Transverse Instabilities

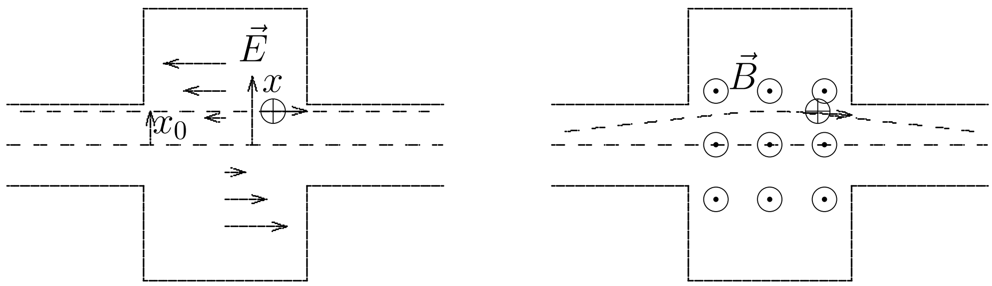
CAS 2005, Trieste; Albert Hofmann

1) Instability mechanisms
2) Transverse impedance
3) Transverse instability with $Q' = 0$
4) Head-tail instability

1) Instability mechanisms

The mechanism of transverse instabilities is in many respect similar to the longitudinal case. The transverse motion of a single particle in a storage ring is determined by the external guide fields consisting here mainly of quadrupole magnets, however also the RF-system, initial conditions and synchrotron radiation have an influence. Many particles in a beam may represent a sizable charge and current which act as a source of electromagnetic fields (self fields). They are modified by boundary conditions imposed by the beam surroundings (vacuum chambers, cavities, etc.) and act back on the beam. However in this case it is the transverse deviation of the beam which represents a dipole moment and excites certain field configuration in cavities which will apply later a transverse force. In case this force increases the original dipole moment we have an instability. Here it is the transverse impedance which describes the relevant properties of the beam surroundings. It is excited by the transverse dipole moment of the beam but is not sensitive to the transverse dimension of the beam or higher order moments. The transverse particle distribution has therefore no or very little influence on the instability. However, this impedance has a fast time response and senses a difference in the dipole moment along the bunch, in particular, between the head and tail of the bunch. This can lead to some new effects, called head-tail instabilities.

2) Transverse impedance



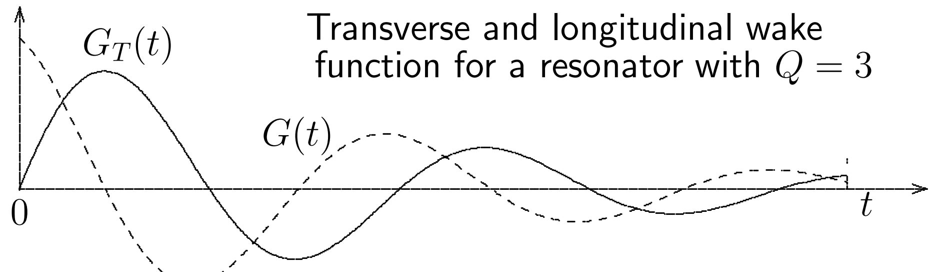
A transverse impedance is excited by the longitudinal bunch motion and gives a deflection field. Example: dipole cavity mode with a longitudinal field E_z vanishing on axis, having a transverse gradient $\partial E_z/\partial x$. It couples only to a beam with transverse dipole moment Ix_0. After 1/4 oscillation $\partial E_z/\partial x$ converts into a B_y which deflects the beam in the x-direction. Maxwell's equation and $E = \hat{E}\mathrm{e}^{j\omega t}$ gives a relation between E and B-fields

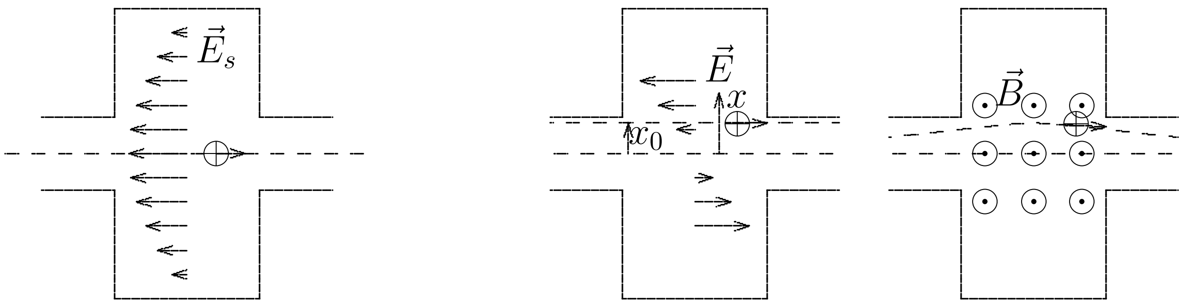
$$\dot{\vec{B}} = -\mathrm{curl}\vec{E} \rightarrow \dot{B}_y = \frac{\partial E_z}{\partial x}\ ,\ B_y(t) = \hat{B}_y \mathrm{e}^{j\omega t} = -\frac{j}{\omega}\frac{\partial \hat{E}}{\partial x}\mathrm{e}^{j\omega t}.$$

In general the ratio between deflecting field and dipole moment is the transverse impedance Z_T

$$Z_T(\omega) = j\frac{\int\left(\vec{E}(\omega) + [\vec{v} \times \vec{B}(\omega)]\right)_T ds}{Ix(\omega)} = -\frac{\omega\int\left(\vec{E}(\omega) + [\vec{v} \times \vec{B}(\omega)]\right)_T ds}{I\dot{x}(\omega)}.$$

If deflecting field and dipole moment are in phase no energy transfer, therefore factor 'j', more physical to give ratio with respect to transverse velocity. A transverse wake function $G_t(T)$ gives deflecting potential felt by second charge traversing impedance at time t behind first one.





longitudinal transverse transverse

In our cavity mode Ix induces E_z giving a longitudinal impedance Z_L. Excitation at distance x_0 gives a gradient of the $\partial E_z/\partial x$ related to Ix_0 by a factor k

$$\frac{\partial E_z}{\partial x} = kIx_0 \text{ and } E_z(x) = \frac{dE_z}{dx}x = kIx_0x \ \ , \ \ E_z(x_0) = kIx_0^2$$

$$Z_L(x_0) = -\frac{\int E_z(x_0)dz}{I} = -kx_0^2\ell \ , \ \ \frac{d^2Z_L^2}{dx_0^2} = -2k\ell$$

$\ell =$ cavity length. Maxwell's eq. $\int \dot{\vec{B}} d\vec{a} = -\oint \vec{E}d\vec{s}$ transform $\partial E_z/\partial x$ into B_y. With $I(t) = \hat{I}e^{j\omega t}$ we get

$$\dot{B}_y = \hat{B}j\omega e^{j\omega t} = -\frac{d\hat{E}_z}{dx}e^{j\omega t} \ \ , \ \ B_y = -\frac{j}{\omega}\frac{\partial E_z}{\partial x} = -\frac{jkIx_0}{\omega}$$

$$Z_T(\omega) = j\frac{\int [\vec{v} \times \vec{B}(\omega)]_T ds}{Ix(\omega)} = -j\frac{cB_y\ell}{Ix_0} = \frac{ck\ell}{\omega} = \frac{-c}{2\omega}\frac{d^2Z_L}{dx^2}.$$

This connection between transverse and longitudinal impedances of the **same** mode gives symmetry relation

longitudinal $\quad Z_r(-\omega) = Z_r(\omega) \ , \ Z_i(-\omega) = -Z_i(\omega)$

transverse $\quad Z_{Tr}(-\omega) = -Z_{Tr}(\omega) \ , \ Z_{Ti}(-\omega) = Z_{Ti}(\omega)$

In a ring of radius R and vacuum chamber radius b the impedances, averaged over **different** modes, have a ratio

$$Z_T(\omega) \approx \frac{2R}{b^2}\frac{Z(\omega)}{\omega/\omega_0)}$$

3) Transverse instabilities with $Q' = 0$

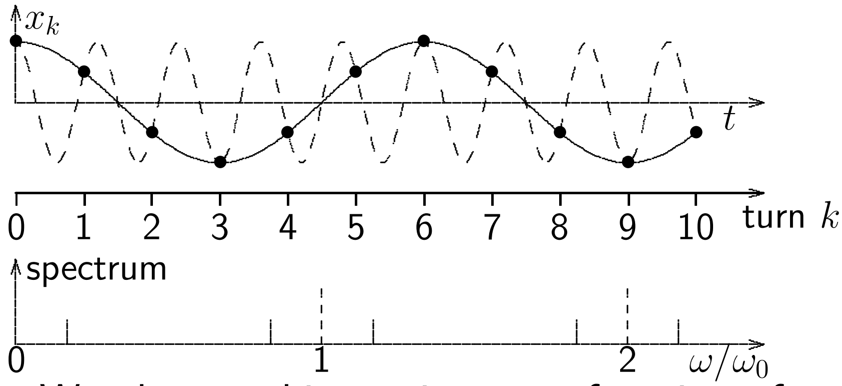
Transverse dynamics

Due to the transverse focusing a particle executes a betatron motion around the orbit. This is an oscillation which is locally harmonic but has a complicated phase advance around the ring. We approximate this by a smooth focusing given by

$$\ddot{x} + \omega_0^2 Q_x^2 x = 0$$

with revolution frequency ω_0 and betatron tune Q_x.

A stationary observer, or the impedance, sees the particle position x_k only at one location each turn k and has no information what the particle does in the rest of the ring

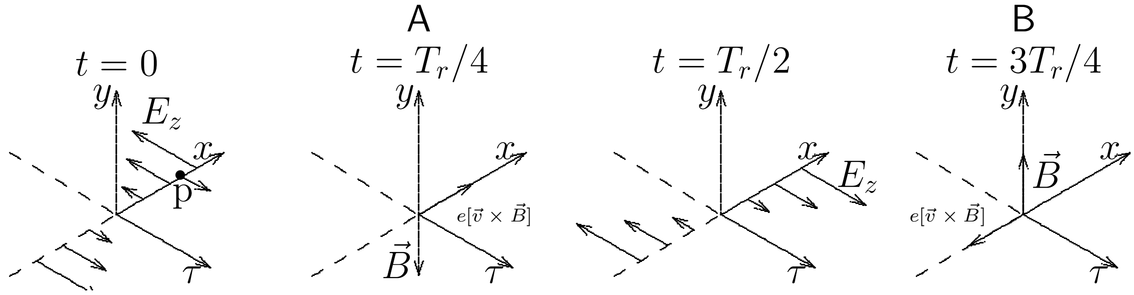


$$x_k = \hat{x}\cos(2\pi q k)$$

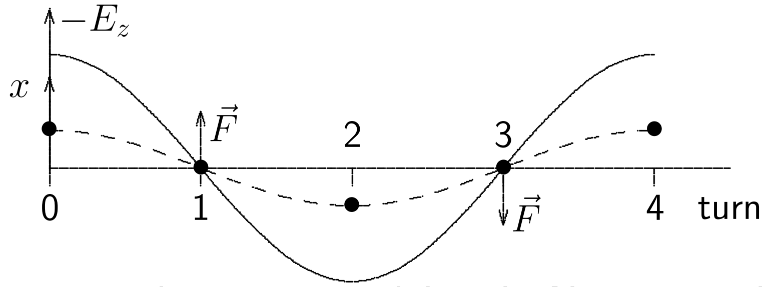
$$x'_k = -\frac{\hat{x}}{\beta_x}\sin(2\pi q k).$$

We observe this motion as a function of turn k. We can make a harmonic fit, i.e. a Fourier analysis. For a single bunch circulating in the machine we find at the revolution harmonic $p\omega_0$ an upper and lower sideband. The distance of the sideband is given by the tune $Q_x =$ integer $+q$. The fractional part q is the only part which matters since the integer cannot be observed. For a very short bunch these sidebands will extend to very high frequencies, for longer bunches they level off. A transverse impedance (or a position monitor) is sensitive to the dipole moment Ix of the current and does not see the revolution harmonics.

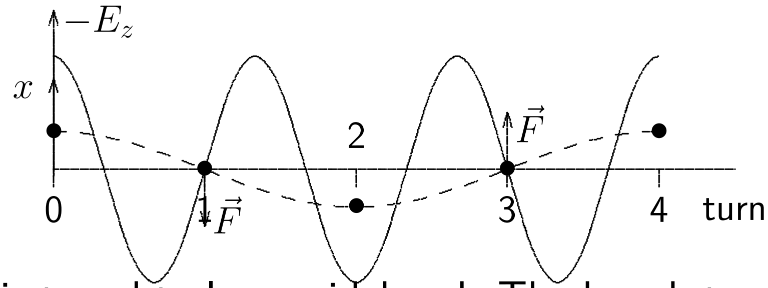
Multi-traversal instability of a single bunch



A bunch p traverses a cavity with off-set x, excites a field $\vec{E}$ which converts after $T_r/4$ into a field $-\vec{B}$, then into $-\vec{E}$ and after into $\vec{B}$. The bunch oscillates with tune Q having a fractional part $q = 1/4$ seen as sidebands at $\omega_0(\text{integer} \pm q)$ by a stationary observer.



A) A cavity is tuned to upper sideband. Next turn the bunch traverses it in the situation 'A', $t = T_r(k + 1/4)$ with a velocity in $-x$-direction and gets by B_y a force in $+x$-direction which damps the oscillation.



B) A cavity is tuned to lower sideband. The bunch traverses it next turn in situation 'B', $t = T_r(k'+3/4) = T_r(k'+1-1/4)$ with negative velocity and a force in same direction. This increases its velocity and leads to instability.

The resistive impedance at the upper sideband damps, the one at the lower sideband excites the oscillation. If we have a more general impedance extending over several sidebands $\omega_0(p+q)$ and $\omega_0(p-q)$ we expect that the growth or damping rate of the oscillation is given by an expression of the form

$$\frac{1}{\tau_s} \propto \sum_p \left(I_{p+}^2 (Z_{Tr}(\omega_{p+}) - I_{p-}^2 Z_{Tr}(\omega_{p-}) \right)$$

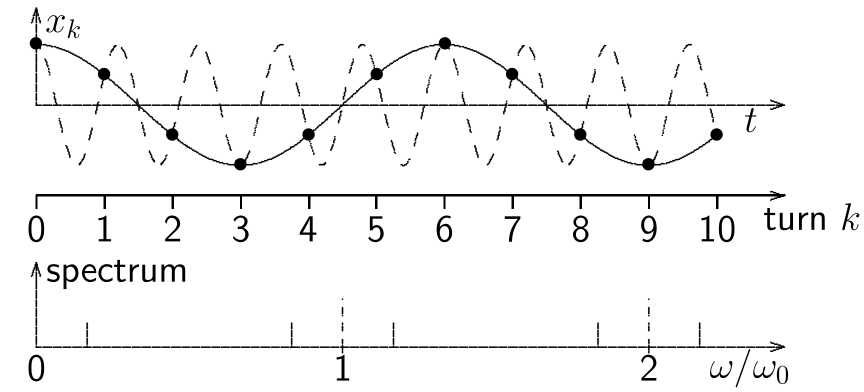
with $\omega_{p\pm} = \omega_0 \, (p \pm q)$ where $I_{p\pm}$ is the Fourier component of the beam current at the upper or lower sidebands. It appears here as the square I_p^2 since the instability is driven by the energy transfer from the longitudinal to the transverse motion.

Single bunch spectrum

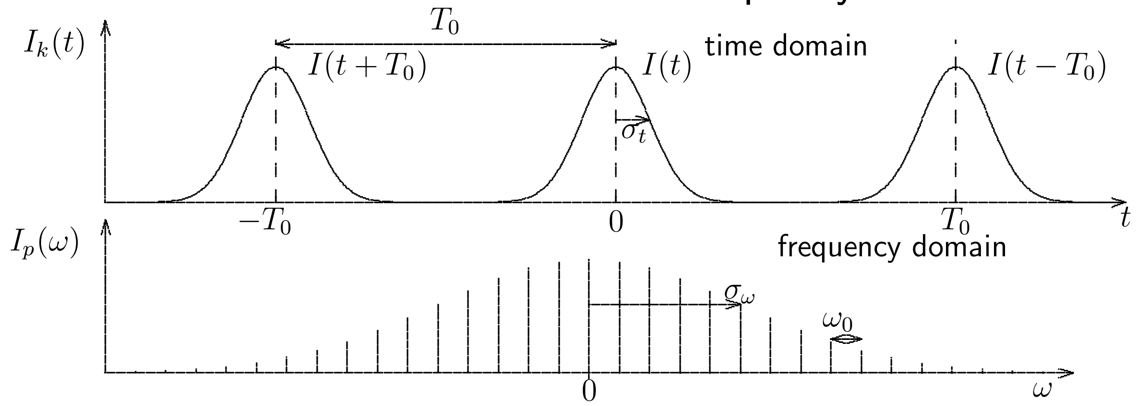
For a qualitative treatment we take a bunch oscillating transversely with tune $Q = n + q$ of fractional part $q = 1/6$. Its displacement versus turns k is $x_k = \hat{x}\cos(2\pi Q_x k) = \hat{x}\cos(2\pi q k)$ can be fitted with a spectrum

$$x_k = \hat{x}\cos(2\pi q k)\ ,\ x_k = \hat{x}\cos(2\pi(1-q)k)\cdots\ .$$

having side-bands $(p \pm q)\omega_0$ around $p\omega_0$ without carriers.

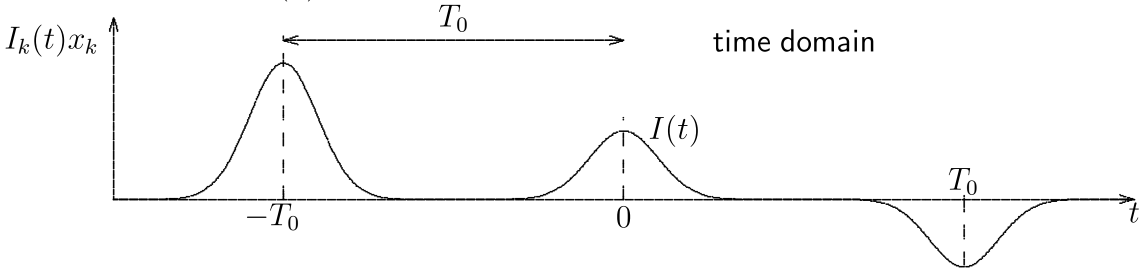


Multi turn bunch current in time and frequency domain



$$I_k(t) = \sum_{-\infty}^{\infty} I(t - kT_0) = \sum_{p=-\infty}^{\infty} I_p e^{jp\omega_0 t} = I_0 + 2\sum_{p=1}^{\infty} I_p \cos(p\omega_0 t)$$

Dipole moment $I_k(t)x_k$ of a circulating, oscillating bunch



$$D_k(t) = I_k(t)x_k = \sum_{-\infty}^{\infty} I(t - kT_0)\hat{x}\cos(2\pi q k).$$

$$D_k(t) = x_k I_k(t) = \hat{x} \sum_{-\infty}^{\infty} \cos(2\pi qk) I(t - kT_0)$$

To express it in a series we **Fourier transform**

$$\begin{aligned} \tilde{I}_k(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} I(t - kT_0) \mathrm{e}^{-j\omega t} dt \\ &= \tilde{I}(\omega) \sum_{k=-\infty}^{\infty} \mathrm{e}^{-jk\omega T_0} \end{aligned}$$

$$\begin{aligned} \tilde{D_x}(\omega) &= \hat{x}\tilde{I}(\omega) \sum_{-\infty}^{\infty} \cos(2\pi qk) \mathsf{e}^{-jk\omega T_0} \\ &= \frac{\hat{x}\tilde{I}(\omega)}{2} \sum_{-\infty}^{\infty} \left[\mathsf{e}^{-jk(\omega T_0 + 2\pi q)} + \mathsf{e}^{-jk(\omega T_0 - 2\pi q)}\right] \end{aligned}$$

Sums are ∞ if exponent is $n2\pi$ and vanish otherwise

$$\sum_{k=-\infty}^{\infty} \mathsf{e}^{-jkx} = 2\pi \sum_{p=-\infty}^{\infty} \delta(x - 2\pi p) \;\; \text{and} \;\; \delta(ax) = \frac{1}{a}\delta(x) \text{ gives}$$

$$\tilde{D}_k(\omega) = \hat{x}\frac{\omega_0 \tilde{I}(\omega)}{2} \sum_{-\infty}^{\infty} \left[\delta(\omega - (p-q)\omega_0)) + \delta(\omega - (p+q)\omega_0)\right]$$

Inverse Fourier transform gives dipole moment in time

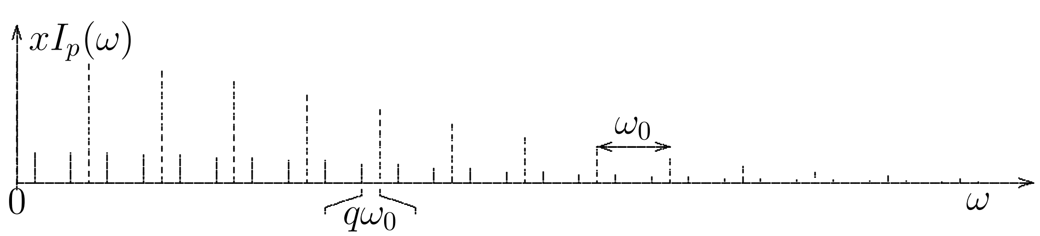
$$D_k(t) = \frac{\omega_0 \hat{x}}{2\sqrt{2\pi}} \sum_{-\infty}^{\infty} \left[I_{p+} \mathsf{e}^{j\omega_p^+ t} + I_{p-} \mathsf{e}^{j\omega_p^- t}\right]$$

with $\omega_p^+ = (p+q)\omega_0,\ \omega_p^- = (p-q)\omega_0,\ I_{p\pm} = \dfrac{\omega_0}{\sqrt{2\pi}} \tilde{I}(\omega_{p\pm})$

$$D_k(t) = \frac{\hat{x}}{2} \sum_{p=-\infty}^{\infty} \left[I_{p+} \mathsf{e}^{j(t\omega_p^+)} + I_{p-} \mathsf{e}^{j(t\omega_p^-)}\right].$$

Combining terms $p > 0$ from first, $p < 0$ from second part and vice versa, using $\tilde{I}(\omega) = \tilde{I}(-\omega)$

$$D_k(t) = \hat{x} \sum_{\omega > 0}^{\infty} [I_{p+} \cos(\omega_p^+ t) + I_{p-} \cos(\omega_p^- t)].$$



Effect of fields induced by dipole moment

A charge e going through the impedance element at turn k feels a transverse force changing its momentum $\Delta p_{ke} = F_T \Delta t \approx F_T \Delta s/c$

$$\Delta p_{ke} = \frac{e}{c}\int \left[\vec{E}(\omega) + [\vec{v}\times\vec{B}(\omega)]\right]_T ds = \frac{-jeD_k(t)Z_T}{c}.$$

Momentum change of **whole** bunch is a convolution of its charge distribution given by **single traversal** current $I(t)$ with momentum change $\Delta p_x(t + kT_0)$ in turn k

$$\begin{aligned}\Delta p_k &= -j\frac{1}{c}\int_{-\infty}^{\infty} I(t) D_k(t+kT_0) Z_T dt \\ &= -j\frac{\hat{x}}{2c}\sum_{p=-\infty}^{\infty}\int_{-\infty}^{\infty} I(t)\left[I_{p+}Z_T(\omega_p^+)\mathrm{e}^{j\omega_p^+(t+kT_0)t}\right. \\ &\qquad\qquad \left. + I_{p-}Z_T(\omega_p^-)\mathrm{e}^{j\omega_p^-(t+kT_0)}\right] dt\end{aligned}$$

$$\int_{-\infty}^{\infty} I(t)\mathrm{e}^{-j(t+kT_0)\omega_p^+} dt = \sqrt{2\pi}\mathrm{e}^{-jT_0k\omega_p^+}\tilde{I}(\omega_p^+) = \frac{2\pi}{\omega_0}\mathrm{e}^{-j2\pi qk} I_{p+}$$

$$\Delta p_k = -j\frac{\hat{c}T_0}{2c}\sum_{-\infty}^{\infty}\left[I_{p+}^2 Z_T(\omega_p^+)\mathrm{e}^{-j2\pi qk} + I_{p-}^2 Z_T(\omega_p^-)\mathrm{e}^{j2\pi qk}\right].$$

Combining terms $p > 0$, $p < 0$ from the two parts, using $Z_{Tr}(\omega) = Z_{Tr}(-\omega)$, $Z_{Ti}(\omega) = Z_{Ti}(-\omega)$ gives

$$\begin{aligned}\Delta p_k = -\frac{T_0}{c}\sum_{\omega>0} & \left[\left(I_{p+}^2 Z_{Tr}(\omega_p^+) - I_{p-}^2 Z_{Tr}(\omega_p^-)\right)\hat{x}\sin(2\pi qk)\right. \\ & - \left.\left(I_{p+}^2 Z_{Ti}(\omega_p^+) + I_{p-}^2 Z_{Ti}(\omega_p^-)\right)\hat{x}\cos(2\pi qk)\right].\end{aligned}$$

$$\begin{aligned}\text{with } x_k &= \hat{x}\cos(2\pi qk),\ x'_k = \frac{\dot{x}}{c} = -\frac{\hat{x}}{\beta_x}\sin(2\pi qk) \\ \Delta p_k &= \frac{T_0}{c^2}\sum_{\omega>0}\left[\left(I_{p+}^2 Z_{Tr}(\omega_p^+) - I_{p-}^2 Z_{Tr}(\omega_p^-)\right)\beta_x\dot{x}_k\right. \\ &\qquad + \left.\left(I_{p+}^2 Z_{Ti}(\omega_p^+) + I_{p-}^2 Z_{Ti}(\omega_p^-)\right) c x_k\right]\Big].\end{aligned}$$

Growth rate and tune change

Transverse velocity and angle change with the momentum

$$\Delta x'_k = \frac{\Delta \dot{x}_k}{c} = \frac{\Delta p_k}{N_0 m_0 \gamma c} = \frac{e \Delta p_k}{m_0 \gamma c I_0 T_0}$$

$$\Delta \dot{x}_k = \frac{e}{m_0 c^2 \gamma I_0} \sum_{\omega>0} \Big[\Big(I_{p+}^2 Z_{Tr}(\omega_p^+) - I_{p-}^2 Z_{Tr}(\omega_p^-)\Big) \beta_x \dot{x}_k) \\ + \Big(I_{p+}^2 Z_{Ti}(\omega_p^+) + I_{p-}^2 Z_{Ti}(\omega_p^-)\Big) c x_k\Big].$$

The velocity change has a part proportional to $\dot{x}$ and Z_{Tr} giving growth/damping and one to x and Z_{Ti} changing Q. Smoothing the first part gives acceleration $\ddot{x} = \Delta \dot{x} \omega_0 / 2\pi$ which we ad to the one of the beam optics focusing

$$\ddot{x} + 2a\dot{x} + Q_x^2 \omega_0^2 = 0, \; x = x_0 \mathrm{e}^{-at} \cos(Q_x \omega_0 t + \phi) \;\text{ if }\; a \ll Q_x \omega_0$$

$$a = \frac{1}{\tau} = \frac{e \omega_0 \beta_x}{4\pi m_0 c^2 \gamma I_0} \sum_{\omega>0} \Big(I_{p+}^2 Z_{Tr}(\omega_p^+) - I_{p-}^2 Z_{Tr}(\omega_p^-)\Big).$$

$\omega_- = (p-q)\omega_0 = -(-p+q)\omega_0 = -(|p|+q)\omega_0$ for $p < 0$ and $Z_{Tr}(\omega) = -Z_{Tr}(-\omega)$ gives sum over $\pm$frequencies

$$a = \frac{1}{\tau} = \frac{e \omega_0 \beta_x}{4\pi m_0 c^2 \gamma I_0} \sum_{p=-\infty}^{\infty} I_{p+}^2 Z_{Tr}(\omega_p^+).$$

The reactive impedance changes angle $\Delta x'_k = \Delta \dot{x}_k / c$ proportional to x_k which is a focusing element of strength

$$\frac{1}{f} = -\frac{\Delta x'_k}{x_k} = -\frac{e}{m_0 c^2 \gamma I_0} \sum_{\omega_\pm>0} \Big(I_{p+}^2 Z_{Ti}(\omega_p^+) + I_{p-}^2 Z_{Ti}(\omega_p^-)\Big) x_k$$

which results in a tune change $\Delta Q_x = \beta_x / (4\pi f)$

$$\Delta \omega_\beta = -\frac{e \omega_0 \beta_x}{4\pi m_0 c^2 \gamma I_0} \sum_{\omega_\pm>0} \Big(I_{p+}^2 Z_{Ti}(\omega_p^+) + I_{p-}^2 Z_{Ti}(\omega_p^-)\Big)$$

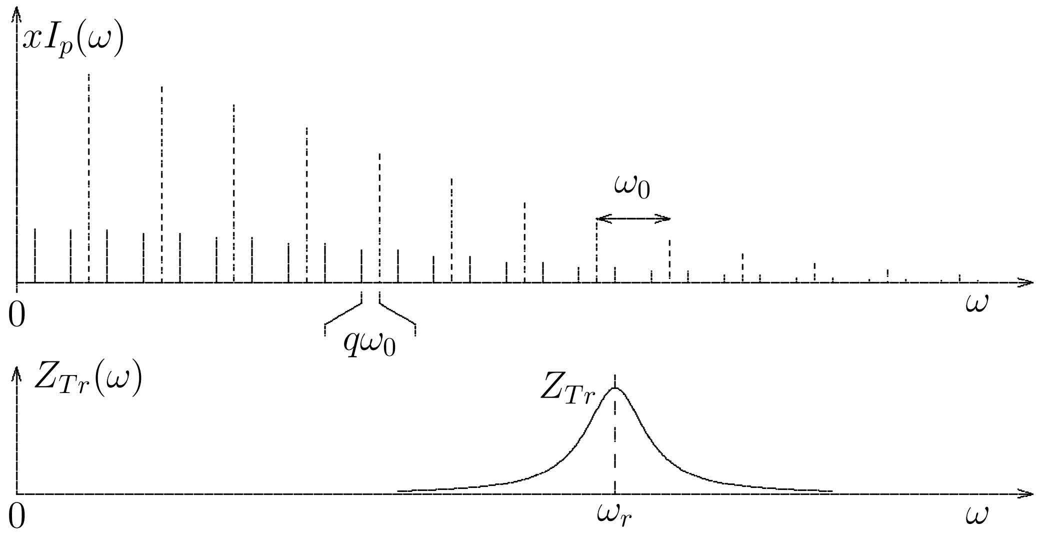
$$= -\frac{e \omega_0 \beta_x}{4\pi m_0 c^2 \gamma I_0} \sum_{p=-\infty}^{\infty} I_{p+}^2 Z_{Ti}(\omega_p^+).$$

Inductive imped.$Z_{Ti} > 0$ defocuses, negative tune shift.

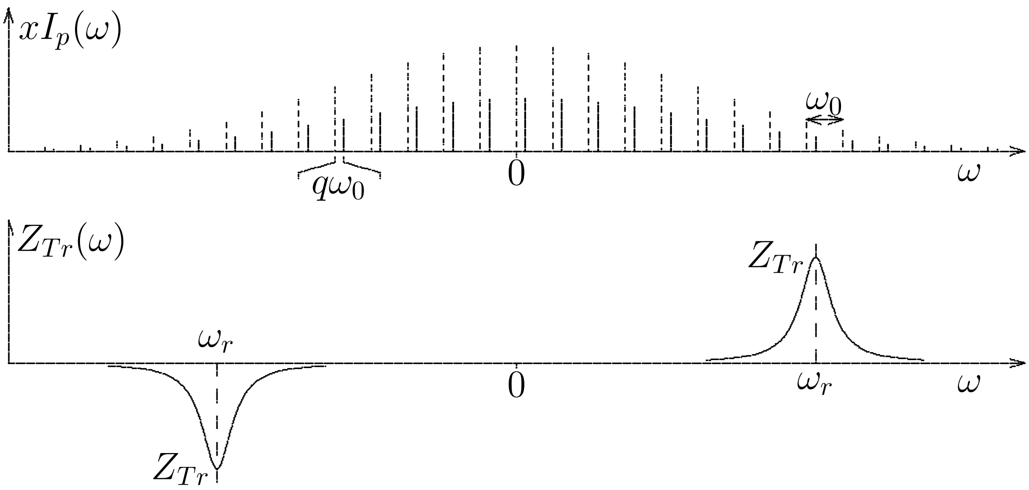
Instability due to the resistive impedance

$$
\begin{aligned}
x &= x_0 \mathrm{e}^{-at} \cos((Q_x\omega_0 + \Delta\omega_\beta)t + \phi) \quad \text{if} \quad a \ll Q_x\omega_0 \\
a &= \frac{1}{\tau} = \frac{e\omega_0\beta_x}{4\pi m_0 c^2 \gamma I_0} \sum_{\omega>0} \left(I_{p+}^2 Z_{Tr}(\omega_p^+) - I_{p-}^2 Z_{Tr}(\omega_p^-) \right).
\end{aligned}
$$

For a distributed impedance we replace local beta function by average $\beta_x \approx \langle\beta_x\rangle \approx R/Q$ with R = average ring radius. Single strong impedances, RF-cavities, are best located at a small beta function.



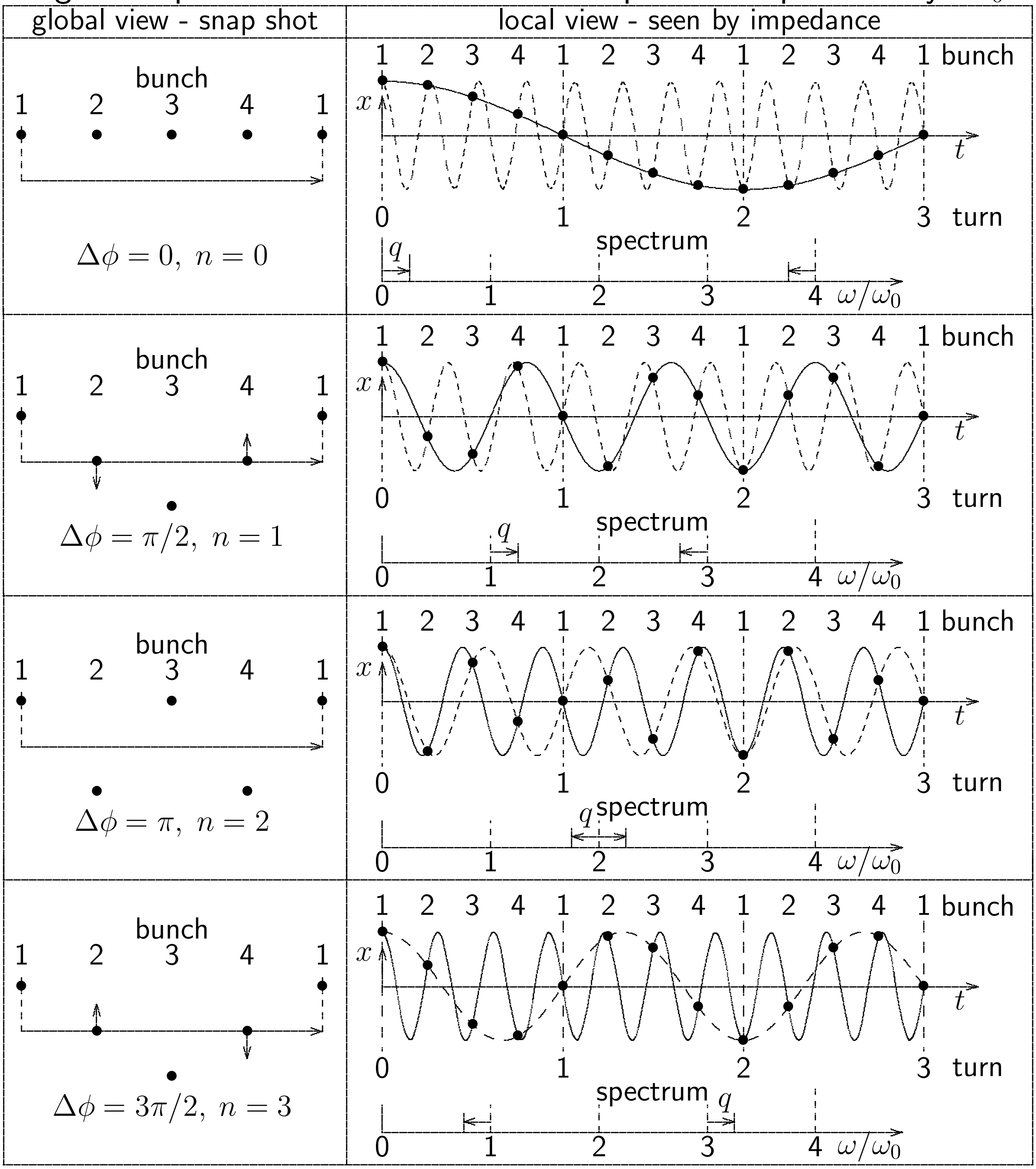
$$
a = \frac{e\omega_0\beta_x}{4\pi m_0 c^2 \gamma I_0} \sum_{p=-\infty}^{\infty} I_{p+}^2 Z_{Tr}(\omega_p^+).
$$



To drive this instability we need a narrow band impedance with a memory lasting at least for one turn.

Transverse instability of many bunches

M circulating bunches can oscillate in M independent modes $n = M\Delta\phi/2\pi$ with phase $\Delta\phi$ between bunches as shown in the global view where all are seen at once. For a local observer the bunches pass by with increasing time delay shown by the bullets which are fitted by an upper (solid line) and lower (dashed line) side-band frequency. Higher frequencies can be fitted and the spectrum repeats every $4\omega_0$.



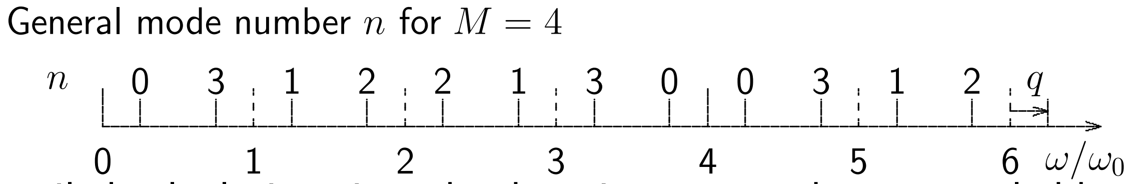
The side-band frequencies of M bunches oscillating in a mode n, obtained by a graphical method, are

$$\omega_{p\pm} = \omega_0 \left(pM \pm (n+q)\right)$$

with bunches number M, mode n, fractional tune q and running integer p. We get the lowest frequencies of $n = 3$ with $p = 0$ for the upper and $p = 1$ for the lower side band

$$\omega_{p+} = \omega_0(3+q)\ ,\ \ \omega_{p-} = \omega_0(1-q).$$

Increasing p gives all higher frequency of the spectrum. A simple picture, shown for $M = 4$ can be quickly give the locations of the side-bands for a mode n

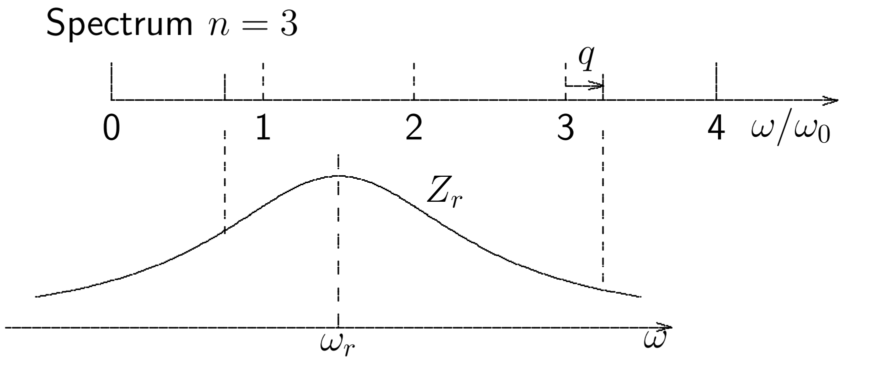


A detailed calculation gives the damping or growth rate coupled bunch oscillations

$$a = \frac{1}{\tau} = \frac{e\omega_0\beta_x}{4\pi m_0 c^2 \gamma I_0} \sum_{\omega>0} \left(I_{p+}^2 Z_{Tr}(\omega_p^+) - I_{P-}^2 Z_{Tr}(\omega_p^-)\right)$$

$$= \frac{e\omega_0\beta_x}{4\pi m_0 c^2 \gamma I_0} \sum_{\omega>0} I_p^2 \left(Z_{Tr}(\omega_p^+) - Z_{Tr}(\omega_p^-)\right).$$

With $x(t) \propto \mathrm{e}^{-at}$ stability if $a > 0,\ Z_{Tr}(\omega_p^+) > Z_{Tr}(\omega_p^-)$. We sum over all side-bands of a given mode n.

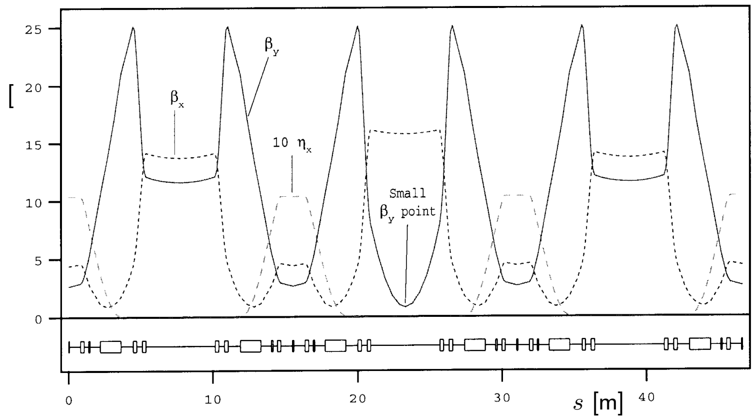


Dependance of the transverse instability on β_y

The transverse instability growth rate is $\propto \beta_y$ since a deflection at high β gives larger oscillation amplitude.

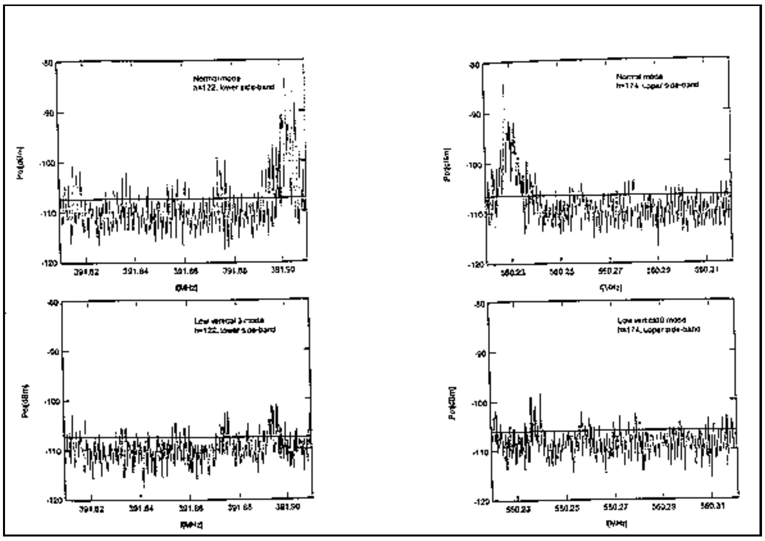
$$a = \frac{1}{\tau} = \frac{e\omega_0\beta_y}{4\pi m_0 c^2 \gamma I_0} \sum_{\omega>0} \left(I_{p+}^2 Z_{Tr}(\omega_p^+) - I_{p-}^2 Z_{Tr}(\omega_p^-) \right).$$

To observe an exponential instability growth one must inject a large current or turn a feed-back system off. For a slow accumulation the instability saturates the current and some unstable betatron lines are seen as shown at LNLS (Laboratòrio Nacional de Luz Síndrotron, Brasil). Here β_y can be reduced in the RF section which eliminated these lines indicating that the offending impedance is the RF-cavity.



normal beta

low beta

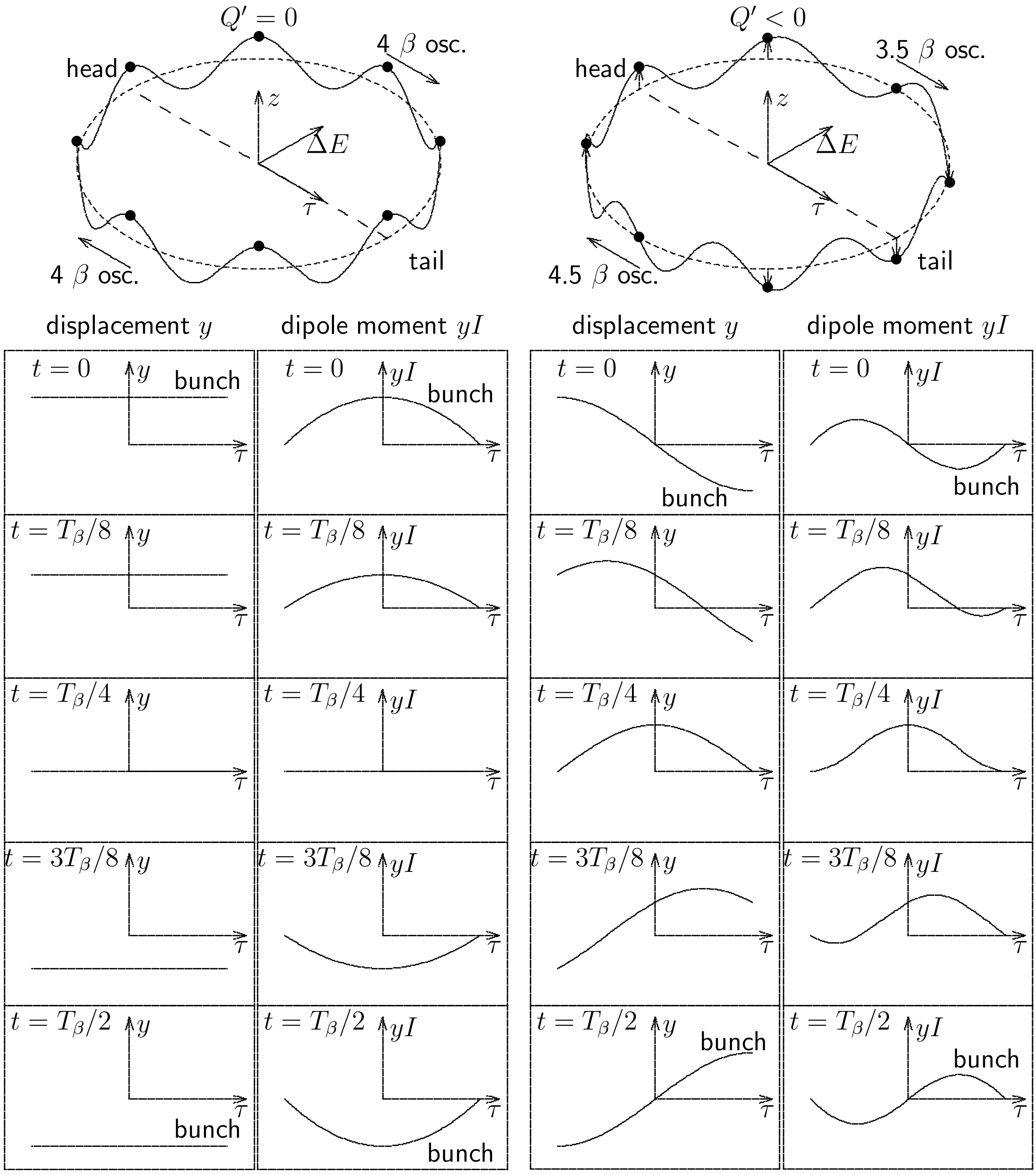


$n+q$ $n-q$

4) Head-tail instability

Head-tail mode oscillation

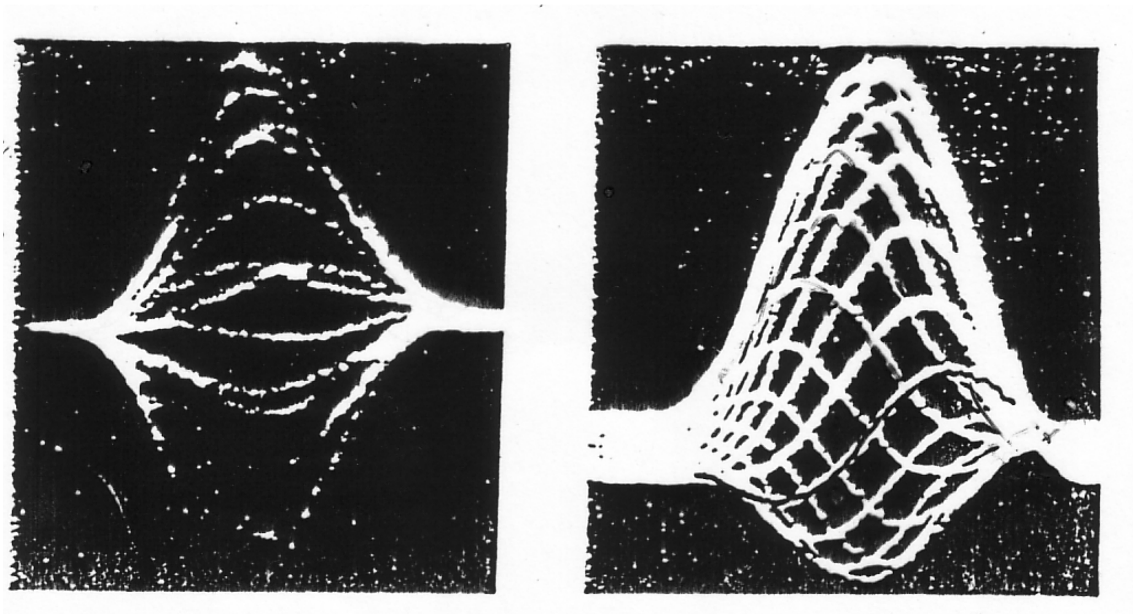
The synchrotron motion of a particle in energy and time deviation ΔE and τ influences the transverse motion via chromaticity $Q' = dQ/(dp/p)$. For $\gamma > \gamma_T$ it has an excess energy moving from head to tail and a lack moving from tail to head. For $Q' > 0$, the phase advances in the first and lags in the second step; vice versa for $Q' < 0$ or $\gamma < \gamma_T$.



Betatron motion observed in steps of its period $T_\beta = T_0/q$

Observation of the head-tail mode in the CERN Booster

The head-tail mode oscillation of relatively long bunches can be observed directly with a fast position monitor. The figure shows such a measurement of the head-tail mode at vanishing and a finite chromaticity taken by J. Gareyte and F. Sacherer in the CERN Booster. I shows several traces each corresponding to a turn of the oscillating bunch passing through the transverse position monitor which gives a signal proportional the instantaneous dipole moment $x(t)I(t)$.



$Q' = 0$ $Q' > 0$

Head-tail mode $m = 0$

Head-tail instability

A broad band impedance is excited by oscillating particles A at the bunch head which in turn excite particles B at the tail with a phase shifted by $\Delta\phi$ compared to the head. Half a synchrotron oscillation later particles B are at the head and while particles A are at the tail oscillating with phase $-\Delta\phi$ compared to B (assuming $Q' = 0$). The excitation by the head has the wrong phase to keep oscillation growing unless $Q' \neq 0$ producing a phase shift during a motion from head to tail or vice versa.
The wake field excited by the head affects the tail later which will oscillate with a phase lag. To keep the oscillation growing the head particle must undergo a relative phase delay while moving to the tail and the tail particle a relative phase advance moving to the head. We expect an instability if $Q' < 0$ for $\gamma > \gamma_T$ or if $Q' > 0$ for $\gamma < \gamma_T$.
The 'wiggle' of the head-tail motion is seen by the impedance as an oscillation with the chromatic frequency ω_ξ.

$$\Delta p/p = \Delta\hat{p}/p \sin(\omega_s t) \;\;,\;\; \tau = -\hat{\tau}\cos(\omega_s t)\;,\;\; \hat{\tau} = \frac{\omega_s}{\eta_c}\frac{\Delta\hat{p}}{p}$$

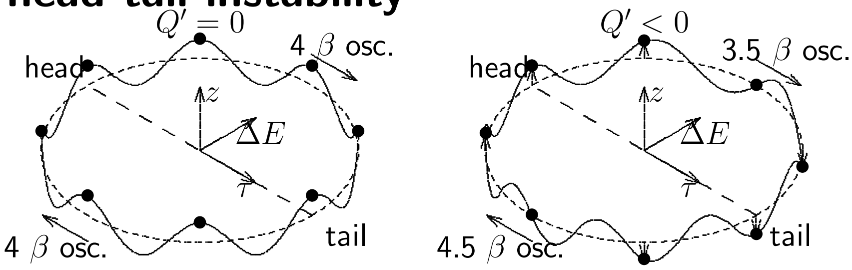
The relative betatron phase shift of a particle while executing part of a synchrotron oscillation is

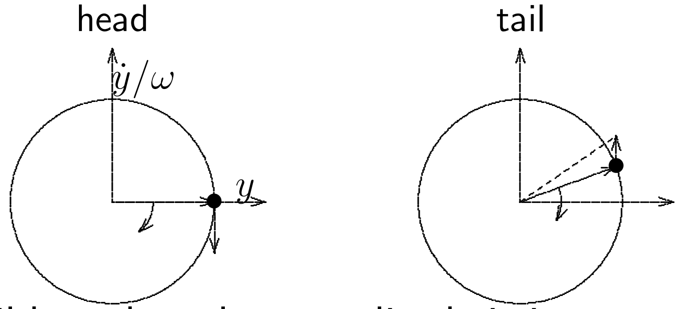
$$\begin{aligned} \Delta\phi_\beta &= \omega_0 \int_{t_1}^{t_2} \Delta Q dt = \omega_0 Q' \frac{\Delta\hat{p}}{p} \int_{t_1}^{t_2} \sin(\omega_s t) dt \\ &= -\omega_0 Q' \frac{\Delta\hat{p}}{p} \left(\cos(\omega_s t_2) - \cos(\omega_s t_2)\right) = \frac{\omega_0 Q'}{\eta_c}(\tau_2 - \tau_1) \end{aligned}$$

This gives for the chromatic frequency

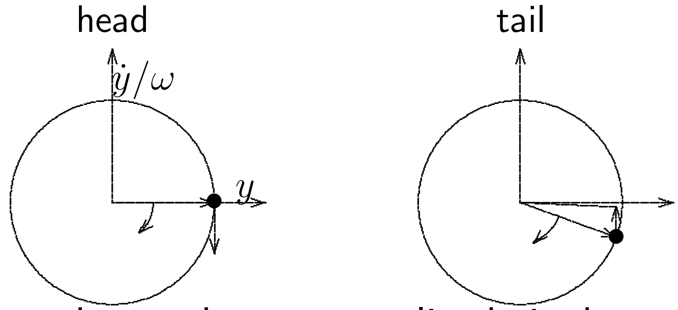
$$\omega_\xi = \frac{\Delta\phi_\beta}{\Delta\tau} = \frac{\omega_0 Q'}{\eta_c}.$$

Model of head-tail instability

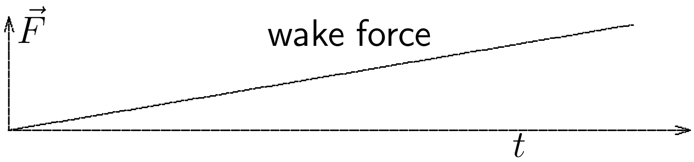




Tail has phase lag, amplitude is increased



Tail has phase advance, amplitude is decreased



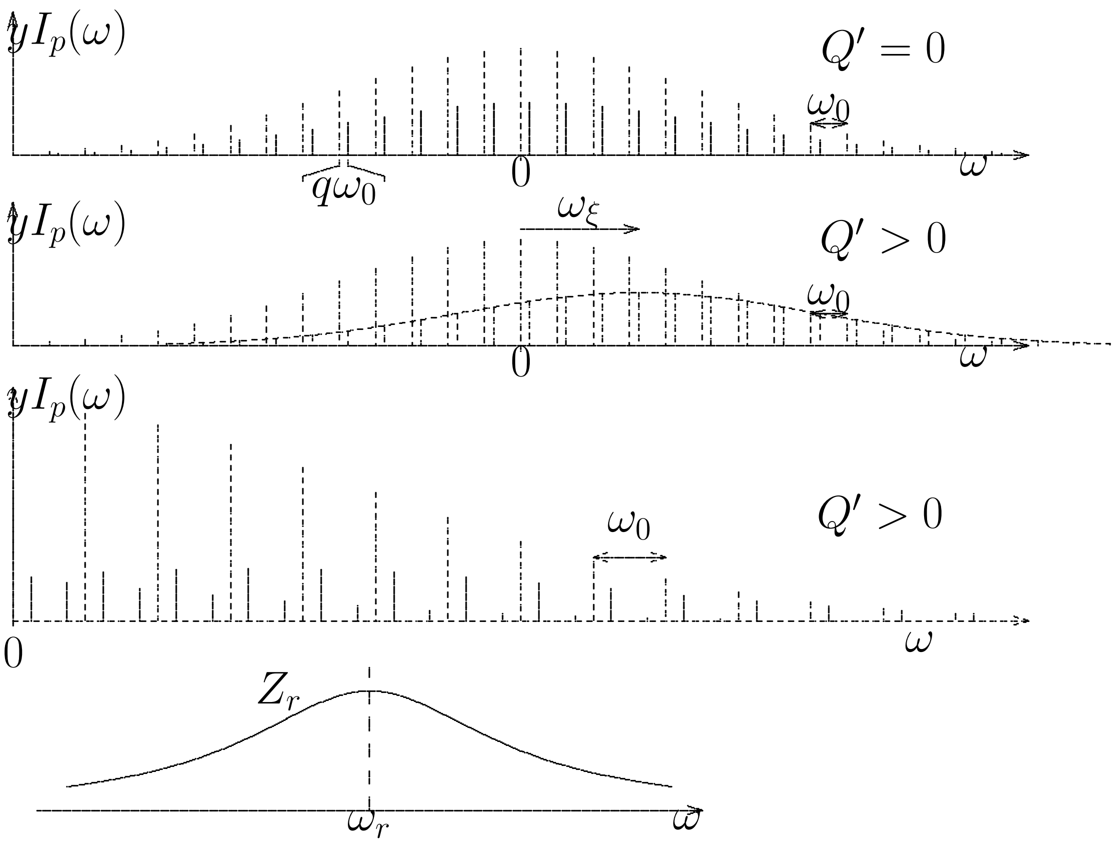
Above transition energy:
$Q' = 0$: Going from head to tail or from tail to head has same phase change. Phase lag and advance between head an tail interchange, neither damping nor growth.
$Q' > 0$: Going from head to tail there is a gain in phase, going from tail to head a loss, giving a systematic phase lag between head and tail and in average damping.
$Q' < 0$: Going from head to tail there is a loss in phase, going from tail to head a gain (picture), giving a systematic phase advance between head and tail and in average growth.
The situation is reversed below transition energy.

The 'wiggle' of the head-tail mode shifts the envelope of the side-bands by the chromatic frequency $\omega_\xi = Q'\omega_0/\eta_c$ and we have current components

$$I_{p\xi\pm} = \frac{\omega_0}{\sqrt{2\pi}}\tilde{I}(\omega_{p\pm} + \omega_\xi),\ \omega_{p\pm} = \omega_0\left(pM \pm (n+q)\right)$$

which can be very different adjacent sidebands. A broad band impedance can give an instability with growth ($a < 0$) or damping ($a > 0$) rate

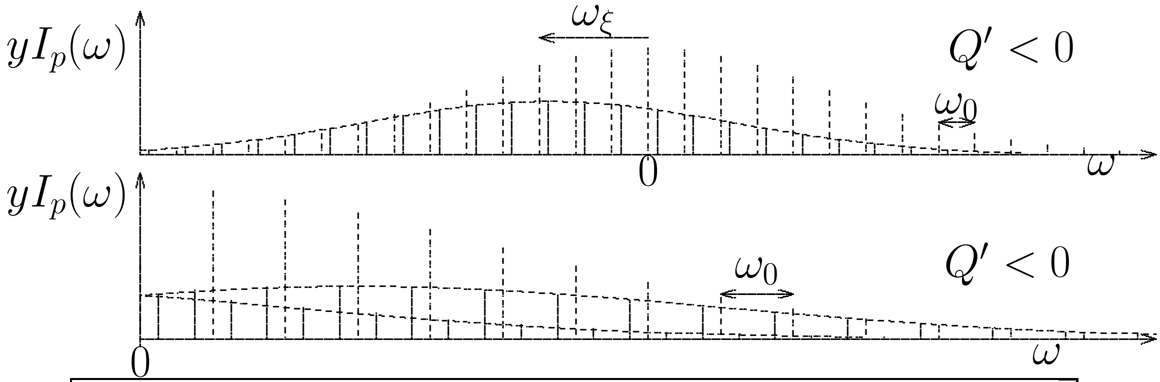
$$a = \frac{e\omega_0\beta_x}{4\pi m_0 c^2\gamma I_0}\sum_{\omega>0}\left(I_{p\xi+}^2 Z_{Tr}(\omega_p^+) - I_{p\xi-}^2 Z_{Tr}(\omega_p^-)\right).$$

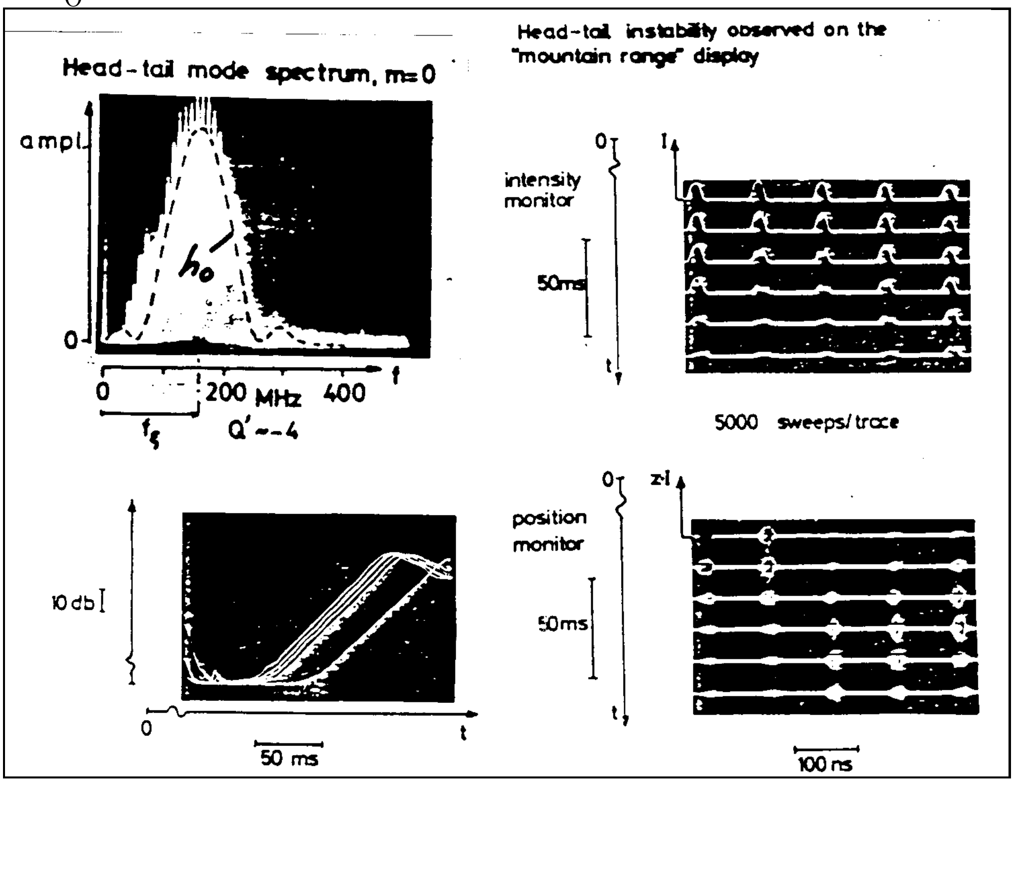


Head-tail instability seen in the CERN ISR

We inject proton bunches with $Q' < 0$ and observe with intensity (u.r.) and position (l.r.) monitors. The first shows decaying bunches, the second growing head-tail oscillations. The filtered betatron signal (l.l.) increases linearly on a logarithmic scale indicating exponential growth up to beam loss. A transverse spectrum 'snap shot' (u.l) during growth shows the envelope of betatron lines shifted by ω_ξ. The current components and frequencies are

$$I_{p\xi\pm} = \frac{\omega_0}{\sqrt{2\pi}}\tilde{I}(\omega_{p\pm} + \omega_\xi),\ \omega_{p\pm} = \omega_0\left(pM \pm (n+q)\right),\ \frac{\omega_\xi}{\omega_0} = \frac{Q'}{\eta_c}$$





Summary

Nearly all longitudinal and transverse instabilities for bunched beams can be treated in frequency domain using the formalism of the Robinson instability. This contains a condition that the resistive impedance at the upper, Z^+, and lower, Z^-, side-band has to fulfill a **stability condition**:

	above transition	below transition
longitudinal	$Z_r^+ < Z_r^-$	$Z_r^+ > Z_r^-$
transverse	$Z_{Tr}^+ > Z_{Tr}^-$	$Z_{Tr}^+ > Z_{Tr}^-$