

Concept of Luminosity

(in particle colliders)

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(</afs/ictp/home/w/wfherr/public/CAS/doc/luminosity.pdf>)

(http://cern.ch/lhc-beam-beam/talks/Trieste_luminosity.pdf)

Why colliding beams ?

■ Two beams: $E_1, \vec{p}_1, E_2, \vec{p}_2, m_1 = m_2 = m$

■ $E_{cm} = \sqrt{(E_1 + E_2)^2 - (\vec{p}_1 + \vec{p}_2)^2}$

■ Collider versus fixed target:

Fixed target: $\vec{p}_2 = 0 \rightarrow E_{cm} = \sqrt{2m^2 + 2E_1m}$

Collider: $\vec{p}_1 = -\vec{p}_2 \rightarrow E_{cm} = E_1 + E_2$

■ LHC (pp): 14000 GeV versus ≈ 115 GeV

■ LEP (e^+e^-): 210 GeV versus ?



Collider performance issues

■ Available energy

■ Number of interactions per second (useful collisions)

■ Total number of interactions

■ Secondary issues:

- Time structure of interactions (how often and how many at the same time)
- Space structure of interactions (size of interaction region)
- Quality of interactions (background, dead time etc.)



Luminosity:

■ We want:

→ Proportionality factor between cross section σ_p and number of interactions per second $\frac{dR}{dt}$

$$\frac{dR}{dt} = \mathcal{L} \times \sigma_p \quad (\rightarrow \text{units : cm}^{-2}\text{s}^{-1})$$

→ Relativistic invariant

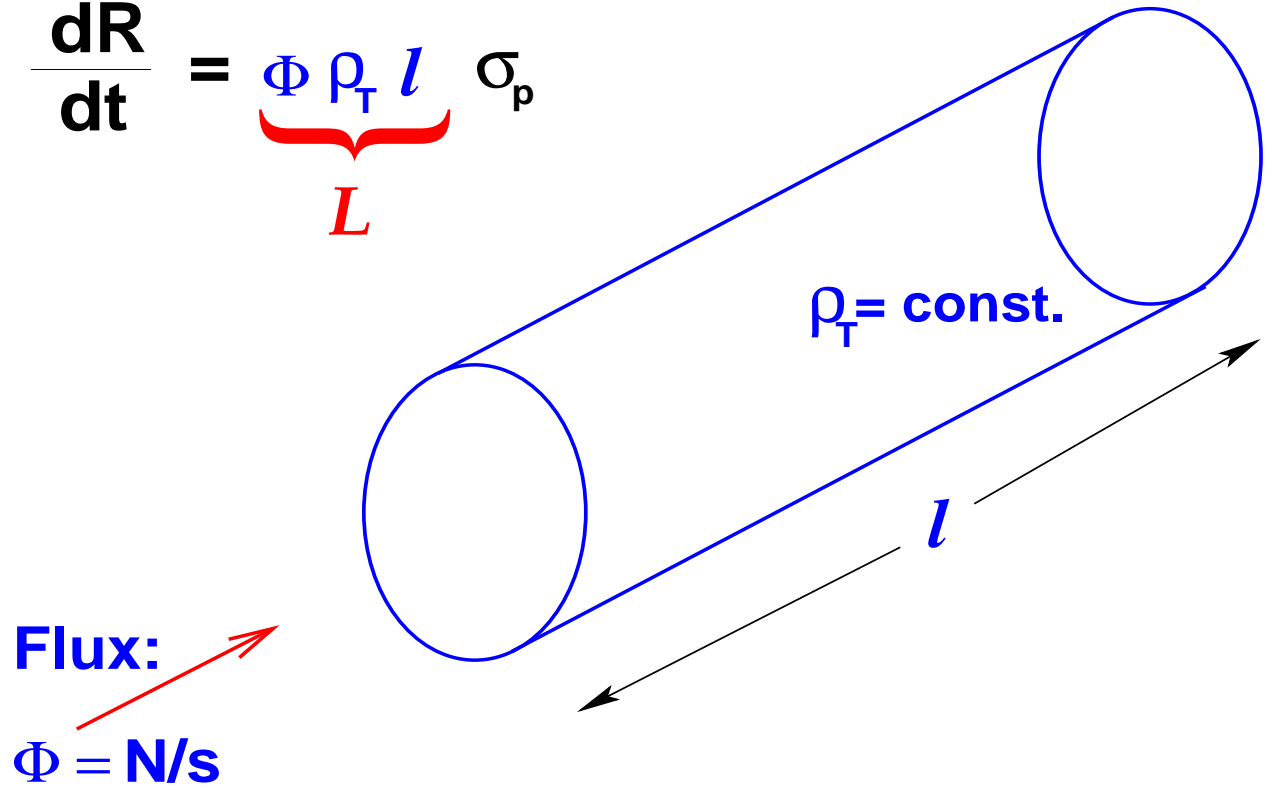
→ Independent of the physical reaction

→ Reliable procedures to **compute** and **measure**



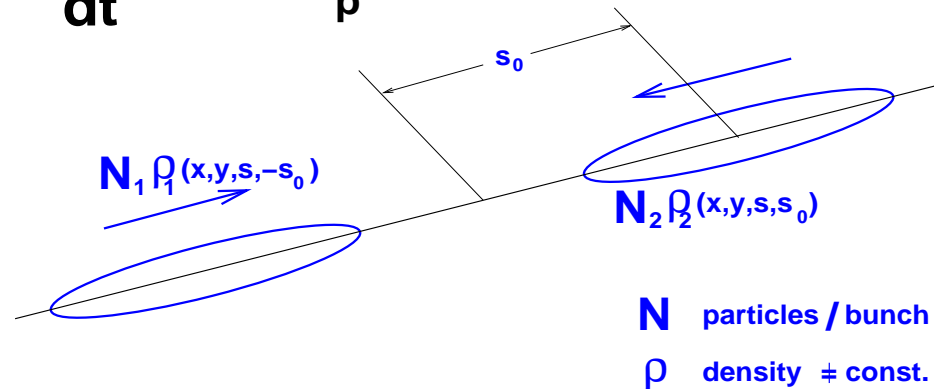
Fixed target luminosity

$$\frac{dR}{dt} = \underbrace{\Phi \rho_T l}_L \sigma_p$$



Collider luminosity (per bunch crossing)

$$\frac{dR}{dt} = L \sigma_p$$



$$\mathcal{L} \propto K \cdot \int \int \int \int_{-\infty}^{+\infty} \rho_1(x, y, s, -s_0) \rho_2(x, y, s, s_0) dx dy ds ds_0$$

s_0 is "time"-variable: $s_0 = c \cdot t$

Kinematic factor: $K = 1/c \sqrt{(\vec{v}_1 - \vec{v}_2)^2 - (\vec{v}_1 \times \vec{v}_2)^2 / c^2}$

Collider luminosity (per beam)

■ Assume uncorrelated densities in all planes

→ factorize: $\rho(x, y, s, s_0) = \rho_x(x) \cdot \rho_y(y) \cdot \rho_s(s \pm s_0)$

■ For head-on collisions ($\vec{v}_1 = -\vec{v}_2$) we get:

$$\mathcal{L} = 2 \cdot N_1 N_2 \cdot f B \int \int \int \int_{-\infty}^{+\infty} dx dy ds ds_0 \\ \rho_{1x}(x) \rho_{1y}(y) \rho_{1s}(s - s_0) \cdot \rho_{2x}(x) \rho_{2y}(y) \rho_{2s}(s + s_0)$$

■ In principle: should know all distributions

→ Use Gaussian ρ for analytic calculation
(in general: they are a good approximation for most realistic distributions)



Gaussian distribution functions

■ $\rho_{iz}(z) = \frac{1}{\sigma_z \sqrt{2\pi}} \exp\left(-\frac{z^2}{2\sigma_z^2}\right) \quad i = 1, 2, \quad z = x, y$

■ $\rho_{is}(s \pm s_0) = \frac{1}{\sigma_s \sqrt{2\pi}} \exp\left(-\frac{(s \pm s_0)^2}{2\sigma_s^2}\right)$

■ For non-Gaussian profiles not always possible to find analytic form, need a numerical integration



Luminosity for two beams (1 and 2)

■ Simplest case : equal beams

→ $\sigma_{1x} = \sigma_{2x}, \quad \sigma_{1y} = \sigma_{2y}, \quad \sigma_{1s} = \sigma_{2s}$

→ but: $\sigma_{1x} \neq \sigma_{1y}, \quad \sigma_{2x} \neq \sigma_{2y}$ is allowed

■ Further: no dispersion at collision point



Integration (head-on)

for $\sigma_1 = \sigma_2 \rightarrow \rho_1 \rho_2 = \rho^2$:

$$\mathcal{L} = \frac{2 \cdot N_1 N_2 f B}{(\sqrt{2\pi})^6 \sigma_s^2 \sigma_x^2 \sigma_y^2} \int \int e^{-\frac{x^2}{\sigma_x^2}} e^{-\frac{y^2}{\sigma_y^2}} e^{-\frac{s^2}{\sigma_s^2}} e^{-\frac{s_0^2}{\sigma_s^2}} dx dy ds ds_0$$

integrating over s and s_0 , using:

$$\int_{-\infty}^{+\infty} e^{-at^2} dt = \sqrt{\pi/a}$$

$$\mathcal{L} = \frac{2 \cdot N_1 N_2 f B}{8(\sqrt{\pi})^4 \sigma_x^2 \sigma_y^2} \int \int e^{-\frac{x^2}{\sigma_x^2}} e^{-\frac{y^2}{\sigma_y^2}} dx dy$$

finally after integration over x and y : $\Rightarrow \mathcal{L} = \frac{N_1 N_2 f B}{4\pi \sigma_x \sigma_y}$

Luminosity for two (equal) beams (1 and 2)

■ Simplest case : $\sigma_{1x} = \sigma_{2x}, \sigma_{1y} = \sigma_{2y}, \sigma_{1s} = \sigma_{2s}$

or: $\sigma_{1x} \neq \sigma_{2x} \neq \sigma_{1y} \neq \sigma_{2y}$, *but* : $\sigma_{1s} \approx \sigma_{2s}$

■ Further: no dispersion at collision point

$$\Rightarrow \mathcal{L} = \frac{N_1 N_2 f B}{4\pi\sigma_x\sigma_y} \left(\mathcal{L} = \frac{N_1 N_2 f B}{2\pi\sqrt{\sigma_{1x}^2 + \sigma_{2x}^2}\sqrt{\sigma_{1y}^2 + \sigma_{2y}^2}} \right)$$




Examples

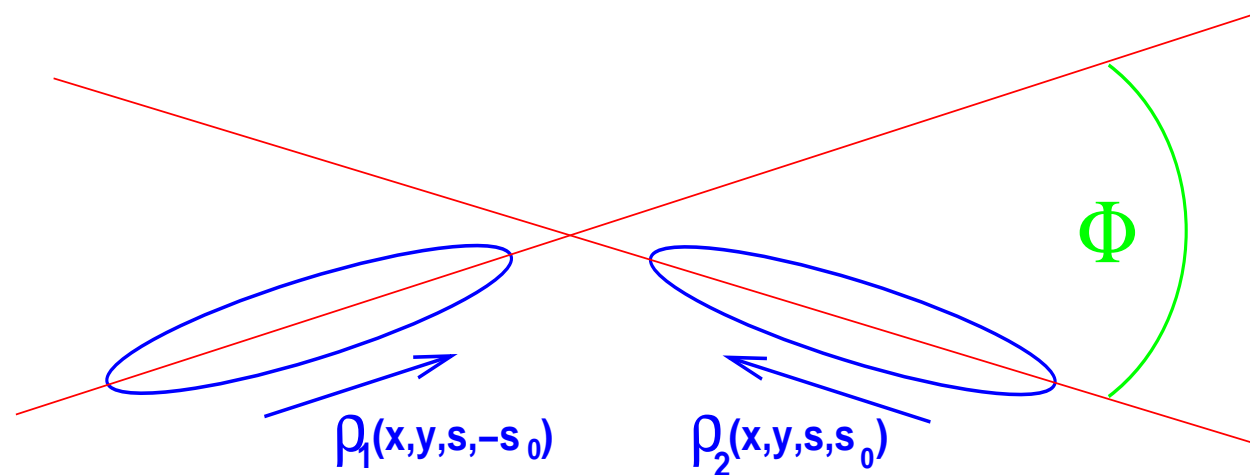
	Energy (GeV)	\mathcal{L}_{max} $\text{cm}^{-2}\text{s}^{-1}$	rate s^{-1}	σ_x/σ_y $\mu\text{m}/\mu\text{m}$	Particles per bunch
SPS (p\bar{p})	315x315	6 10³⁰	4 10⁵	60/30	$\approx 10 10^{10}$
Tevatron (p\bar{p})	1000x1000	100 10³⁰	7 10⁶	30/30	$\approx 30/8 10^{10}$
HERA (e⁺p)	30x920	40 10³⁰	40	250/50	$\approx 3/7 10^{10}$
LHC (pp)	7000x7000	10000 10³⁰	10⁹	17/17	11 10¹⁰
LEP (e⁺e⁻)	105x105	100 10³⁰	≤ 1	200/2	$\approx 50 10^{10}$
PEP (e⁺e⁻)	9x3	8000 10³⁰	NA	150/5	$\approx 2/6 10^{10}$



Complications

- Crossing angle
 - Hour glass effect
 - Collision offset (wanted or unwanted)
 - Non-Gaussian profiles
 - Dispersion at collision point
 - $\delta\beta^*/\delta s = \alpha^* \neq 0$
 - Strong coupling
 - etc.
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Collisions at crossing angle

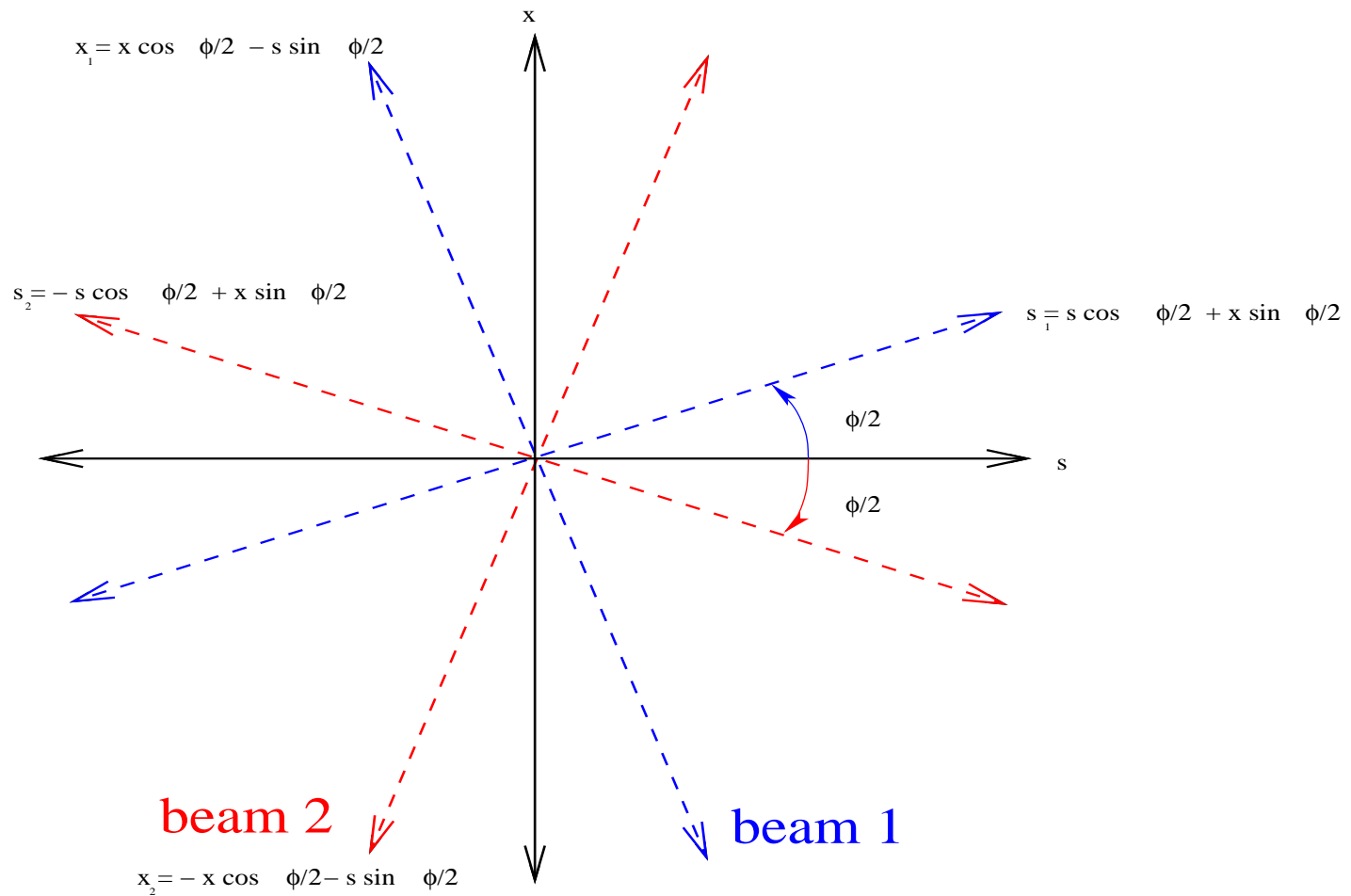


■ Needed to avoid unwanted collisions

→ For colliders with many bunches: LHC, CESR, KEKB

→ For colliders with coasting beams

Collisions angle geometry (horizontal plane)



Crossing angle

Assume crossing in **horizontal (x, s)-** plane.
Transform to new coordinates:

$$\begin{cases} x_1 = x \cos \frac{\phi}{2} - s \sin \frac{\phi}{2}, & s_1 = s \cos \frac{\phi}{2} + x \sin \frac{\phi}{2}, \\ x_2 = x \cos \frac{\phi}{2} + s \sin \frac{\phi}{2}, & s_2 = s \cos \frac{\phi}{2} - x \sin \frac{\phi}{2} \end{cases}$$

$$\mathcal{L} = 2 \cos^2 \frac{\phi}{2} N_1 N_2 f B \int \int \int \int_{-\infty}^{+\infty} dx dy ds ds_0 \\ \rho_{1x}(x_1) \rho_{1y}(y_1) \rho_{1s}(s_1 - s_0) \rho_{2x}(x_2) \rho_{2y}(y_2) \rho_{2s}(s_2 + s_0)$$

Integration (crossing angle)

use as before:

$$\int_{-\infty}^{+\infty} e^{-at^2} dt = \sqrt{\pi/a}$$

and:

$$\int_{-\infty}^{+\infty} e^{-(at^2+bt+c)} dt = \sqrt{\pi/a} \cdot e^{\frac{b^2-ac}{a}}$$

Further:

- Since σ_x , x and $\sin(\phi/2)$ are small: drop all terms $\sigma_x^k \sin^l(\phi/2)$ or $x^k \sin^l(\phi/2)$ for all: $k+l \geq 4$
- Approximate: $\sin(\phi/2) \approx \tan(\phi/2) \approx \phi/2$

Crossing angle

- Crossing Angle \Rightarrow

$$\mathcal{L} = \frac{N_1 N_2 f B}{4\pi \sigma_x \sigma_y} \cdot S$$

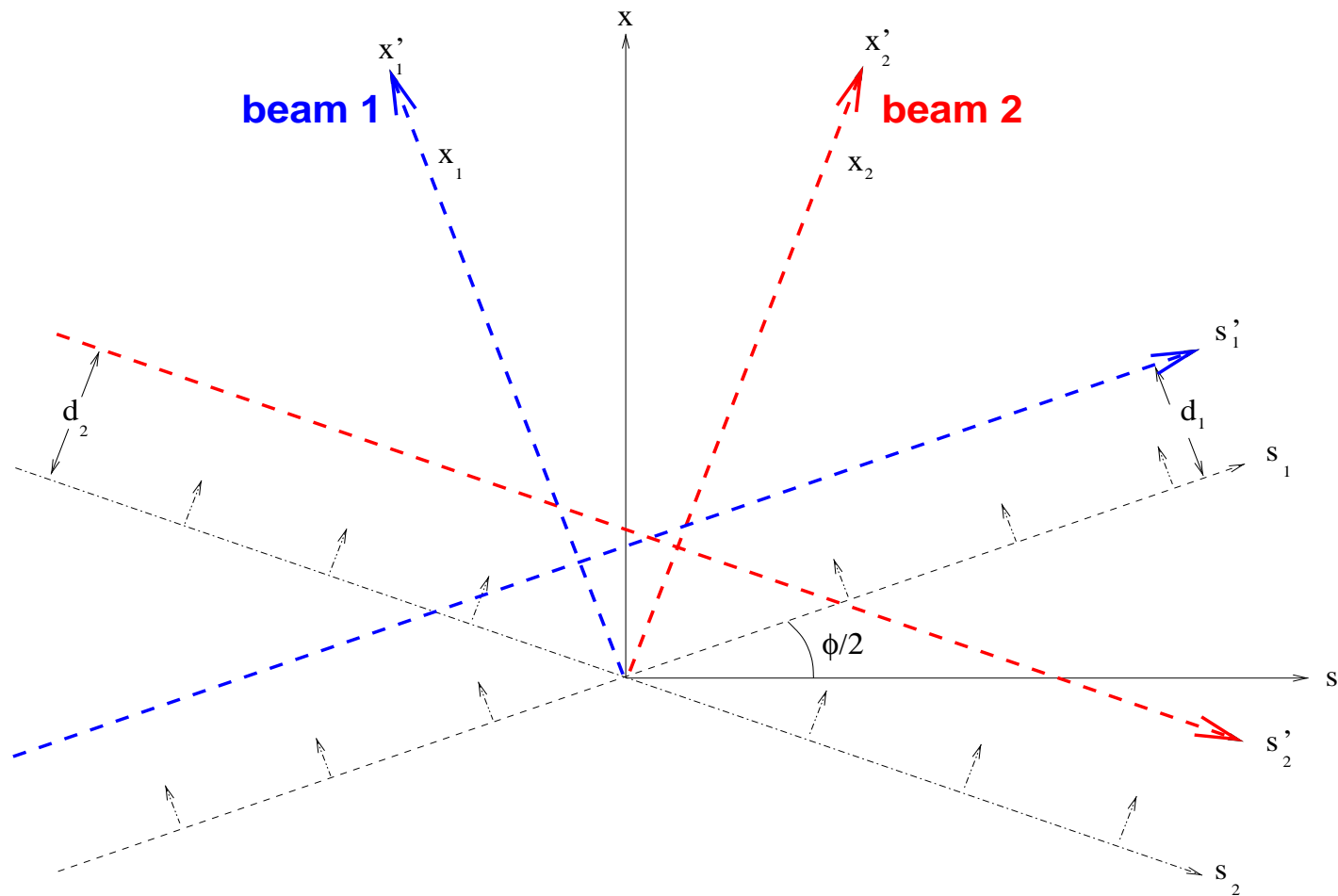
- S is the reduction factor
- For small crossing angles and $\sigma_s \gg \sigma_{x,y}$

- $\Rightarrow S = \frac{1}{\sqrt{1 + \left(\frac{\sigma_s}{\sigma_x} \tan \frac{\phi}{2}\right)^2}} \approx \frac{1}{\sqrt{1 + \left(\frac{\sigma_s}{\sigma_x} \frac{\phi}{2}\right)^2}}$

Example LHC:

$$\Phi = 285 \mu\text{rad}, \sigma_s = 7.5 \text{ cm}, S = 0.81$$

Offset and crossing angle



Offset and crossing angle

- Transformations with offsets in crossing plane:

$$\begin{cases} x_1 = d_1 + x \cos \frac{\phi}{2} - s \sin \frac{\phi}{2}, & s_1 = s \cos \frac{\phi}{2} + x \sin \frac{\phi}{2}, \\ x_2 = d_2 + x \cos \frac{\phi}{2} + s \sin \frac{\phi}{2}, & s_2 = s \cos \frac{\phi}{2} - x \sin \frac{\phi}{2} \end{cases}$$

- Gives after integration over y and s_0 :

$$\mathcal{L} = \frac{\mathcal{L}_0}{2\pi\sigma_s\sigma_x} 2 \cos^2 \frac{\phi}{2} \int \int e^{-\frac{x^2 \cos^2(\phi/2) + s^2 \sin^2(\phi/2)}{\sigma_x^2}} e^{-\frac{x^2 \sin^2(\phi/2) + s^2 \cos^2(\phi/2)}{\sigma_s^2}} \times e^{-\frac{d_1^2 + d_2^2 + 2(d_1 + d_2)x \cos(\phi/2) - 2(d_2 - d_1)s \sin(\phi/2)}{2\sigma_x^2}} dx ds.$$

Offset and crossing angle

After integration over x :

$$\mathcal{L} = \frac{N_1 N_2 f B}{8\pi^{\frac{3}{2}} \sigma_s} \cdot 2 \cos \frac{\phi}{2} \int_{-\infty}^{+\infty} W \cdot \frac{e^{-(As^2+2Bs)}}{\sigma_x \sigma_y} ds$$

with:

$$A = \frac{\sin^2 \frac{\phi}{2}}{\sigma_x^2} + \frac{\cos^2 \frac{\phi}{2}}{\sigma_s^2} \quad B = \frac{(d_2 - d_1) \sin(\phi/2)}{2\sigma_x^2}$$


and $W = e^{-\frac{1}{4\sigma_x^2}(d_2-d_1)^2}$

\Rightarrow Luminosity with correction factors

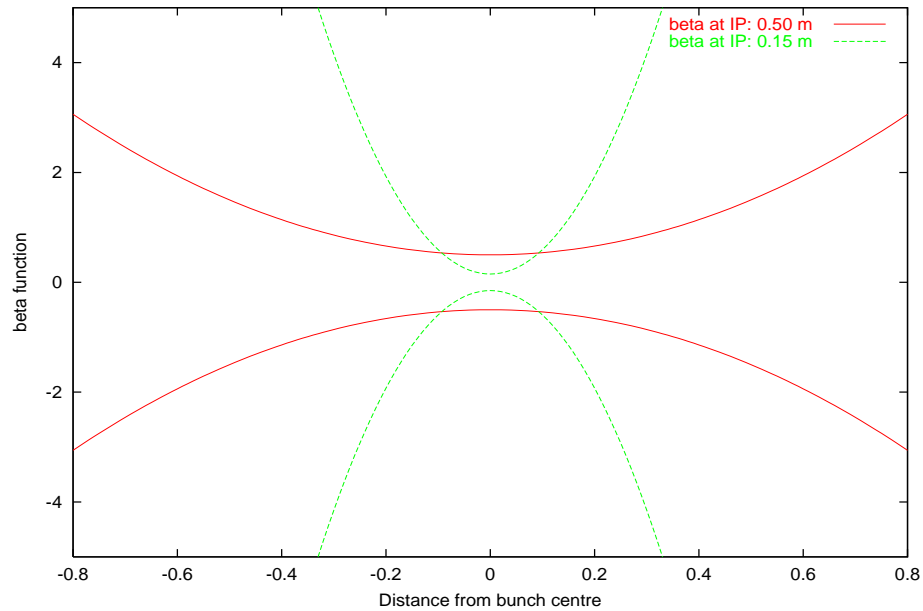


Luminosity with correction factors

$$\mathcal{L} = \frac{N_1 N_2 f B}{4\pi\sigma_x\sigma_y} \cdot W \cdot e^{\frac{B^2}{A}} \cdot S$$

- W : correction of beam offset
 - S : correction for crossing angle
 - $e^{\frac{B^2}{A}}$: correction for crossing angle **and** offset
- 

Hour glass effect



● β -functions depends on position s



Hour glass effect

- β -functions depends on position s
- Usually: $\beta(s) = \beta^* \left(1 + \left(\frac{s}{\beta^*}\right)^2\right)$
 - i.e. $\sigma \implies \sigma(s) \neq \text{const.}$
 - $\sigma(s) = \sigma^* \sqrt{\left(1 + \left(\frac{s}{\beta^*}\right)^2\right)}$
- Important when β^* comparable to the r.m.s. bunch length σ_s

Hour glass effect

→ Take it easy: $\beta_x^* = \beta_y^*$, crossing angle, but no offset


→ Replace σ by $\sigma(s)$ in standard formulae

$$\bullet \quad \mathcal{L} = \left(\frac{N_1 N_2 f B}{8\pi} \right) \frac{2 \cos \frac{\phi}{2}}{\sqrt{\pi} \sigma_s} \int_{-\infty}^{+\infty} \frac{e^{-s^2 A}}{\sigma_x^* \sigma_y^* \left[1 + \left(\frac{s}{\beta^*} \right)^2 \right]} ds$$

$$\bullet \quad A = \frac{\sin^2 \frac{\phi}{2}}{(\sigma_x^*)^2 \left[1 + \left(\frac{s}{\beta^*} \right)^2 \right]} + \frac{\cos^2 \frac{\phi}{2}}{\sigma_s^2}$$

→ Numerical Integration

Calculations for the LHC

- $N_1 = N_2 = 1.15 \times 10^{11}$ particles/bunch
 - $B = 2808$ bunches/beam
 - $f = 11.2455$ kHz, $\phi = 285$ μ rad
 - $\beta_x^* = \beta_y^* = 0.55$ m
 - $\sigma_x^* = \sigma_y^* = 16.6$ μ m, $\sigma_s = 7.7$ cm
- 

- **Simplest case (Head on collision):**

$$\mathcal{L} = 1.200 \times 10^{34} \text{ cm}^{-2}\text{s}^{-1}$$

- **Effect of crossing angle:**

$$\mathcal{L} = 0.973 \times 10^{34} \text{ cm}^{-2}\text{s}^{-1}$$

- **Effect of crossing angle & Hourglass:**

$$\mathcal{L} = 0.969 \times 10^{34} \text{ cm}^{-2}\text{s}^{-1}$$



If the beams are not Gaussian ??

Exercise:

- Assume flat distributions (normalized to 1)

$$\rho_1 = \rho_2 = \frac{1}{2a}, \quad \text{for } [-a \leq z \leq a], \quad z = x, y$$

Calculate r.m.s. in x and y:

$$\langle (x, y)^2 \rangle = \int_{-\infty}^{+\infty} (x, y)^2 \cdot \rho(x, y) dx dy$$

and

$$\mathcal{L} = \int_{-\infty}^{+\infty} \rho_1(x, y) \rho_2(x, y) dx dy$$

- Compute: $\mathcal{L} \cdot \sqrt{\langle x^2 \rangle \cdot \langle y^2 \rangle}$
- Repeat for various distributions and compare

Integrated luminosity

- $\mathcal{L}_{\text{int}}(s) = \int_0^T \int_{-s}^{+s} \mathcal{L}(s', t) ds' dt$
 - **The real figure of merit:**
 $\mathcal{L}_{\text{int}}(s) \cdot \sigma_p = \text{number of events}$
 - **Experiments:** continuous recording of \mathcal{L}
 - **For studies:** assume some life time behaviour.
E.g. $\mathcal{L}(t) \longrightarrow \mathcal{L}_0 \exp\left(-\frac{t}{\tau}\right)$
 - **Contributions to life time from:** intensity decay, emittance growth etc.
-

Integrated luminosity

- Knowledge of preparation time allows optimization of \mathcal{L}_{int}



Integrated luminosity

- Typical run times LEP:
 $t_r \approx 8 - 10$ hours
- For LHC long preparation time t_p expected
- Optimum combination of t_r and t_p gives maximum luminosity
- t_r is usually a "free" parameter, i.e. can be chosen



Maximising Integrated Luminosity

- Assume exponential decay of luminosity

$$\mathcal{L}(t) = \mathcal{L}_0 \cdot e^{t/\tau}$$

- Average luminosity $\langle \mathcal{L} \rangle$

$$\langle \mathcal{L} \rangle = \frac{\int_0^{t_r} dt \mathcal{L}(t)}{t_r + t_p} = \mathcal{L}_0 \cdot \tau \cdot \frac{1 - e^{-t_r/\tau}}{t_r + t_p}$$

- (Theoretical) maximum for:

$$t_r \approx \tau \cdot \ln(1 + \sqrt{2t_p/\tau + t_p/\tau})$$

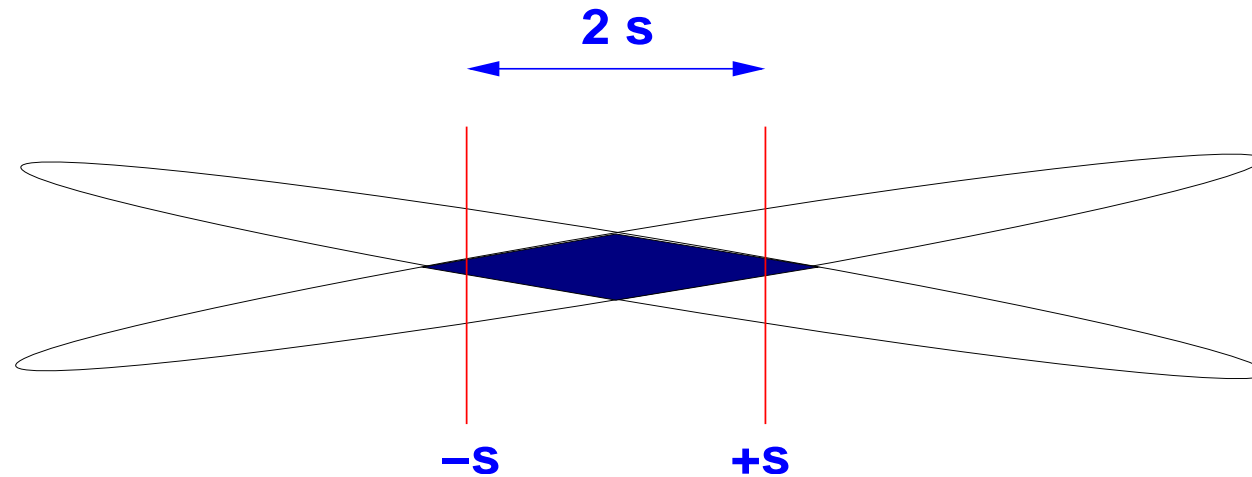
- Example LHC (OB): $t_p \approx 10\text{h}$, $\tau \approx 15\text{h}$,

$$\Rightarrow t_r \approx 15\text{h}$$

- Exercise: Would you improve τ (long t_r) or t_p ?



Luminous region



- Density distribution of interaction vertices
- Fraction of collisions occur $\pm s$ from the IP ?
- Important for experiments !



Luminous region


- Depends on σ_x , σ_y , σ_s and crossing angle ϕ
- Integrate only along a finite longitudinal length:

$$\mathcal{L}_0 = \int_{-\infty}^{+\infty} \mathcal{L}(s') ds' \longrightarrow \mathcal{L}(S) = \int_{-S}^{+S} \mathcal{L}(s') ds'$$


$$\mathcal{L}(S) = \left(\frac{N_1 N_2 f B}{8\pi\sigma_x^* \sigma_y^*} \right) \frac{2 \cos \frac{\phi}{2}}{\sqrt{\pi}\sigma_s} \sqrt{\frac{\pi}{A}} \operatorname{erf} \left(\sqrt{A} S \right)$$



Some results for LHC

- $\sigma_s = 7.7$ cm, $\beta^* = 0.55$ m, $\phi = 285$ μ rad:
 - 100% lumi $\rightarrow s = \pm 12$ cm
 - 90% lumi $\rightarrow s = \pm 7$ cm
 - 80% lumi $\rightarrow s = \pm 5.5$ cm
- 

Interactions per crossing

- Luminosity/ $fB \propto N_1 N_2$
 - In LHC: crossing every 25 ns
 - Per crossing approximately 20 interactions
 - May be undesirable (pile up in detector)
 - \implies more bunches B, or smaller N ??
- Beware: maximum (peak) luminosity
 \mathcal{L}_{max} is not the whole story ... !
- 


Luminosity measurement

- One needs to get a signal proportional to interaction rate → **Beam diagnostics**
- Large dynamic range:
 $10^{27} \text{ cm}^{-2}\text{s}^{-1}$ to $10^{34} \text{ cm}^{-2}\text{s}^{-1}$
- Very fast, if possible for individual bunches
- Used for optimization
- For absolute luminosity need calibration

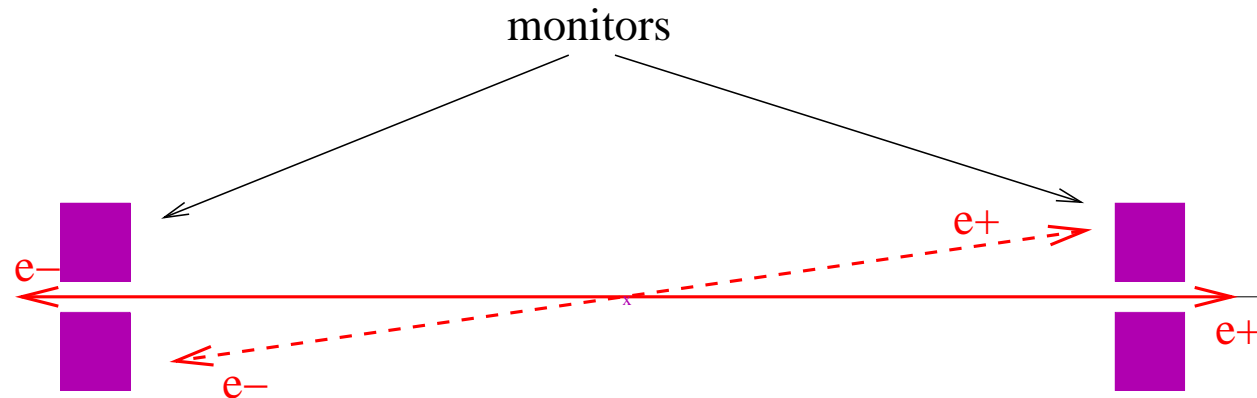


Luminosity calibration

(e^+e^-)

- Use well known and calculable process
 - $e^+e^- \rightarrow e^+e^-$ elastic scattering (Bhabha scattering)
 - Have to go to small angles ($\sigma_{el} \propto \Theta^{-3}$)
 - Small rates at high energy ($\sigma_{el} \propto \frac{1}{E^2}$)
- 

Luminosity calibration



- Measure coincidence at small angles
- Low counting rates, in particular for high energy !
- Background may be problematic



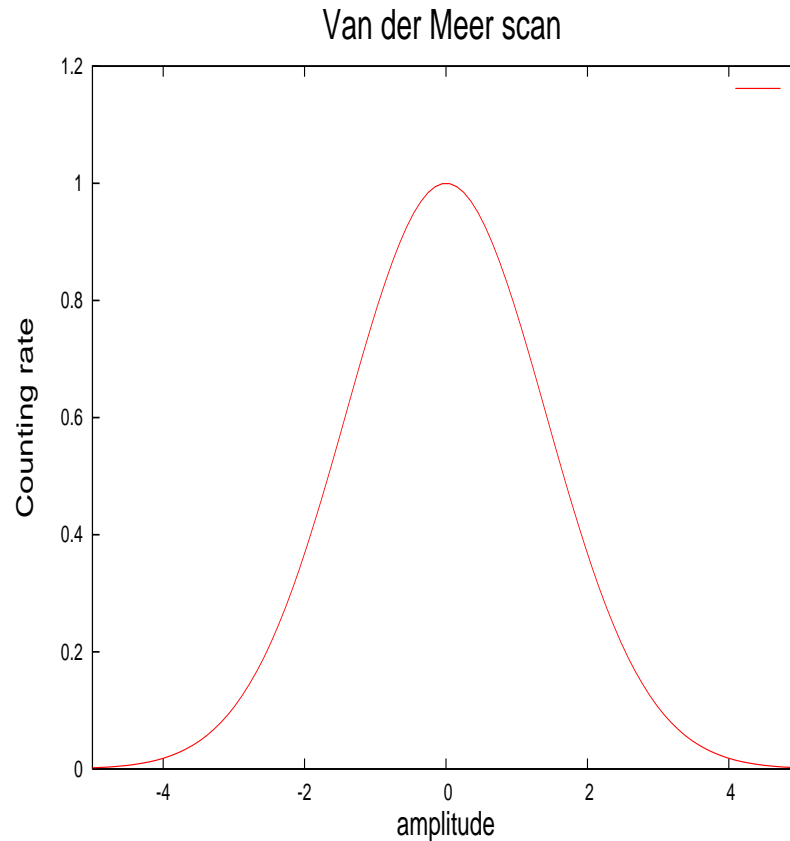
Luminosity calibration

(hadrons, e.g. pp or $p\bar{p}$)

- Must measure beam current and beam sizes
- Beam size measurement:
 - Wire scanner or synchrotron light monitors
 - Measurement with beam ... → remember luminosity with offset
 - Move the two beams against each other in transverse planes (van der Meer scan)



Luminosity optimization



Record counting rates $R(d)$ as function of movement d

Since $R(d)$ is proportional to luminosity $L(d)$

Get ratio of luminosity $L(d)/L(0)$

Luminosity optimization

- From ratio of luminosity $\mathcal{L}(d)/\mathcal{L}_0$
- Remember: $W = e^{-\frac{1}{4\sigma^2}(d_2-d_1)^2}$
- Determines σ
- ... and centres the beams !
- ”beam-beam deflection scans ...” \implies



Absolute value of \mathcal{L} (pp or $p\bar{p}$)

- By total rate and optical theorem

(also: luminosity independent determination of σ_{tot}):

- $\sigma_{tot} \cdot \mathcal{L} = N_{inel} + N_{el}$ (Total counting rate)

- $\lim_{t \rightarrow 0} \frac{d\sigma_{el}}{dt} = (1 + \rho^2) \frac{\sigma_{tot}^2}{16\pi} = \frac{1}{\mathcal{L}} \frac{dN_{el}}{dt} \Big|_{t=0}$

→
$$\mathcal{L} = \frac{(1 + \rho^2) (N_{inel} + N_{el})^2}{16\pi (dN_{el}/dt)_{t=0}}$$

- Luminosity determined from experimental rates
-

Absolute value of \mathcal{L} (pp or $p\bar{p}$)

● By Coulomb normalization:

● Coulomb amplitude exactly calculable:

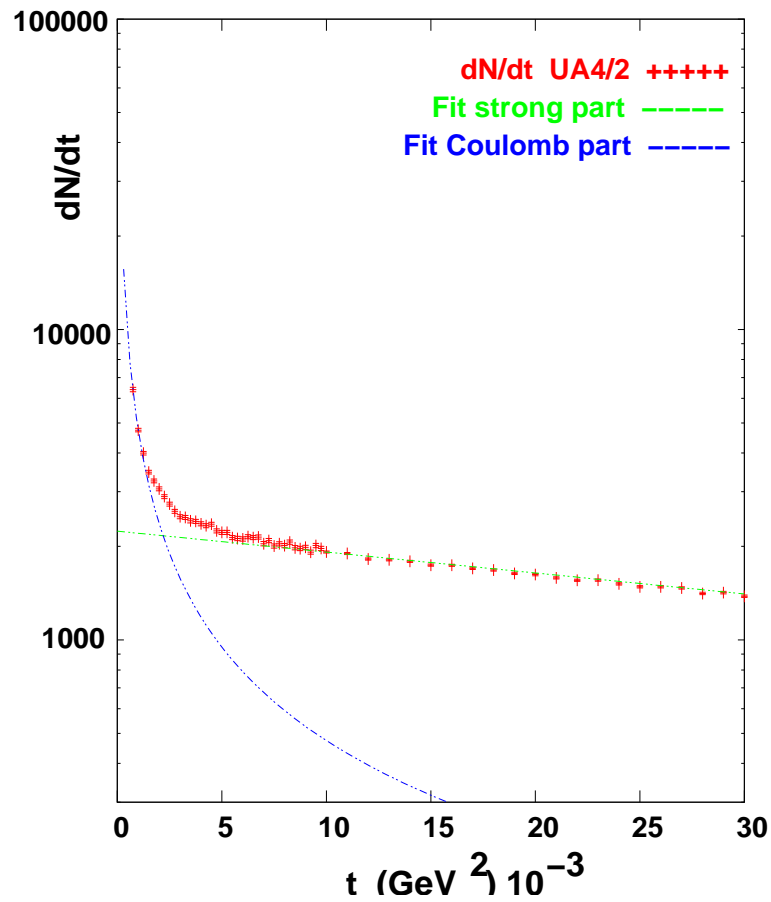
$$\begin{aligned} \bullet \lim_{t \rightarrow 0} \frac{d\sigma_{el}}{dt} &= \frac{1}{\mathcal{L}} \frac{dN_{el}}{dt} \Big|_{t=0} = \pi |f_C + f_N|^2 \\ &\simeq \pi \left| \frac{2\alpha_{em}}{-t} + \frac{\sigma_{tot}}{4\pi} (\rho + i) e^{B\frac{t}{2}} \right|^2 \simeq \frac{4\pi\alpha_{em}^2}{t^2} \Big|_{|t| \rightarrow 0} \end{aligned}$$

● Fit gives: σ_{tot}, ρ, B and \mathcal{L}

● Can be done measuring **only** elastic scattering
(No N_{inel} needed !)

● \implies Roman pots

Differential elastic cross section



- Measure dN/dt at small t ($0.01 < \text{GeV}^2$) and extrapolate to $t = 0.0$
- Needs special optics to allow measurement at very small t
- Measure total counting rate $N_{\text{el}} + N_{\text{inel}}$
Needs good detector coverage
- Often use slightly modified method, precision 1 – 2 %



Not treated :

- Coasting beams (e.g. ISR)
- Asymmetric colliders (e.g. PEP, HERA)
- Linear colliders (SLC, TESLA etc.)



How to cook high Luminosity ?

- Get high intensity
- Get small beam sizes (small ϵ and β^*)
- Get short bunches
- Get many bunches
- Get exact head-on collisions

