LANDAU DAMPING

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1) Introduction

Mechanism of Landau damping

An undamped single oscillator with resonant frequency ω_r reacts to a pulse excitation with an free oscillation lasting for a long time. A harmonic excitation with frequency ω, starting at a time $t = 0$ and lasting for a long time, results in a complicated transient motion and ends up in forced oscillation at the same frequency ω but a phase which depends on the difference $\omega_r - \omega$. For $\omega = \omega_r$ the oscillation amplitude grows linearly with time.

We consider a set of oscillators having different resonant frequencies ω_{ri} with distribution $f(\omega_r)$. A pulse excitation results in an oscillation of each oscillator with the same initial velocity $\dot{x}(0)$ followed by a free oscillation with individual frequencies ω_{rj}. For instabilities or beam observation the **center-of-mass motion** of the particles is relevant. Due to their different ω_{ri} the freely oscillating particles change their phase with respect to each other and the center-of-mass motion is slowly reduced.

This represents a kind of damping where the **coherent** center-of-mass motion is reduced compared to the **incoherent** motion of the particles. This damping is faster the wider the width of the distribution, i.e. the spread of resonant frequencies. It differs in many respect from other damping mechanisms. The decay of the center of mass oscillation is usually not exponential but follows a function which depends on the form of the resonant frequency distribution $f(\omega_r)$.

In case of a harmonic excitation the phases of the individual particle oscillations are different and depend on the distance $\omega - \omega_{rj}$ between the excitation and individual resonant frequency. This leads to some cancelation which reduces the amplitude of the center-of-mass motion compared to the one of the individual particles.

Treatment of Landau damping

Landau damping can be understood from different points of view and presented in different ways. We treat it here in a manner close to beam observation and experiments.

We calculate the center-of-mass response of a beam with resonant frequency distribution $f(\omega_r)$ to a pulse or harmonic excitation and compare it with experiments.

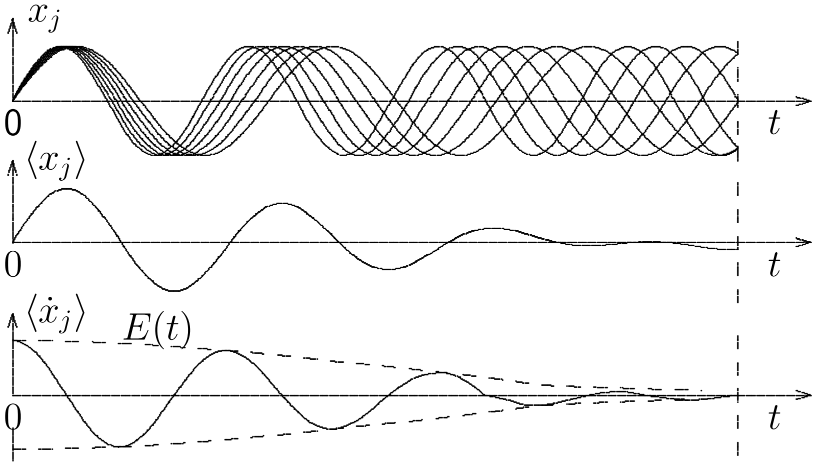
This center-of-mass motion induces fields in the surroundings which act back on the beam and can enhance the excitation. The electrical properties of the components surrounding the beam, relevant for this effect, can be expressed by an impedance. The fields induced in this impedance can be sufficiently large to keep this process going leading to a self excitation. This leads to an instability having a threshold determined by beam response and impedance. Below this threshold the frequency spread eliminates any coherent motion at **infinitesimally small amplitudes** before it can grow; we have stability. Above, the voltage induced in the resistive part of the impedance leads to an increase of initial coherent motion and we have an instability.

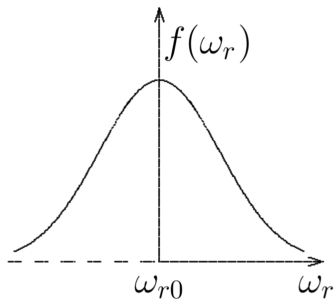
The amount of Landau damping depends on the frequency distribution $f(\omega_r)$ or its derivative at the frequency ω at which the instability occurs. It can happen that the coherent (center-of-mass) motion has a different frequency than the incoherent individual particle frequencies. In this case Landau damping might become ineffective and we can get an instability for a very small resistive impedance.

We will calculate the beam response and Landau damping for transverse and longitudinal oscillations of a coasting (un-bunched) beam. From this we can determine the maximum transverse and longitudinal impedance which still does not create an instability and represent this in the so-called stability diagram. Finally, the dependence of the resonant frequency on amplitude due to a non-linearity is considered which produces a frequency spread leading to Landau damping for bunched beams.

1) Response of an oscillator set to excitation

Response to a pulse excitation





A set of oscillators j of resonant frequencies ω_{rj} get at $t = 0$ a kick with same initial velocity $\dot{x}_j(0^+) = \dot{x}_0$ and make after a free oscillation with different ω_{ej} and constant amplitude $\hat{x}_j = \dot{x}_0/\omega_{rj}$

$$\dot{x}_j(t) = \dot{x}_0 \cos(\omega_{rj} t) \ , \ x_j = \hat{x}_j \sin(\omega_{rj} t)$$

Observers see only the center-of-mass motion $\langle x_j(t) \rangle = \frac{1}{N} \Sigma \, x_j(t)$. We take a normalized, narrow frequency distribution centered around ω_{r0}

$$f(\omega_r) \ , \ \int f(\omega_r) dr = 1 \ , \ \Delta\omega_r = \omega_r - \omega_{r0} \ll \omega_{r0}.$$

This center-of-mass displacement and velocity, given below, are 'damped'

$$\begin{aligned} \langle \dot{x}(t) \rangle &= \dot{x}_0 \int f(\omega_r) \cos(\omega_r t) d\omega_r \\ \langle x(t) \rangle &= \dot{x}_0 \int \frac{f(\omega_r)}{\omega_r} \sin(\omega_r t) d\omega_r \approx \frac{\dot{x}_0}{\omega_{r0}} \int f(\omega_r) \sin(\omega_r t) d\omega_r \end{aligned}$$

Expressing the velocity response by the difference frequency $\Delta\omega_r$

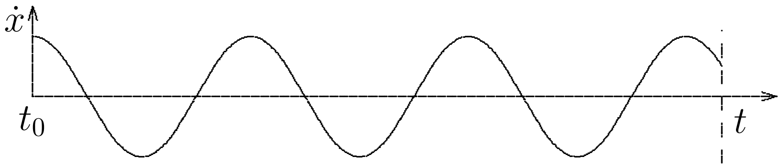
$$g(t) = \langle \dot{x}(t) \rangle / \dot{x}_0 = \cos(\omega_{r0} t) I_1(t) + \sin(\omega_{r0}) I_2(t) = \cos(\omega_{r0} t - \phi) E(t)$$

with I_1 and I_2 representing inverse Fourier integrals

$$\begin{aligned} I_1(t) &= \int f(\Delta\omega_r) \cos(\Delta\omega_r t) d\omega_r, \ I_2(t) = -\int f(\Delta\omega_r)) \sin(\Delta\omega_r t) d\omega_r \\ g(t) &\propto \mathcal{F}^{-1}_{\cos}(f(\Delta\omega_r)) \cos(\omega_{r0} t) - \mathcal{F}^{-1}_{\sin}(f(\Delta\omega_r)) \sin(\omega_{r0} t) \text{ and} \\ E(t) &= \sqrt{I_1^2(t) + I_2^2(t)} \text{ being the envelope of the oscillating response.} \end{aligned}$$

The center-of-mass velocity response $g(t)$ of an oscillator set with resonance frequency distribution $f(\Delta\omega_r)$ to a pulse excitation (Green function) is proportional to the inverse Fourier transform of this distribution times an oscillation with the central frequency ω_{r0}.

Response of a single oscillator to a harmonic excitation



Before we got the velocity response of a single undamped oscillator with resonant frequency ω_r due to a pulse excitation at a time t_0

$$\dot{x}(t) = \dot{x}_0 \cos(\omega_r(t - t_0)).$$

We consider now a harmonic excitation at frequency ω starting at time t_1 and lasting up to the present observation time t. We treat this as a series of infinitesimal kicks with a harmonic modulation

$$d\dot{x}_0 = \frac{d\hat{\dot{x}}}{dt_0}\cos(\omega t_0)dt_0 = G dt_0 = \hat{G}\cos(\omega t_0)dt_0$$

where G is an acceleration. The velocity obtained at the time t is

$$\dot{x}(t) = \hat{G}\int_{t_1}^{t} \cos(\omega t_0)\cos(\omega_r(t - t_0))dt_0.$$

Calling $T = t - t_0,\ T_1 = t - t_1$ and developing $\cos(\omega(t - T))$ gives

$$\begin{aligned}
\frac{\dot{x}(t)}{\hat{G}} &= \int_0^{T_1} \cos(\omega(t - T))\cos(\omega_r T)dT, \\
&= \int_0^{T_1} (\cos(\omega t)\cos(\omega T) + \sin(\omega t)\sin\omega T))\cos(\omega_r T)dT \\
&= \frac{T_1}{2}\left[\cos(\omega t)\left(\frac{\sin((\omega_r - \omega)T_1)}{(\omega_r - \omega)T_1} + \frac{\sin((\omega_r + \omega)T_1)}{(\omega_r + \omega)T_1}\right)\right. \\
&\qquad \left. - \sin(\omega t)\left(\frac{1 - \cos((\omega_r - \omega)T_1)}{(\omega_r - \omega)T_1} + \frac{1 - \cos((\omega + \omega_r)T_1)}{(\omega_r + \omega)T_1}\right)\right]
\end{aligned}$$

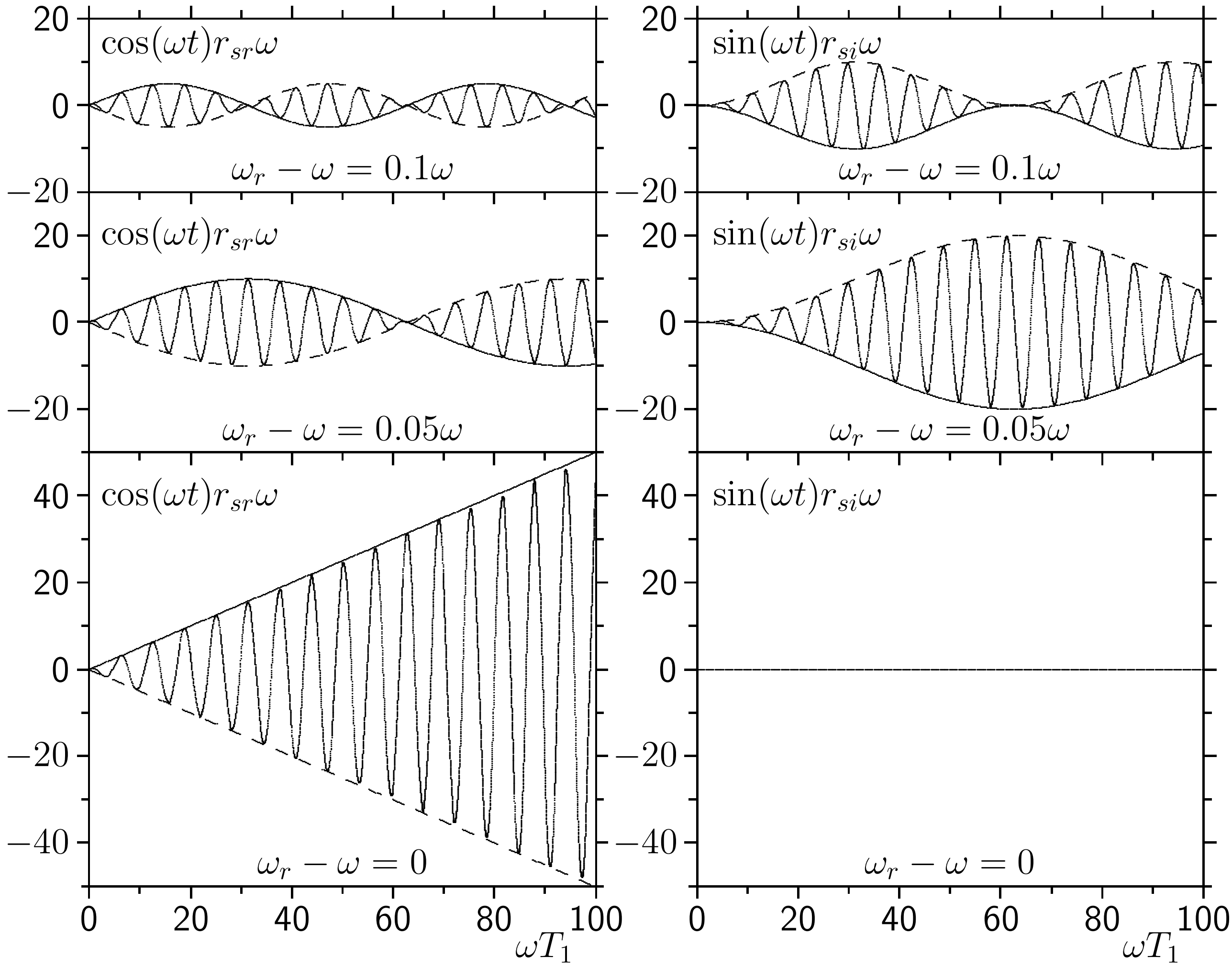
For $\omega \approx \omega_{r0}$ we neglect in each line the 2^{nd} term compared to the 1^{st}

$$\begin{aligned}
\frac{\dot{x}(t)}{\hat{G}} &= \frac{1}{2}\left[\cos(\omega t)\frac{\sin((\omega_r - \omega)T_1)}{\omega_r - \omega} - \sin(\omega t)\frac{1 - \cos((\omega_r - \omega)T_1)}{\omega_r - \omega}\right] \\
&= \cos(\omega t) r_{sr}(\omega_r) + \sin(\omega t) r_{si}(\omega_r).
\end{aligned}$$

Velocity and acceleration are in phase for the 1^{st}, resistive, term and energy can be absorbed, but out of phase for the 2^{nd}, reactive, term.

$$
\begin{aligned}
\frac{\dot{x}(t)}{\hat{G}} &= \frac{T_1}{2}\left[\cos(\omega t)\frac{\sin((\omega_r-\omega)T_1)}{(\omega_r-\omega)T_1} - \sin(\omega t)\frac{1-\cos((\omega_r-\omega)T_1)}{(\omega_r-\omega)T_1}\right] \\
&= \cos(\omega t) r_{sr}(\omega_r-\omega) + \sin(\omega t) r_{si}(\omega_r-\omega).
\end{aligned}
$$

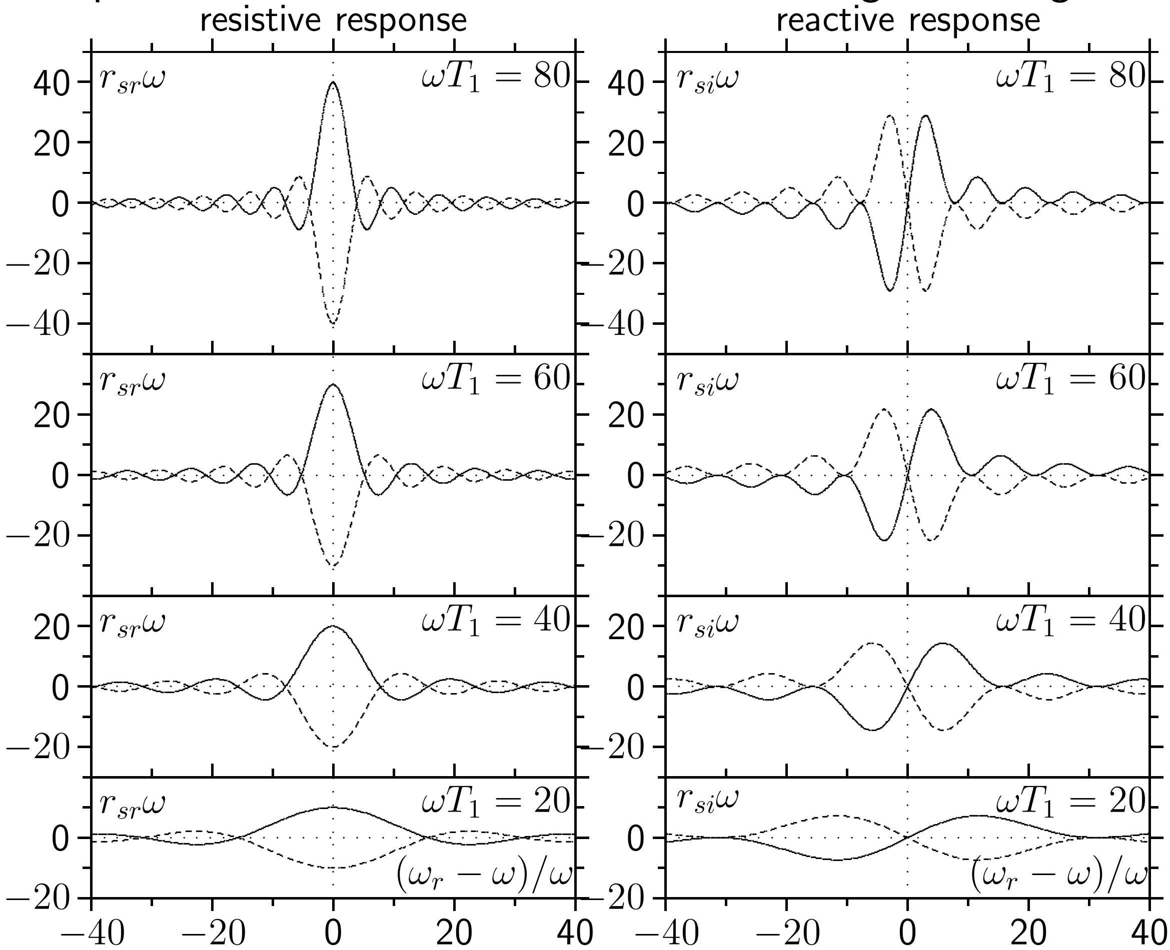
At $\omega_r = \omega$ the first term increases linearly with excitation $r_{sr}(0) = T_1/2$ because $\sin x/x = 1$ for $x \to 0$ and the second term vanishes $r_{si}(0) = 0$. For the general case the response increases with T_1 initially with the same slope, but reaches a maximum value being smaller for larger frequency differences $\omega_r - \omega$ and oscillates slowly around the extreme values due to an interference between resonant and excitation frequency. The reactive part has a similar behavior but the initial increase is quadratic. We multiply these responses with ω to make them dimensionless and plot them as a function of excitation time T_1 for different frequency differences $\omega_r - \omega$



It is instructive to plot the single oscillator response versus the frequency difference $\Delta\omega = \omega_r - \omega$ for different excitation times T_1

$$\begin{aligned}\frac{\dot{x}(t)}{\hat{G}} &= \frac{1}{2}\left[\cos(\omega t)\frac{\sin((\omega_r-\omega)T_1)}{\omega_r-\omega} - \sin(\omega t)\frac{1-\cos((\omega_r-\omega)T_1)}{\omega_r-\omega}\right] \\ &= \cos(\omega t) r_{sr}(\omega_r-\omega) + \sin(\omega t) r_{si}(\omega_r-\omega).\end{aligned}$$

This response is more and more concentrated around $\omega_r = \omega$ where its resistive part has a maximum and the reactive one goes through zero.



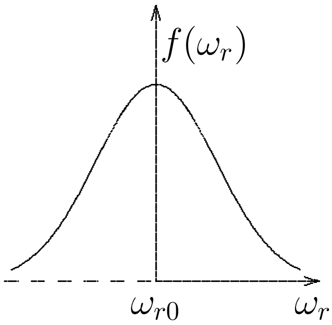
The fast oscillation of the response for large T_1 and $\omega_r - \omega$ gets averaged while integrating over $f(\omega_r)$ giving with $\int_{-\infty}^{\infty} dx \sin(ax)/x = 1$

$$\frac{\sin(\omega_r - \omega T_1)}{(\omega_r-\omega)} \approx \begin{cases}\infty & \text{if } \omega = \omega_r \\ 0 & \text{if } \omega \neq \omega_r\end{cases} \quad \rightarrow \quad r_{sr} \approx \frac{\pi}{2}\delta(\omega_r - \omega)$$

$$\frac{1-\cos(\omega_r - \omega T_1)}{(\omega_r-\omega)} \approx \begin{cases}0 & \text{if } \omega = \omega_r \\ 1/(\omega_r-\omega) & \text{if } \omega \neq \omega_r\end{cases}$$

$$\dot{x}(t) \approx \frac{\hat{G}}{2}\left[\cos(\omega t)\pi\delta(\omega_r-\omega) - \sin(\omega t)\left(\frac{1}{\omega_r-\omega}\right)_{\omega\neq\omega_r}\right]$$

Response of an oscillator set to a harmonic excitation



$$f(\omega_r) = \frac{1}{N}\frac{dN}{d\omega_r}$$

$$\int_0^\infty f(\omega_r) d\omega_r = 1$$

We take many oscillators with a normalized distribution $f(\omega_r)$. The velocity of the center-of-mass motion is obtained by taking the single oscillator response

$$\dot{x}(t) \approx \frac{\hat{G}}{2}\left[\cos(\omega t)\pi\delta(\omega_r - \omega) - \sin(\omega t)\left(\frac{1}{\omega_r - \omega}\right)_{\omega \neq \omega_r}\right]$$

and integrating over this distribution

$$\begin{aligned}
\langle \dot{x}(t) \rangle &= \int_0^\infty \dot{x}(t, \omega_r) f(\omega_r) d\omega_r \\
\frac{\langle \dot{x}(t) \rangle}{\hat{G}} &= \frac{1}{2}\left(\cos(\omega t)\pi f(\omega) - \sin(\omega t) \text{PV} \int_{-\infty}^{\infty} \frac{f(\omega_r) d\omega_r}{\omega_r - \omega}\right) \\
&= \cos(\omega t) r_r(\omega) + \sin(\omega t) r_i(\omega)
\end{aligned}$$

with the 'principle value integral'

$$\text{PV} \int \frac{f(\omega_r)}{\omega - \omega_r} d\omega_r = \lim_{\epsilon \to 0}\left[\int_{-\infty}^{\omega - \epsilon} \frac{f(\omega_r)}{\omega_r - \omega} + \int_{\omega + \epsilon}^{\infty} \frac{f(\omega_r)}{\omega_r - \omega}\right] d\omega_r$$

This response to harmonic excitation is called **transfer function**. Its resistive part is proportional to the distribution at ω and vanishes therfore if the excitation frequency lies outside the distribution $f(\omega_r)$. By integrating over t we get the displacement response

$$\begin{aligned}
\frac{\langle x(t) \rangle}{\hat{G}} &= \frac{1}{2\omega}\left(\sin(\omega t)\pi f(\omega) + \cos(\omega t) \text{PV} \int_{-\infty}^{\infty} \frac{f(\omega_r) d\omega_r}{\omega_r - \omega}\right) \\
&= \frac{1}{\omega}\left[\sin(\omega t) r_r - \cos(\omega t) r_i\right]
\end{aligned}$$

Short derivation using complex notation

We have derived the response to a harmonic excitation using real functions and positive frequencies. We use now complex notation with

$$\cos(\omega t) = (\mathrm{e}^{j\omega t} + \mathrm{e}^{-j\omega t})/2\ ,\ 0 \le \omega \le \infty \to \mathrm{e}^{j\omega t}/2\ ,\ -\infty \le \omega \le \infty$$

For oscillators with $f(\omega_r) = f(-\omega_r)$, no damping, complex notation

$$\ddot{x} + \omega_r^2 x = \frac{\hat{G}}{2}\mathrm{e}^{j\omega t}\ ,\ (-\omega^2 + \omega_r^2)x = \frac{\hat{G}}{2}\mathrm{e}^{j\omega t}.$$

The displacement response of a single oscillator is

$$x = \frac{\hat{G}\mathrm{e}^{j\omega t}}{2(\omega_r^2 - \omega^2)} = \frac{\hat{G}\mathrm{e}^{j\omega t}}{2(\omega_r - \omega)(\omega_r + \omega)} = \frac{\hat{G}\mathrm{e}^{j\omega t}}{4\omega}\left(\frac{1}{\omega_r - \omega} - \frac{1}{\omega_r + \omega}\right)$$

For $\omega > 0$ only the first and for $\omega < 0$ only the second term is large. The first, integrated over $f(\omega_r)$, gives center-of mass response

$$\frac{\langle x\rangle_+}{\hat{G}} = \frac{\mathrm{e}^{j\omega t}}{4\omega}\int_0^\infty \frac{f(\omega_r)}{\omega_r - \omega}d\omega_r\ ,\quad \frac{\langle \dot{x}\rangle_+}{\hat{G}} = \frac{j\mathrm{e}^{j\omega t}}{4}\int_0^\infty \frac{f(\omega_r)}{\omega_r - \omega}d\omega_r.$$

This integration over a pole, treated in theory of functiond. gives a PV (principle value) integral plus an imaginary residue. The sign ambiguity, due to undefined initial value, is resolved assuming $\dot{x}(-\infty) = 0$

$$\int \frac{f(\omega_r)d\omega_r}{\omega_r - \omega} = \pm j\pi f(\omega) + \mathsf{PV}\int \frac{f(\omega_r)d\omega_r}{\omega_r - \omega} = -j\pi f(\omega) + \mathsf{PV}\int \frac{f(\omega_r)d\omega_r}{\omega_r - \omega}$$

$$\text{giving for } \omega > 0\ \frac{\langle \dot{x}\rangle_+}{\hat{G}} = \frac{\mathrm{e}^{j\omega t}}{4}\left[\pi f(\omega) + j\mathsf{PV}\int_{-\infty}^\infty \frac{f(\omega_r)}{\omega_r - \omega}d\omega_r\right]$$

$$\text{and for } \omega < 0\ \frac{\langle \dot{x}\rangle_-}{\hat{G}} = \frac{\mathrm{e}^{-j\omega t}}{4}\left[\pi f(\omega) - j\mathsf{PV}\int_{-\infty}^\infty \frac{f(\omega_r)}{\omega_r - \omega}d\omega_r\right]$$

$$\frac{\langle \dot{x}\rangle_+}{\hat{G}} + \frac{\langle \dot{x}\rangle_-}{\hat{G}} = \frac{1}{2}\left[\cos(\omega t)\pi f(\omega) - \sin(\omega t)\mathsf{PV}\int_{-\infty}^\infty \frac{f(\omega_r)}{\omega_r - \omega}d\omega_r\right]$$

which agrees with the previous result.

Relation between pulse and harmonic excitation

The center-of-mass velocity response of an oscillator set with resonant frequency distribution $f(\Delta\omega_r)$ to a pulse and harmonic excitation is

$$g(t) = \cos(\omega_{r0}t)\int f(\Delta\omega_r)\cos(\Delta\omega_r t)d\omega_r - \sin(\omega_{r0}t)\int f(\Delta\omega_r)\sin(\Delta\omega_r t)d\omega_r$$

where $\Delta\omega_r = \omega_r - \omega_{r0}$ is deviation from distribution center.

$$\begin{aligned}\frac{\langle \dot{x}(t)\rangle}{\hat{G}} &= \frac{1}{2}\left(\cos(\omega t)\pi f(\omega) - \sin(\omega t)\text{PV}\int_{-\infty}^{\infty}\frac{f(\omega_r)d\omega_r}{\omega_r - \omega}\right)\\ &= \cos(\omega t)r_r(\omega) + \sin(\omega t)r_i(\omega)\end{aligned}$$

We Fourier transform $g(t)$ using a factor $1/(2\pi)$ instead $1/\sqrt{2\pi}$

$$\tilde{g}_{\cos}(\Delta\omega) = \frac{1}{2\pi}\int_0^{\infty} dt\int d\Delta\omega_r\cos(\Delta\omega_r)\cos(\Delta\omega t)$$

with $\Delta\omega = \omega - \omega_{r0}$. The integral $\int_0^{\infty}\cos(\Delta\omega_r t)\cos(\Delta\omega t)dt$ is infinite for $\Delta\omega_r = \Delta\omega$ and vanishes otherwise, giving the Dirac delta function

$$\int_0^{\infty}\cos(\Delta\omega_r t)\cos(\Delta\omega t)dt = \pi\delta(\omega_r - \omega).$$

This gives for the cosine Fourier transform

$$\tilde{g}_{\cos}(\Delta\omega) = \frac{1}{2}\int_0^{\infty} f(\Delta\omega_r)\delta(\omega_r - \omega)d\omega_r = \frac{1}{2}f(\Delta\omega).$$

The sine Fourier transform integral we got before

$$\begin{aligned}\int_0^{\infty}\cos(\Delta\omega_r t)\sin(\Delta\omega t)dt &= \left.\frac{1-\cos((\omega_r-\omega)T)}{2(\omega_r-\omega)}\right|_{T\to\infty}\\ &= \begin{cases} 0 & \text{if } \omega_r = \omega \\ 1/(\omega_r-\omega) & \text{if } \omega \neq \omega_r\end{cases}\end{aligned}$$

$$\tilde{g}_{\sin}(\Delta\omega) = \frac{1}{2}\ \text{PV}\int_0^{\infty}\frac{f(\Delta\omega_r)}{\omega_r - \omega}d\omega_r.$$

The cosine and sine Fourier transforms of the pulse response $g(t)$, (Green function), equal the resistive and reactive parts of the harmonic response, (transfer function).

Response of oscillators with a Gaussian distribution

$$f(\Delta\omega_r) = \frac{1}{\sqrt{2\pi}\sigma_\omega} \mathrm{e}^{-(\Delta\omega_r/\sigma_\omega)^2/2} \quad \text{with} \quad \int_{-\infty}^{\infty} f(\Delta\omega_r) d\omega_r = 1$$

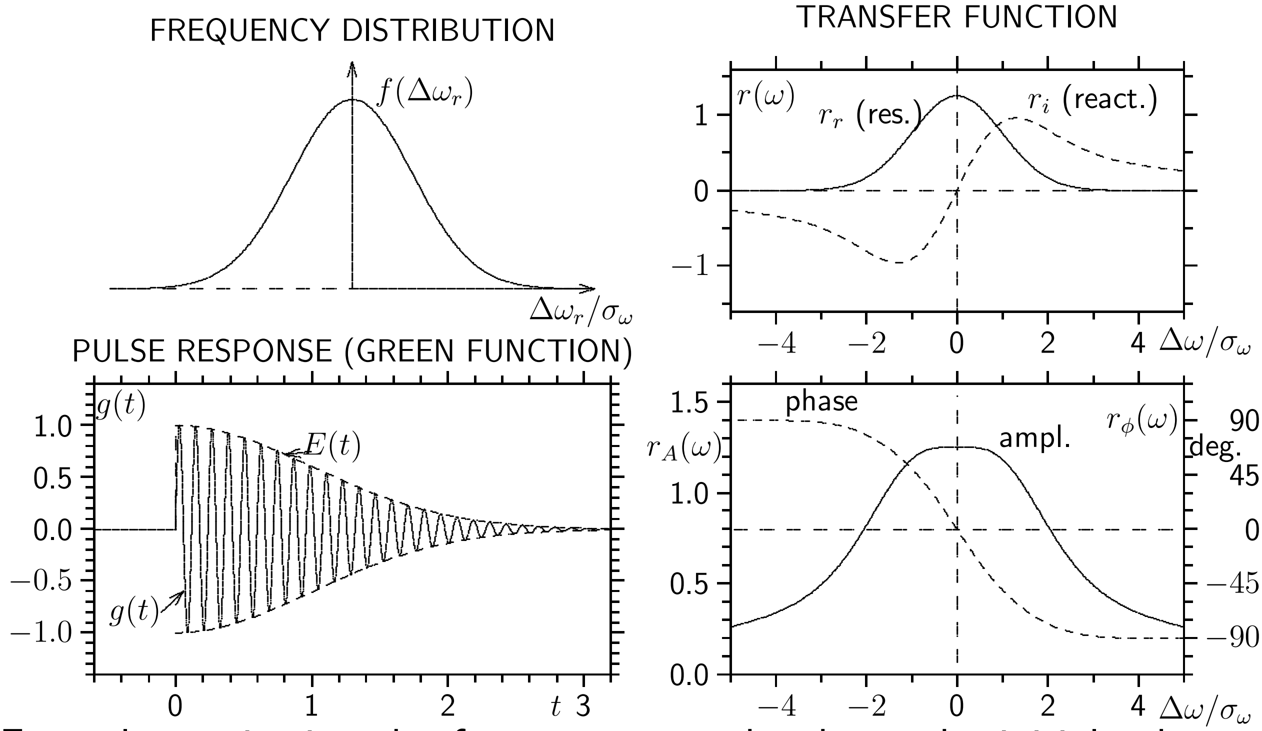
Pulse response $g(t)$ and its envelope $E(t)$ are

$$\begin{aligned} g(t) &= \frac{\cos(\omega_{r0}t)}{\sqrt{2\pi}\sigma_\omega} \int_{-\infty}^{\infty} \mathrm{e}^{-\Delta\omega_r^2/2\sigma_\omega^2} \cos(\Delta\omega_r t) d\omega_r \\ &= \mathrm{e}^{-\sigma_\omega^2 t^2/2} \cos(\omega_{r0}t) = E(t)\cos(\omega_{r0}t). \end{aligned}$$

The transfer function is obtained by a Fourier transform of $E(t)$

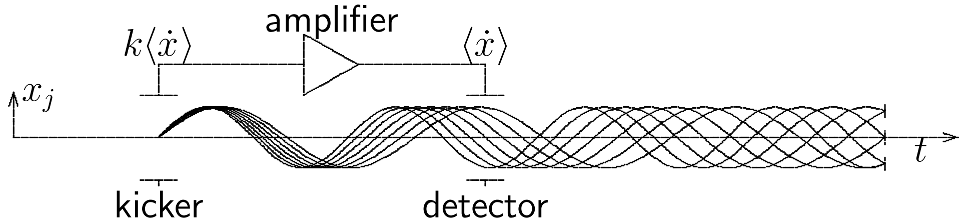
$$\begin{aligned} r_r(\omega) &= \int_0^{\infty} \mathrm{e}^{-\sigma_\omega^2 t^2/2} \cos(\Delta\omega t) dt = \frac{\pi}{\sqrt{2\pi}\sigma_\omega} \mathrm{e}^{-\Delta\omega^2/2\sigma_\omega^2} \\ r_i(\omega) &= \int_0^{\infty} \mathrm{e}^{-\sigma_\omega^2 t^2/2} \sin(\Delta\omega t) dt = \frac{\sqrt{2}}{\sigma_\omega} \mathrm{e}^{-(\Delta\omega/\sigma_\omega)^2/2} \int_0^{\Delta\omega/(\sqrt{2}\sigma_\omega)} \mathrm{e}^{t'^2} dt' \end{aligned}$$

The integral on the right is called Dawson integral.



For pulse excitation the frequency spread reduces the initial coherent motion while individual oscillations continue with fixed amplitudes. In harmonic excitation the oscillators respond with different phases but a few, being on resonance, have growing amplitudes and absorb energy. Both lead to finite incoherent oscillations. Landau damping works by reducing the coherent motion at small levels before finite amplitudes are reached.

Landau damping of oscillator set



Based on the center of mass response of a set of oscillators we illustrate how the frequency spread leads to Landau damping of coherent oscillation which would otherwise grow.
The velocity of center of mass motion of a set of oscillators is measured by a detector, the signal is amplified and fed to a kicker to produce an acceleration G_s in phase with the velocity which should lead to a growing oscillation, i.e. a negative feed-back system. The center of mass velocity response to an external acceleration $G = \hat{G}\exp(j\omega t)$ is in general and for a Gaussian distribution

$$\begin{aligned}\langle\dot{x}\rangle &= G[r_r + jr_i] = G\frac{1}{2}\left(\pi f(\omega) + jPV\int\frac{f(\Delta\omega_r)}{\omega_r - \omega}d\omega_r\right)\\ &= G\left(\frac{\pi}{\sqrt{2\pi}\sigma_\omega}\mathrm{e}^{-\Delta\omega^2/2\sigma_\omega^2} + j\frac{\sqrt{2}}{\sigma_\omega}\mathrm{e}^{-(\Delta\omega/\sigma_\omega)^2/2}\int_0^{\Delta\omega/(\sqrt{2}\sigma_\omega)}\mathrm{e}^{t'^2}dt'\right).\end{aligned}$$

We assume now that the excitation happens at the central frequency for the Gaussian distribution $\Delta\omega = 0$ for which $r_i = 0$

$$\langle\dot{x}\rangle = G\frac{\pi}{\sqrt{2\pi}\sigma_\omega}$$

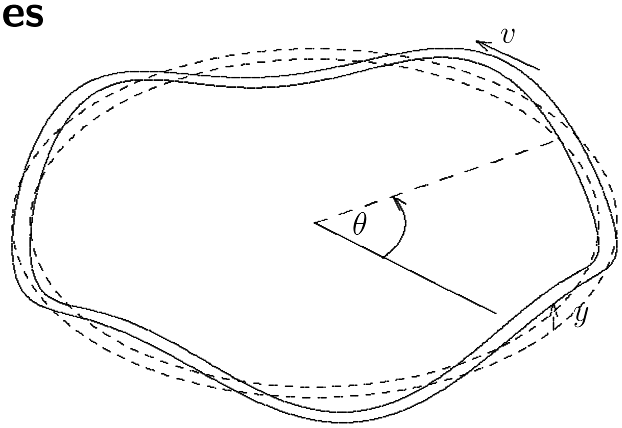
We replace the external excitation G by the one of the feed-back $G_s = k\langle\dot{x}\rangle$ and assume a gain k just sufficient to keep the oscillation going. This is the limit of stability since a slightly larger gain would increase the oscillation leading to an exponential growth

$$\langle\dot{x}\rangle = k\langle\dot{x}\rangle\frac{\pi}{\sqrt{2\pi}\sigma_\omega} \;\rightarrow\; k \le \sqrt{\frac{2}{\pi}}\sigma_\omega.$$

This maximum gain k still giving stability is proportional to the frequency spread. Landau damping works by making an accidental coherent oscillation incoherent at infinitesimal levels without having first a growth reaching finite amplitudes. It does not lead to a growth of incoherent oscillations.

3) Transverse coasting beam instability

Oscillation modes



A uniform coasting beam of N particles circulates with revolution frequency ω_0, current $I = eN\omega_0/(2\pi)$ in a ring of uniform focusing. Each particle executes a betatron oscillation of frequency $Q\omega_0$

$$\theta_i = \theta_{0i} + \omega_0 t \quad , \quad y_i(t) = \hat{y}\cos(Q\omega_0(t - t_i)).$$

Depending on the phases $Q\omega_0 t_i$ between adjacent particle we have different modes. We choose a set having a form as seen at a fixed location θ

$$y(t) = \hat{y}\cos(n\theta - \omega t) \ , \ y(0) = \hat{y}\cos(n\theta).$$

Frozen in time $t = 0$ we have a closed wave with n periods. Following a particle $\theta_s(t) = \theta_0 + \omega_0 t$ give us the betatron oscillation with frequency $Q\omega_0$.

$$y_s = \hat{y}\cos(n\theta_0 - (\omega - n\omega_0)t) = \hat{y}\cos(n\theta_s - Q\omega_0 t)$$

giving for the frequency ω seen by a stationary observer

$$\omega = (n + Q)\omega_0 = \omega_\beta \ \text{ with } \ -\infty < n < \infty.$$

We divide modes into fast and slow waves according to the sign of the phase difference between adjacent particle

$$\begin{aligned} \omega_{\beta f} &= (n_f + Q)\omega_0 \ , \ n_f > -Q \\ \omega_{\beta s} &= (n_s - Q)\omega_0 \ , \ n_s > Q. \end{aligned}$$

Effect of momentum spread

The betatron frequencies of a beam with nominal momentum are:

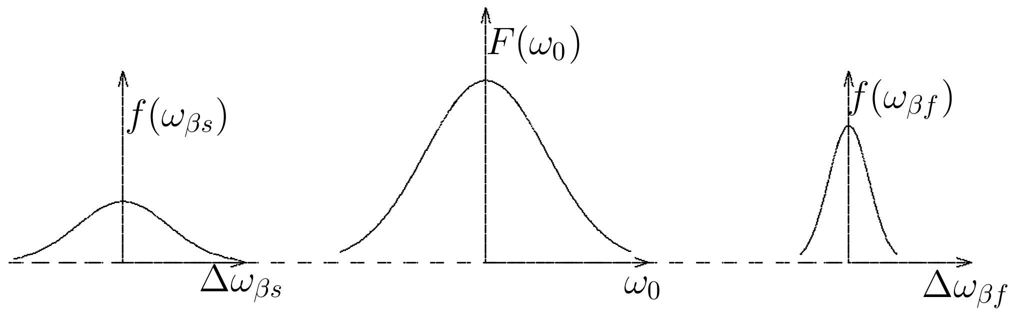
$$\omega_{\beta f} = (n_f + Q)\omega_0 \;\;, \;\; \omega_{\beta s} = (n_s - Q)\omega_0 .$$

Through $\dfrac{\Delta E}{E} = \beta^2 \dfrac{\Delta p}{p} = -\dfrac{\beta^2}{\eta_c}\dfrac{\Delta\omega_0}{\omega_0}$, and $\Delta Q = Q' \dfrac{\Delta p}{p}$.

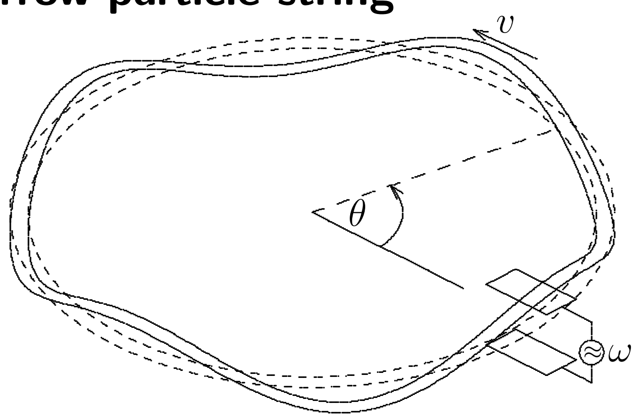
they are affected by a momentum deviation

$$\begin{aligned}\Delta\omega_{\beta f} &= (Q' - \eta_c(n_f + Q))\omega_0 \frac{\Delta p}{p} \\ \Delta\omega_{\beta s} &= (Q' - \eta_c(n_s - Q))\omega_0 \frac{\Delta p}{p}.\end{aligned}$$

resulting in two frequency distributions $f(\omega_{\beta f}),\; f(\omega_{\beta s})$.



Response of narrow particle string



All particle have the same momentum forming uniform ring. We excite them

$$\ddot{y} + \omega_0 Q^2 y = \hat{G}\cos(\omega)$$

and seek a solution $y(t) = \hat{y}\cos(n\theta - \omega)$. To excite such a mode n each particle has to be driven by the proper phase corresponding to its longitudinal position θ. Therefore, we expect to find a excitation frequency which is not $\omega_0 Q$ but close to the fast or slow wave frequencies $\omega_{\beta f} = (n_f + Q)\omega_0$ or $\omega_{\beta s} = (n_s - Q)\omega_0$.

Substituting the desired solution form in the differential equation form gives

$$\left(-(n\omega_0 - \omega)^2 + Q^2\omega_0^2\right)\hat{y}\cos(n\theta - \omega t) = \hat{G}\cos(\omega t).$$

We assume excitation and observation is done at the location $\theta = 0$

$$\begin{aligned}\hat{y} &= \frac{\hat{G}}{\omega_0^2 Q^2 - (n\omega_0 - \omega)^2} = \frac{-\hat{G}}{(\omega - \omega_0(n+Q)(\omega - \omega_0(n-Q)} \\ &= \frac{-\hat{G}}{(\omega - \omega_{\beta f})(\omega - \omega_{\beta s})} = \frac{\hat{G}}{2\omega_0 Q}\left(\frac{1}{\omega - \omega_{\beta s}} - \frac{1}{\omega - \omega_{\beta f}}\right).\end{aligned}$$

to excite the fast wave we use $\omega \approx (n_f + Q)\omega_0$ and the first term is much smaller than the second one. Correspondingly for the slow wave we use $\omega \approx (n_s - Q)\omega_0$ and the second term is much smaller than the first one. We approximate for the two waves

$$\left(\frac{\hat{y}}{\hat{G}}\right)_f \approx -\frac{1}{2\omega_0 Q}\left(\frac{1}{\omega - \omega_{\beta f}}\right),\ \left(\frac{\hat{y}}{\hat{G}}\right)_s \approx \frac{1}{2\omega_0 Q}\left(\frac{1}{\omega - \omega_{\beta s}}\right).$$

The two responses have opposite sign, this will be discussed later.

Response of the whole beam

The whole beam has frequency distribution $f(\omega_{\beta f})$ and $f(\omega_{\beta s})$ The center-of-mass responses in displacement and velocity are related by $\langle \dot{y} \rangle = j\omega \langle y \rangle$

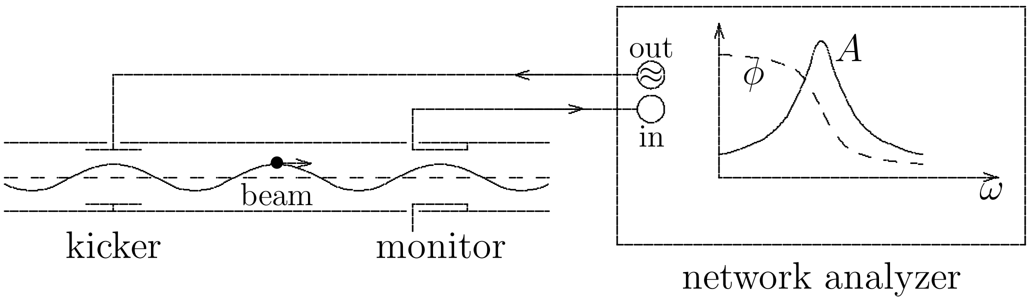
$$\langle \hat{\dot{y}} \rangle_f = -\frac{\hat{G}\omega}{2Q\omega_0} \int \frac{f(\omega_{\beta f})}{\omega_{\beta f} - \omega} d\omega_{\beta f}) = -\frac{\hat{G}\omega}{2Q\omega_0} \left(\pi f(\omega) - jPV \int \frac{f(\omega_{\beta f})}{\omega - \omega_{\beta f}} \right) d\omega_{\beta f}.$$

$$\langle \hat{\dot{y}} \rangle_s = \frac{\hat{G}\omega}{2Q\omega_0} \int \frac{f(\omega_{\beta s}}{\omega_{\beta s} - \omega} d\omega_{\beta s} = \frac{\hat{G}\omega}{2Q\omega_0} \left(\pi f(\omega) - jPV \int \frac{f(\omega_{\beta s})}{\omega - \omega_{\beta s}} \right) d\omega_{\beta s}.$$

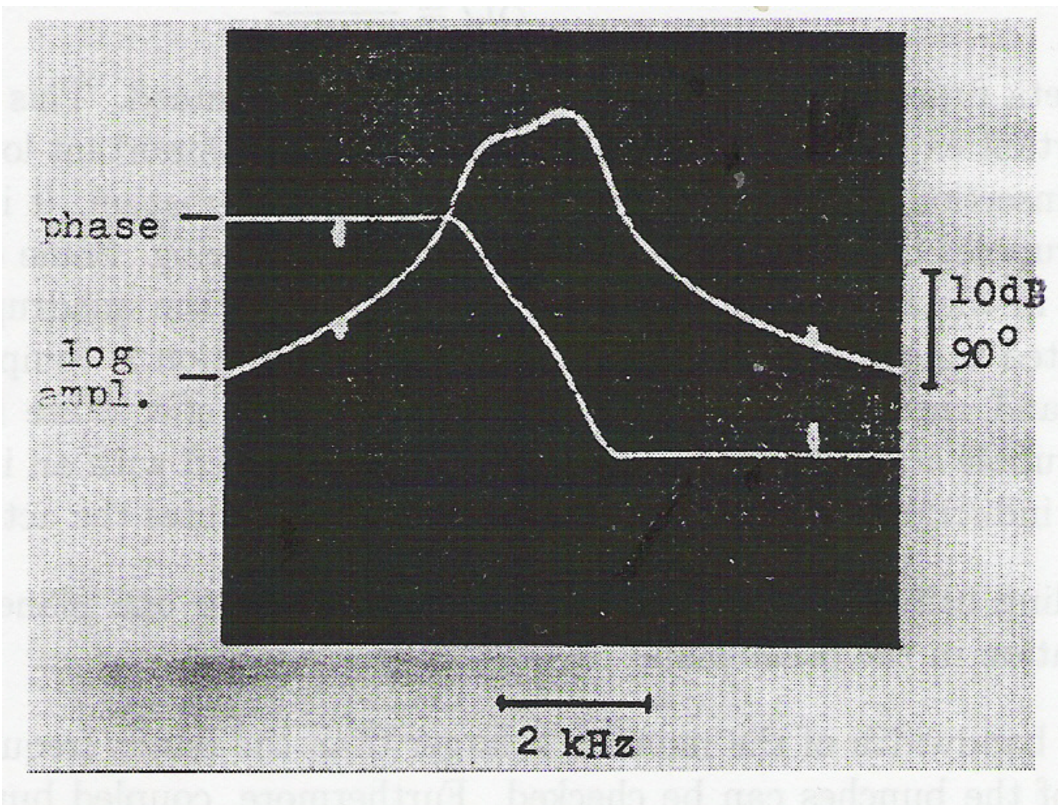
The term $\pi f(\omega)$ is real, exciting acceleration and responding velocity are in phase resulting in an absorption of energy and damping, called Landau damping. It is only present if the excitation frequency ω is within the frequency distribution of the individual particles. The second term is imaginary and gives the out-of-phase response being of less interest.

The spread in betatron frequencies is given by the momentum spread and the dependence of revolution frequency ω_0 and betatron tune Q on momentum deviation $\Delta p/p$. It is therefore determined by an **external parameter** which is not affected by the excitation of betatron oscillations.

Measuring the beam response

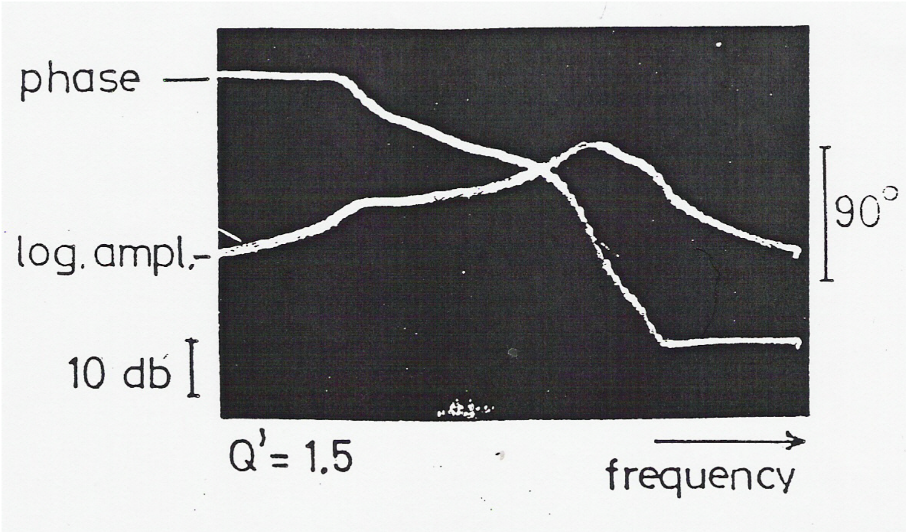


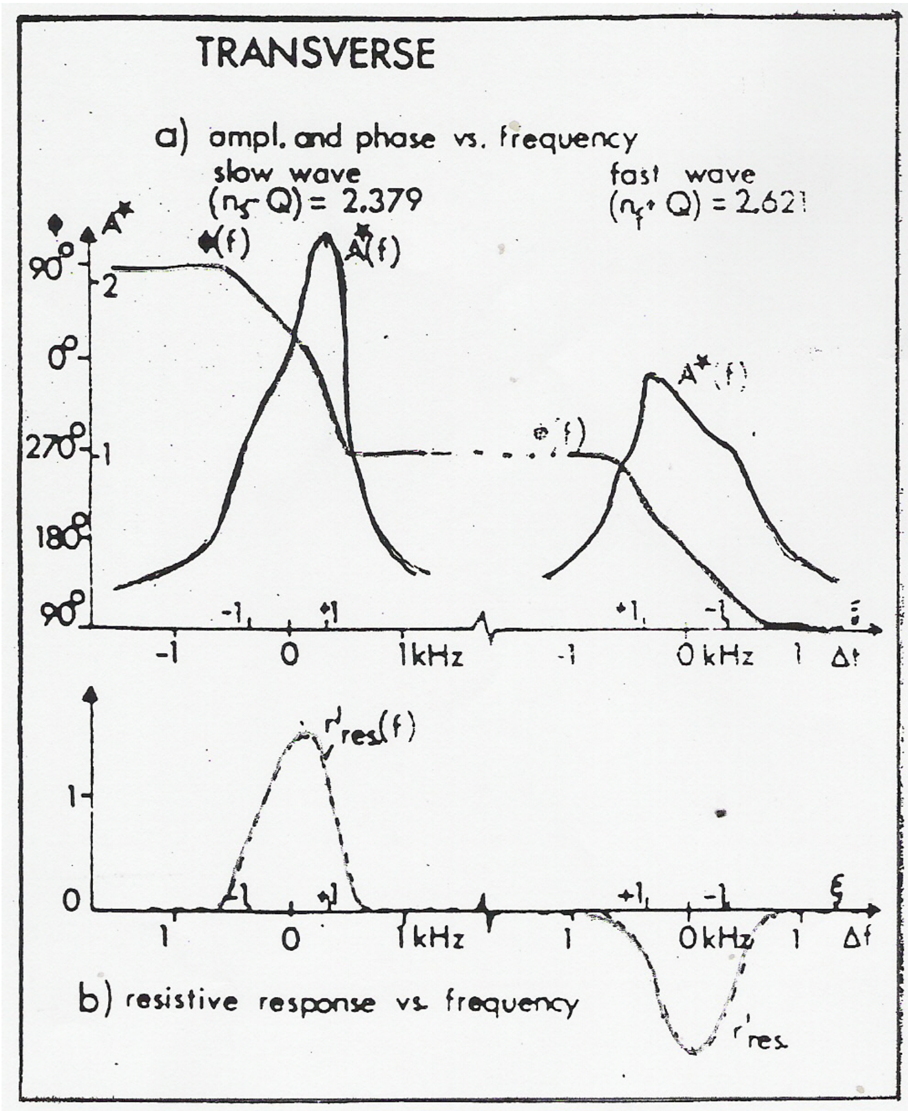
The center-of-mass displacement response can directly be measured with a network analyzer. Here, we derived the velocity response which is more transparent for understanding the resistive and reactive behavior of the beam. In measurements the displacement is observed and our equation have to be converted to analyze the results. Due to cable delays the real and imaginary part of the response are often mixed. It is easier to measure amplitude and phase response and correct the latter off-line.



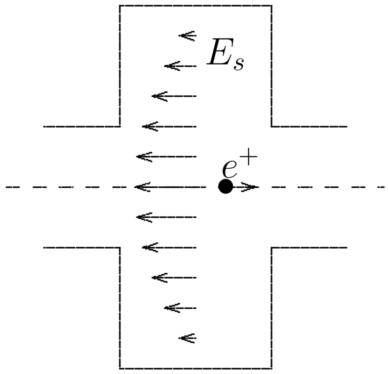
Vertical TF of an unbunched beam in the ISR

Measurement of upper and lower side-band





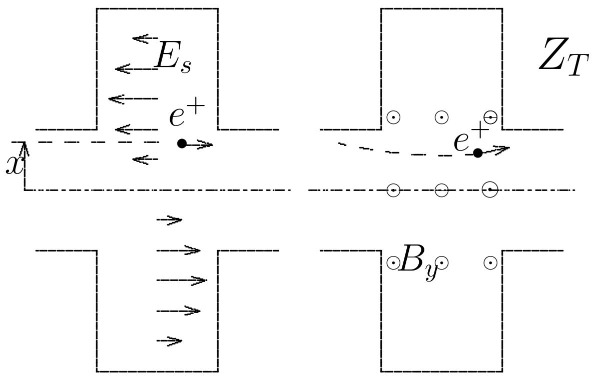
Longitudinal impedance



$$Z(\omega) = -\frac{\int \tilde{E}_s(\omega) ds}{\tilde{I}(\omega)} = \frac{\tilde{V}(\omega)}{\omega}.$$

The longitudinal impedance is the ratio between the integrated longitudinal field and the exciting current. It has a real (resistive) part for which voltage and current are in phase and an imaginary (reactive) part for which they are out of phase. It is measured in Ohm=V/A.

Transverse impedance



$$\begin{aligned} Z_T(\omega) &= j\frac{\int \left(\tilde{\mathbf{E}}(\omega + [\mathbf{v} \times \tilde{\mathbf{B}}(\omega)]\right)_T ds}{\widetilde{Ix}(\omega)} \\ &= \frac{-\omega \int \left(\tilde{\mathbf{E}}(\omega + [\mathbf{v} \times \tilde{\mathbf{B}}(\omega)]\right)_T ds}{\widetilde{I\dot{x}}(\omega)}. \end{aligned}$$

The transverse impedance is the ratio between a longitudinal integral over the transversely deflecting fields and the dipole moment of the current which excites it. It is illustrated by a cavity mode having a transverse electric field with a gradient $\partial E_s/\partial x$ which is first induced by the dipole moment of the current. After 1/4 oscillation this is converted into a transverse magnetic field B_y which produces a deflection in the x-direction. The 'j' in front of the first definition indicates that the exciting dipole moment and the deflecting field are out of phase. However, the second definition relates the transverse deflection to the transverse velocity is real indicating the transfer of energy. Like in the longitudinal case the transverse impedance has a real (resistive) and an imaginary (reactive) part, furthermore it has a horizontal and vertical component. It is measured in units of Ohm/m=V/(A m).

Stability limit

The oscillating beam can induce a voltage in a transverse impedance which in turn applies a self acceleration G_s to the beam

$$Z_T(\omega) = -\frac{\omega}{I\dot{y}(\omega)} \oint \left(\vec{E}(\omega) + [\vec{\beta} \times \vec{B}(\omega)]\right)_T ds \quad , \quad \hat{G}_s = -\frac{eZ_T I\langle\dot{y}\rangle}{\gamma m_0 2\pi R\omega}$$

If $\hat{G}_s = \hat{G}$ we can have a steady self sustained oscillation without external excitation, i.e. a threshold of an instability. Introducing this into the response we get for this threshold

$$1 = -\frac{jecIZ_T(\omega)}{4\pi QE} \int \frac{f(\omega_{\beta f})}{\omega - \omega_{\beta f}} d\omega_{\beta f} = -\frac{ecIZ_T(\omega)}{4\pi QE} \left(\pi f(\omega) - jPV \int \frac{f(\omega_{\beta f})}{\omega - \omega_{\beta f}} d\omega_{\beta f}\right).$$

$$1 = \frac{jecIZ_T(\omega)}{4\pi QE} \int \frac{f(\omega_{\beta s})}{\omega_{\beta s} - \omega} d\omega_{\beta s} = \frac{ecIZ_T(\omega)}{4\pi QE} \left(\pi f(\omega) + jPV \int \frac{f(\omega_{\beta s})}{\omega - \omega_{\beta s}} d\omega_{\beta s}\right).$$

These equations represent relations between the complex impedance and the complex beam response to an excitation. We plot this as a stability diagram shown for a Gaussian distribution. If the impedances lies inside the central curve we have stability, outside an instability. The curve itself represents the threshold. Its shape is determined by the frequency distribution of the particles.

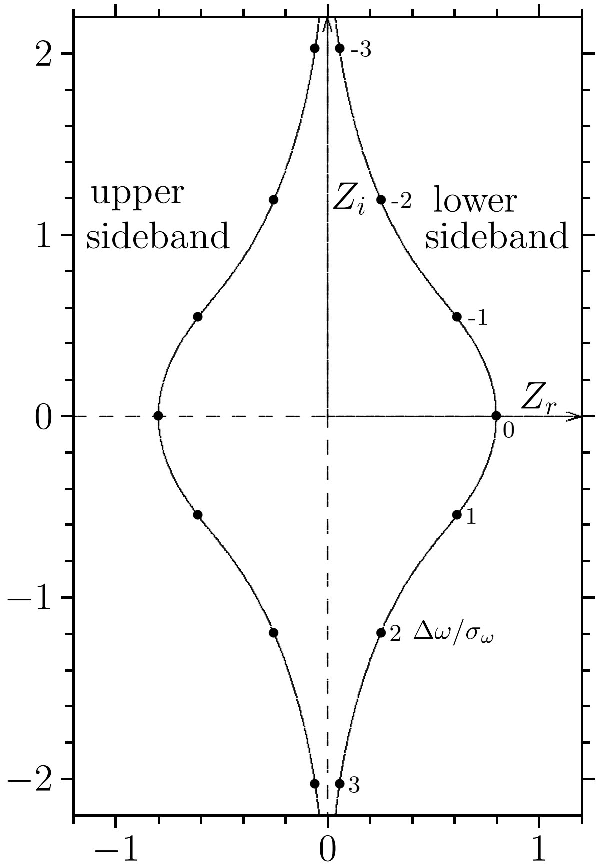
The stability diagram is the inverse response of the beam, i.e. inverse amplitude plotted against the negative phase, it is an inverse Nyquist diagram.

lower sideband

$$\frac{jecIZ_T(\omega)}{4\pi QE} \leq \frac{1}{\int \frac{f(\omega_{\beta s})}{\omega - \omega_{\beta s}} d\omega_{\beta s}}$$

upper sideband

$$\frac{jecIZ_T(\omega)}{4\pi QE} \leq -\frac{1}{\int \frac{f(\omega_{\beta f})}{\omega - \omega_{\beta f}} d\omega_{\beta f}}$$



Response in the presence of an impedance

The beam response to an external acceleration is for the lower side band

$$\langle\hat{\dot{y}}\rangle_s = \frac{\hat{G}\omega}{2Q\omega_0}\int\frac{f(\omega_{\beta s})}{\omega_{\beta s}-\omega}d\omega_{\beta s}$$

The oscillating beam can induce a voltage in a transverse impedance which in turn applies a self acceleration G_Z to the beam

$$Z_T(\omega) = -\frac{\omega}{I\dot{y}(\omega)}\oint\left(\vec{E}(\omega)+[\vec{\beta}\times\vec{B}(\omega)]\right)_T ds \ \ , \ \ \hat{G}_Z = -\frac{eZ_T I\langle\dot{y}\rangle}{\gamma m_0 2\pi R\omega}$$

This self excitation has to be added to the external one. We take the inverse response (stability diagram) due to both

$$\frac{(\hat{G}+\hat{G}_Z)}{\langle\hat{\dot{y}}\rangle_s} = \frac{\omega}{2Q\omega_0\int\frac{f(\omega_{\beta s})}{\omega_{\beta s}-\omega}d\omega_{\beta s}}$$

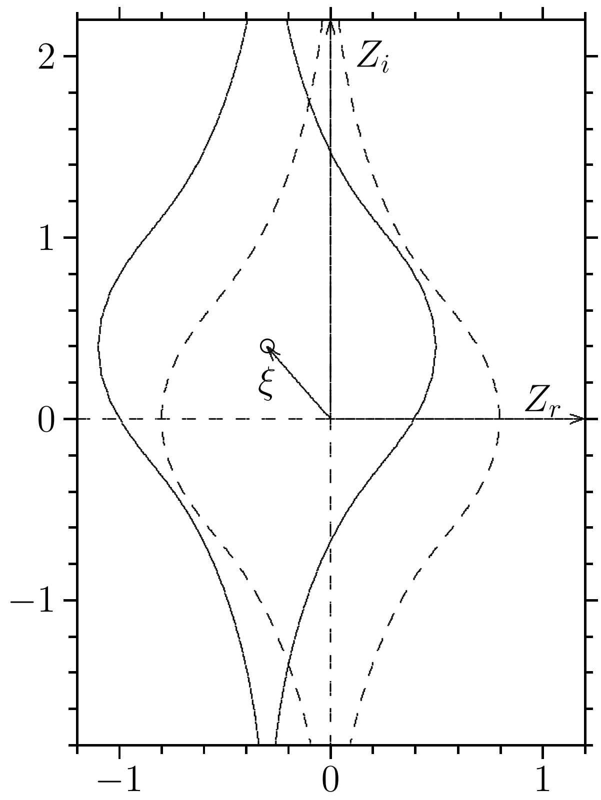
However, we know only the external excitation and would like the relation of the response to it. The inverse of this response is

$$\frac{\hat{G}}{\langle\hat{\dot{y}}\rangle_s} = \frac{\omega}{2Q\omega_0\int\frac{f(\omega_{\beta s})}{\omega_{\beta s}-\omega}d\omega_{\beta s}} - \frac{\hat{G}_Z}{\langle\hat{\dot{y}}\rangle_s} = \frac{\omega}{2Q\omega_0\int\frac{f(\omega_{\beta s})}{\omega_{\beta s}-\omega}d\omega_{\beta s}} - \frac{eZ_T I}{\gamma m_0 2\pi R\omega}$$

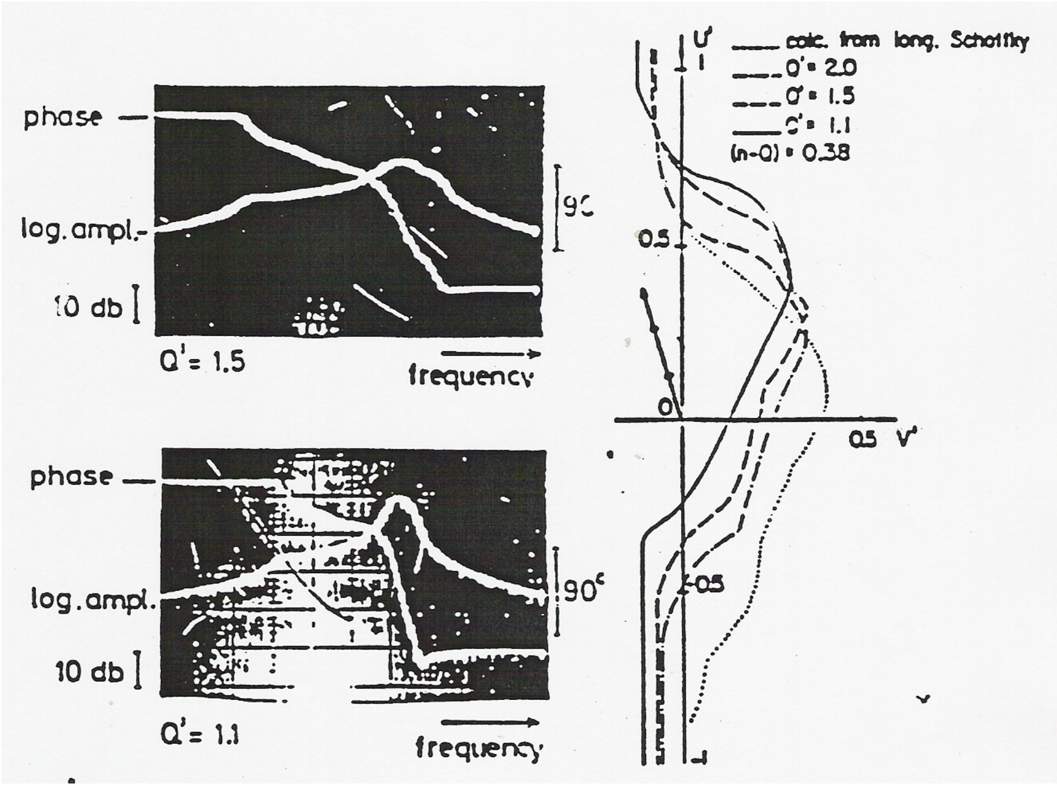
The presence of an impedance shifts the stability diagram by a vector

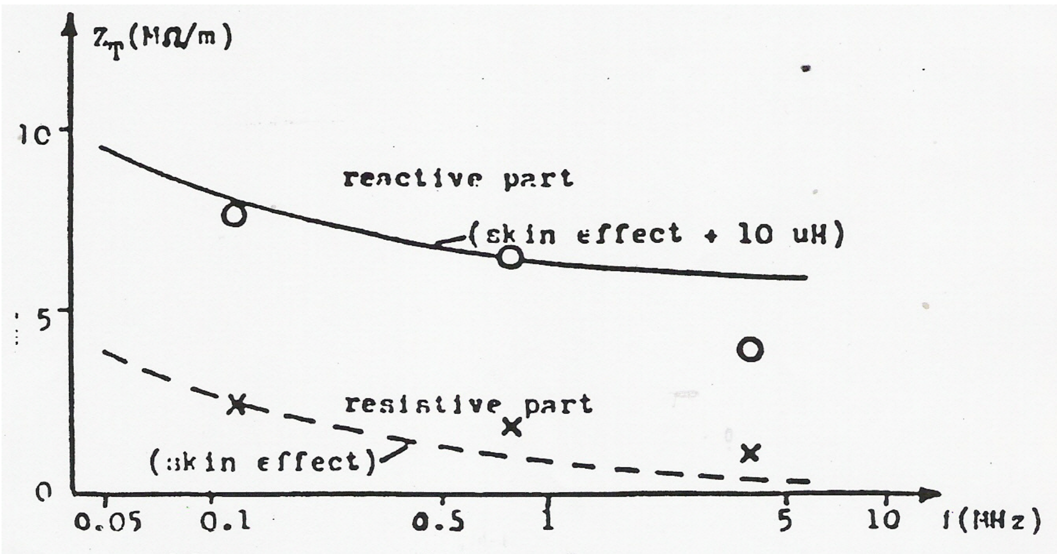
$$\xi = -\frac{eZ_T I}{2piR\gamma m_0\omega}$$

which is proportional to the negative complex impedance.



Transverse beam response in the presence of an impedance





4) Longitudinal coasting beam instability

Dynamics

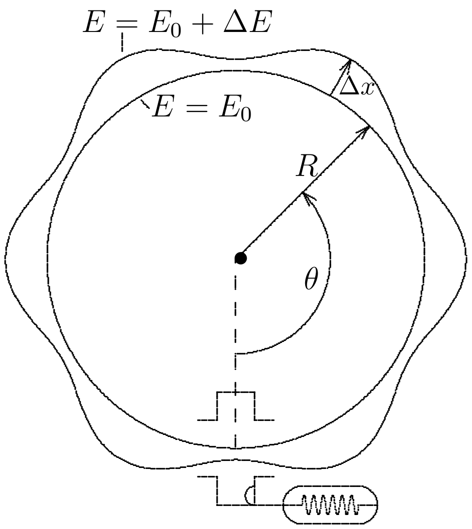
The longitudinal coasting beam dynamics is governed by relations between deviations Δp in momentum and $\Delta\omega_0$ in revolution frequency

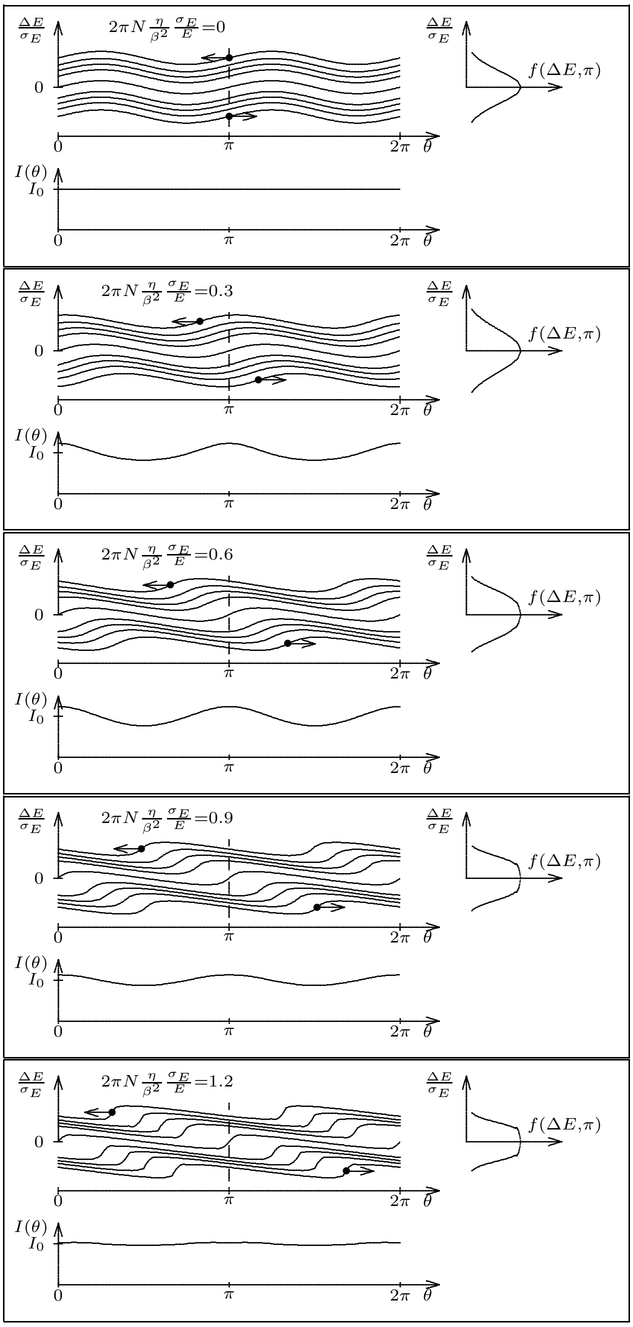
$$\frac{\Delta p}{p} = \frac{1}{\beta^2}\frac{\Delta E}{E} \approx \frac{\Delta E}{E} - \frac{1}{\eta_c}\frac{\Delta\omega_0}{\omega_0}\ \text{, with}\ \ \eta_c = \alpha_c - \frac{1}{\gamma^2},\ \Delta\omega_0 = \omega_r - \omega_0$$

giving an equilibrium distribution in E and ω_r around E_0 and ω_0

$$f_0(\Delta E) = \frac{1}{N}\frac{d^2N}{d\theta dE}\ \rightarrow F_0(\Delta\omega_0) = \frac{1}{N}\frac{d^2N}{d\theta d\omega_0},\ \int d\theta \int F_0(\Delta\omega_0)d\omega_r = 1.$$

Pulse response

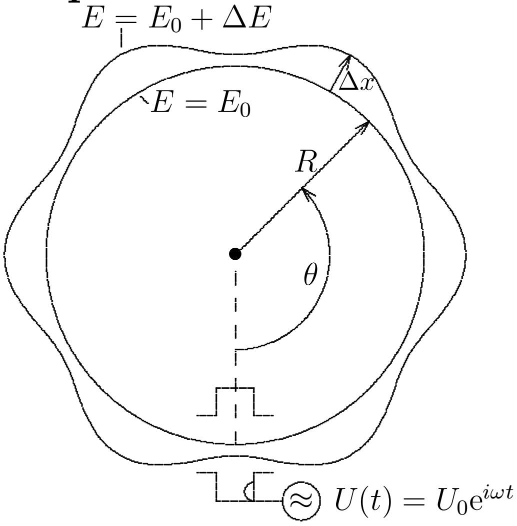




At $t = 0$ we pulse excite a mode

$$\begin{aligned}
\delta E &= \delta E_0 \cos(n\theta) \\
f(0^+) &= f_0(\Delta E + \delta E) \\
&\approx f_0(\Delta E) + \frac{df_0(\Delta E)}{dE}\delta E \\
&= f_0 + \frac{df_0}{dE}\delta E_0 \cos(n\theta), \\
f(t) &= f_0 + \frac{df_0}{dE}\cos(n\theta - \omega_r t) \\
\text{with } \omega_r &= \omega_0 - \omega_0\eta_c\Delta E/E \\
I(t) &= Ne\omega_0 \int f(\Delta E)dE \\
&= I_0 + I_1(t) \quad \text{current.}
\end{aligned}$$

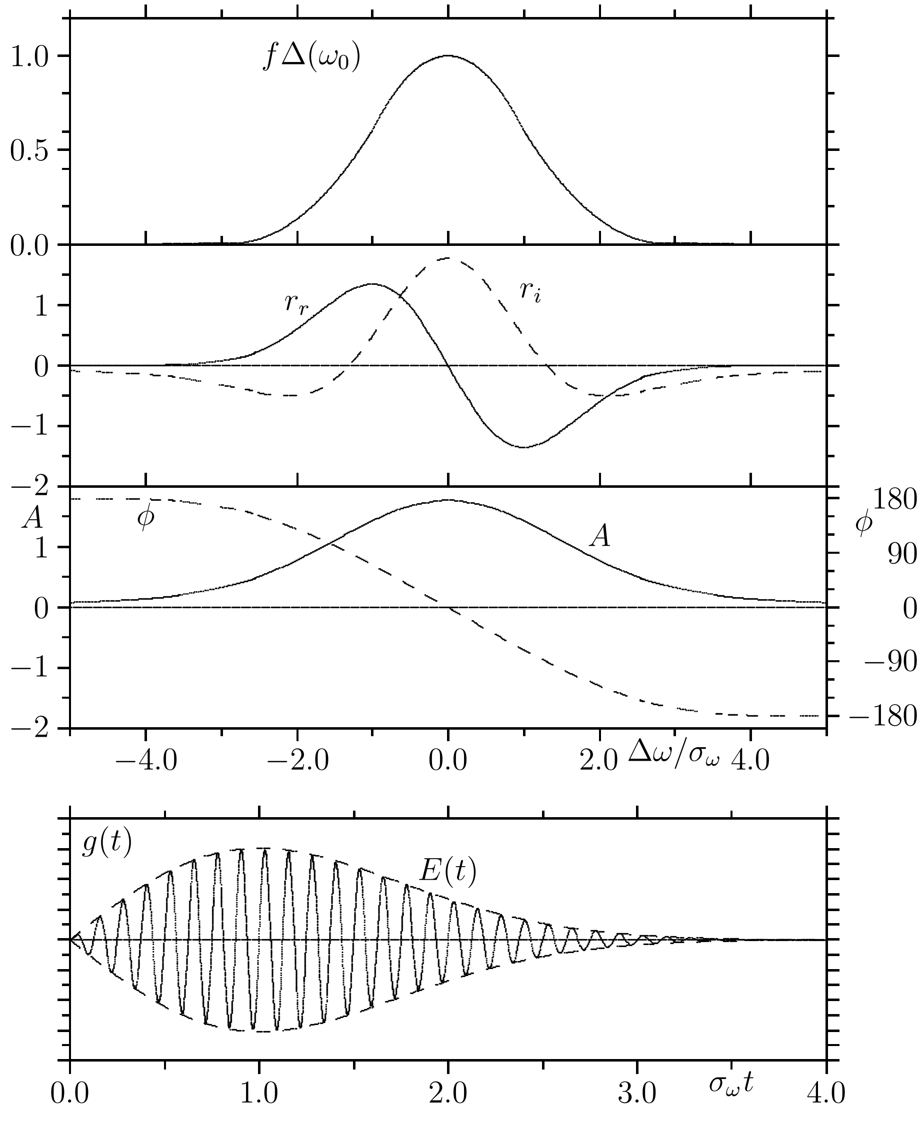
Response to a harmonic excitation



$$\frac{\Delta E}{E} = -\frac{1}{\eta_c}\frac{\Delta\omega_0}{\omega_0}, \quad \Delta\omega_0 = \omega_r - \omega_0$$
$$I(t) = Ne\omega_0 \int f(\Delta E)dE = I_0 + I_1(t)$$
$$I_1(t) \propto e^{i\omega t}.$$

The response is a perturbed current $I_1(t) = U(t)(r_r(\omega) + jr_i(\omega))$

$$I_1(t) = \frac{-jNe^2\omega_0^3 U(t)}{2\pi\beta^2 E}\int \frac{dF_0/d\omega_0}{\omega - n\omega_0}d\omega_0 = \frac{Ne^2\omega_0^3 U}{2\pi\beta^2 E}\left(\pi\frac{dF_0}{d\omega_0}(\omega) - jPV\int\right).$$



Longitudinal Landau damping of a coasting beam

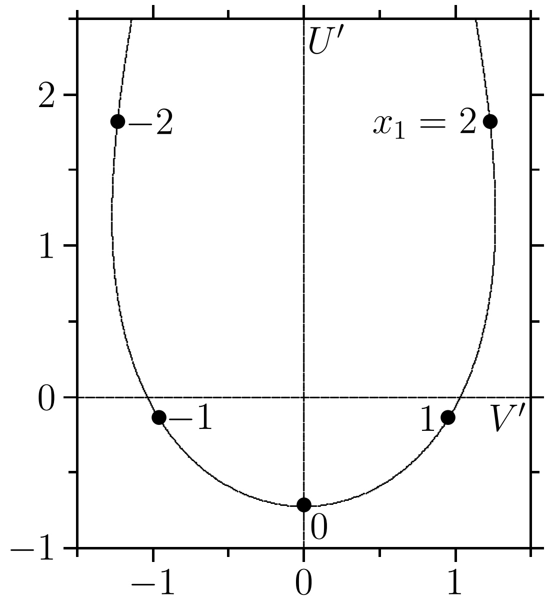
The response is a perturbed current $I_1(t) = U(t)(r_r(\omega) + jr_i(\omega))$

$$I_1(t) = \frac{-jNe^2\omega_0^3 U(t)}{2\pi\beta^2 E}\int \frac{dF_0/d\omega_0}{\omega - n\omega_0}d\omega_0 = \frac{Ne^2\omega_0^3 U}{2\pi\beta^2 E}\left(\pi\frac{dF_0}{d\omega_0}(\omega) - jPV\int\right).$$

To find the limit of stability we consider a complex longitudinal impedance in which the perturbed current $I_1(t)$ induces just the voltage we used to excite the beam and get this current $U(t) = I_1(t)Z(\omega)$

$$1 \;=\; \frac{N\mathrm{e}^2\omega_0^3\eta(Z_r(\omega) + jZ_i(\omega))}{2\pi\beta^2 E}\left(\frac{\pi dF_0}{d\omega_0}(\omega) - jPV\int\frac{dF_0(\omega_0)/dt}{\omega - n\omega_0}d\omega_0\right).$$

This represents a mapping between two complex quantities which can be represented by a stability diagram. We separate beam parameters from the integral which depends on the distribution form.



Separate physics from distribution with

$$\begin{aligned}
I_0 &= \frac{Ne\omega_0}{2\pi}\\
\delta p &= \text{half width at half height}\\
S &= \eta\omega_0\frac{\delta p}{p}\ \text{frequency spread}\\
x &= \frac{\omega_r - \omega_0}{S}\,,\quad x_1 = \frac{\omega - n\omega_0}{nS}\\
g_0(x) &= \frac{2\pi S F_0(\omega_r)}{N}\,,\quad \int g_0(x)dx = 1.
\end{aligned}$$

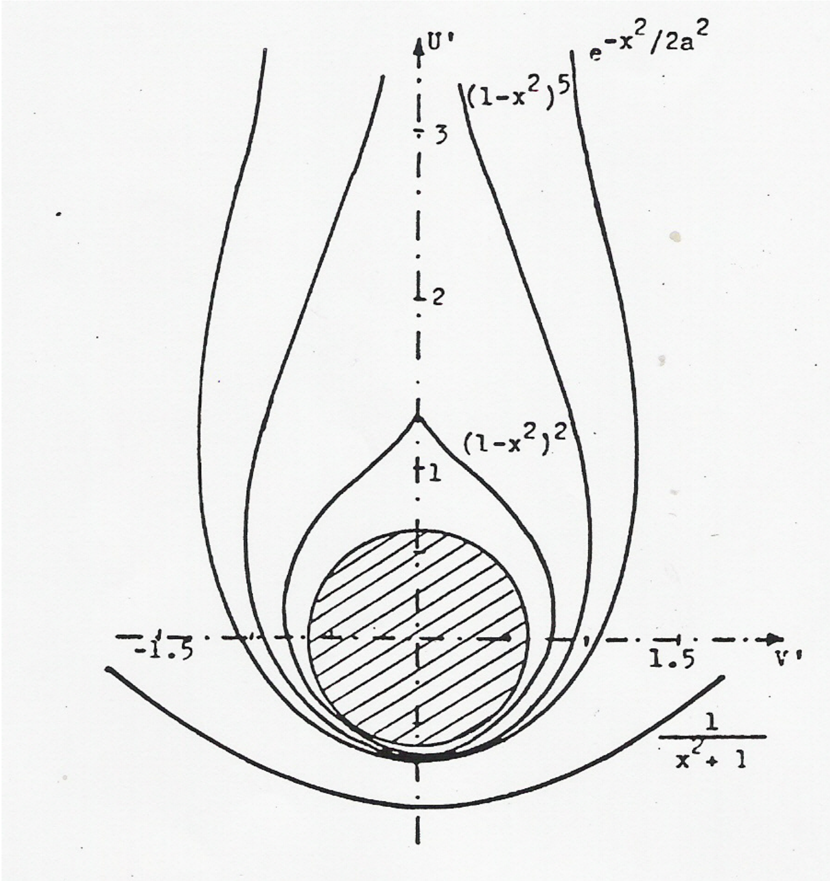
$$1 \;=\; -\frac{eI_0 Z(\omega)/n}{2\pi\beta^2 E_0\eta(\Delta p/p)^2}\left[\frac{\pi dg_0}{dx}(x_1) - iPV\int\frac{\frac{dg_0}{dx}}{x - x_1}dx\right]$$

$$V' + iU' \;=\; \frac{eI_0\,(Z_r + iZ_i)}{2\pi\beta^2 E_0 n\eta(\Delta p/p)^2} = -\left[\pi\frac{dg_0}{dx}(x_1) - iPV\int\frac{\frac{df}{dx}}{x - x_1}dx\right]^{-1}$$

Gaussian: $\sigma_\omega = S/\sqrt{2\log 2} = S/a,\ g_0(x) = a/\sqrt{2\pi}\exp -a^2x^2/2$

$$V' + iU' = \left[\frac{a^2}{\sqrt{2\pi}}\left(\pi a x_1\,\mathrm{e}^{-a^2x_1^2/2} + j\mathrm{PV}\int\frac{ax\,\mathrm{e}^{-a^2x^2/2}}{x - x_1}dx\right)\right]^{-1}$$

Longitudinal stability criterion (Keil-Schnell)



Stability diagrams for different distributions, (A. Ruggiero, V. Vaccaro)

We separated the dependence on the distribution form from the one on measurable beam and accelerator parameters E_0, I_0, $\Delta p/p$, $Z(\omega)$ and η_c and got the normalized stability diagram

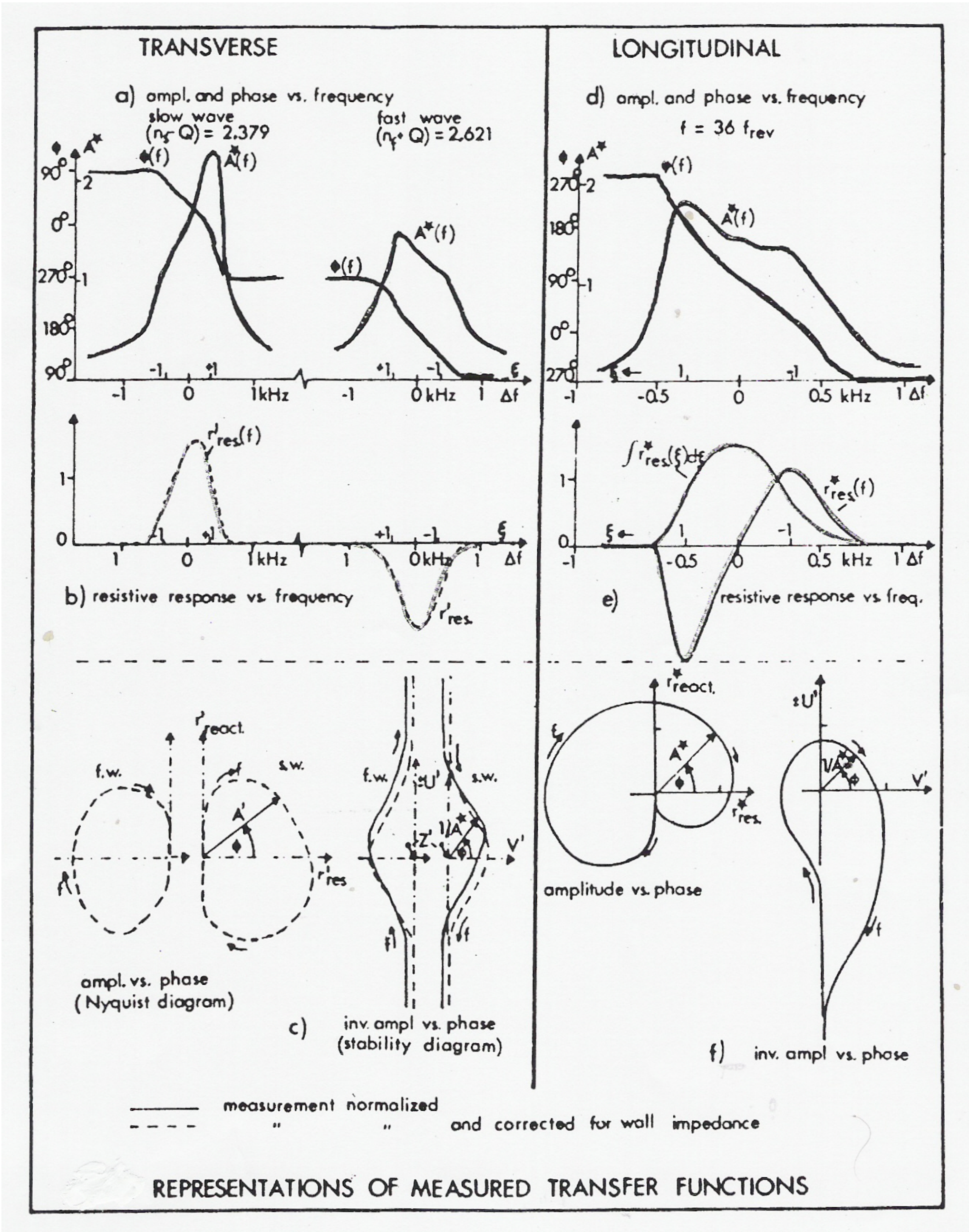
$$V' + iU' = \frac{eI_0 Z(\omega)/n}{2\pi\beta^2 E_0 \eta_c (\delta p/p)^2} = -\left[\pi \frac{dg_0}{dx}(x_1) - iPV \int \frac{\frac{df}{dx}}{x - x_1} dx\right]^{-1}$$

We approximate such diagrams of different distributions by a circle radius of 0.6 and get a condition for the absolute impedance divided by mode number, called Keil-Schnell stability criterion

$$\left|\frac{Z}{n}\right| \leq \frac{2\pi\beta^2 E\eta_c (\delta p/p)^2}{eI_0}.$$

Important is the strong dependence on the momentum spread, or the connected frequency spread, which gives rise to Landau damping.

Measured transverse and longitudinal coasting beam responses



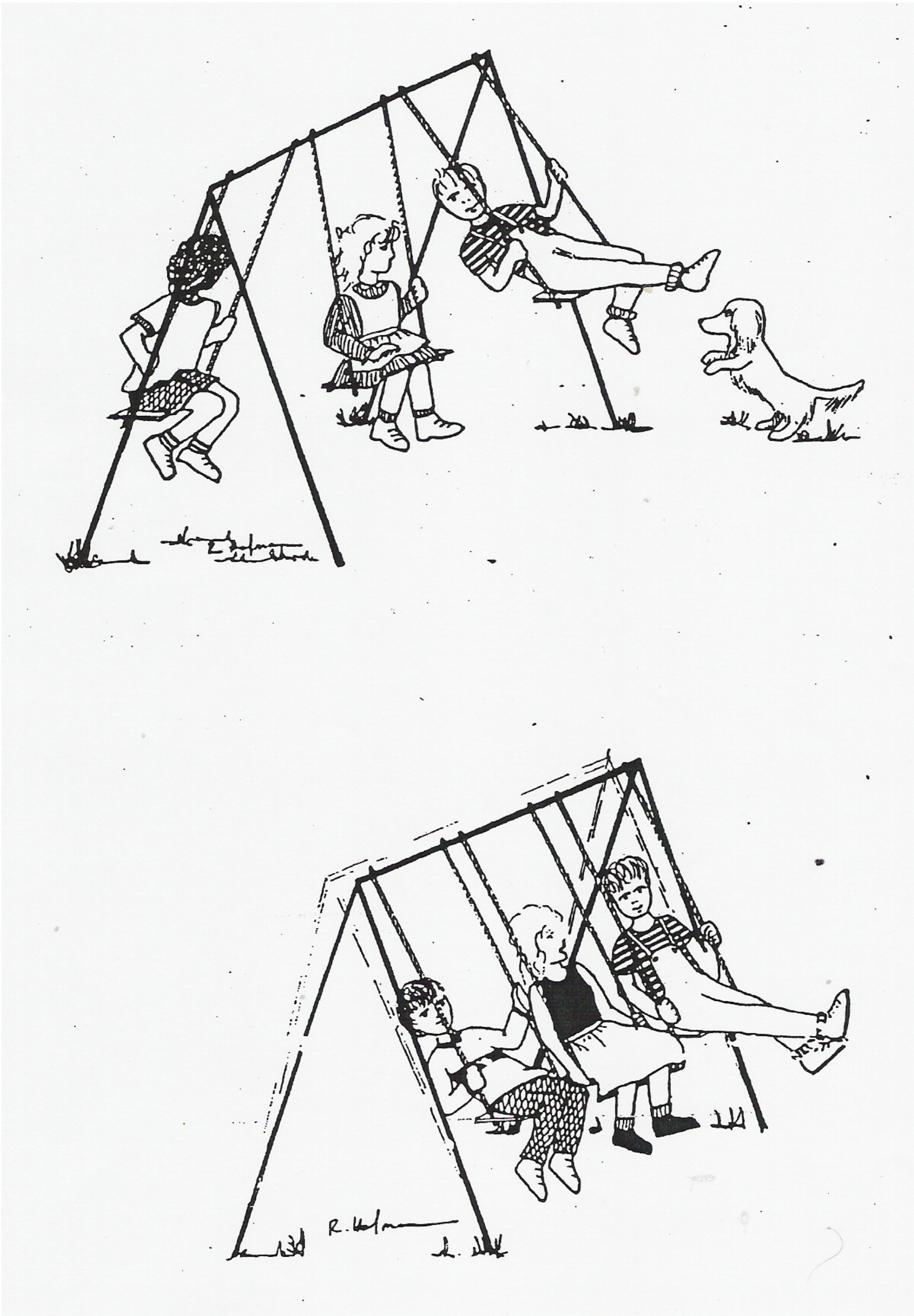
Transverse:

In each side-band the phase changes by π. The resistive response is positive for the slow and negative for the fast wave.

Longitudinal:

Each revolution harmonics gives a 2π phase change.

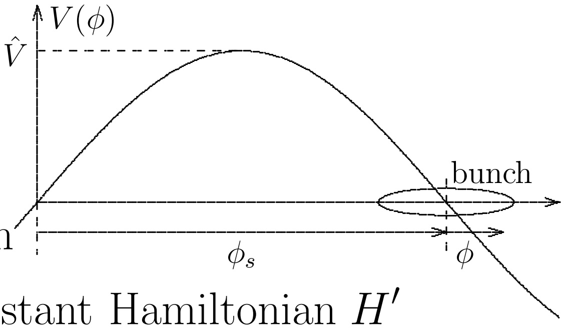
Simple demonstration of frequency spread and shift



5) Landau damping due to non-linearities

In a non-linear oscillator ω_r depends on amplitude or Hamiltonian H'. A distribution $\psi(H')$ gives one in ω_r. Example of synchrotron oscillation with nominal parameters E_0, $T_0 = 2\pi/\omega_0$ and $\phi_s = h\omega_0 t_s = \pi$ with deviations $\epsilon = \Delta E/E_0 \approx \Delta p/p_0$ and $\phi = h\omega_0\tau$.

For a stationary bucket, $\phi_s = \pi$



$$\ddot{\phi} + \omega_{s0}^2 \sin\phi = 0\ ,\ \frac{\omega_{s0}^2}{\omega_0^2} = \frac{\eta_c h e \hat{V}}{2\pi E_0}$$

For $\phi \ll 1 \rightarrow \ddot{\phi} + \omega_{s0}^2\phi = 0$ with solution

$\epsilon = \hat{\epsilon}\cos(\omega_{s0}t)\ ,\ \tau = \hat{\tau}\sin(\omega_{s0}t)$ and constant Hamiltonian H'

$$H' = \frac{\eta_c\epsilon^2}{2} + \frac{\omega_{s0}^2\tau^2}{2\eta_c} = \frac{\eta_c\hat{\epsilon}^2}{2} = \frac{\omega_{s0}^2\hat{\tau}^2}{2\eta_c} \text{ with } \frac{\partial H'}{\partial\epsilon} = \dot{\tau}\ ,\ \frac{\partial H'}{\partial\tau} = \dot{\epsilon}.$$

A stationary distribution is a direct function of H'; for a Gaussian

$$\psi(\epsilon,\tau) = \frac{1}{2\pi\sigma_\epsilon\sigma_\tau}\mathrm{e}^{-\frac{\epsilon^2}{2\sigma_\epsilon^2}}\mathrm{e}^{-\frac{\tau^2}{2\sigma_\tau^2}} = \frac{\mathrm{e}^{-H'/\langle H'\rangle}}{2\pi\sigma_\epsilon\sigma_\tau}\ ,\ \sigma_\epsilon = \frac{\omega_{s0}\sigma_\tau}{\eta_c}\ ,\ \langle H'\rangle = \eta_c\sigma_\epsilon^2$$

Next approximation $\phi < 1$; $\sin\phi \approx \phi - \phi^3/6$

$$\ddot{\phi} + \omega_{s0}^2\sin\phi = 0\ \rightarrow\ \ddot{\phi} + \omega_{s0}^2\left(\phi - \phi^3/6\right) = 0.$$

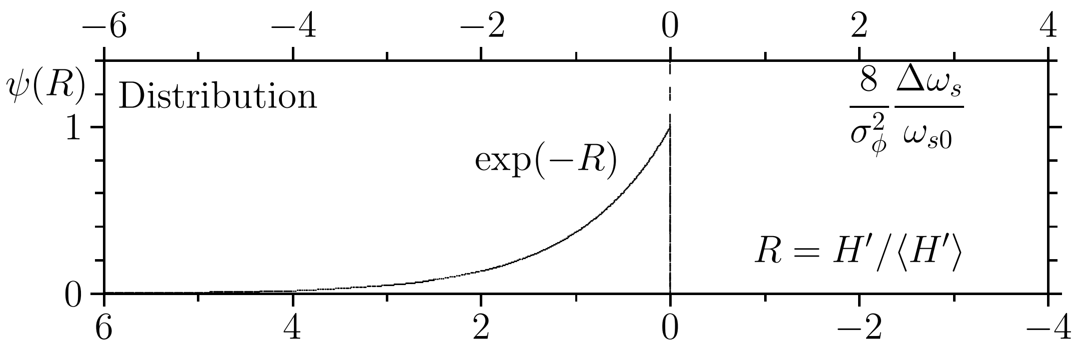
We seek solution $\phi = \hat{\phi}(\cos\omega_s t)$ with $\omega_s \neq \omega_{s0}$, neglect higher harmonics and use $\cos^3 x = (3\cos x + \cos(3x))/4$

$$-\omega_s^2\hat{\phi}\cos(\omega_s t) + \omega_{s0}^2\left(\hat{\phi}\cos(\omega_s t) - \frac{1}{24}\hat{\phi}^3\left(3\cos(\omega_s t) + \cos(3\omega_s t)\right)\right) = 0$$

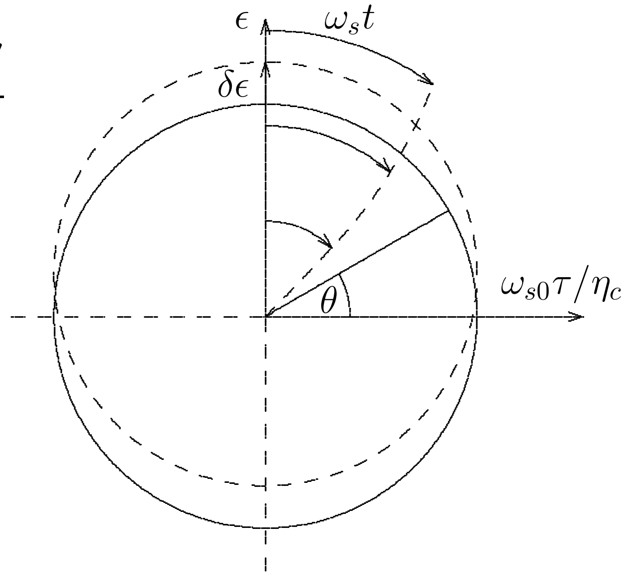
$$\omega_s = \omega_{s0}\sqrt{1 - \frac{1}{8}\hat{\phi}^2} \approx \left(1 - \frac{1}{16}\hat{\phi}^2\right)\ ,\ \frac{\Delta\omega_s}{\omega_{s0}} = -\frac{\hat{\phi}^2}{16} = -\frac{h^2\omega_0^2\eta_c}{\omega_{s0}^2}\frac{H'}{8}.$$

A distribution in H becomes a corresponding one in ω_s.

$$\psi(H') \propto \mathrm{e}^{-H'/\langle H'\rangle}\ \rightarrow\ \psi(\Delta\omega_s) \propto \mathrm{e}^{\Delta\omega_s/\langle|\Delta\omega_s|\rangle}$$



Pulse excitation



$$\frac{\eta_c\epsilon^2}{2}+\frac{\omega_s^2\tau^2}{2\eta_c} = H' \ , \ \frac{\Delta\omega_s}{\omega_{s0}} = -\frac{h^2\omega_0^2\eta_c}{\omega_{s0}^2}\frac{H'}{8}$$

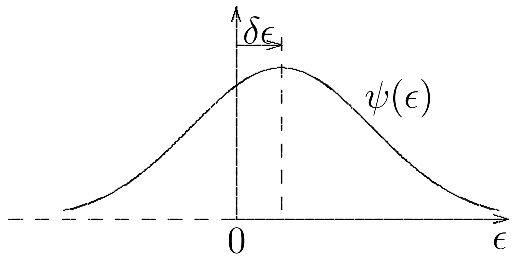
$$\frac{\partial H'}{\partial\epsilon} = \dot{\tau} \ , \ \frac{\partial H'}{\partial\tau} = \dot{\epsilon}$$

$$\psi_0(H') = \frac{\mathrm{e}^{-H'/\langle H'\rangle}}{\langle H'\rangle} \ , \ \langle H'\rangle = \eta_c\sigma_\epsilon^2$$

An energy change $\delta\epsilon$ gives

at $t = 0 : \psi(0^+) = \psi_0(H') + \psi_1(\epsilon,\tau)$

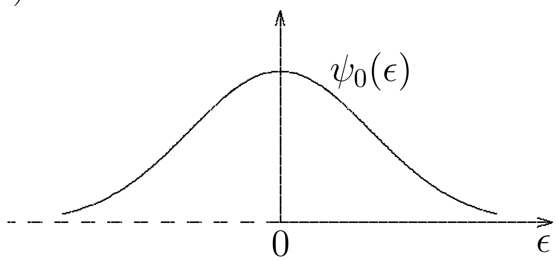
$$\psi_1(0^+) \ = \ \frac{\partial\psi_0}{\partial\epsilon}\delta\epsilon = \frac{d\psi_0}{dH'}\frac{\partial H'}{\partial\epsilon}\delta\epsilon = \frac{d\psi_0}{dH'}\eta_c\epsilon\delta\epsilon.$$



At $t > 0$ rotation with $\omega_s = \omega_{s0} + \Delta\omega_s$
gives also a displacement in τ

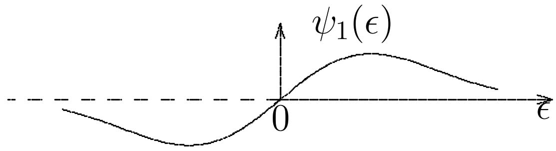
$$\epsilon \ = \ \hat{\epsilon}\sin(\theta - \omega_s t) \ , \ \tau = \hat{\tau}\cos(\theta - \omega_s t)$$

$$\psi_1(t) \ = \ \frac{d\psi_0}{dH'}\eta_c\hat{\epsilon}\sin(\theta - \omega_s t)\delta\epsilon$$



A slice $\psi_1 dH'$ has long. dipole moment

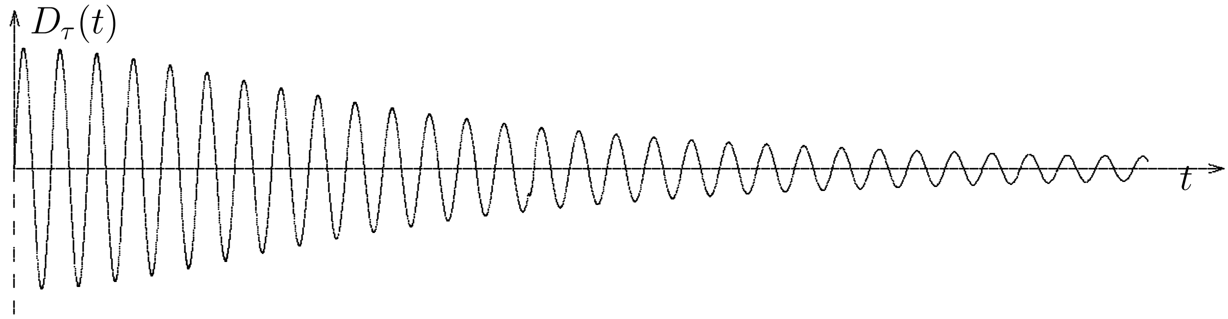
$$dD_\tau = \int d\psi_1\tau d\theta$$



integral over H' gives full dipole moment

$$D_\tau = \int dH' \int d\psi_1\tau d\theta$$

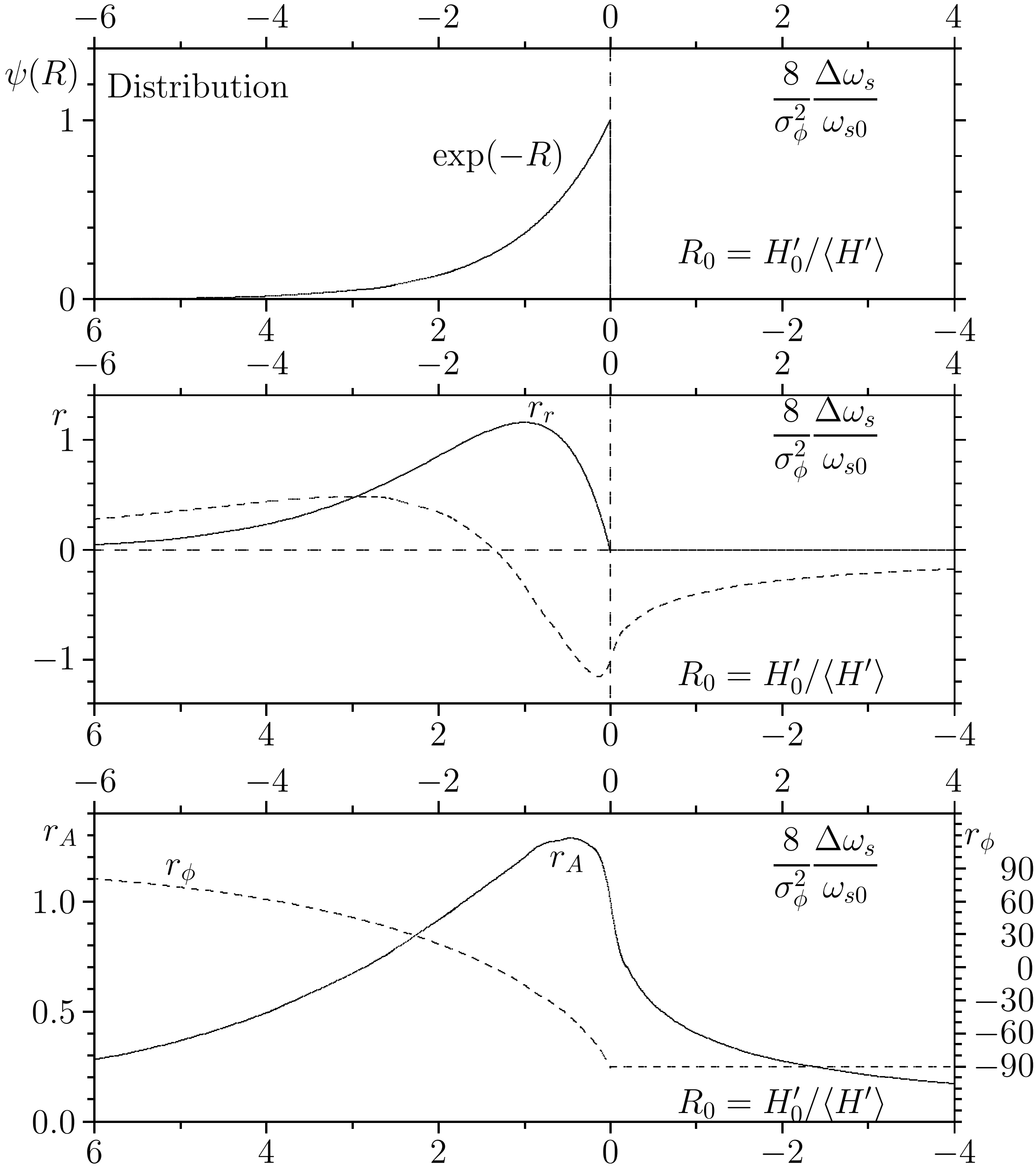
$$\frac{D_\tau(t)}{\delta\epsilon} = \frac{2\pi\eta_c}{\omega_{s0}}\frac{(1-(bt)^2)\sin(\omega_{s0}t) + 2bt\cos(\omega_{s0})}{(1+(bt)^2)^2} \text{ with } b = \frac{\omega_{s0}\sigma_\phi^2}{8}$$



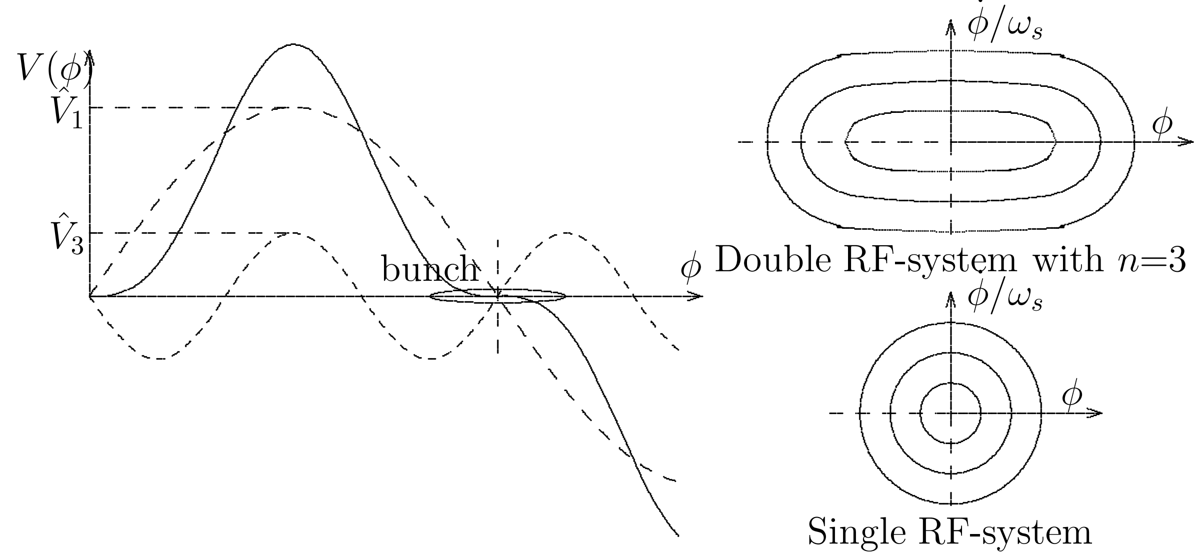
Harmonic excitation

We use a harmonic excitation $C\cos(\omega t_0)dt_0$ with ω corresponding to a Hamiltonian H_0' and get a dipole moment response

$$\frac{D_\tau}{C} = A\left[\pi e^{-R_0} R_0 \cos(\omega t) - PV\int_0^\infty \frac{e^{-R} R dR}{(R_0 - R)} \sin(\omega t)\right].$$



Increase Landau damping
Double RF-system



Two RF-systems with ω_{RF} and $n\omega_{RF}$ approximated for $\phi_s \approx \pi$,

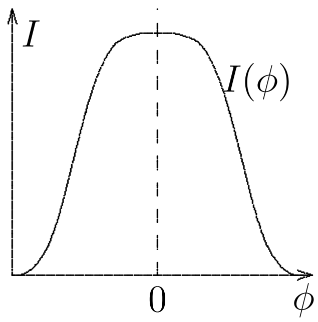
$$V(\phi) = -\hat{V}_1 \sin\phi + \hat{V}_n \sin(n\phi) \text{ with } \hat{V}_n/ = \hat{V}_1/n$$

$$V(\phi) \approx \hat{V}_1 \left(-\left(\phi - \frac{\phi^3}{6}\right) + \frac{1}{n}\left(n\phi - \frac{n^3\phi^3}{6}\right)\right) \approx -\hat{V}_1 \frac{n^2-1}{6}\phi^3$$

$$\ddot{\phi} + \omega_{s0}^2 \frac{n^2-1}{6}\phi^3 = 0 \ , \ \omega_s \propto \hat{\phi}$$

$$\frac{\dot{\phi}^2}{2} + \omega_{s0}^2 \frac{n^2-1}{24}\phi^4 = H = \text{constant}$$

$$I(\phi) = \hat{I} \exp\left[-\omega_{s0}^2 \frac{n^2-1}{24\sigma_{\phi}^2}\phi^4\right]$$



ω_{s0} and ω_s are synchrotron frequencies of the basic and the double RF-system. The latter depends strongly on amplitude and gives large spread and Landau damping. The flat voltage leads to a long bunch. Successful operation at ELETTRA, G. Penco et al.

Octupoles for transverse Landau damping

Octupoles give a restoring force $\propto x^3$ and make the betatron frequency dependent on amplitude resulting in a spread. Since they also produce non-linear resonances they should be distributed. The beam-beam force in collider represents a non-linear lens which gives Landau damping.

Bibliography

Landau damping is a field which can be presented in many different ways. We used her an approach close to beam observation and to experiments and involved Green and transfer functions. Details of the mathematics involved in this can be found in books on filter theory. Furthermore, the integrals involving residues and the mapping used in the stability diagrams is treated in mathematical books on complex numbers and theory of functions.

In some of the references listed below Landau damping is presented from other points of view.

H.G. Hereward, "The elementary theory of Landau damping"; CERN 1965 - 20, PS-Division.

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D. Sagan, "On the physics of Landau damping"; Amer. J. Physics, 1984.

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