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Ordinary Differential Equations a Refresher

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Outline

1. What is a differential equation?
2. Initial Value Problems
   - Linear first order differential equations
   - Second order differential equations
   - Recasting high order differential equations as a system of first order differential equations
3. Boundary Value Problems
4. Solution techniques for nonlinear differential equations
   - Power series solutions
5. Stability Analysis
6. Numerical Solution with the Runge-Kutta Method

Differential Equations: The Basics I

- Ordinary differential equations are used to model change over an independent variable (for our purposes it will usually be t for time or x for a space like variable) without using partial derivatives. So we have equation involving the derivatives of an unknown function y of a single variable t over an interval $t \in (I)$.
- Differential equations contain three types of variables: an independent variable, at least one dependent variable (these will be functions of the independent variable), and the parameters.
- ODE's can contain multiple iterations of derivatives. They are named accordingly (i.e. if there are only first derivatives, then the ODE is called a first order ODE).

Differential Equations: The Basics II

- If the function F is linear in the variables $a_0, a_1, \ldots, a_n$ the ODE is said to be **linear**. If, in addition, F is homogeneous then the ODE is said to be homogeneous.

The general n-th order linear ODE can be written

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = b(x).$$

General Solution of a Linear Differential Equation

- It represents the set of all solutions, i.e., the set of all functions which satisfy the equation in the interval (I).
- For example, the general solution of the differential equation $y' = 3x^2$ is $y = x^3 + C$ where C is an arbitrary constant. The constant C is the value of y at $x = 0$. This **initial condition** completely determines the solution.

A System of ODE's I

$$y_1' = G_1(x, y_1, y_2, \ldots, y_n) \tag{1}$$
$$y_2' = G_2(x, y_1, y_2, \ldots, y_n) \tag{2}$$
$$\vdots \tag{3}$$
$$y_n' = G_n(x, y_1, y_2, \ldots, y_n) \tag{4}$$

An n-th order ODE of the form $y^{(n)} = G(x, y, y', \ldots, y^{n-1})$ can be transformed in the form of the system of first order DE's. If

A System of ODE's II

we introduce dependant variables $y_1 = y, y_2 = y', \ldots, y_n = y^{n-1}$
we obtain the equivalent system of first order equations

$$
\begin{aligned}
y_1' &= y_2, \\
y_2' &= y_3, \\
&\vdots \\
y_n' &= G(x, y_1, y_2, \ldots, y_n).
\end{aligned}
$$

For example, the ODE $y'' = y$ is equivalent to the system

$$
\begin{aligned}
y_1' &= y_2, \\
y_2' &= y_1.
\end{aligned}
$$

A System of ODE's III

In this way the study of n-th order equations can be reduced to the study of systems of first order equations. Some times, one called the latter as the **normal form** of the n-th order ODE.

Systems of equations arise in the study of the motion of particles. For example, if $P(x, y)$ is the position of a particle of mass m at time t, moving in a plane under the action of the force field $(f(x, y), g(x, y))$, we have

$$m\frac{d^2x}{dt^2} = f(x, y),$$
$$m\frac{d^2y}{dt^2} = g(x, y).$$

A System of ODE's IV

The general first order ODE in normal form is

$$y' = F(x, y).$$

If F and $\frac{\partial F}{\partial y}$ are continuous one can show that, given a, b, there is a unique solution with $y(a) = b$.

A Simple Example: Population Modeling

Population growth is commonly modelled with differential equations. In the following equation: $t =$ time, $P =$ population and $k =$ proportionality constant. k represents the constant ratio between the growth rate of the population and the size of the population.

$$\frac{dP}{dt} = kP$$

In this particular equation, the left hand side represents the growth rate of the population being proportional to the size of the population P. This is a very simple example of a first order, ordinary differential equation.

Initial Value Problems I

An initial value problem consists of a differential equation and an initial condition. So, going back to the population example, the following is an example of an initial value problem:

$$\frac{dP}{dt} = kP, \quad P(0) = P_0$$

The solution to this set of equations is a function, call it $P(t)$, that satisfies both equations.

Initial Value Problems II

Ansatz:

$$P(t) = Ce^{kt}$$

General Solutions to a Differential Equation

Let's look at a simple example and walk through the steps of finding a general solution to the following equation

$$\frac{dy}{dt} = (ty)^2$$

We will simply "separate" the variables then integrate the both sides of the equation to find the general solution.

$$\begin{aligned} \frac{dy}{dt} &= t^2 y^2 \\ \frac{1}{y^2}\, dy &= t^2\, dt \\ \int \frac{1}{y^2}\, dy &= \int t^2\, dt \end{aligned}$$

$$
\begin{aligned}
-y^{-1} &= \frac{t^3}{3} + c \\
-\frac{1}{y} &= \frac{t^3}{3} + c \\
\Rightarrow y(t) &= -\frac{1}{\frac{t^3}{3} + c}
\end{aligned}
$$

where $c \in \Re$ is any real number.

Linear First Order Differential Equations I

Initial value problems consist of a differential equation and an initial value. We will work through the example below:

$$\frac{dx}{dt} = -xt; \qquad x(0) = \frac{1}{\sqrt{\pi}}$$

First we will need to find the general solution to $\frac{dx}{dt} = -xt$, then use the initial value $x(0) = \frac{1}{\sqrt{\pi}}$ to solve for c. Since we do not

Linear First Order Differential Equations II

know what $x(t)$ is, we will need to "separate" the equation before integrating.

$$\begin{aligned} \frac{dx}{dt} &= -xt \\ -\frac{1}{x}\,dx &= t\,dt \\ \int -\frac{1}{x}\,dx &= \int t\,dt \end{aligned}$$

Linear First Order Differential Equations Continued

$$\begin{aligned} -\ln x &= \frac{t^2}{2} + c \\ x &= e^{-(\frac{t^2}{2}+c)} \\ x &= e^{-(\frac{t^2}{2})}e^{-c} \\ x &= ke^{-\frac{t^2}{2}} \end{aligned}$$

The above function of t is the general solution to $\frac{dx}{dt} = -xt$ where k is some constant. Since we have the initial value $x(0) = \frac{1}{\sqrt{\pi}}$, we can solve for k.

Solving Initial Value Problems

Thus we can see that the solution to the initial value problem

$$\frac{dx}{dt} = -xt, \quad x(0) = \frac{1}{\sqrt{\pi}}$$

is

$$x(0) = \frac{1}{\sqrt{\pi}} = ke^{-\frac{0^2}{2}}$$

$$x(t) = \frac{1}{\sqrt{\pi}}\, e^{-\frac{t^2}{2}}$$

Second Order Differential Equations I

Second order differential equations simply have a second derivative of the dependent variable. The following is a common example that models a simple harmonic oscillator:

$$\frac{d^2y}{dt^2} + \frac{k}{m}y = 0$$

where m and k are determined by the mass and spring involved. This second order differential equation can be rewritten as the following first order differential equation:

$$\frac{dv}{dt} = -\frac{k}{m}y$$

Second Order Differential Equations II

where v denotes velocity.
If $v(t)$ is velocity, then $v = \frac{dy}{dt}$. Thus, we can substitute in $\frac{dv}{dt}$ into our second order differential equation and essentially turn it into a first order differential equation.

$$\frac{d^2y}{dt^2} = -\frac{k}{m}y \Leftrightarrow \frac{dv}{dt} = -\frac{k}{m}y$$

Now we have the following system of first order differential equations to describe the original second order differential equation:

Second Order Differential Equations III

$$
\begin{aligned}
\frac{dy}{dt} &= v \\
\frac{dv}{dt} &= -\frac{k}{m} y
\end{aligned}
$$

With $k/m = 1$ consider the following initial value problem:

$$\frac{d^2y}{dt^2} + y = 0$$

with $y(0) = 0$ and $y'(0) = v(0) = 1$. Let's show that $y(t) = \sin(t)$ is a solution.

Second Order Differential Equations IV

$$\begin{aligned}
\frac{dy}{dt} &= \frac{d}{dt}\sin(t) = \cos(t) = v \\
\frac{dv}{dt} &= -\sin(t) = -y \\
\Rightarrow \frac{d^2y}{dt^2} &= -\sin(t) \\
\Rightarrow \frac{d^2y}{dt^2} + y &= \frac{d^2(\sin(t))}{dt^2} + \sin(t) \\
&= -\sin(t) + \sin(t) = 0
\end{aligned}$$

Reminder: Lie transformations I

A Lie transformation is written as:

$$\mathcal{M} = e^{-t:H:} \tag{5}$$

where the Lie operator $:H:$ is defined by:

$$:H: = \frac{\partial H}{\partial \vec{q}} \frac{\partial}{\partial \vec{p}} - \frac{\partial H}{\partial \vec{p}} \frac{\partial}{\partial \vec{q}}. \tag{6}$$

$\vec{q}$ are the coordinates and $\vec{p}$ the conjugate momenta; h is a function of $\vec{q}$ and $\vec{p}$. The exponential operator is defined in terms of its series expansion:

$$e^{-t:H:} = 1 - t{:}H{:} + \frac{t^2}{2}{:}H{:}^2 - \frac{t^3}{3!}{:}H{:}^3 + \cdots \tag{7}$$

Reminder: Lie transformations II

If H is the Hamiltonian of the system, then the evolution of any function of the phase space variables is given by:

$$\frac{df}{dt} = -:H:f, \qquad f(t) = e^{-t:H:}f(0). \tag{8}$$

Lie transformation

The operator $e^{:g:}$ is called a *Lie transformation.*
To see how this works, consider the example of a familiar system: a simple harmonic oscillator in one degree of freedom.
The Hamiltonian is:

$$H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2. \qquad (9)$$

Suppose we want to find the coordinate q as a function of time t. Of course, in this case, we could simply write down the equations of motion (from Hamilton's equations) and solve them (because the Hamiltonian is integrable). However, we can also write:

$$q(t) = e^{-t:H:}q(0). \qquad (10)$$

Lie transformation example: harmonic oscillator

To evaluate the Lie transformation, we need $:H\!:q$.

$$:H\!:q = \frac{\partial H}{\partial q}\frac{\partial q}{\partial p} - \frac{\partial H}{\partial p}\frac{\partial q}{\partial q} = -\frac{\partial H}{\partial p} = -p. \tag{11}$$

Similarly, we find:

$$:H\!:p = \omega^2 q. \tag{12}$$

This means that:

$$:H\!:^2 q \quad = \quad :H\!:(-p) = -\omega^2 q, \tag{13}$$

$$:H\!:^3 q \quad = \quad :H\!:(-q) = \omega^2 p, \tag{14}$$

and so on.

Lie transformation example: harmonic oscillator

Using the above results, we find:

$$
\begin{aligned}
q(t) &= q(0) - t{:}H{:}q(0) + \frac{t^2}{2}{:}H{:}^2 q(0) - \frac{t^3}{3!}{:}H{:}^3 q(0) + \frac{t^4}{4!}{:}H{:}^4 q(0) \cdots && (15) \\
&= q(0) + tp(0) - \omega^2 \frac{t^2}{2} q(0) - \omega^2 \frac{t^3}{3!} p(0) + \omega^4 \frac{t^4}{4!} q(0) \cdots && (16)
\end{aligned}
$$

Collecting together even and odd powers of t, we see that equation (16) can be written:

$$
q(t) = q(0)\cos(\omega t) + \frac{p(0)}{\omega}\sin(\omega t). \tag{17}
$$

Power Series Solutions I

To demonstrate how to use power series to solve a nonlinear differential equation we will look at Hermite's Equation [1]:

$$\frac{d^2y}{dt^2} - 2t\frac{dy}{dt} + 2py = 0$$

We will use the following power series and its first and second derivatives to make a guess:

Power Series Solutions II

$$y(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + ... = \sum_{n=0}^{\infty} a_n t^n \tag{18}$$

$$\frac{dy}{dt} = a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + ... = \sum_{n=1}^{\infty} n a_n t^{n-1} \tag{19}$$

$$\frac{d^2y}{dt^2} = 2a_2 + 6a_3 t + 12a_4 t^2 + ... = \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2} \tag{20}$$

From the previous equations we can conclude that

Power Series Solutions III

$$\begin{aligned} y(0) &= a_0 \\ y'(0) &= a_1 \end{aligned}$$

Next we will substitute (18), (19) and (20) into Hermite's Equation and collect matching terms.

$$\begin{aligned} \frac{d^2y}{dt^2} - 2t\frac{dy}{dt} + 2py = 0 = (2a_2 + 6a_3t + 12a_4t^2 + \ldots) \\ -2t(a_1 + 2a_2t + 3a_3t^2 + 4a_4t^3 + \ldots) \\ +2p(a_0 + a_1t + a_2t^2 + a_3t^3 + \ldots) \\ \Rightarrow (2pa_0 + 2a_2) + (2pa_1 - 2a_1 + 6a_3)t + \\ (2pa_2 - 4a_2 + 12a_4)t^2 + (2pa_3 - 6a_3 + 20a_5)t^3 = 0 \end{aligned}$$

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Power Series Solutions IV

Then from here, we will set all coefficients equal to 0 since the equation is equal to 0 and $t \neq 0$. We get the following sequence of equations:

$$\begin{aligned} 2pa_0 + 2a_2 &= 0 \\ 2pa_1 - 2a_1 + 6a_3 &= 0 \\ 2pa_2 - 4a_2 + 12a_4 &= 0 \\ 2pa_3 - 6a_3 + 20a_5 &= 0 \end{aligned}$$

Then will several substitutions we arrive at the following set of equations:

Power Series Solutions V

$$
\begin{aligned}
\Rightarrow a_2 &= -pa_0 \\
a_3 &= -\frac{p-1}{3}a_1 \\
a_4 &= -\frac{p-2}{6}a_2 = \frac{(p-2)p}{6}a_0 \\
a_5 &= -\frac{p-3}{10}a_3 = \frac{(p-3)(p-1)}{30}a_1
\end{aligned}
$$

[1] Klein-Gordon equation, travelling wave solutions

Stability Analysis I

Many realistic models of physical systems require mathematics that are intractable, yet we still would like information about the system. One of the most important pieces of information of interest to us is the stability of a dynamical system, a system that changes with time t. We use a general second-order differential equation for stability analysis.

$$\frac{d^2y}{dt^2} + \gamma_1(t)\frac{dy}{dt} + \gamma_0(t)y = 0$$

The second-order differential equation is transformed to a set of first-order differential equations by defining

$$\Lambda_1 = y, \quad \Lambda_2 = \frac{dy}{dt}$$

Stability Analysis II

and arrive at the set of first-order differential equations

$$\frac{d\Lambda_1}{dt} = \Lambda_2 \equiv f_1 \quad \frac{d\Lambda_2}{dt} = -\Lambda_0\Lambda_2 - \Lambda_1\Lambda_2 \equiv f_2 \tag{21}$$

Equations 21 can be written in matrix form:

$$\frac{d\mathbf{\Lambda}}{dt} = \dot{\mathbf{\Lambda}} = \mathbf{A}\mathbf{\Lambda} \equiv \mathbf{f} \tag{22}$$

with

$$\dot{\mathbf{\Lambda}} = \frac{\mathbf{d}}{\mathbf{dt}}\begin{pmatrix}\Lambda_1\\ \Lambda_2\end{pmatrix}, \quad \mathbf{f} = \begin{pmatrix}f_1\\ f_2\end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 0 & 1 \\ -\Lambda_0 & -\Lambda_1 \end{pmatrix}.$$

Stability Analysis III

The solution x of Eq. (21) represents the state of the dynamical system as t changes. An equilibrium state $\mathbf{\Lambda_e}$ is a state of the system which satisfies the equation

$$\dot{\mathbf{\Lambda}} = \mathbf{0}$$

The stability of a dynamical system can be determined by calculating *what happens to the system when it is slightly perturbed* from an equilibrium state. Stability calculations are relatively straightforward for linear systems, but can be very difficult or intractable for nonlinear problems. Since many dynamical models are nonlinear, approximation techniques must be used to analyze their stability. One way to analyze the stability of a nonlinear, dynamical model is to first linearize the

Stability Analysis IV

problem. As a first approximation, the nonlinear problem is linearized by performing a Taylor series expansion of Eq. (21) about an equilibrium point.
Let

$$\mathbf{u} \equiv \mathbf{\Lambda} - \mathbf{\Lambda_e}$$

the displacement of the system from its equilibrium state, then result is

$$\mathbf{u} = \mathbf{J}\mathbf{u} + \xi(\mathbf{u}), \tag{23}$$

and $\xi(\mathbf{u})$ contains terms of second-order or higher from the Taylor series expansion. The Jacobian matrix $\mathbf{J}$ is evaluated at the equilibrium point $\mathbf{\Lambda_e}$ thus

$$\mathbf{J} = \left(\begin{array}{cc} \frac{\partial f_1}{\partial \Lambda_1} & \frac{\partial f_1}{\partial \Lambda_2} \\ \frac{\partial f_2}{\partial \Lambda_1} & \frac{\partial \partial f_2}{\partial \Lambda_2} \end{array} \right)_{\mathbf{\Lambda_e}} = \left(\begin{array}{cc} 0 & 1 \\ -\Lambda_0 & -\Lambda_1 \end{array} \right)_{\mathbf{\Lambda_e}}.$$

Stability Analysis V

In this case, the matrices **A** and **J** are equal. Neglecting higher-order terms in Eq. (23) gives the linearized equation

$$\dot{\mathbf{u}} = \mathbf{J}\mathbf{u}. \tag{24}$$

We solve Eq. (24) by trying a solution with the exponential time dependence

$$\dot{\mathbf{u}} = \mathbf{e}^{\lambda \mathbf{t}}\mathbf{g} \tag{25}$$

where **g** is a nonzero vector and λ indicates whether or not the solution will return to equilibrium after a perturbation. Substituting Eq. (25) into Eq. (24) gives an eigenvalue problem of the form

Stability Analysis VI

$$(\mathbf{J} - \lambda \mathbf{I})\mathbf{g} = \mathbf{0}. \tag{26}$$

The eigenvalues λ are found from the characteristic equation

$$\det(\mathbf{J} - \lambda \mathbf{I}) = \mathbf{0}. \tag{27}$$

where det denotes the determinant. The following summarizes the interpretation of λ if we assume that the independent variable t is monotonically increasing:

EV	Interpretation
$\lambda > 0$	Diverges from equilibrium solution
$\lambda = 0$	Transition point
$\lambda < 0$	Converges to equilibrium solution

Stability Analysis VII

The linearized form of Eq. (21), namely, Eq. (24), exhibits stability when the product lt is less than zero because the difference $\mathbf{u} \rightarrow \mathbf{0}$ as $\lambda t \rightarrow -\infty$ in Eq. (25). If the product lt is greater than zero, the difference u diverges. This does not mean the solution of the nonlinear problem is globally divergent because of our linearization assumption. It does imply that a perturbation of the solution from its equilibrium value is locally divergent. Thus an estimate of the stability of the system is found by calculating the eigenvalues from the characteristic equation.
EXERCISE: calculate λ for our particular case.

Numerical Solution with the Runge-Kutta Method I

Systems of linear ODEs may be solved numerically using techniques such as the Runge-Kutta fourth-order numerical algorithm. Suppose the initial conditions are

$$\mathbf{x}(t_0) = \mathbf{x}_0$$

at $t = t_0$ for the system of equations

$$\frac{d}{dt}\mathbf{x} = f(\mathbf{x}, t).$$

Values of $\mathbf{x}$ as functions of t are found by incrementally stepping forward in t. The fourth-order Runge-Kutta method calculates new values of $\mathbf{x}_{n+1}$ from old values $\mathbf{x}_n$ using the algorithm

Numerical Solution with the Runge-Kutta Method II

$$\mathbf{x}_{n+1} = \mathbf{x}_n + \frac{h}{6}(w_1 + 2w_2 + 2w_3 + w_4) + \mathcal{O}(h^5)$$

where h is an incremental step size $0 < h < 1$. The terms of the algorithm are

$$
\begin{aligned}
t_{n+1} &= t_n + h && (28)\\
w_1 &= f(\mathbf{x}_n, t_n) && (29)\\
w_2 &= f(\mathbf{x}_n + \frac{1}{2}hw_1, t_n + \frac{1}{2}h) && (30)\\
w_3 &= f(\mathbf{x}_n + \frac{1}{2}hw_2, t_n + \frac{1}{2}h) && (31)\\
w_4 &= f(\mathbf{x}_n + hw_3, t_n + h) && (32)
\end{aligned}
$$

Numerical Solution with the Runge-Kutta Method III

The calculation begins at $n = 0$ and proceeds iteratively. At the end of each step, the new values are defined as present values at the n^{th} level and another iteration is performed.

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Numerical Solution with the Runge-Kutta Method IV

Exercice:
Suppose we want to solve a system of two first-order ODEs of the form

$$\frac{dx_1}{dt} = x_2 \quad (33)$$

$$\frac{dx_2}{dt} = -x_1 \quad (34)$$

with inital conditions

$$\mathbf{x}(t_0) = (0,1)^T.$$

Write a Python RK-4 program to solve this ODE.