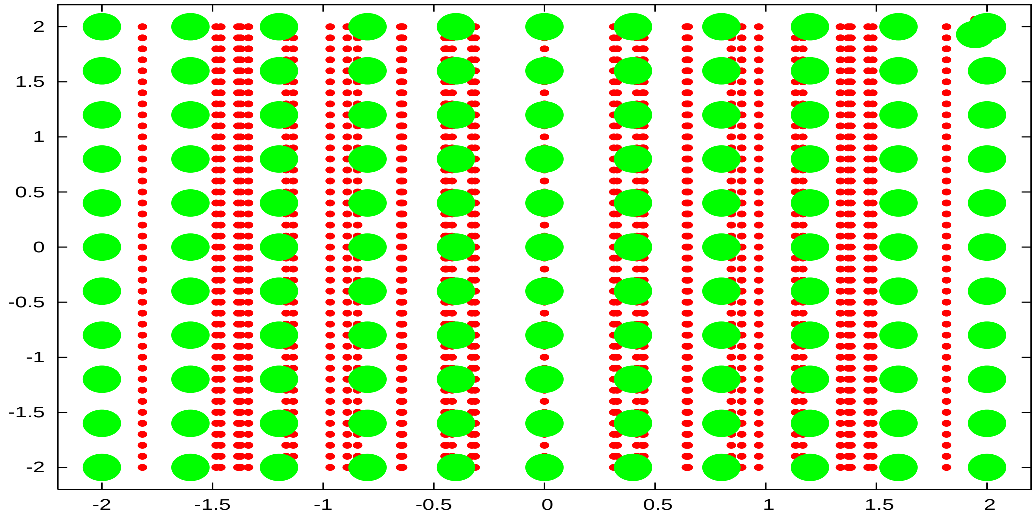
- **Most "problems" are fictitious**
- **Not coming from the physics of the process**
- **Appear in specific mathematical treatment and versions of theory**

Plasma oscillations



- **Plasma without disturbance: ions (●) and electrons (•)**

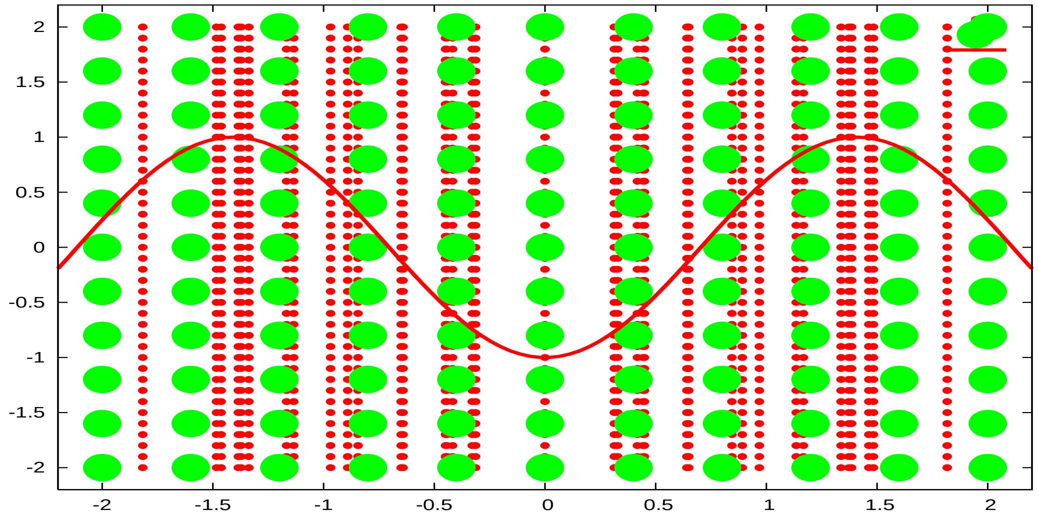
Plasma oscillations



- **Plasma: stationary ions (●) with displaced electrons (•)**
- **Restoring force: oscillate at plasma frequency** $\omega^2 = \frac{ne^2}{m\epsilon_0}$

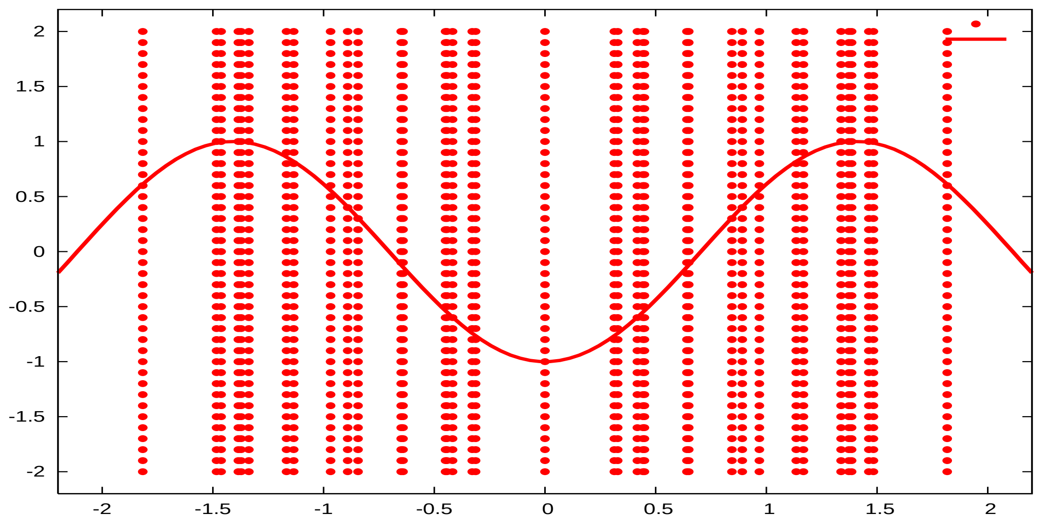
i.e. a stationary plane wave solution (Langmuir, 1929)

Plasma oscillations



- **Restoring force: oscillate at plasma frequency** $\omega^2 = \frac{ne^2}{m\epsilon_0}$
- **Produces field (mode) of the form:**
  $E(x,t) = E_0 \sin(kx - \omega t)$ **(or** $E(x,t) = E_0 e^{i(kx-\omega t)}$**)**

Plasma oscillations



- **Electrons interact with the field they produce**
- **Field (mode) of the form:**
  $E(x,t) = E_0 \sin(kx - \omega t)$ **(or** $E(x,t) = E_0 e^{i(kx-\omega t)}$**)**

Plasma oscillations

- **Individual particles interact with the field produced by all particles**
  - → **Changes behaviour of the particles**
  - → **Can change the field producing the forces**
  - → **Particles may have different velocities !**
- **Self-consistent treatment required**

If we allow ω to be complex ($\omega = \omega_r + i\omega_i$):

$$E(x,t) = E_0 e^{i(kx-\omega t)} \;\Rightarrow\; E(x,t) = E_0 e^{i(kx-\omega_r t)} \cdot e^{\omega_i t}$$

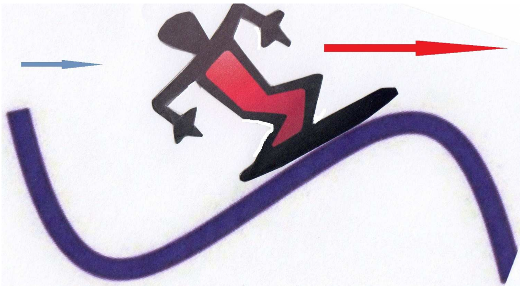
we can have a damped oscillation for $\omega_i < 0$

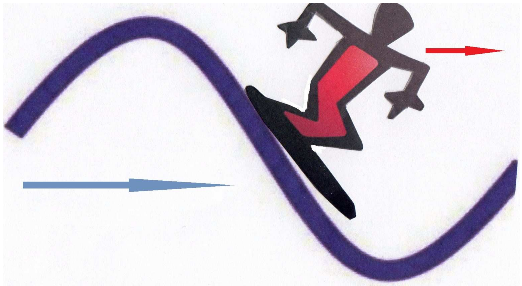
Resonance damping

- Interaction with a "mode" -

- **Surfer gains energy from the mode (wave)**

Interaction with a "mode"





- If Surfer faster than wave:

  mode gains energy from the surfer

- If Surfer slower than wave:

  mode loses energy to the surfer

- Does that always work like that ?

Interaction with a "mode"

- **NO, consider two extreme cases:**
  - **Surfer very fast: "jumps" across the wave crests, little interaction with the wave (water skiing)**
  - **Surfer not moving: "oscillates" up and down with the waves**

➡ **Wave velocity and Surfer velocity must be similar ... !!**

➡ **Surfer is "trapped" by the wave**

Interaction with a "mode"

- Remember: particles may have different velocities !
- If more particles are moving slower than the wave:
  - Net absorption of energy <u>from</u> the wave
  - Wave is damped !
- If more particles are moving faster than the wave:
  - Net absorption of energy <u>by</u> the wave
  - Wave is anti-damped !
- Always: the <u>slope</u> of the particle distribution at the wave velocity is important !

→ Have to show that now (with some theory)

Liouville theorem

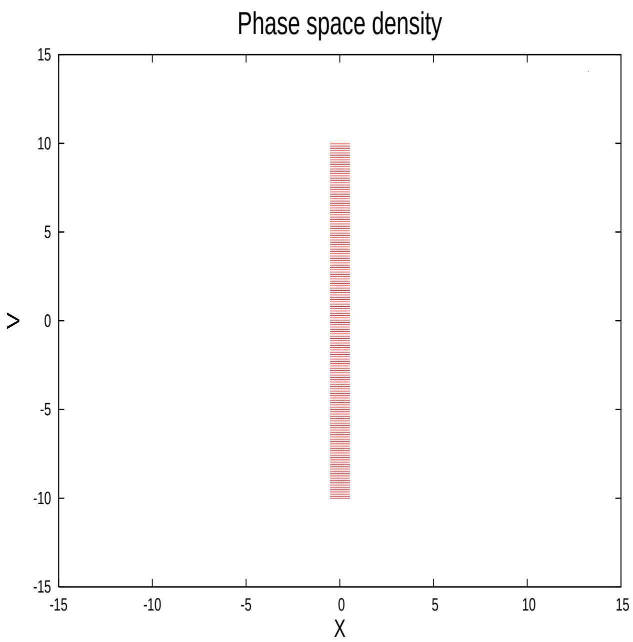
- **Consider an ensemble of particles**
- **Phase space moves like incompressible fluid**
- **Density is always conserved**
- **Described by a density distribution function** $\psi(\vec{x}, \vec{v}, t)$**:**

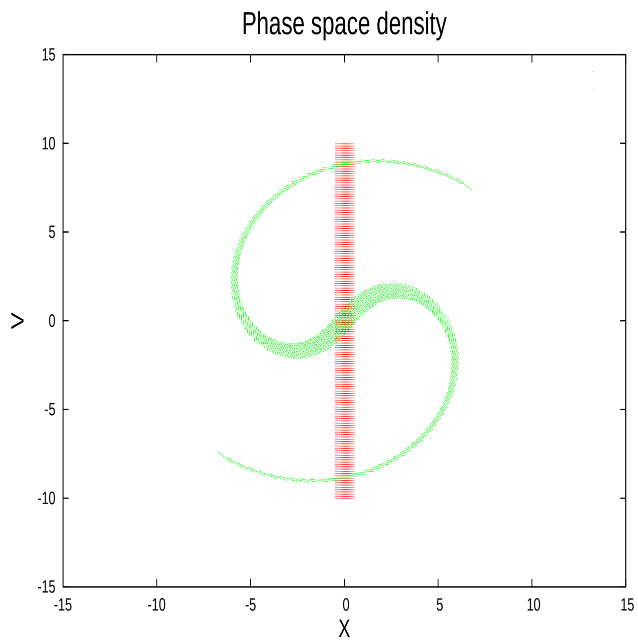
  $\int \psi(\vec{x}, \vec{v}, t) dx dv = N$

- **If the distribution function is** **<u>stationary</u>** $\rightarrow$

  $\psi(\vec{x}, \vec{v}, t) \Rightarrow \psi(\vec{v})$

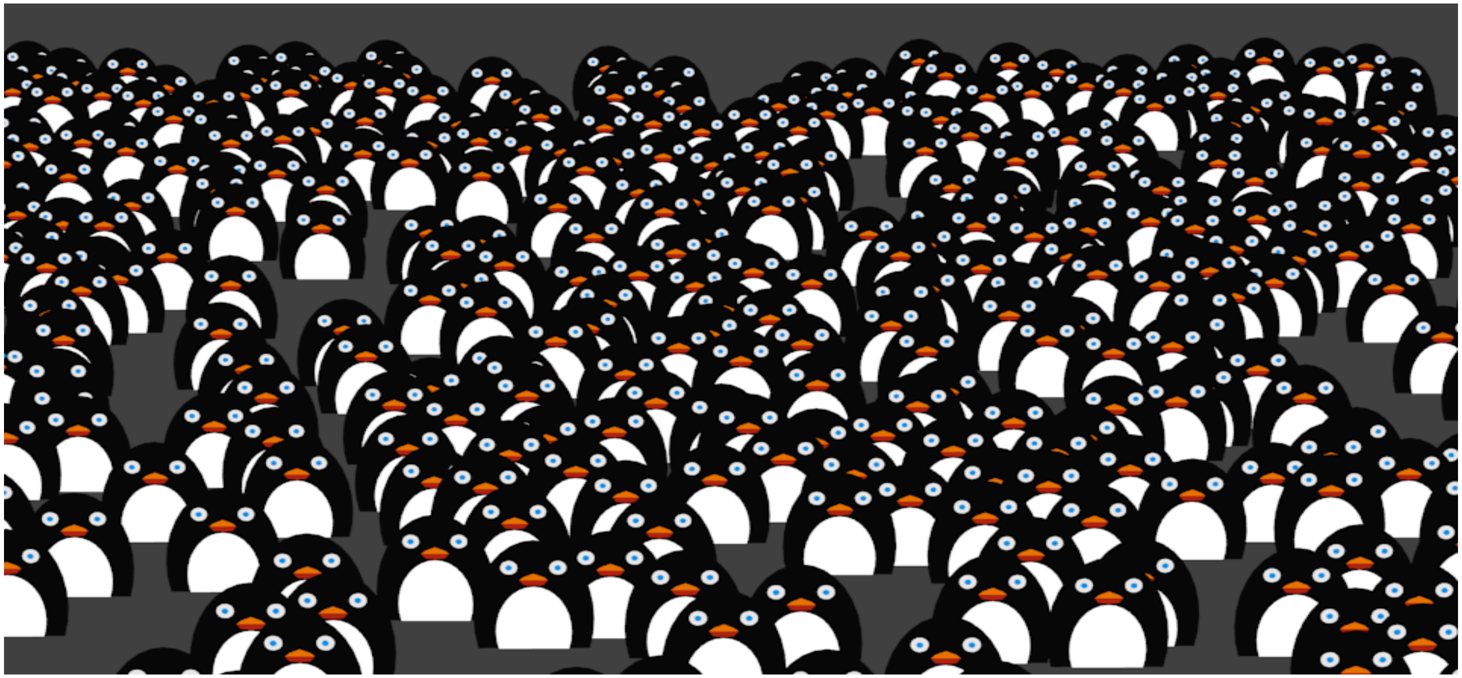
Phase space density





- **Form of phase space distorted by non-linearity**
- **Local phase space density is conserved**
- **Global density is changed (e.g. beam size)**

Phase space density



- **Local** phase space density is conserved (number of neighbours)
- How do we describe the evolution of the distribution ?

Boltzmann equation

Time evolution of $\psi(\vec{x}, \vec{v}, t)$:

$$\frac{d\psi}{dt} = \underbrace{\frac{\partial\psi}{\partial t}}_{\text{time change}} + \underbrace{\vec{v}\cdot\frac{\partial\psi}{\partial\vec{x}}}_{\text{space change}} + \underbrace{\frac{1}{m}\vec{F}(\vec{x},t)\cdot\frac{\partial\psi}{\partial\vec{v}}}_{\text{v change, force F}} + \underbrace{\Omega(\psi)}_{\text{collision}}$$

Without collisions and stationary, it becomes Vlasov-equation:

$$\frac{d\psi}{dt} = \frac{\partial\psi}{\partial t} + \vec{v}\cdot\frac{\partial\psi}{\partial\vec{x}} + \frac{1}{m}\vec{F}(\vec{x},t)\cdot\frac{\partial\psi}{\partial\vec{v}} = 0$$

Vlasov equation

In physical coordinates $\psi(\vec{x}, \vec{v}, t)$:

$$\frac{d\psi}{dt} = \frac{\partial\psi}{\partial t} + \vec{v} \cdot \frac{\partial\psi}{\partial\vec{x}} + \frac{1}{m}\vec{F}(\vec{x}, t) \cdot \frac{\partial\psi}{\partial\vec{v}} = 0$$

$\vec{F}(\vec{x}, t)$ **is force of the field (mode) on the particles**

No binary collisions between particles

INTERLUDE

Why is the Vlasov equation useful ?

$$\frac{d\psi}{dt} = \frac{\partial\psi}{\partial t} + \vec{v}\cdot\frac{\partial\psi}{\partial\vec{x}} + \frac{1}{m}\vec{F}(\vec{x},t)\cdot\frac{\partial\psi}{\partial\vec{v}} = 0$$

$\vec{F}(\vec{x},t)$ **can be forces introduced by impedances, beam-beam effects, etc. From the solution one can determine whether a disturbance is growing (instability, negative imaginary part of frequency) or decaying (stability, positive imaginary part of frequency).**

INTERLUDE

Why is the Vlasov equation useful ?

$$\frac{d\psi}{dt} \quad = \quad \frac{\partial\psi}{\partial t} + \vec{v}\cdot\frac{\partial\psi}{\partial\vec{x}} + \frac{1}{m}\vec{F}(\psi,\vec{x},t)\cdot\frac{\partial\psi}{\partial\vec{v}} \quad = \quad 0$$

strictly speaking: $\vec{F}(\vec{x},t)$ are given by <u>external</u> forces. When a particle interacts strongly with the <u>collective</u> forces produced by the other particles, they can be treated the same as external forces.

INTERLUDE

Why is the Vlasov equation useful ?

$$\frac{d\psi}{dt} = \frac{\partial\psi}{\partial t} + \vec{v}\cdot\frac{\partial\psi}{\partial\vec{x}} + \frac{1}{m}\vec{F}(\vec{x},t)\cdot\frac{\partial\psi}{\partial\vec{v}} = 0$$

- **This is the basis for treatment of collective effects**
- **Warning, it does not apply for:**

  Dissipative forces (gas scattering, IBS, ...)

  Random forces (e.g. radiation ..)
- **Would need "Fokker-Planck" equation**

Back to Plasma Oscillations

For our problem we need:

for the force $\vec{F}$ (depending on field $\vec{E}$):

$$\vec{F} = e \cdot \vec{E}$$

for the field $\vec{E}$ (depending on potential Φ):

$$\vec{E} = -\nabla\Phi$$

for the potential Φ (depending on distribution ψ):

$$\Delta\Phi = -\frac{\rho}{\epsilon_0} = -\frac{e}{\epsilon_0}\int \psi dv$$

Plasma oscillations

Therefore:

$$\frac{d\psi}{dt} \quad = \quad \frac{\partial \psi}{\partial t} + \vec{v} \cdot \frac{\partial \psi}{\partial \vec{x}} + \frac{1}{m} \vec{E}(\vec{x}, t) \cdot \frac{\partial \psi}{\partial \vec{v}} \quad = \quad 0$$

and:

$$\Delta \Phi = \frac{e}{\epsilon_0} \int \psi dv$$

Coupled equations: perturbation produces field which acts back on perturbation.
Do we find a solution ?

Plasma oscillations

Assume a small non-stationary perturbation ψ_1 on the stationary distribution $\psi_0(\vec{v})$:

$$\psi(\vec{x}, \vec{v}, t) = \psi_0(\vec{v}) + \psi_1(\vec{x}, \vec{v}, t)$$

Then we get:

$$\frac{d\psi}{dt} = \frac{\partial \psi_1}{\partial t} + \vec{v} \cdot \frac{\partial \psi_1}{\partial \vec{x}} + \frac{1}{m} \vec{E}(\vec{x}, t) \cdot \frac{\partial \psi_0}{\partial \vec{v}} = 0$$

and:

$$\Delta \Phi = -\frac{\rho}{\epsilon_0} = -\frac{e}{\epsilon_0} \int \psi_1 dv$$

Plasma oscillations

$$\psi_1(\vec{x}, \vec{v}, t) \implies \vec{E}(\vec{x}, t) \implies \psi_1(\vec{x}, \vec{v}, t) \implies \ldots$$

- **Density perturbation produces electric field**
- **Electric field acts back and changes density perturbation**
- **Change with time ..**
- **How can we attack that ?**

Plasma oscillations - Vlasov's approach

Expand as double Fourier transform:$^{*)}$

$$\psi_1(\vec{x},\vec{v},t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\tilde{\psi}_1(k,\vec{v},\omega)e^{i(kx-\omega t)}dkd\omega$$

$$\Phi(\vec{x},\vec{v},t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\tilde{\Phi}(k,\vec{v},\omega)e^{i(kx-\omega t)}dkd\omega$$

and apply to Vlasov equation

$^{*)}$ **Remember: we assumed the field (mode) of the form:**
$E(x,t) = E_0 e^{i(kx-\omega t)}$

Plasma oscillations

Assuming a perturbation as above, the condition for a solution is:

$$1 + \frac{e^2}{\epsilon_0 m k} \int \frac{\partial \psi_0 / \partial v}{(\omega - kv)} dv = 0$$

This is the Dispersion Relation for plasma waves
i.e. relation between frequency (ω) and wavelength (k)

Plasma oscillations

Looking at this relation:

- **It depends on the (velocity) distribution** ψ
- **It depends on the slope of the distribution** $\partial\psi_0/\partial v$
- **The effect is strongest for velocities close to the wave velocity, i.e.** $v \approx \frac{\omega}{k}$

Plasma oscillations

Looking at this relation:

- **It depends on the (velocity) distribution** ψ
- **It depends on the slope of the distribution** $\partial\psi_0/\partial v$
- **The effect is strongest for velocities close to the wave velocity, i.e.** $v \approx \frac{\omega}{k}$

→ **There seems to be a complication (singularity) at** $v \equiv \frac{\omega}{k}$

Can we deal with this problem ?

Dealing with the singularity

- **Handwaving argument (Vlasov):**
  - **In practice ω is never real (collisions !)**
- **Optimistic argument (Bohm et al.):**
  - $\partial\psi_0/\partial v \;=\; 0$ **where** $v \equiv \frac{\omega}{k}$
- **Alternative approach (van Kampen):**
  - **Search for stationary solutions (normal mode expansion)**
  - **Continous versus discrete modes (not treated here)**
- **Better argument (Landau):**
  - **Initial value problem with perturbation $\psi_1(\vec{x}, \vec{v}, t)$ at $t \;=\; 0$, (time dependent solution with complex ω)**
  - **Solution: in time domain use Laplace transformation**

    in space domain use Fourier transformation

Plasma oscillations - Landau's approach

Fourier transform in space domain:

$$\tilde{\psi}_1(k, \vec{v}, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \psi_1(\vec{x}, \vec{v}, t) e^{i(kx)} dx$$

$$\tilde{E}(k, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} E(\vec{x}, t) e^{i(kx)} dx$$

and Laplace transform in time domain:

$$\Psi_1(k, \vec{v}, p) = \int_{0}^{+\infty} \tilde{\psi}_1(k, \vec{v}, t) e^{(-pt)} dt$$

$$\mathcal{E}(k, p) = \int_{0}^{+\infty} \tilde{E}(k, t) e^{(-pt)} dt$$

Plasma oscillations

In Vlasov equation and after some algebra (see books) this leads to the modified dispersion relation:

$$1 + \frac{e^2}{\epsilon_0 m k}\left[P.V.\int \frac{\partial\psi_0/\partial v}{(\omega - kv)}dv - \frac{i\pi}{k}\left(\frac{\partial\psi_0}{\partial v}\right)_{v=\omega/k}\right] = 0$$

P.V. refers to "Cauchy Principal Value"
Second term only in Landau's treatment → responsible for damping

Plasma oscillations

Evaluating the term:

$$-\frac{i\pi}{k}\left(\frac{\partial\psi}{\partial v}\right)_{v=\omega/k}$$

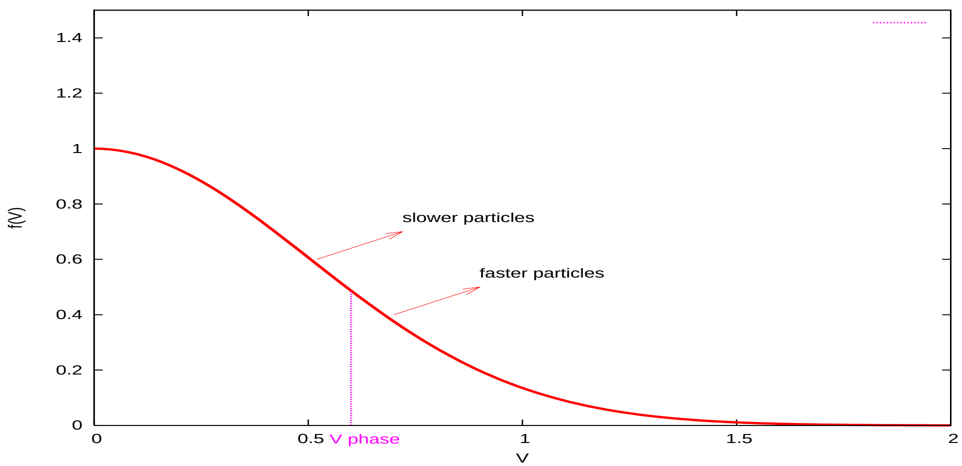
- ω **is complex and the imaginary part becomes:**

$$Im(\omega) = \omega_i = \frac{\pi}{2}\frac{\omega_p e^2}{\epsilon_0 m k^2}\left(\frac{\partial\psi}{\partial v}\right)_{v=\omega/k}$$

- **Get a damping (without collisions) if:** $\left(\frac{\partial\psi}{\partial v}\right)_{v=\omega/k} < 0$

→ **Landau Damping**

Velocity distribution



- **Distribution of particle velocities (e.g. Maxwellian distribution)**
- **More "slower" than "faster" particles ➡ damping**
- **More "faster" than "slower" particles ➡ anti-damping**

Warning: a paradox

For a bar discussion →

If this is true:

Should it not be possible to go to a Lorentz frame which is moving relative to the particles faster than the wave (phase-) velocity ?

In this frame we have always anti-damping !!

Is this true ???

Now what about accelerators ???

- **Landau damping in plasmas, all right**
- **Physical origin rather simple**
- **How to apply it in accelerators ?**
- **We have:**
  - **No plasmas but beams**
  - **No distribution of velocity, but tune**
  - **No electrons, but ions (e.g. p)**
  - **Also transverse oscillations**

Now what about accelerators ???

- **How to apply it in accelerators ?**
- **Can be formally solved using Vlasov equation, but physical interpretation very fuzzy (and still debated ..)**
- **Different (more intuitive) treatment (following Chao, Hofmann, Hereward, Sagan)**
- **Look now at:**
  - **Beam response to excitation**
  - **Beam transfer function and stability diagrams**
  - **Phase mixing**
  - **Conditions and tools for stabilization, problems**

Response of a beam to excitations

How does a beam respond to an external excitation?

Consider a harmonic, linear oscillator with frequency ω driven by an external sinusoidal force $f(t)$ with frequency Ω: The equation of motion is:

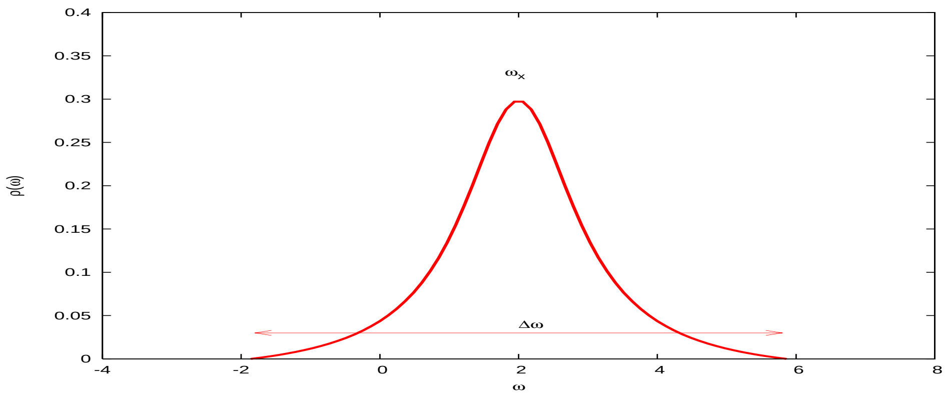
$$\ddot{x} + \omega^2 x = A\ cos\Omega t = f(t)$$

for initial conditions $x(0) = 0$ and $\dot{x}(0) = 0$ the solution is:

$$x(t) = -\frac{A}{(\Omega^2 - \omega^2)}(cos\ \Omega t \quad \underbrace{-\ cos\ \omega t}_{x(0)=0,\dot{x}(0)=0}\ )$$

Response of a beam to excitations

The beam consists of an ensemble of oscillators with different frequencies ω **with a distribution** $\rho(\omega)$ **and a spread** $\Delta\omega$**.**
Number of particles per frequency band:
$\rho(\omega) \; = \; \frac{1}{N} dN/d\omega$ **with** $\int_{-\infty}^{\infty} \rho(\omega) d\omega = 1$



reminder: for a transverse (betatron motion) ω_x **is the tune !**

IMPORTANT MESSAGE !

- $\rho(\omega)$ **is distribution of <u>external focusing frequencies</u> !**
  - **Transverse, bunched and unbunched beams: betatron tune**
  - **Longitudinal, bunched beams: synchrotron tune**
  - **Longitudinal, unbunched beams: ??? (see later !)**
- $\Delta\omega$ **is spread of <u>external focusing frequencies</u> !**

Response of a beam to excitations

The beam consists of an ensemble of oscillator with different frequencies ω **with a distribution** $\rho(\omega)$**, number of particles per frequency band:**

$\rho(\omega) \ = \ \frac{1}{N} dN/d\omega$ **with** $\int_{-\infty}^{\infty} \rho(\omega) d\omega = 1$

The average beam response (centre of mass) is then:

$$< x(t) >= \int_{-\infty}^{\infty} x(t) \rho(\omega) d\omega =$$

$$< x(t) >= - \int_{-\infty}^{\infty} \left[\frac{A}{(\Omega^2 - \omega^2)} (cos\ \Omega t \ - \ cos\ \omega t) \right] \rho(\omega) d\omega$$

Response of a beam to excitations

We can re-write (simplify) the expression

$$< x(t) >= -\int_{-\infty}^{\infty} \left[\frac{A}{(\Omega^2 - \omega^2)} (cos\ \Omega t\ -\ cos\ \omega t) \right] \rho(\omega) d\omega$$

for a narrow beam spectrum around a frequency ω_x (tune) and the driving force near this frequency $\Omega \approx \omega_x$ *)

$$< x(t) >= -\frac{A}{2\omega_x} \int_{-\infty}^{\infty} \left[\frac{1}{(\Omega - \omega)} (cos\ \Omega t\ -\ cos\ \omega t) \right] \rho(\omega) d\omega$$

For the further evaluation we transform variables from ω to $u = \omega - \Omega$, and assume that Ω is complex: $\Omega = \Omega_r + i\Omega_i$

*) **justified later ... (but you may already guess !)**

Response of a beam to excitations

We get now two contributions to the integral:

$$< x(t) > \quad = \quad -\frac{A}{2\omega_x} cos(\Omega t) \int_{-\infty}^{\infty} du \;\, \rho(u+\Omega) \frac{1 - cos(ut)}{u}$$
$$+\frac{A}{2\omega_x} sin(\Omega t) \int_{-\infty}^{\infty} du \;\, \rho(u+\Omega) \frac{sin(ut)}{u}$$

This avoids singularities for $u = 0$

We are interested in long term behaviour,

i.e. $t \rightarrow \infty$**, so we use:**

$$\lim_{t\rightarrow\infty} \frac{sin(ut)}{u} = \pi\delta(u)$$

$$\lim_{t\rightarrow\infty} \frac{1 - cos(ut)}{u} = P.V. \left(\frac{1}{u}\right)$$

Response of a beam to excitations

and obtain for the asymptotic behaviour (back to ω, Ω)$^{*)}$:

$$< x(t) >= \frac{A}{2\omega_x}\left[\pi\rho(\Omega)sin(\Omega t) + cos(\Omega t)P.V.\int_{-\infty}^{\infty} d\omega \frac{\rho(\omega)}{(\omega - \Omega)}\right]$$

The response or Beam Transfer Function has a:

- **Resistive part: absorbs energy from oscillation ➡ damping (would not be there without the term $-cos\ \omega t$)**
- **Reactive part: "capacitive" or "inductive", depending on sign of term relative to driving force**

$^{*)}$ Assuming Ω is complex, we integrate around the pole and obtain a 'principal value P.V.' and a 'residuum' (Sokhotski-Plemelj formula)

Response of a beam to excitations

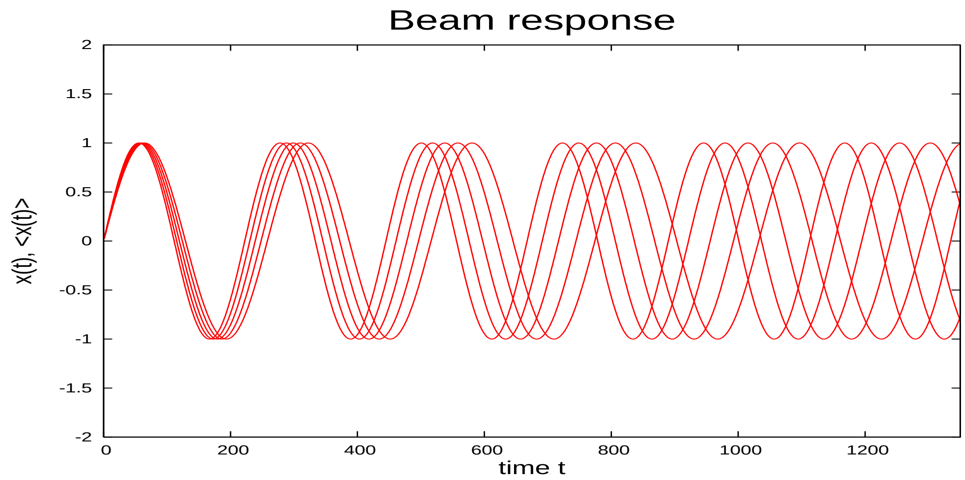
What do we see:

- **The "damping" part only appeared because of the initial conditions !!!**
- **With other initial conditions, we get additional terms in the beam response**
- **I.e. for $x(0) \neq 0$ and $\dot{x}(0) \neq 0$ we may add:**

$$x(0) \int d\omega \rho(\omega) cos(\omega t) + \dot{x}(0) \int d\omega \rho(\omega) \frac{sin(\omega t)}{\omega}$$

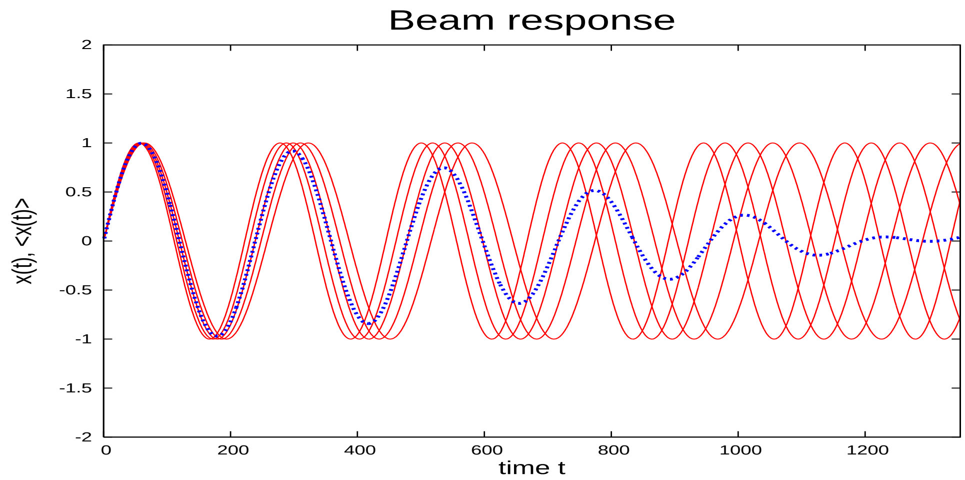
- **Do not participate in the dynamics, what do they do ?**

Particle motion



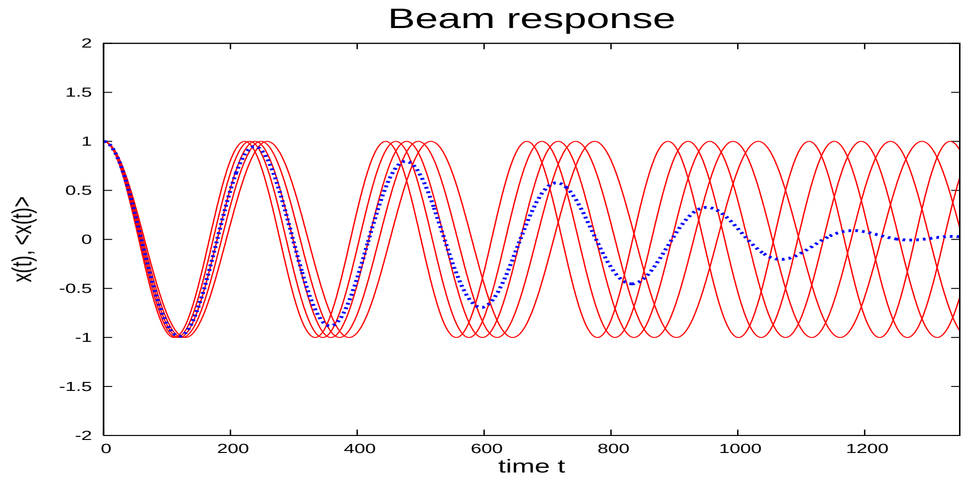
- **Oscillation of particles with different tunes**
- **Initial conditions:** $x(0) = 0$ **and** $\dot{x}(0) \neq 0$

Particle motion



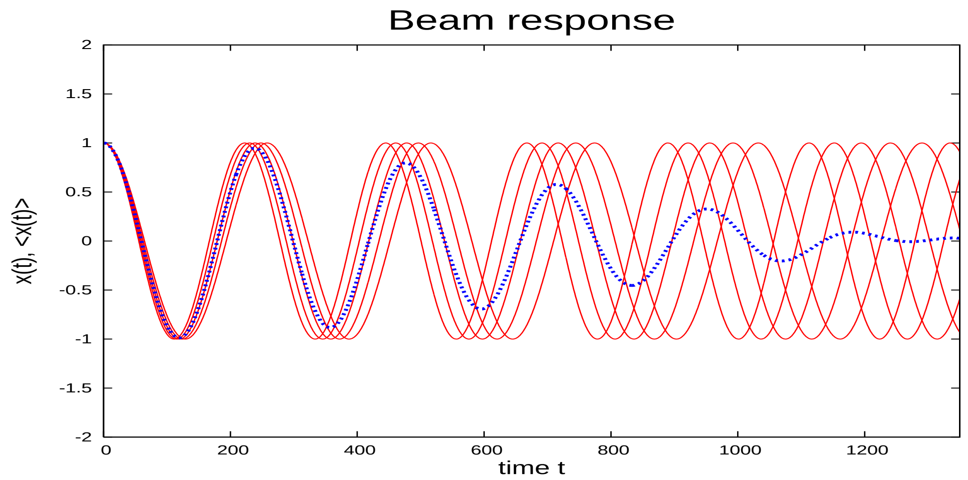
- **Oscillation of particles with different tunes**
- **Initial conditions:** $x(0) = 0$ **and** $\dot{x}(0) \neq 0$
- **Average over particles, centre of mass motion**

Particle motion



- **Oscillation of particles with different tunes**
- **Initial conditions:** $x(0) \neq 0$ **and** $\dot{x}(0) = 0$
- **Average over particles, centre of mass motion**

Particle motion



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- **Average over particles, centre of mass motion**

This is NOT Landau Damping !!

Physics of Landau Damping

Part 2

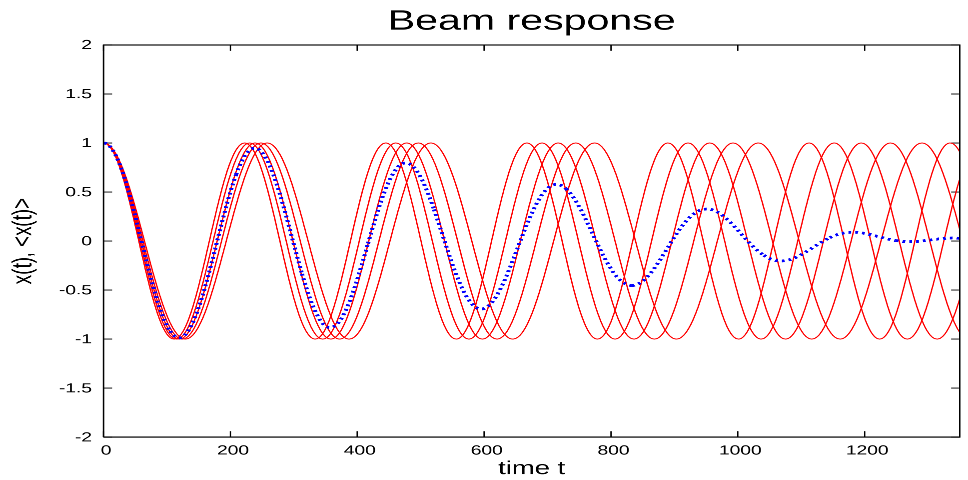
Werner Herr
CERN

http://cern.ch/Werner.Herr/CAS2013/lectures/Trondheim_landau.pdf

Interpretation of Landau Damping

- **Initial conditions:** $x(0) = 0$ **and** $\dot{x}(0) = 0$**, beam is quiet**
- **Spread of frequencies** $\rho(\omega)$
- **When an excitation is applied:**
  - **Particles cannot organize into collective response (phase mixing)**
  - **Average response is zero**
  - **The beam is kept stable, i.e. stabilized**

Particle motion



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Interpretation of Landau Damping

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→ **Next : quantitative analysis**

Response of a beam to excitations

For this, we re-write (simplify) the response in complex notation:

$$< x(t) >= \frac{A}{2\omega_x}\left[\pi\rho(\Omega)sin(\Omega t) + cos(\Omega t)P.V.\int_{-\infty}^{\infty} d\omega\frac{\rho(\omega)}{(\omega - \Omega)}\right]$$

becomes:

$$< x(t) >= \frac{A}{2\omega_x}e^{-i\Omega t}\left[P.V.\int d\omega\frac{\rho(\omega)}{(\omega - \Omega)} + i\pi\rho(\Omega)\right]$$

First part describes oscillation with complex frequency Ω

Response of a beam to excitations

Reminds us a few things ....

Since we know the collective motion is described as $e^{(-i\Omega t)}$

For an oscillating solution Ω **must fulfill the relation**

$$1 + \frac{1}{2\omega_x}\left[P.V.\int d\omega \frac{\rho(\omega)}{(\omega - \Omega)} + i\pi\rho(\Omega)\right] = 0$$

This is again a dispersion relation, i.e. condition for oscillating solution.

What do we do with that ??

Well, look where $\Omega_i < 0$ **provides damping !!**

Note: no contribution to damping when Ω **outside spectrum !!**

Simplify by moving to normalized parametrization.
Following Chao's proposal, in the expression:

$$< x(t) >= \frac{A}{2\omega_x} e^{-i\Omega t} \left[P.V. \int d\omega \frac{\rho(\omega)}{(\omega - \Omega)} + i\pi\rho(\Omega) \right]$$

we use again u, but normalized to frequency spread $\Delta\omega$:

with $\quad u = (\omega_x - \Omega) \quad \rightarrow \quad u = \frac{(\omega_x - \Omega)}{\Delta\omega}$

and introduce two functions $f(u)$ and $g(u)$:

$$f(u) = \Delta\omega P.V. \int d\omega \frac{\rho(\omega)}{\omega - \Omega}$$

$$g(u) = \pi\Delta\omega\rho(\omega_x - u\Delta\omega) = \pi\Delta\omega\rho(\Omega)$$

The response with the driving force discussed above:

$$< x(t) >= \frac{A}{2\omega_x \ \Delta\omega} \ e^{-i\Omega t} \left[f(u) + i \cdot g(u) \right]$$

where $\Delta\omega$ is the frequency spread of the distribution.

The expression $f(u) + i \cdot g(u)$ is the Beam Transfer Function

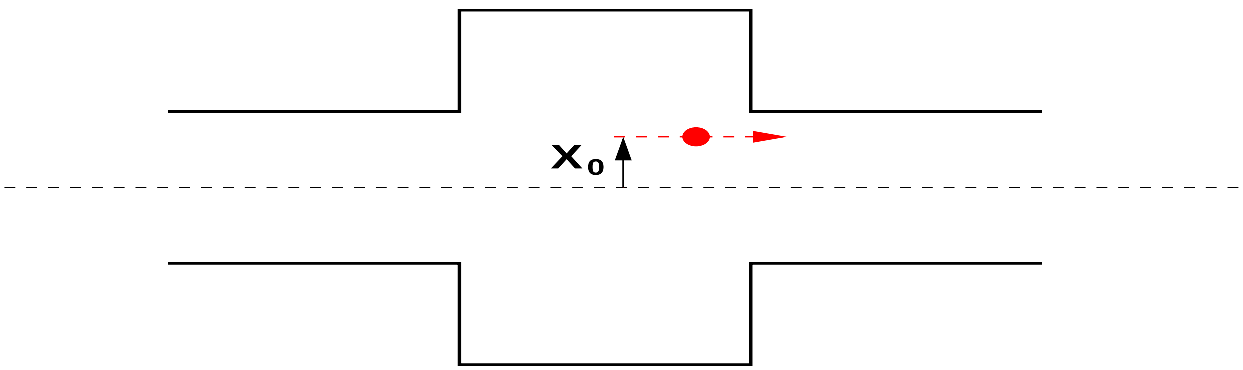
Easier with this to evaluate the different cases and examples ....

For important distributions $\rho(\omega)$ analytical functions f(u) and g(u) exist (see e.g. Chao, Tigner, "Handbook ..")

Will lead us to stability diagrams.

Response of a beam in presence of wake fields

Example: the driving force comes from the displacement of the beam as a whole, i.e. $< x > \; = \; X_0$ **! For example driven by a wakefield or impedance.**



The equation of motion for a particle is then something like:

$$\ddot{x} + \omega^2 x = f(t) = K \cdot < x >$$

where K **is a "coupling coefficient"**

Coupling coefficient K depends on nature of wake field:

- **Purely real:**
  - **Force in phase with the displacement**
  - **e.g. image space charge in perfect conductor**
- **Purely imaginary:**
  - **Force in phase with the velocity**
- **In practice, have both and we can write:**

$$K = 2\omega_x(U - iV)$$

Response of a beam in presence of wake fields

Interpretation:

- **A beam travelling off centre through an impedance induces transverse fields**
- **Transverse fields kick back on all particles in the beam, via:**

$$\ddot{x} + \omega^2 x = f(t) = K \cdot < x >$$

- **If beam moves as a whole (in phase, collectively !) this can grow for** $V > 0$
- **The coherent frequency** Ω **becomes complex and shifted by** $(\Omega - \omega_x)^{*)}$

*) **without impedance:** $\Omega = \omega_x$ **(betatron frequency, i.e.tune)**

For a beam without frequency spread ($\rho(\omega) = \delta(\omega - \omega_x)$) we can easily sum over all particles and for the centre of mass motion $< x >$ we get:

$$< \ddot{x} > + \Omega^2 < x > = f(t) = -2\omega_x(U - iV) \cdot < x >$$

- **For the original coherent motion with frequency Ω this means**
  - **In-phase component U changes the frequency**
  - **Out-of-phase component V creates growth ($V > 0$) or damping ($V < 0$)**

For any $V > 0$ the beam is unstable (even if very small) !!

Response of a beam in presence of wake fields

What happens for a beam with an frequency spread ?

The response (and therefore the driving force) was:

$$< x(t) >= \frac{A}{2\omega_x \ \Delta\omega} \ e^{-i\Omega t} \left[f(u) + i \cdot g(u) \right]$$

Response of a beam in presence of wake fields

The (complex) frequency Ω is now determined by the condition:

$$-\frac{(\Omega - \omega_x)}{\Delta\omega} \quad = \quad \frac{1}{(f(u) + ig(u))}$$

All information about stability contained in this relation !

- **The (complex) frequency difference $(\Omega - \omega_x)$ contains impedance, intensity, γ, ... (see lecture by G. Rumolo).**
- **The right hand side contains information about the frequency spectrum (see definitions for $f(u)$ and $g(u)$).**

Without Landau damping (no frequency spread):

- **If** $\Im(\Omega - \omega_x) \; < \; 0$ **beam is stable**
- **If** $\Im(\Omega - \omega_x) \; > \; 0$ **beam is unstable (growth rate** τ^{-1} **!)**

With Landau damping we have a condition for stability:

$$-\frac{(\Omega - \omega_x)}{\Delta\omega} \quad = \quad \frac{1}{(f(u) + ig(u))}$$

How to proceed to find limits ?

Could find the complex Ω **at edge of stability (**$\tau^{-1} \; = \; 0$ **!)**

→ **Can do a bit more ...**

Stability diagram

Look at the right hand side first.

Take the (real) parameter u in

$$D_1 = \frac{1}{(f(u) + ig(u))}$$

1 Scan u from $-\infty$ to $+\infty$

2 Plot the real and imaginary part of D_1 in complex plane

Why is this formulation interesting ???

The expression:

$$(f(u) + ig(u))$$

is actually the Beam Transfer Function,

i.e. it can be measured !!

- **With its knowledge (more precise: its inverse) we have conditions on $(\Omega - \omega_x)$ for stability**
- **Intensities, impedances, ...**

Example: rectangular distribution:

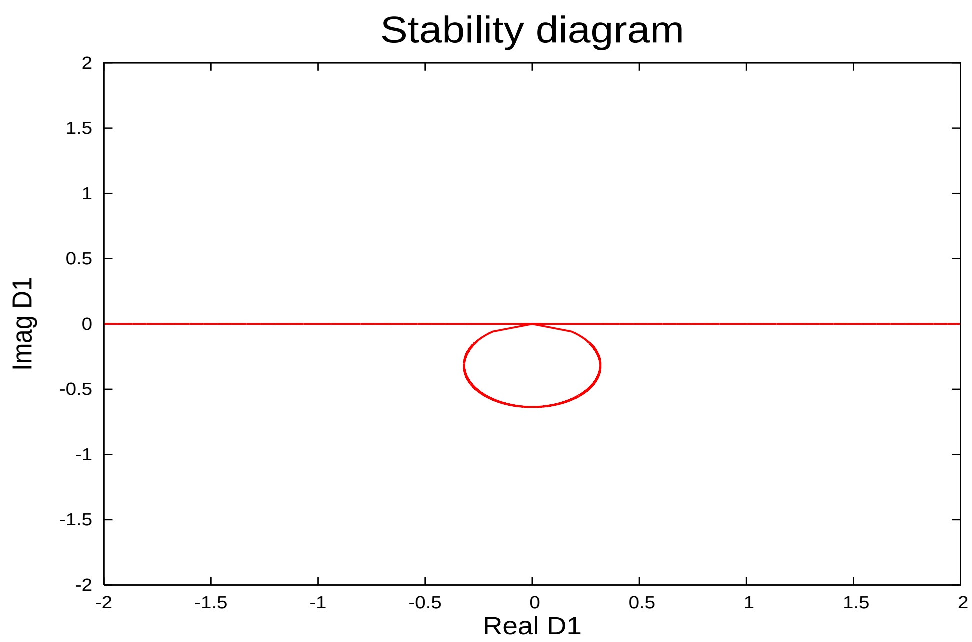
$$\rho(\omega) = \begin{cases} \frac{1}{2\Delta\omega} & \text{for} |\omega - \omega_x| \leq \Delta\omega \\ 0 & \text{otherwise} \end{cases}$$

Step 1: Compute f(u) and g(u) (or look it up, e.g. Chao, Tigner, "Handbook of ...")

$$f(u) = \frac{1}{2} \ln \left| \frac{u+1}{u-1} \right| \qquad g(u) = \frac{\pi}{2} \cdot H(1 - |u|)$$

Step 2: Plot the real and imaginary part of D_1

Stability diagram



- **Real**(D_1) **versus Imag**(D_1) **for rectangular** $\rho(\omega)$
- **This is a Stability Boundary Diagram**
- **Separates stable from unstable regions (stability limit)**

Stability diagram

Take the (real) parameter u in

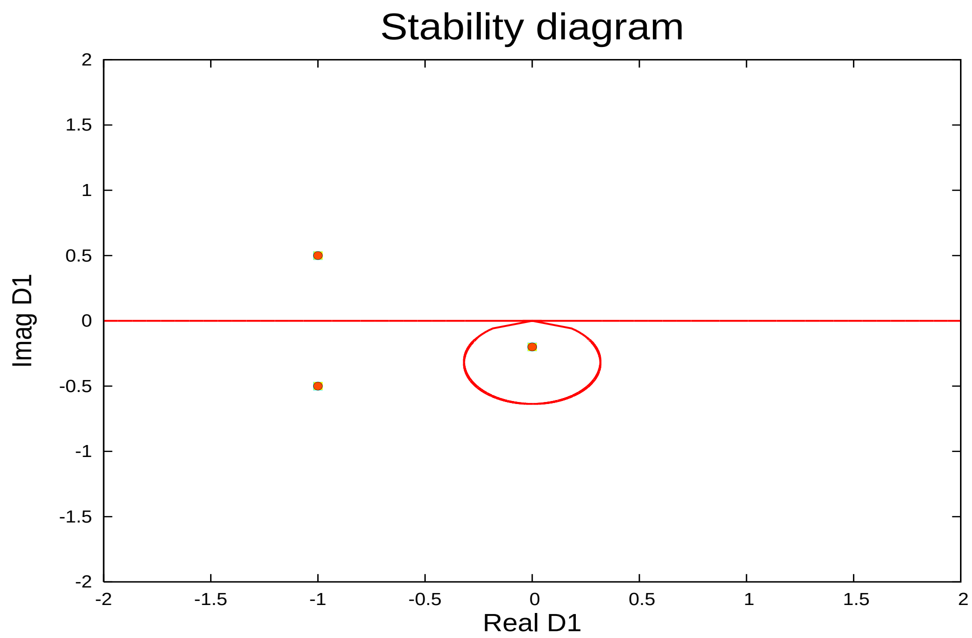
$$D_1 = \frac{1}{(f(u) + ig(u))}$$

1 Scan u from $-\infty$ to $+\infty$

2 Plot the real and imaginary part of D_1 in complex plane

→ **Plot the complex expression of $-\frac{(\Omega - \omega_x)}{\Delta\omega}$ in the same plane as a point (this point depends on impedances, intensities ..)**

Stability diagram



- This is a **Stability Boundary Diagram**
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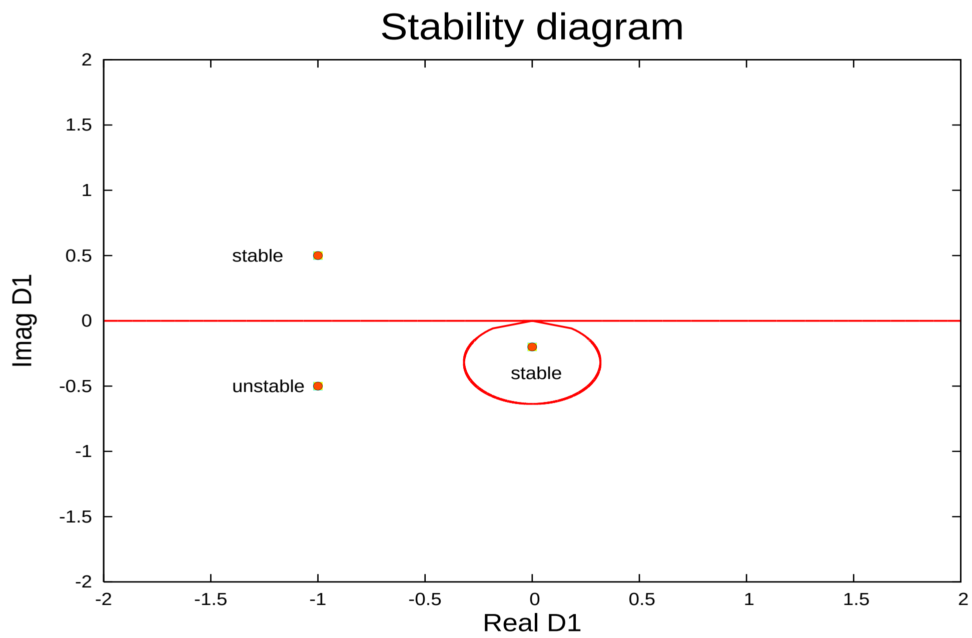
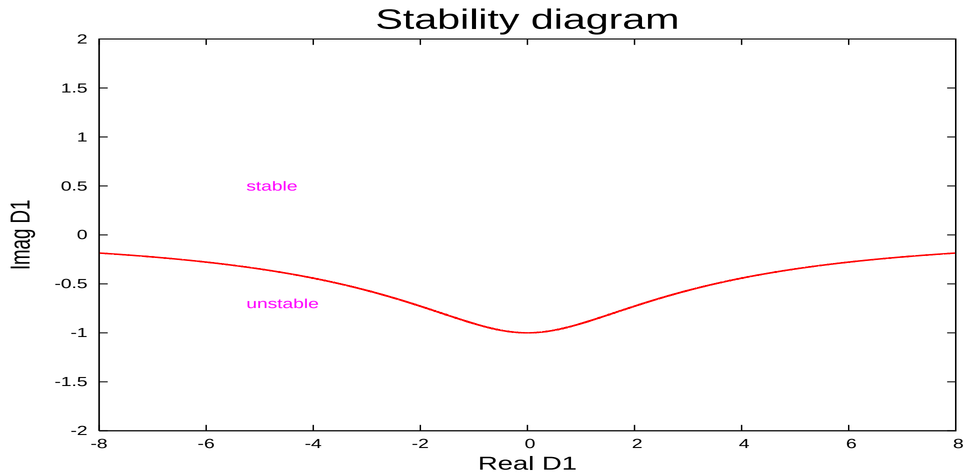
Stability diagram



- **This is a Stability Boundary Diagram**
- **Separates stable from unstable regions**

Stability diagram

For other types of frequency distributions, example:



Real(D_1) **versus Imag**(D_1) **for bi-Lorentz distribution** $\rho(\omega)$
In all cases: half of the complex plane is stable without Landau Damping

Now: transverse instability of unbunched beams

The technique applies directly. Frequency (tune) spread from:

- **Change of revolution frequency with energy spread (momentum compaction)**
- **Change of betatron frequency with energy spread (chromaticity)**

but oscillation depends on mode number n (number of oscillations around the circumference C):

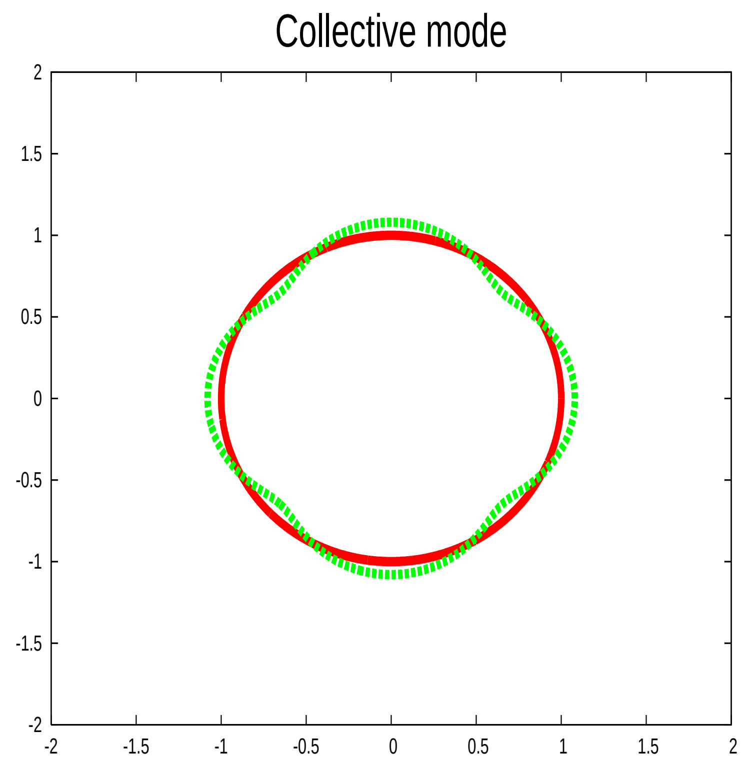
$$\propto \quad exp(-i\Omega t + in(s/C))$$

and the variable u should be written:

$$u \;=\; (\omega_x + n \cdot \omega_0 - \Omega)/\Delta\omega$$

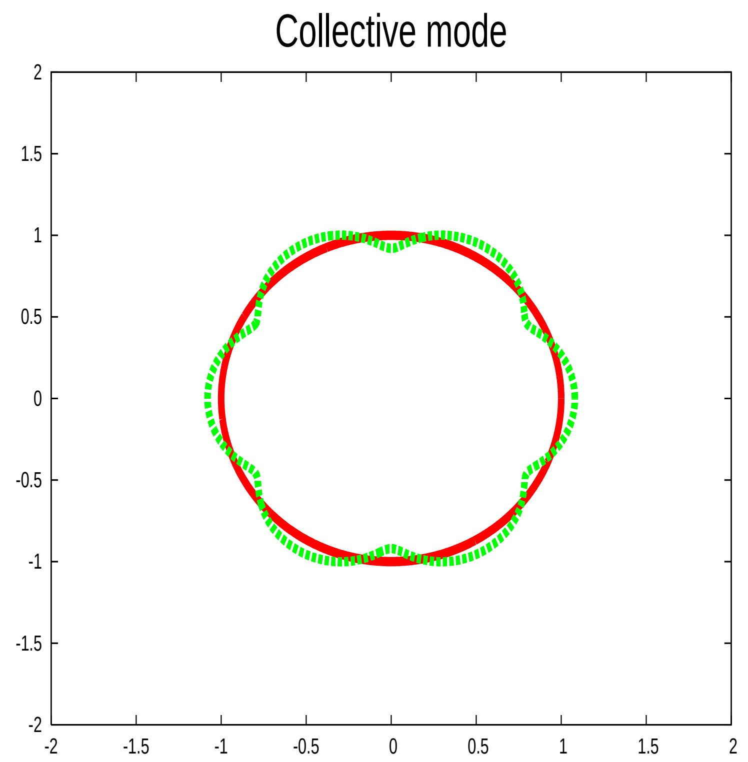
the rest is the same treatment.

Examples: transverse instability of unbunched beams



- **Transverse collective mode with mode index** $n = 4$

Examples: transverse instability of unbunched beams



- **Transverse collective mode with mode index** $n = 6$

What about longitudinal instability of unbunched beams

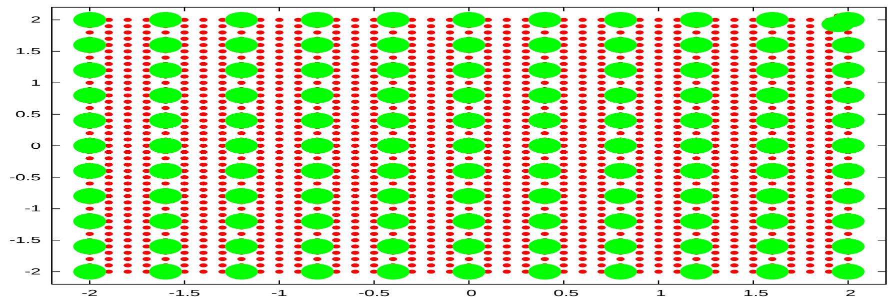
- **No external focusing !**
- **No spread $\Delta\omega$ of focusing frequencies !**
  - **Spread in revolution frequency: related to energy**
  - **Energy excitations directly affect frequency spread**

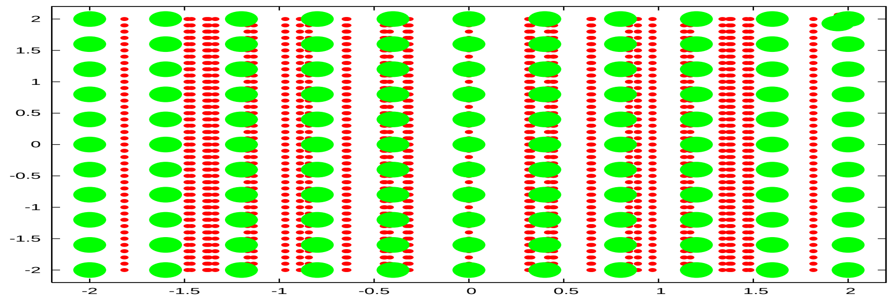
$$\frac{\Delta\omega_{rev}}{\omega_0} = -\frac{\eta}{\beta^2}\frac{\Delta E}{E_0}$$

Frequency distribution by:

$$\rho(\omega_{rev}) \quad \text{and} \quad \Delta\omega_{rev}$$

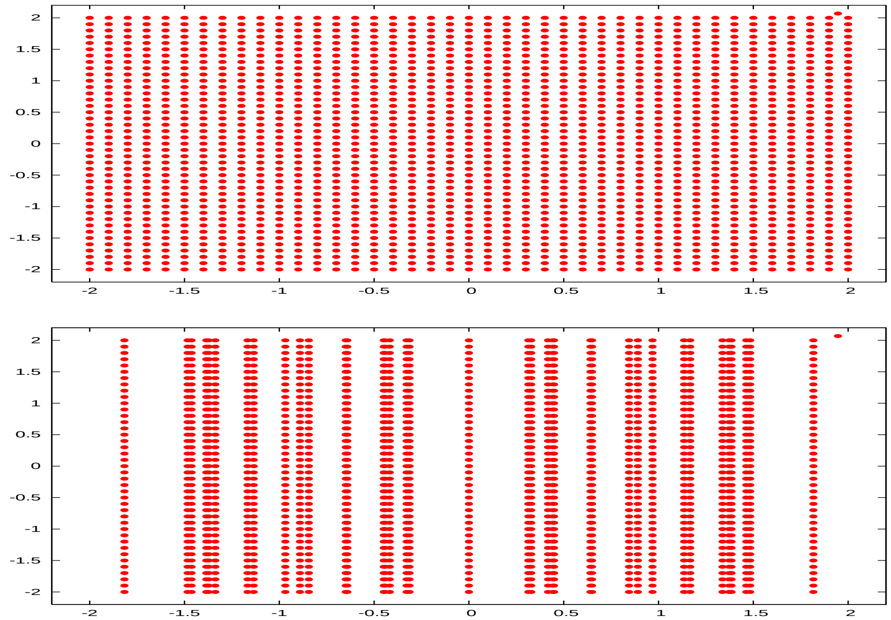
What about longitudinal instability of unbunched beams





- **With and without perturbation in a plasma**

What about longitudinal instability of unbunched beams



- **With and without longitudinal modulation in a beam**

What about longitudinal instability of unbunched beams

No external focusing ($\omega_x = 0$):

$$u = \frac{(\omega_x + n \cdot \omega_0 - \Omega)}{\Delta\omega} \quad \Rightarrow \quad u = \frac{(n \cdot \omega_0 - \Omega)}{n \cdot \Delta\omega}$$

$$-\frac{(\Omega - n \cdot \omega_0)^2}{n^2 \Delta\omega^2} = \frac{1}{(F(u) + iG(u))} = D_1$$

and introduce two new functions $F(u)$ and $G(u)$:

$$F(u) = n \cdot \Delta\omega^2 P.V. \int d\omega_0 \frac{\rho'(\omega_0)}{n \cdot \omega_0 - \Omega}$$

$$G(u) = \pi \Delta\omega^2 \rho'(\Omega/n)$$

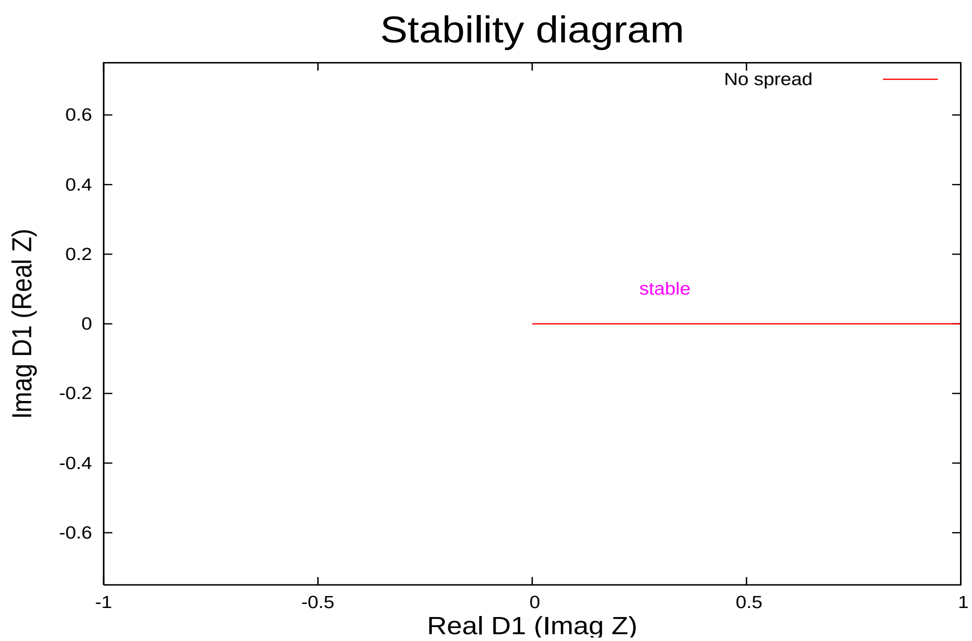
IMPORTANT MESSAGE !

$$-\frac{(\Omega - n \cdot \omega_0)^2}{n^2 \Delta\omega^2} \quad = \quad \frac{1}{(F(u) + iG(u))} \quad = \quad D_1$$

- **The impedance now related to the square of the complex frequency shift** $(\Omega - n \cdot \omega_0)^2$
- **Consequence: no more stable in one half of the plane !**
- **Landau damping always required**

Longitudinal stability - unbunched beam

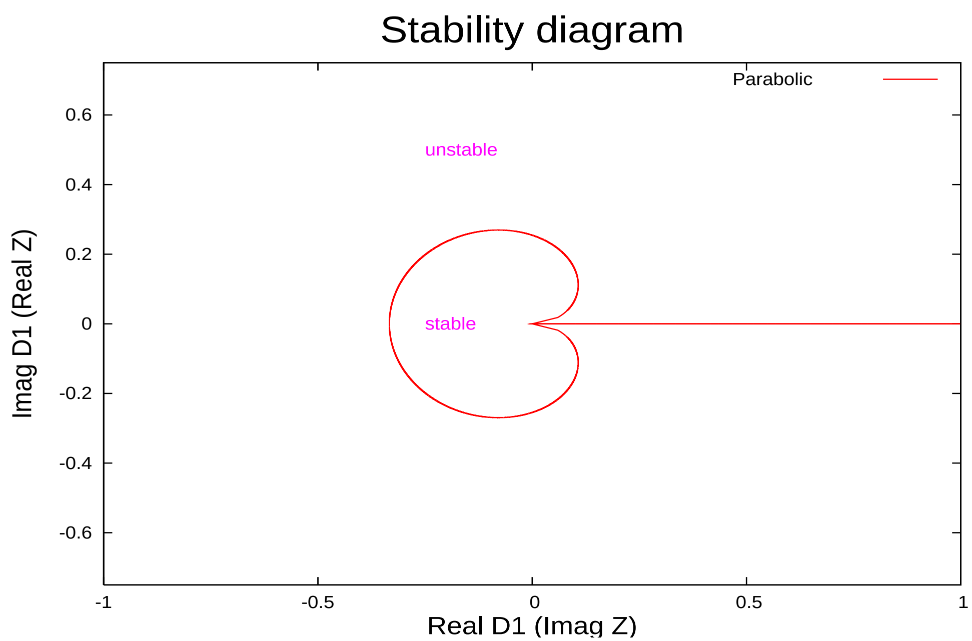
Stability diagram for unbunched beams, longitudinal, no spread:



Real(D_1) **versus Imag**(D_1) **unbunched beam without spread**

Longitudinal stability - unbunched beam

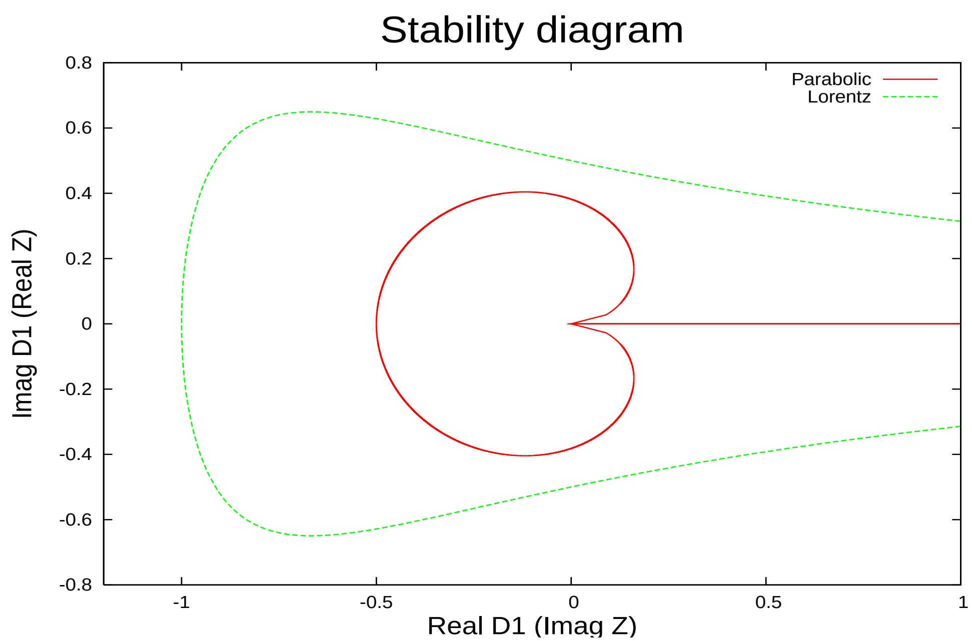
Stability diagram for unbunched beams, longitudinal:



Real(D_1) versus Imag(D_1) for parabolic $\rho(\omega)$ and unbunched beam

Longitudinal stability - unbunched beam

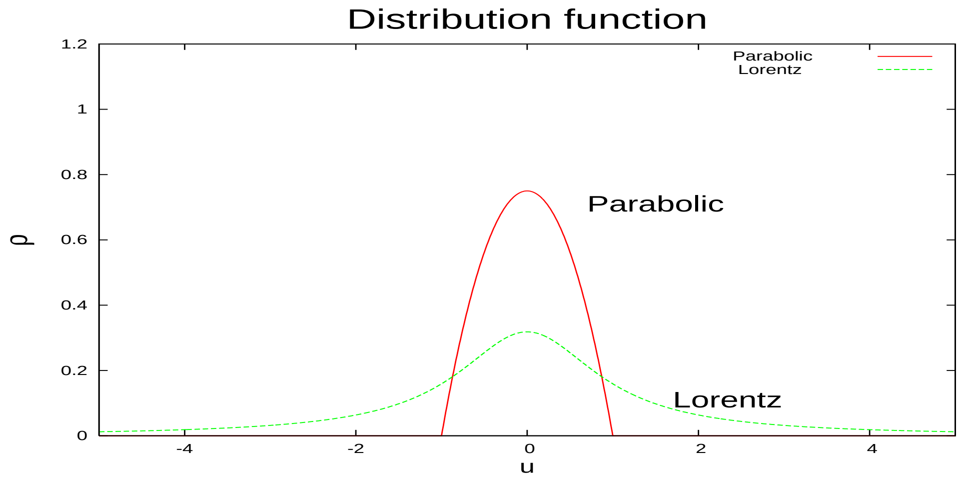
Stability diagram for unbunched beams, longitudinal:



Real(D_1) **versus Imag**(D_1) **for parabolic and Lorentz distribution** $\rho(\omega)$ **and unbunched beam**

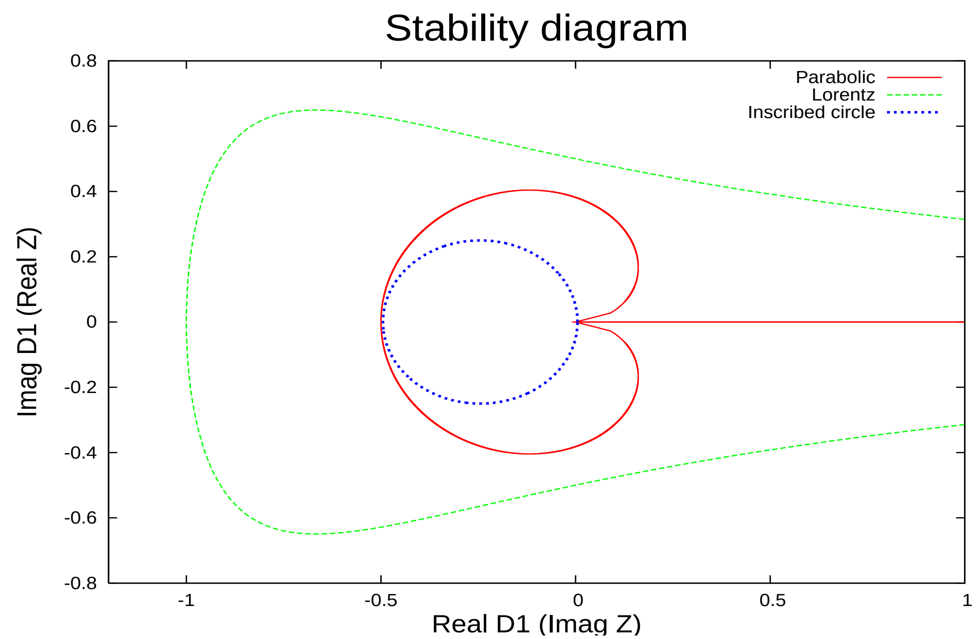
Longitudinal stability - unbunched beam

Why so different stability region:



- **Larger stability provided by tail of frequency distribution ....**
- **What if we do not know exactly the distribution function ?**

Longitudinal stability - unbunched beam



- **Stability boundary relates Z, I, etc. with frequency spread**
- **Can derive criteria for stable or unstable beams**
- **Simplified criterion: inscribe $\approx$ circle as estimate**

Longitudinal stability - unbunched beam

- **For longitudinal stability/instability:**

$$\frac{|Z_{\parallel}|}{n} \leq F \frac{\beta^2 E_0 |\eta_c|}{qI} \left(\frac{\Delta p}{p}\right)^2$$

- **This is the Keil-Schnell criterion, frequency spread from momentum spread and momentum compaction** η_c
- **For given beam parameters define maximum impedance** $\frac{|Z_{\parallel}|}{n}$
- **Can derive similar criteria for other instabilities (see lecture by G. Rumolo)**

Effect of the simplifications

- **We have used a few simplifications in the derivation:**
  - **Oscillators are linear**
  - **Movement of the beam is rigid (i.e. beam shape and size does not change)**
- **What if we consider the "real" cases ?**

  i.e. non-linear oscillators

The case of non-linear oscillators

Consider now a bunched beam, because of the synchrotron oscillation: revolution frequency and betatron spread (from chromaticity) average out !

- **Source of frequency spread: non-linear force**
  - **Longitudinal: sinusoidal RF wave**
  - **Transverse: octupolar or high multipolar field components**

Can we use the same considerations as for an ensemble of linear oscillators ?

The case of non-linear oscillators

NO !

The excited betatron oscillation will change the frequency distribution $\rho(\omega)$ (frequency depends on amplitude) !!

Complete derivation through Vlasov equation.

The equation:

$$< x(t) >= \frac{A}{2\omega_x} e^{-i\Omega t} \left[P.V. \int d\omega \frac{\rho(\omega)}{(\omega - \Omega)} + i\pi\rho(\Omega) \right]$$

becomes:

$$< x(t) >= \frac{A}{2\omega_x} e^{-i\Omega t} \left[P.V. \int d\omega \frac{\partial\rho(\omega)/\partial\omega}{(\omega - \Omega)} + i\pi\partial\rho(\Omega)/\partial\Omega \right]$$

Response in the presence of non-linear fields

Study this configuration for instabilities in the transverse plane

Since the frequency ω depends now on the particles amplitudes J_x and J_y *):

$$\omega_x(J_x, J_y) = \frac{\partial H}{\partial J_x}$$

is the amplitude dependent betatron tune (similar for ω_y). We then have to write:

$$\rho(\omega) \quad \Rightarrow \quad \rho(J_x, J_y)$$

*) **see e.g. "Tools for Non-Linear Dynamics" (W.Herr, this school)**

Response in the presence of non-linear fields

Assuming a periodic force in the horizontal (x) plane and using now the tune (normalized frequency) $Q = \frac{\omega}{\omega_0}$:

$$F_x = A \cdot exp(-i\omega_0 Qt)$$

the dispersion integral can be written as:

$$1 = -\Delta Q_{coh} \int_0^\infty \mathrm{d}J_x \int_0^\infty \mathrm{d}J_y \frac{J_x \frac{\partial \rho(J_x, J_y)}{\partial J_x}}{Q - Q_x(J_x, J_y)}$$

➡ **Then proceed as before to get stability diagram ...**

What happens when bunches are not rigid ?

If particle distribution changes (often as a function of time), obviously the frequency distribution $\rho(\omega)$ changes as well. :

- **Examples:**
  - **Higher order modes**
  - **Coherent beam-beam modes**
- **Treatment requires solving the Vlasov equation (perturbation theory or numerical integration)**

→ **Pragmatic approach (20-20 hindsight): use <u>unperturbed stability region</u> and <u>perturbed complex tune shift</u> ...**

Landau damping as a cure

If the boundary of

$$D_1 \quad = \quad \frac{1}{(f(u) + ig(u))}$$

determines the stability, can we:

- **Increase the stable region by:**
  - **Modifying the frequency distribution** $\rho(\omega)$**, i.e.** $\rho(J_x, J_y)$
  - **Introducing tune spread artificially (octupoles, other high order fields)**

The tune dependence of an octupole (k_3**) can be written as**$^{*)}$**:**

$$Q_x(J_x, J_y) = Q_0 + a \cdot k_3 \cdot J_x + b \cdot k_3 \cdot J_y$$

$^{*)}$ see e.g. "Tools for Non-Linear Dynamics" (W.Herr, this school)

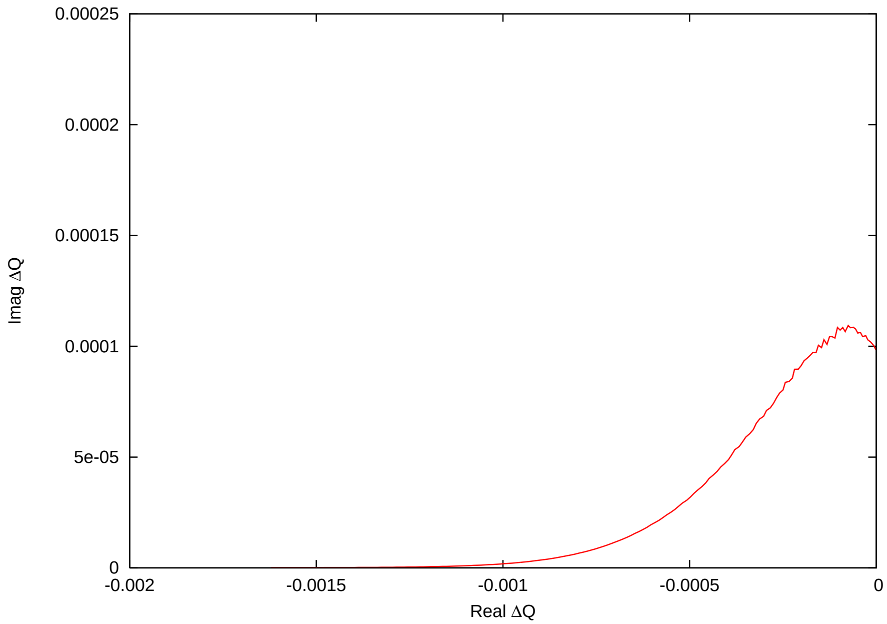
Landau damping as a cure

- **Other sources to introduce tune spread:**
  - **Space charge**
  - **Chromaticity**
  - **High order multipole fields**
  - **Beam-beam effects (colliders only)**

Landau damping as a cure

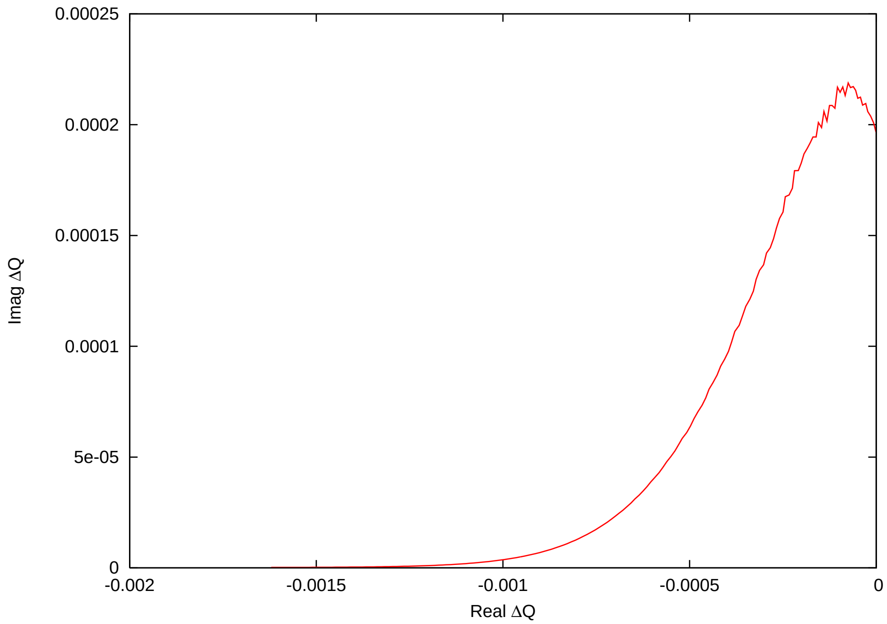
- **Recipe for "generating" Landau damping:**
  - **For a multipole field, compute detuning** $Q(J_x, J_y)$
  - **Given the distribution** $\rho(J_x, J_y)$
  - **Compute the stability diagram by scanning frequency**

Stability diagram with octupoles



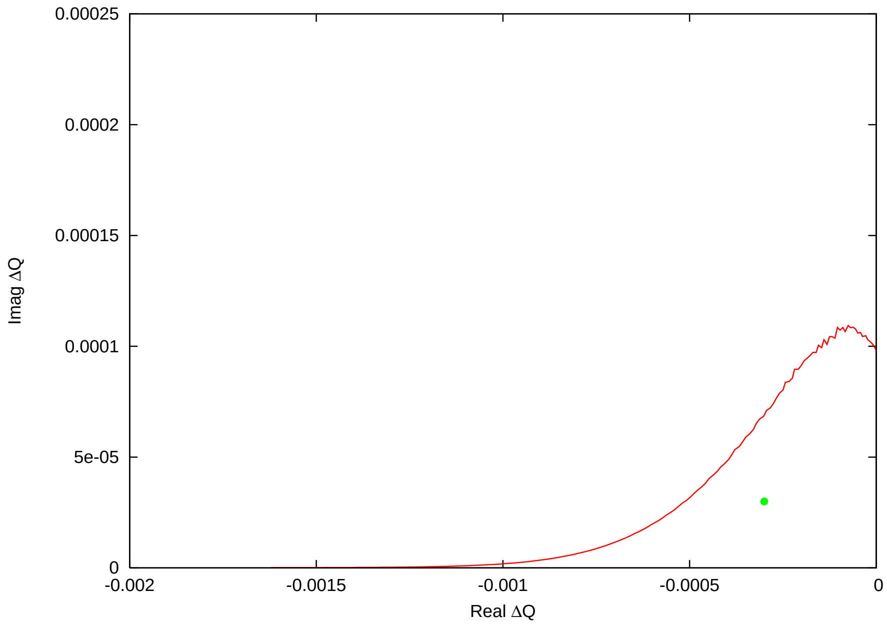
- **Stabilization with octupoles**

Stability diagram with octupoles



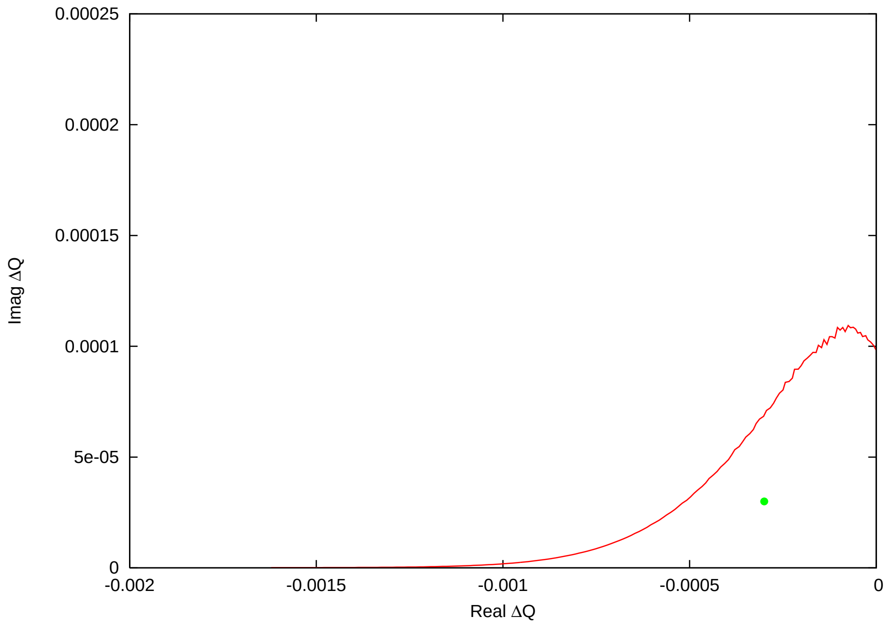
- **Stabilization with octupoles, increased strengths**

Stability diagram with octupoles



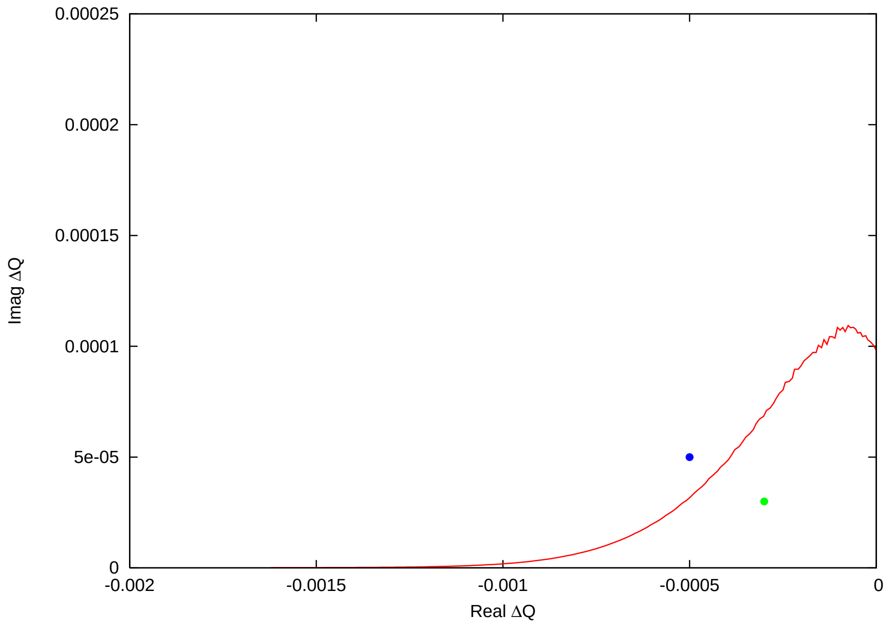
- **Complex coherent tune of an unstable mode**
- **Now in the stable region**

Stability diagram with octupoles



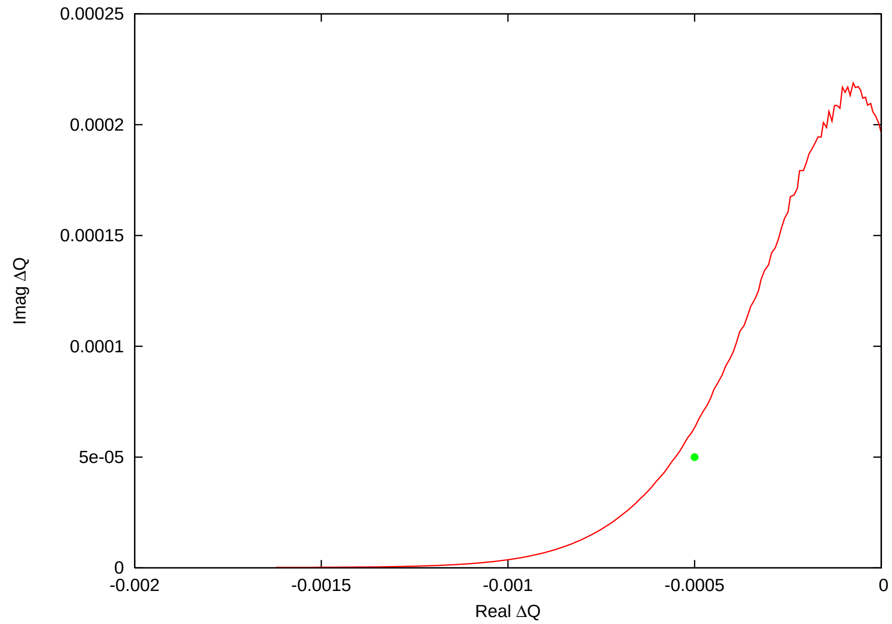
- **Complex coherent tune of an unstable mode**
- **What if we increase the impedance (or intensity) ?**

Stability diagram with octupoles



- **Complex coherent tune of an unstable mode**
- **Now in the unstable region**

Stability diagram with octupoles



- **Complex coherent tune of an unstable mode**
- **Increased octupole strength makes it stable again**

Stability diagram with octupoles

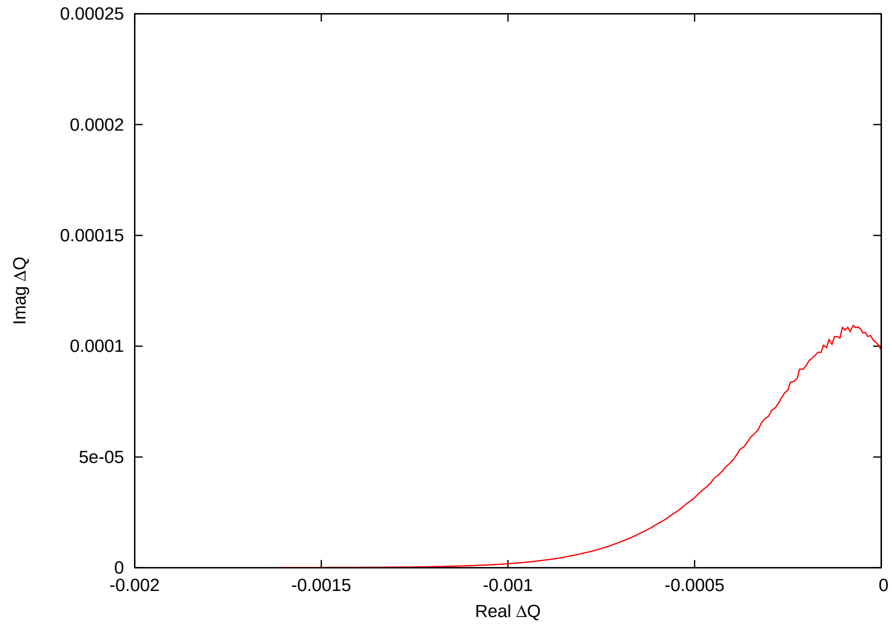
- **Can we increase the octupole strength as we like ??**
- **The downside:**
  - **Octupoles introduce strong non-linearities at large amplitudes**
  - **Not many particles at large amplitudes: requires large strengths**
  - **Can cause reduction of dynamic aperture and life time**
  - **They can change the chromaticity !**
- **The lesson: use them if you have no choice ....**

Another example: Head-Tail modes

(see e.g. Lecture G. Rumolo)

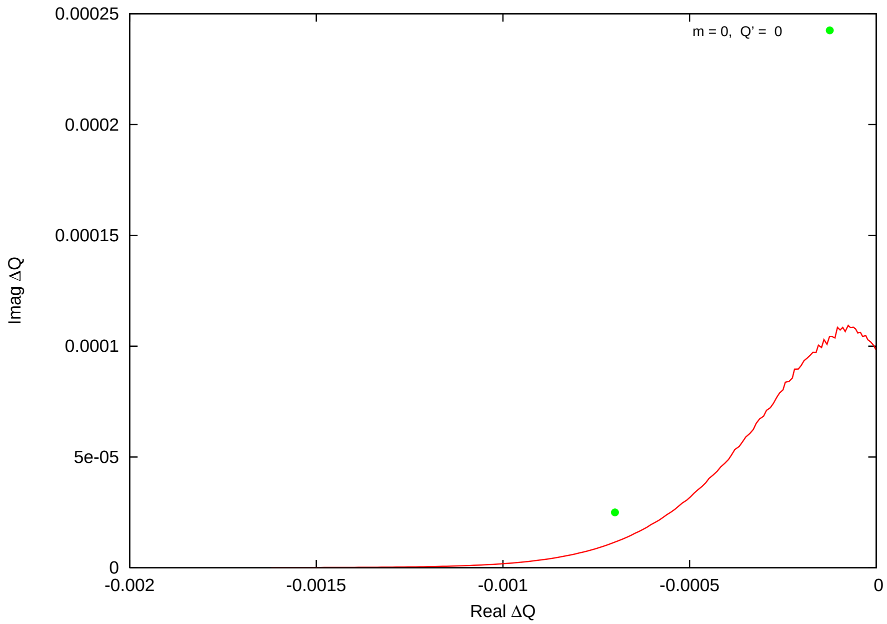
- **For short range wake fields**
- **Broad band impedance**
- **Growth and damping controlled with chromaticity Q'**
  - **Some modes need positive Q'**
  - **Some modes need negative Q'**
  - **Some modes can be damped by feedback (m = 0)**
- **In the control room: juggle with octupoles and Q' ....**

Stability diagram and head-tail modes



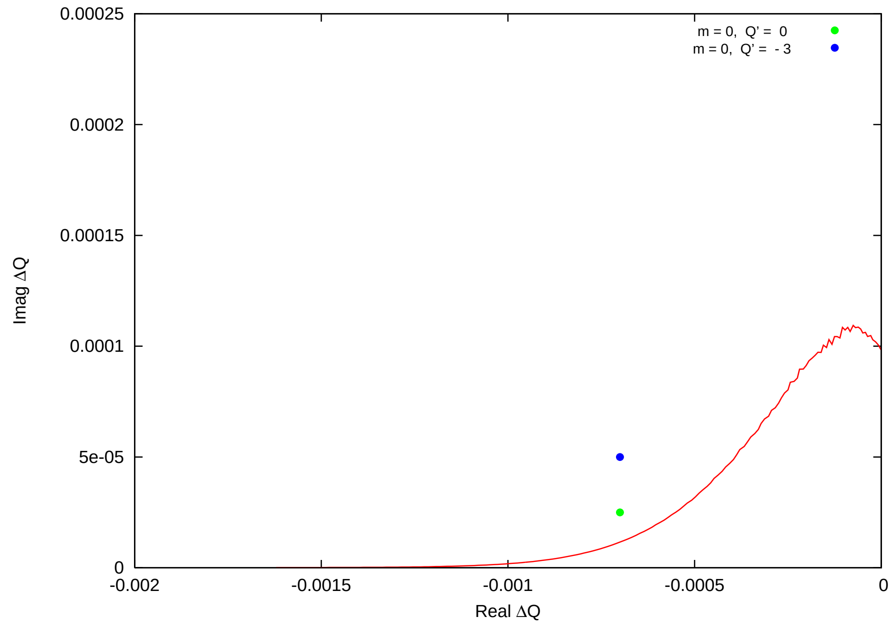
- **Stability region and head-tail modes for different chromaticity**
- **Stabilization with octupoles**

Stability diagram and head-tail modes



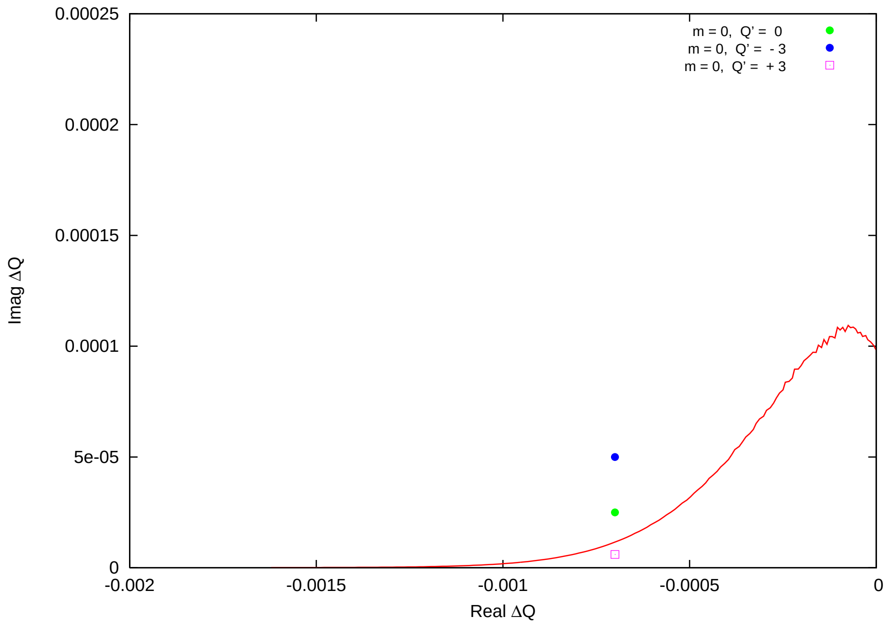
- **Stability region and head-tail modes for different chromaticity**
- **Stabilization with octupoles**

Stability diagram and head-tail modes



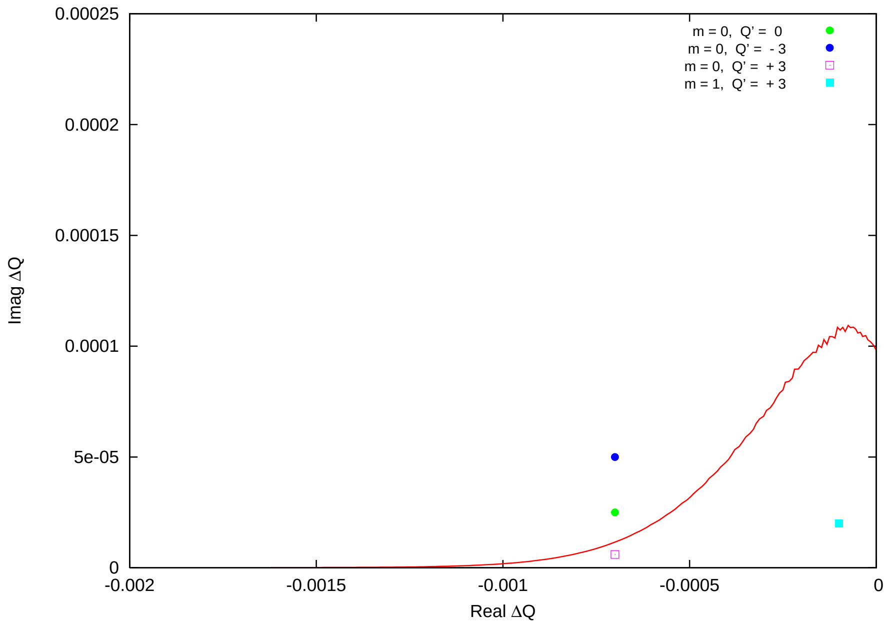
- **Stability region and head-tail modes for different chromaticity**
- **Stabilization with octupoles**

Stability diagram and head-tail modes



- **Head-tail mode m = 0 stabilized with positive chromaticity, (what about higher orders ?)**
- **Stabilization with octupoles**

Stability diagram and head-tail modes

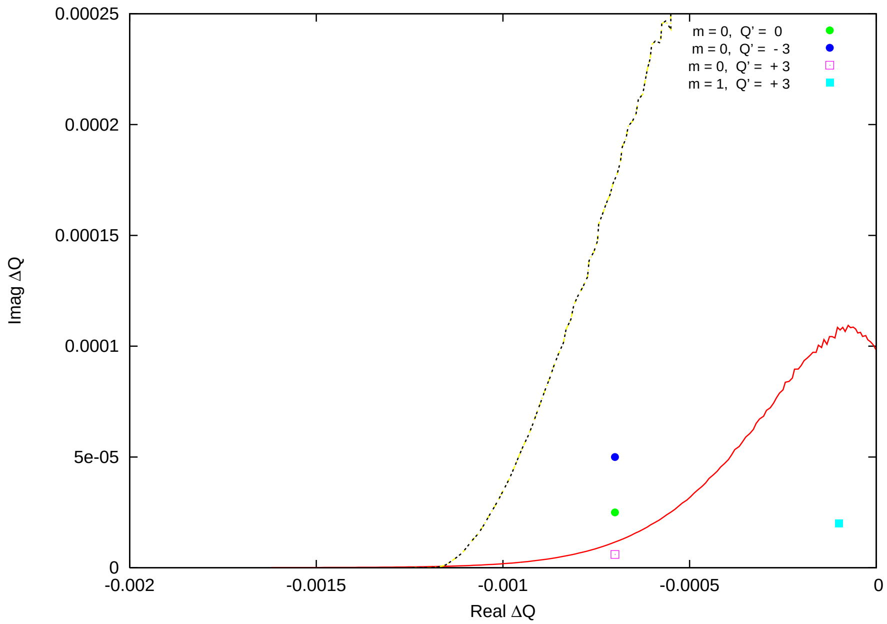


- **Head-tail mode m = 0 stabilized with positive chromaticity, (what about higher orders ?)**
- **Stabilization with octupoles**

Stability diagram with octupoles

- **Would need very large octupole strength for stabilization**
- **The known problems:**
  - **Can cause reduction of dynamic aperture and life time**
  - **Life time important when beam stays in the machine for a long time**
  - **Colliders: life time more than 10 - 20 hours needed ...**
- **Is there another option ?**

Stability diagram and head-tail modes



- **Stability region and head-tail modes for different chromaticity**
- **Stabilization with octupoles or colliding beams**

What makes the difference ... ?

The tune dependence of an octupole can be written as:

$$Q_x(J_x, J_y) = Q_0 + aJ_x + bJ_y$$

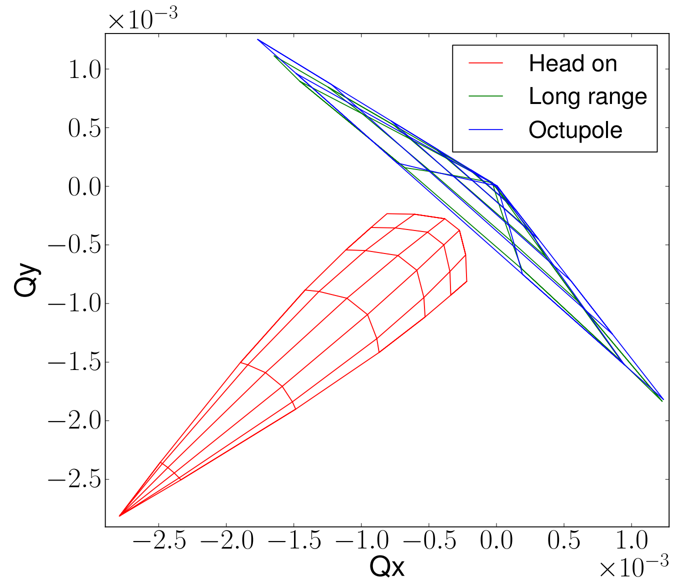
linear in the action (for coefficients, see Appendix).

The tune dependence of a head-on beam-beam collision can be written as*):

with $\alpha = \frac{x}{\sigma^*}$ **we get** $\Delta Q/\xi = \frac{4}{\alpha^2}\left[1 - I_0(\frac{\alpha^2}{4}) \cdot e^{\frac{-\alpha^2}{4}}\right]$

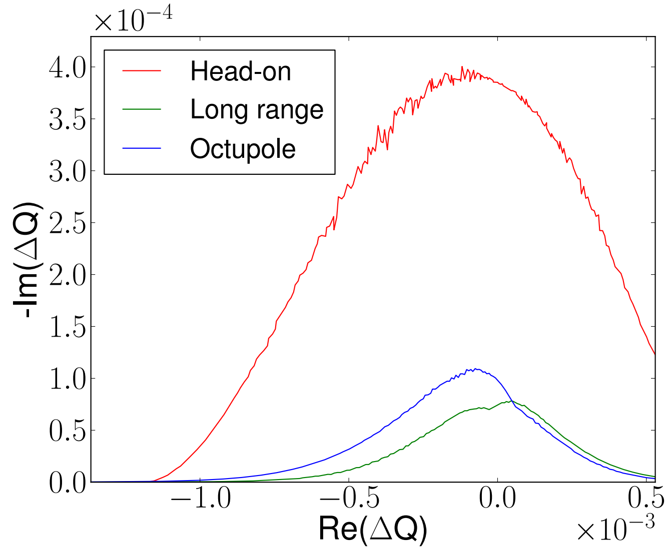
*) see e.g. "Beam-Beam effects" (Tatiana Pieloni, this school)

Response in the presence of non-linear fields



- **Tune footprints for beam-beam and octupoles**
- **Overall tune spread always the same !**
- **But: for beam-beam largest effect for <u>small</u> amplitudes**

Response in the presence of non-linear fields



- **Stability diagrams for beam-beam and octupoles**
- **Stability region very different !**

The Good, the Bad, and the Weird ...

Landau Damping with non-linear fields: Are there any side effects ?

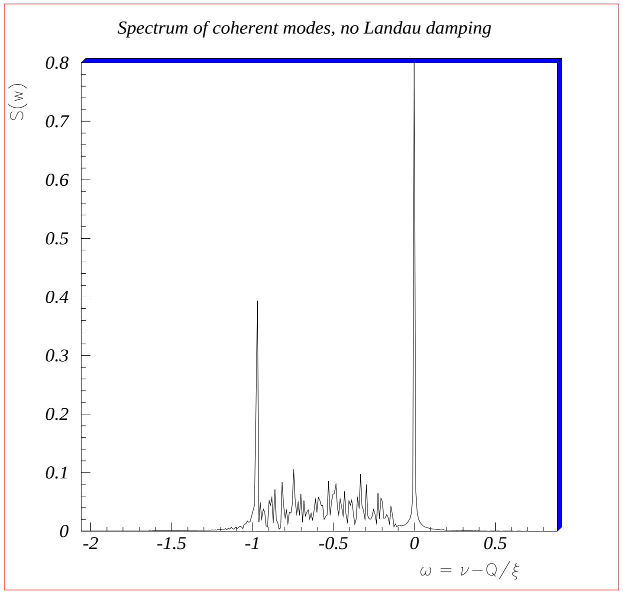
- **Good:**
  - **Stability region increased**
- **Bad:**
  - **Non-linear fields introduced (resonances !)**
  - **Changes optical properties, e.g. chromaticity ... (feed-down !)**
- **Weird:**
  - **Non-linear effects for large amplitudes (octupoles)**
  - **Much better: head-on beam-beam (but only in colliders ...)**

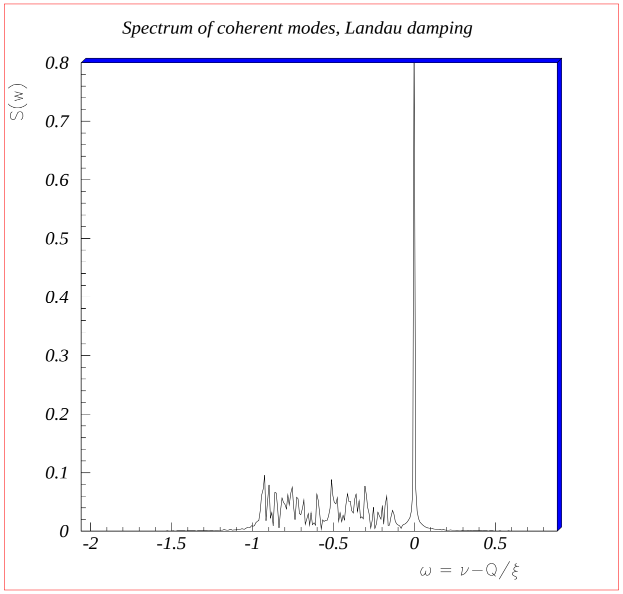
Conditions for Landau "damping"

- **Presence of incoherent frequency (tune) spread**
- **Coherent mode must be inside this spread**

remember the convolution integral of the surfer ...

Beams with and without damping





- **Coherent mode inside or outside the incoherent spectrum**
- **Landau damping restored**

Conditions for Landau "damping"

- **Presence of incoherent frequency (tune) spread**
- **Coherent mode must be inside this spread**

  remember the convolution integral of the surfer ...
- **The same particles must be involved !**

Summary

- **Long history**
- **Different approaches to the mathematical treatment, (needed for rigorous treatment of different configurations)**
- **Many applications (plasmas, accelerators, wind waves, bio-physics, ...)**
- **Very important for hadron accelerators, but should be used with care ...**
- **It works ! It is not a mystery !**

APPENDIX:

Tune shift of an octupole:

The tune dependence of an octupole can be written as:

$$Q_x(J_x, J_y) = Q_0 + aJ_x + bJ_y$$

for the coefficients:

$$\Delta\ Q_x = \left[\frac{3}{8\pi}\int \beta_x^2 \frac{K_3}{B\rho} ds\right] J_x - \left[\frac{3}{8\pi}\int 2\beta_x\beta_y \frac{K_3}{B\rho} ds\right] J_y$$

$$\Delta\ Q_y = \left[\frac{3}{8\pi}\int \beta_y^2 \frac{K_3}{B\rho} ds\right] J_y - \left[\frac{3}{8\pi}\int 2\beta_x\beta_y \frac{K_3}{B\rho} ds\right] J_x$$