Nonlinear Dynamics (see also CAS Advanced Course)

(with strong emphasis on implementation)

The menu:

- **Linear case (later generalization)**
- **Non-linear elements and power series, symplecticity (now non-linear elements)**
- **Linear maps (usually matrices) $\rightarrow$ non-linear maps**
- **Action-angle variables, invariants of motion and Liouville Theorem**
- **Thin lenses and symplectic integration**

Nonlinear Dynamics (see also CAS Advanced Course)

(with strong emphasis on implementation)

The ambition:

- **Treat nonlinear dynamics without too many handwaving arguments**

→ **Find a formalism that can be "easily" extended to deal nonlinear dynamics !!**

(extended means that it is the same for both cases)

Recommended Bibliography (and acknowledgement):

[AW] A. Wolski, Beam Dynamics in High Energy Particle Accelerators, Imperial College Press, 2014.

[AC1] A. Chao, Lecture Notes on Topics in Accelerator Physics, SLAC, 2001.

[WH] W. Herr, Mathematical and Numerical Methods for Nonlinear Dynamics in Proceedings of the CERN Accelerator School: Advanced Accelerator Physics, Trondheim, Norway, 18-29 August 2013, CERN-2014-009, pp. 157-198.

[EF1] E. Forest, Beam Dynamics - A New Attitude and Framework, Harwood Academic Publishers, 1998.

[EF2] E. Forest, From Tracking Code to Analysis, Springer, 2016.

[AD] A. Dragt, Lie Methods for Nonlinear Dynamics with Applications to Accelerator Physics

[MB] M. Berz, Modern Map Methods in Beam Physics, Academic Press, 1999.

For what follows one should (!) always use canonical variables !

In Cartesian coordinates: $R \;=\; (X, P_X, Y, P_Y, Z, P_Z, t)$

If the energy is constant (i.e. P_Z **= const.), we use:** (X, P_X, Y, P_Y, Z, t)

This system is rather inconvenient, what we want is the description of the particle in the neighbourhood of the reference orbit/trajectory:

$$R_d \;=\; (X, P_X, Y, P_Y, Z, t)$$

which are considered now the deviations from the reference and which are zero for a particle on the reference trajectory

Very important: it is the <u>reference</u> not the <u>design</u> trajectory !

(so far it is a straight line along the Z-direction)

The independent variable is usually the time t (Newton)

Problem: particles with different initial conditions generally require different times to pass through an element. Better to measure the progress using a longitudinal coordinate Z.

We therefore replace time t by Z and eventually by s using:

$s = Z + ct$ → $\boxed{R = (X, P_X, Y, P_Y, s)}$ s is the distance along reference path

Non-trivial: strictly speaking requires the Hamiltonian formalism

→ **using s is Hamiltonian in disguise ...**

For a "curved" trajectory, in general not circular, with a local radius of curvature $\rho(s)$ in the horizontal (X - Z plane), we transform to a new coordinate system (x, y, s) (co-moving frame) with (see e.g. [AW]):

$$X = (x+\rho)\cos\left(\frac{s}{\rho}\right) - \rho \qquad \text{(needed tomorrow)}$$

$$Y = y$$

$$Z = (x+\rho)\sin\left(\frac{s}{\rho}\right)$$

The new canonical momenta become:

$$p_x = P_X \cos\left(\frac{s}{\rho}\right) + P_Z \sin\left(\frac{s}{\rho}\right)$$

$$p_y = P_Y$$

$$p_s = P_Z\left(1 + \frac{x}{\rho}\right)\cos\left(\frac{s}{\rho}\right) - P_X\left(1 + \frac{x}{\rho}\right)\sin\left(\frac{s}{\rho}\right)$$

finally for the transverse coordinates: $\boxed{r = (x, p_x, y, p_y)}$

Some clarification (again):

F.A.Q.: Phase Space ($x, p_x, ...$) or Trace Space ($x, x', ...$)

- **Beam dynamics is strictly correct only with ($x, p_x, ...$), but in general quantities cannot be measured easily**
- **Beam dynamics with ($x, x', ...$) needs special precaution, but quantities much easier to measure or more relevant (e.g. crossing angle, bumps, ...)**
- **Some quantities are different (e.g. emittance)**

Be aware of that when you do the calculations ...

Usual starting point: Linear dynamics in synchrotrons

Each element at position s acts as a source of forces, i.e. we must write for the forces K $\Rightarrow$ $K(s)$ (so long harmonic oscillator ..... !)

To justify the Courant-Snyder ansatz:
linear (uncoupled !) optics in rings often introduced using 1D*) Hill type equation where $K(s)$ is assumed to be a periodic function in s:

$$\frac{d^2 x(s)}{ds^2} + \underbrace{\left(a_0 + 2\sum_{n=1}^{\infty} a_n \cdot \cos(2ns)\right)}_{K(s)} x(s) = 0 \qquad \text{and} \qquad \underbrace{K(s+C) \;=\; K(s)}_{ring...}$$

Solution of a Boundary Value Problem (rings !) must be periodic too !

Not applicable in the general case (e.g. Linacs, Beamlines, FFAG, Recirculators, ...), much better to treat it as an Initial Value Problem

*) **What about 2D ??**

First: For **any** linear, 1st order equation of the type

$$\frac{dx(s)}{ds} = K(s)\, x(s) \qquad \text{(and initial values at } s_0)$$

the solution can always be written as (Floquet, Hamilton, e.g. [AD]):

$$\begin{aligned} x(s) &= a \cdot x(s_0) + b \cdot x'(s_0) \\ x'(s) &= c \cdot x(s_0) + d \cdot x'(s_0) \end{aligned} \Longrightarrow \begin{pmatrix} x \\ x' \end{pmatrix}_s = \overbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}^{A} \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$

(now $K(s)$ does not have to be periodic)

Second: The determinant of A is always 1

Third: No need for an "ansatz" (i.e. knowing the solution)

→ **Much better to use matrices for our linear systems from the start just have to know what is A between the locations s and s_0**

Real life: adding nonlinear elements (e.g. magnetic fields)

Nonlinear elements can be described by polynomials of higher order:

$$\boxed{\frac{d^2x(s)}{ds^2} + K(s)x(s) = \sum_{i,j\geq 0} p_{ij}(s)x^iy^j}$$

Electromagnetic fields can be described with the multipole expansion:

$$B_y + iB_x = \sum_{n=1} (b_n + ia_n)(x + iy)^{n-1}$$

(in LHC need up to $n = 20$ **!)**

Equations of motions become (here horizontal plane):

$$\boxed{\frac{d^2x(s)}{ds^2} + K(s)x(s) = \frac{F_x(x,y,s)}{v \times p} = -\frac{B_y(x,y,s)}{p}}$$

(Note: we have now coupling between the planes if $i \neq 0$ **and** $j \neq 0$ **!!)**

Some problems with this approach:

- It is rather hopeless to describe a complicated system
- It is totally hopeless to find a closed solution
- Perturbation treatment required, but does not always give satisfactory results and does not fully exploit potential of computing and numerical techniques
- Numerical methods create new problems, hopefully less important
- ...

Many concepts (more or less) valid in 1D become incorrect for 2D, hidden approximations (often inspired by the linear treatment) can lead to misconceptions and eventually permanent brain damage ..

The most reliable tools to study realistic models are simulations (e.g. tracking codes)

Particle Tracking:

.. a numerical solution of the (nonlinear) Initial Value Problem:
It is a "integrator" of the equation of motion
Vast amount of tracking codes available, many analysis tools available
(Examples: Lyapunov, Chirikov, chaos detection, frequency analysis, ...)

Ambition:

Find an approach to link simulations with theoretical analysis, would allow a better understanding of the physics in realistic machines

→ **Based on finite maps i.e. discrete systems**

Watch out for numerical problems !!

Look at the linear treatment first, then generalize to nonlinear theory

Linear optics was already treated in detail, I use the very basics to show the idea and demonstrate the transition

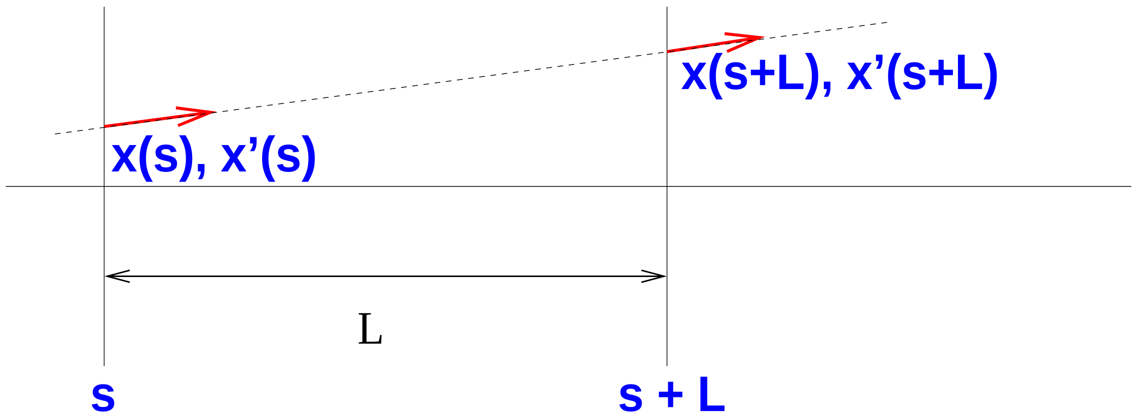
The procedure and formalism is identical

For consistency with some (classical) textbooks and other lectures I sometimes (where not critical) use x, x', y, y' **instead of** x, p_x, y, p_y

→ **Linear maps are usually written as <u>matrices</u>**

→ **Some simple examples (simplified, the full version later this week)**

A drift space (one dimension only) of length L, starting at position s and ending at s + L



The simplest description (1D, using x, x') is (should be in 3D of course):

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s+L} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \circ \begin{pmatrix} x \\ x' \end{pmatrix}_{s} = \begin{pmatrix} x + x' \cdot L \\ x' \end{pmatrix}$$

<u>This is only an approximation, something may go badly wrong, see later ... !</u>

Focusing quadrupole of length L and constant strength k_1 ($k_1 > 0$):

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_2} = \begin{pmatrix} \cos(L \cdot \sqrt{k_1}) & \frac{1}{\sqrt{k_1}} \cdot \sin(L \cdot \sqrt{k_1}) \\ -\sqrt{k_1} \cdot \sin(L \cdot \sqrt{k_1}) & \cos(L \cdot \sqrt{k_1}) \end{pmatrix} \circ \begin{pmatrix} x \\ x' \end{pmatrix}_{s_1}$$

similar for a defocusing quadrupole, i.e. for $k_1 < 0$

(it is the solution of $\frac{d^2x(s)}{ds^2} = K(s)\,x(s)$ when $K(s) = k_1 = const.$)

However: fundamental for the map approach ➡

Can we get the maps:

1. For all elements, including nonlinear (e.g. sextupoles) where no solution exists ?

2. From first principles (i.e. fields), without reference to their use ?
(a particle does not know what the element is (supposed) to do ?)

Starting from a position s_0 and combining all matrices to get the matrix to position $s_0 + L$ (shown for 1D only):

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_0 + L} = \underbrace{\mathcal{M}_N \circ \mathcal{M}_{N-1} \circ \ldots \circ \mathcal{M}_1}_{\mathcal{M}(s_0, L)} \circ \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$

For a ring with circumference C we get the One-Turn-Matrix (OTM) at s_0

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_0 + C} = \underbrace{\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}}_{\mathcal{M}_{OTM}} \circ \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$

Without proof (trust me for a few minutes), the scalar product:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_0} \cdot \mathcal{M}_{OTM} \cdot \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0} = \text{const.} = \text{J}$$

is a constant of the motion: invariant of the One-Turn-Map $\mathcal{M}_{OTM}$

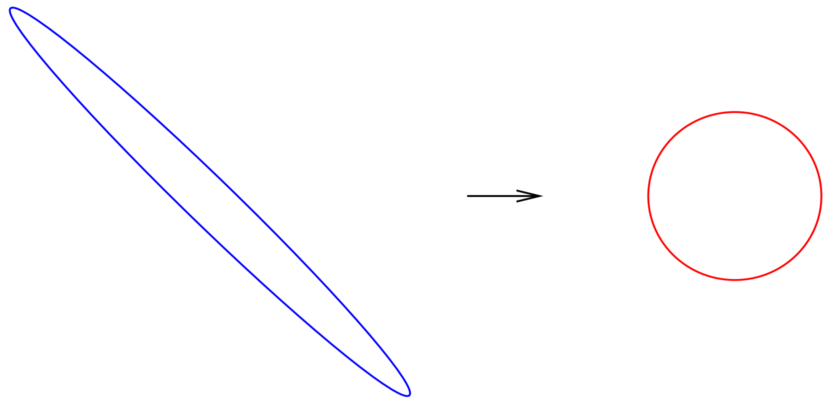
Next: matrices can be transformed into **Normal Forms**

we try to find a (invertible) transformation $\mathcal{A}$ such that (called "similarity transformation"):

$$\boxed{\mathcal{A}\mathcal{M}\mathcal{A}^{-1} = \mathcal{R} \qquad (\text{or}: \qquad \mathcal{A}^{-1}\mathcal{R}\mathcal{A} = \mathcal{M})}$$

- **The matrix $\mathcal{R}$ is:**
  - **A "Normal Form", (or at least a very simplified form of the matrix)**
  - **Example (most important case): $\mathcal{R}$ becomes a pure rotation**
- **The matrix $\mathcal{R}$ describes the same dynamics as $\mathcal{M}$, but:**
  - **All coordinates are transformed by $\mathcal{A}$**
  - **This transformation $\mathcal{A}$ "analyses" the complexity of the motion, it contains the structure of the phase space**

Transformation to Normal Form (pictorially)



$$\color{blue}{M} = \mathcal{A} \circ \color{red}{\mathcal{R}} \circ \mathcal{A}^{-1} \quad \text{or :} \quad \color{red}{\mathcal{R}} = \mathcal{A}^{-1} \circ \color{blue}{M} \circ \mathcal{A}$$

Motion on an ellipse becomes motion on a circle (i.e. a rotation):
$\mathcal{R}$ **is the "simple" part of the map - shape is "dumped" into** $\mathcal{A}$

How to get that (i.e. $\mathcal{A}$**) ? Remember lectures on Linear Algebra (Eigenvectors, Eigenvalues ...), see also backup slides**

We find the two components of the original map:

$$\mathcal{A} = \begin{pmatrix} \sqrt{\beta(s_0)} & 0 \\ -\frac{\alpha(s_0)}{\sqrt{\beta(s_0)}} & \frac{1}{\sqrt{\beta(s_0)}} \end{pmatrix} \quad \text{and} \quad \mathcal{R} = \begin{pmatrix} \cos(\mu_x) & \sin(\mu_x) \\ -\sin(\mu_x) & \cos(\mu_x) \end{pmatrix}$$

The Normal Form transformation gives plenty of information:

- **We have stable oscillations when the eigenvalues μ_x (and μ_y etc.) are real, (forget about the $Tr(\mathcal{M}) \leq 2$ business). This concept is valid also for 2D or any complicated systems, e.g. coherent motion with 6000×6000 matrices etc: many modes !**
- **μ_x is the "tune" $Q_x \cdot 2\pi$ (now we can talk about <u>phase advance</u> !)**
- **$\beta, \alpha, ...$ are the optical parameters and describe the ellipse**
- **The closed orbit (an invariant, identical coordinates after one turn !):**

$$\mathcal{M}_{OTM} \circ (x,\ x')_{co} \equiv (x,\ x')_{co}$$

Note 1:

- The <u>only</u> assumption was that particles make more than one turn !!!
- Matrices $\mathcal{R}$ and $\mathcal{M}$ are called <u>similar</u> (i.e. have the same eigenvalues) (<u>to be equivalent</u> is not sufficient !)

Note 2:

in 2 dimensions the normal form is a 4×4 matrix:

$$\mathcal{R}^{2D} = \begin{pmatrix} \cos(\mu_x) & \sin(\mu_x) & 0 & 0 \\ -\sin(\mu_x) & \cos(\mu_x) & 0 & 0 \\ 0 & 0 & \cos(\mu_y) & \sin(\mu_y) \\ 0 & 0 & -\sin(\mu_y) & \cos(\mu_y) \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

What if the two planes (oscillators) are linearly coupled ?

Assume a one-turn-matrix in 2D (4×4 matrix):

$$R = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \quad \overbrace{\longrightarrow}^{coupling!} \quad T = \begin{pmatrix} M & n \\ m & N \end{pmatrix}$$

M,m,N,n are 2-by-2 block matrices.

In case of coupling: $m \neq 0, n \neq 0$ **we can try to transform as:**

$$T = \begin{pmatrix} M & n \\ m & N \end{pmatrix} = V R' V^{-1}$$

with (same procedure as before, find the simple case):

$$R' = \begin{pmatrix} A' & 0 \\ 0 & B' \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} \gamma I & C \\ -C^t & \gamma I \end{pmatrix}$$

A short comparison of the different approaches (not rigorous)

- **Classical perturbation method:**
  - **Transform/expand solution in terms of distortion parameter**
  - **Analytical/symbolic expression for the solution**
  - **Solution is approximate (eigenvalues inexact, not always useful)**
- **Map/Normal Form approach:**
  - **Transform Differential Equation in terms of distortion parameter (Normal Form) to get an equation that can be solved**
  - **No symbolic expression for the solution**
  - **Requires some approximation of the model**
  - **Procedure to get solutions/eigenvalues is exact**

Using the map/Normal Form approach we get an exact solution at the expense of giving up a closed analytical form for the solution

Impact on a key concept:

A central question in physics (accelerator theory) is to find, understand and quantify invariants:

A property of a system that is unchanged, i.e. conserved as the system evolves (typical examples are: energy, momentum, angular momentum, charge, invariant mass, speed of light, ..)

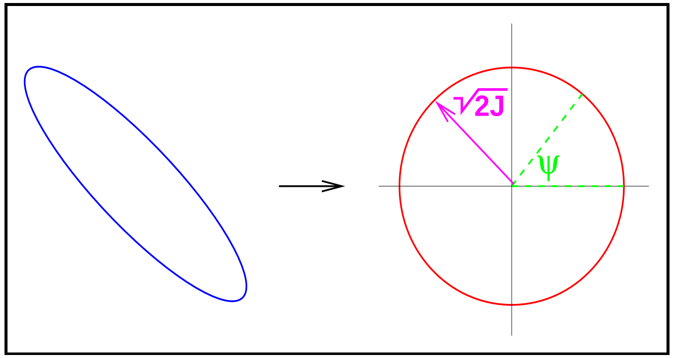
Given a map $\mathcal{M}$ we look for ζ with

$$\boxed{\mathcal{M}\,\zeta \;=\; \zeta}$$

e.g. Special Relativity is <u>all about invariants</u> ("mechanics", electrodynamics, ...

More appropriate for studies: using Action - Angle variables

Once the particles "travel" on a circle (i.e. always !), the motion is better described by the canonical variables action J_x and angle Ψ_x:



$$x = \sqrt{2J_x\beta_x}\ \cos(\Psi_x)$$

$$p_x = -\sqrt{\frac{2J_x}{\beta_x}}\ (\sin(\Psi_x) + \alpha_x\cos(\Psi_x))$$

$$J_x = \frac{1}{2}(\gamma_x x^2 + 2\alpha_x x p_x + \beta_x p_x^2)\ ^{*)}$$

- **Angular position along the ring Ψ becomes the independent variable !**
- **The trajectory of a particle is now <u>independent</u> of the position s !**
- **Constant Radius $\sqrt{2J}$ $\xrightarrow{defines}$ action J (invariant of motion)**

$*$) **Never call that "emittance", this is fraudulent and brain clobbering !**

Interlude: If we have many particles, action is related to beam emittance (this is valid also for sources, electrons, linacs and beam lines, and non-Gaussian beams:

If we can measure $< x^2 >$, $< p_x^2 >$ and $< xp_x >$ of a beam, and define a beam emittance ϵ_x (see e.g. [AW, AC2], also CERN convention):

$$\boxed{\epsilon_x \;=\; < J_x >}$$

this means:

$$\epsilon_x \;=\; \sqrt{< x^2 >< p_x^2 > \;-\; < xp_x >^2}$$

We can use action-angle variables defined before as:

$$x \;=\; \sqrt{2J_x\beta_x}\;\cos(\Psi_x) \qquad\qquad p_x \;=-\sqrt{\frac{2J_x}{\beta_x}}\;(\sin(\Psi_x) + \alpha_x \cos(\Psi_x))$$

and from above we get (Ψ disappears by the averaging)

$$< x^2 > \;=\; \beta_x\epsilon_x, \qquad < xp_x > \;=\; -\,\alpha_x\epsilon_x, \qquad < p_x^2 > \;=\; \gamma_x\epsilon_x$$

Since other definitions often refer to the treatment by Courant and Snyder, here a quote from Courant himself:

"The invariant J is simply related to the area enclosed by the ellipse:

$$\text{Area enclosed} \quad = \quad 2\pi J.$$

In accelerator and storage ring terminology there is a quantity called the emittance which is closely related to this invariant. The emittance, however, is a property of a distribution of particles, not a single particle. Consider a Gaussian distribution in amplitudes. Then the (rms) emittance, ϵ, is given by:

$$(x_{rms})^2 \quad = \quad \beta_x(s) \cdot \epsilon_x.$$

In terms of the action variable, J, this can be rewritten

$$\epsilon_x \quad = \quad < J > .$$

where the bracket indicates an average over the distribution in J."

Note: this is also the CERN and CAS convention ...

Introducing nonlinear elements: various types of nonlinear maps

- **Choice depends on the application, some examples:**
  - **Taylor (Power) maps**
  - **Lie transformations**
  - **Truncated Power Series Algebra (TPSA), can generate maps from tracking (see lectures by Etienne Forest)**
- **Not all maps are allowed !**
  - **Key concept: Symplecticity (again) most relevant for rings !**

Linear first ..

A symplectic matrix $\mathcal{M}$ has to fulfil the condition:

$$\mathcal{M}^T \cdot S \cdot \mathcal{M} = S \qquad \text{with} \qquad S = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$\lim_{n\to\infty} \mathcal{M}^n = \text{finite} \quad \Longrightarrow \quad \text{requires} \quad \det \mathcal{M} = 1^{*)}$$

1. $\mathcal{M}$ is <u>area preserving (x, p)</u> and J is an <u>invariant</u>:

$$\boxed{\mathcal{M}\, J = J}$$

2. <u>All</u> eigenvalues of $\mathcal{M}$ are <u>non-zero</u> and it is <u>invertible</u>

3. Products of symplectic matrices are symplectic

*) **(But note:** det $\mathcal{M} = 1$ **alone is <u>not</u> sufficient, except in 1D)**

Introducing nonlinear elements (e.g. 2nd order)

Effect of a sextupole-like element with strength k_2 is (normal component):

$$\begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_{s_2} = \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_{s_1} + \begin{pmatrix} 0 \\ -\frac{1}{2}k_2 L \cdot (x_{s_1}^2 - y_{s_1}^2) \\ 0 \\ \frac{1}{2}k_2 L \cdot (x_{s_1} \cdot y_{s_1}) \end{pmatrix}$$

- ➡ **Amplitudes appear as second power**
- ➡ **(Normally) Cannot be written as a matrix**

We need something like (here for x-coordinate), i.e. Power Series:

$$x_{new} = \overbrace{R_{11} \cdot x + R_{12} \cdot x' + R_{21} \cdot y + R_{22} \cdot y'}^{\text{matrix part (power 1)}} +$$

$$+\overbrace{T_{111} \cdot x^2 + T_{112} \cdot xx' + T_{122} \cdot x'^2 + T_{113} \cdot xy + T_{114} \cdot xy' + ...}^{\text{sextupole part (power 2)}}$$

$$+\overbrace{U_{1111} \cdot x^3 + U_{1112} \cdot x^2 x' + ...}^{\text{octupole part (power 3)}}$$

$$+.......$$

and the equivalent for $x'_{new}, y_{new}, y'_{new}$ and higher orders

<u>Note:</u> for sextupoles and higher we have coupling terms $x^n y^m$, etc.

Normally, because one could write it as (1D, horizontal plane only):

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{new} = \begin{pmatrix} R_{11} & R_{12} & T_{111} & T_{112} & T_{122} \\ R_{21} & R_{22} & T_{211} & T_{212} & T_{222} \end{pmatrix} \circ \begin{pmatrix} x \\ x' \\ x^2 \\ xx' \\ x'^2 \end{pmatrix}$$

Just a fake, looks good but does not win anything ..

Easier to implement as (here up to 3rd order):

$$z_j^{new} = \sum_{k=1}^{6} R_{jk} z_k + \sum_{k=1}^{6} \sum_{l=1}^{6} T_{jkl} z_k z_l + \sum_{k=1}^{6} \sum_{l=1}^{6} \sum_{m=1}^{6} U_{jklm} z_k z_l z_m \quad \text{for} \quad \text{j} = 1..6$$

Explicit map (2D) for a sextupole with length L and strength k_2:

$$
\begin{aligned}
x_2 &= x_1 + Lx_1' &&- k_2\left(\tfrac{L^2}{4}(x_1^2 - y_1^2) + \tfrac{L^3}{12}(x_1x_1' - y_1y_1') + \tfrac{L^4}{24}(x_1'^2 - y_1'^2)\right)\\
x_2' &= x_1' &&- k_2\left(\tfrac{L}{2}(x_1^2 - y_1^2) + \tfrac{L^2}{4}(x_1x_1' - y_1y_1') + \tfrac{L^3}{6}(x_1'^2 - y_1'^2)\right)\\
y_2 &= y_1 + Ly_1' &&+ k_2\left(\tfrac{L^2}{4}x_1y_1 + \tfrac{L^3}{12}(x_1y_1' + y_1x_1') + \tfrac{L^4}{24}(x_1'y_1')\right)\\
y_2' &= y_1' &&+ k_2\left(\tfrac{L}{2}x_1y_1 + \tfrac{L^2}{4}(x_1y_1' + y_1x_1') + \tfrac{L^3}{6}(x_1'y_1')\right)
\end{aligned}
$$

- **Can this be used in this form ?**
- **This is not a matrix - what about the "symplectic" condition ? How to test it ?**

(if bored: find T_{234}, T_{324} ...)

➡ **It is the associated Jacobian matrix $\mathcal{J}$ (all 1st order partial derivatives) which must fulfil the symplecticity condition:**

$$\mathcal{J}_{ik} = \frac{\partial z_2^i}{\partial z_1^k} \quad \left(\text{e.g. } \mathcal{J}_{xy} = \frac{\partial x_2}{\partial y_1}\right)$$

$\mathcal{J}$ **must fulfil:** $\mathcal{J}^T \cdot \mathcal{S} \cdot \mathcal{J} = \mathcal{S}$

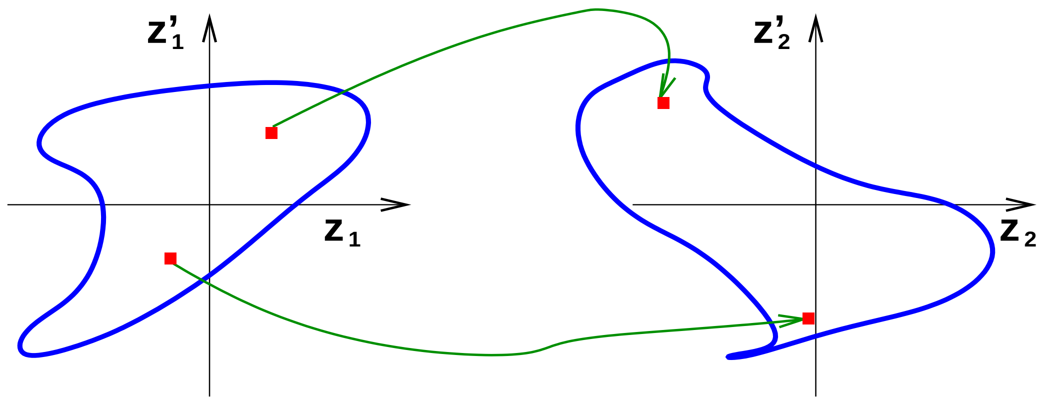
The coordinate z and the phase space dimension can be very high order: (number of particles) · (number of degrees of freedom)

(LHC $\approx 10^{15}$, in most of my examples n = 4)

$$z_2^n = \mathcal{M}\, z_1^n$$

... an interesting consequence ⇨

Transformation of the occupied phase space



$$\boxed{V_2 \;=\; \int_{V_2} d^n z_2 \;=\; \int_{V_1} \left|\frac{\partial z_2}{\partial z_1}\right| d^n z_1 \;=\; \int_{V_1} |\mathcal{M}|\, d^n z_1 \;=\; \int_{V_1} d^n z_1 \;=\; V_1}$$

Under symplectic transformations ⇨ phase space volume is conserved !!

<u>This</u> is Liouville's theorem !!

(not to be mistaken for Poincare invariant: $\int p \cdot dq \;=\; const.$ **)**

There is also a problem:

$$\mathcal{J}_{ik} = \frac{\partial z_2^i}{\partial z_1^k} \quad \left(\text{e.g. } \mathcal{J}_{xy} = \frac{\partial x_2}{\partial y_1}\right)$$

$\mathcal{J}$ **must fulfil:** $\mathcal{J}^T \cdot \mathcal{S} \cdot \mathcal{J} = \mathcal{S}$

In general: $\mathcal{J}_{ik} \neq$ **const** **(i.e. depend on** $x_1, x_1', \ldots$**)**

Confusing ?? o.k. $\rightarrow$ **example sextupoles**

$$
\begin{aligned}
x_2 &= x_1 + Lx_1' &&- k_2 \left(\tfrac{L^2}{4}(x_1^2 - y_1^2) + \tfrac{L^3}{12}(x_1 x_1' - y_1 y_1') + \tfrac{L^4}{24}(x_1'^2 - y_1'^2) \right) \\
x_2' &= x_1' &&- k_2 \left(\tfrac{L}{2}(x_1^2 - y_1^2) + \tfrac{L^2}{4}(x_1 x_1' - y_1 y_1') + \tfrac{L^3}{6}(x_1'^2 - y_1'^2) \right) \\
y_2 &= y_1 + Ly_1' &&+ k_2 \left(\tfrac{L^2}{4} x_1 y_1 + \tfrac{L^3}{12}(x_1 y_1' + y_1 x_1') + \tfrac{L^4}{24}(x_1' y_1') \right) \\
y_2' &= y_1' &&+ k_2 \left(\tfrac{L}{2} x_1 y_1 + \tfrac{L^2}{4}(x_1 y_1' + y_1 x_1') + \tfrac{L^3}{6}(x_1' y_1') \right)
\end{aligned}
$$

$$
\mathcal{J}_{ik} = \begin{pmatrix}
\frac{\partial x_2}{\partial x_1} & \frac{\partial x_2}{\partial x_1'} & \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_1'} \\
\frac{\partial x_2'}{\partial x_1} & \frac{\partial x_2'}{\partial x_1'} & \frac{\partial x_2'}{\partial y_1} & \frac{\partial x_2'}{\partial y_1'} \\
\frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_1'} & \frac{\partial y_2}{\partial y_1} & \frac{\partial y_2}{\partial y_1'} \\
\frac{\partial y_2'}{\partial x_1} & \frac{\partial y_2'}{\partial x_1'} & \frac{\partial y_2'}{\partial y_1} & \frac{\partial y_2'}{\partial y_1'}
\end{pmatrix}
\xrightarrow{k_2=0}
\begin{pmatrix}
1 & L & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & L \\
0 & 0 & 0 & 1
\end{pmatrix}
$$

$$
\begin{aligned}
x_2 &= x_1 + L x_1' &&- k_2 \left(\tfrac{L^2}{4}(x_1^2 - y_1^2) + \tfrac{L^3}{12}(x_1 x_1' - y_1 y_1') + \tfrac{L^4}{24}(x_1'^2 - y_1'^2) \right) \\
x_2' &= x_1' &&- k_2 \left(\tfrac{L}{2}(x_1^2 - y_1^2) + \tfrac{L^2}{4}(x_1 x_1' - y_1 y_1') + \tfrac{L^3}{6}(x_1'^2 - y_1'^2) \right) \\
y_2 &= y_1 + L y_1' &&+ k_2 \left(\tfrac{L^2}{4} x_1 y_1 + \tfrac{L^3}{12}(x_1 y_1' + y_1 x_1') + \tfrac{L^4}{24}(x_1' y_1') \right) \\
y_2' &= y_1' &&+ k_2 \left(\tfrac{L}{2} x_1 y_1 + \tfrac{L^2}{4}(x_1 y_1' + y_1 x_1') + \tfrac{L^3}{6}(x_1' y_1') \right)
\end{aligned}
$$

$$
\mathcal{J}_{ik} = \begin{pmatrix}
\frac{\partial x_2}{\partial x_1} & \frac{\partial x_2}{\partial x_1'} & \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_1'} \\
\frac{\partial x_2'}{\partial x_1} & \frac{\partial x_2'}{\partial x_1'} & \frac{\partial x_2'}{\partial y_1} & \frac{\partial x_2'}{\partial y_1'} \\
\frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_1'} & \frac{\partial y_2}{\partial y_1} & \frac{\partial y_2}{\partial y_1'} \\
\frac{\partial y_2'}{\partial x_1} & \frac{\partial y_2'}{\partial x_1'} & \frac{\partial y_2'}{\partial y_1} & \frac{\partial y_2'}{\partial y_1'}
\end{pmatrix}
\xrightarrow{k_2=0}
\begin{pmatrix}
1 & L & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & L \\
0 & 0 & 0 & 1
\end{pmatrix}
$$

For $k_2 \neq 0$ coefficients depend on initial values, e.g.:

$$
\frac{\partial y_2}{\partial y_1} = 1 + k_2 \left(\frac{L^2}{4} x_1 + \frac{L^3}{12} x_1' \right) \Rightarrow
$$

Power series are not symplectic, cannot be used

Directly using finite power series maps is ruled out ...

Small error for small L, no error for $L = 0$!

Accelerator physicists love Zero length elements:
engineers don't, they are technically difficult

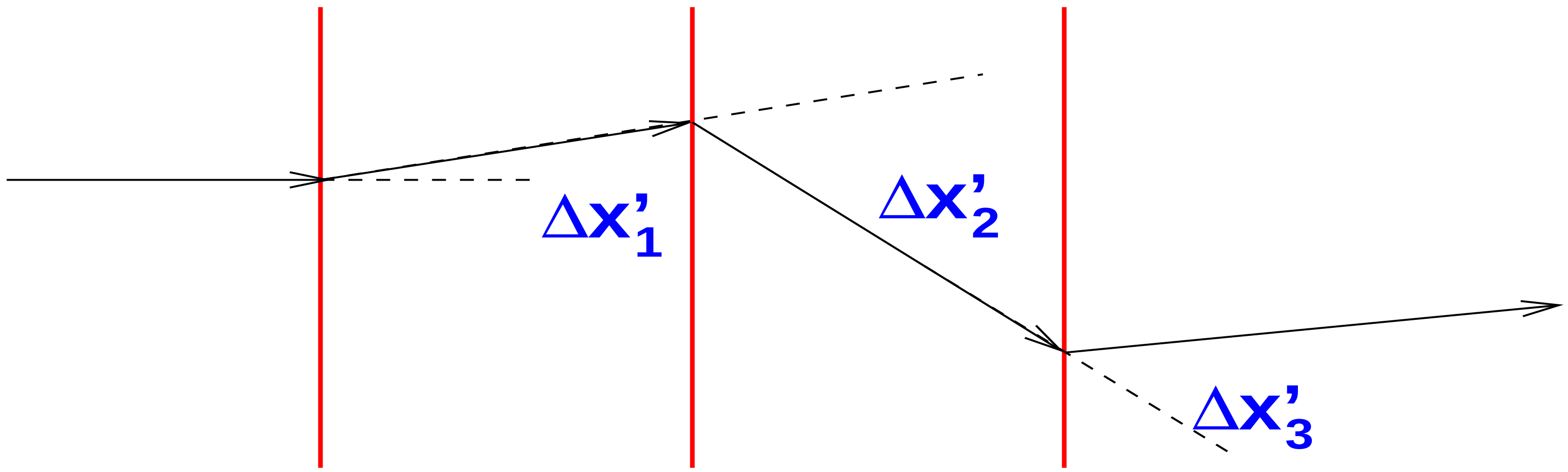
Thick "magnet":
Length and Strength specified for computation

Example sextupole: L and k_2

Thin "magnet":
let the length go to zero, but keep Field Integral
finite (L and k_n are not specified separately):

Example sextupole: $L \cdot k_2$

Moving through thin elements (shown for 1D)



The "momentum" x' receives an amplitude dependent deflection, "kick" $\quad x' \rightarrow x' + \Delta x'$

➡ $\Delta x' = f(x)$ **(polynomials of some - possibly high - order)**

Always symplectic: no change of amplitude inside the element, no dependence on initial angle

Can we approximate a thick element by one or more thin element(s) ?

- Yes, when the length is small or does not matter
- Symplecticity o.k.

- What about accuracy, what have we lost ??

→ Demonstrate with some simple examples

(What follows is valid for all elements and provides the tools !!!)

Check out a quadrupole:

- **Start with "exact"[*)] map, compare with thin quadrupole**

$$\mathcal{M}_{s\to s+L} = \begin{pmatrix} \cos(L\cdot\sqrt{K}) & \frac{1}{\sqrt{K}}\cdot\sin(L\cdot\sqrt{K}) \\ -\sqrt{K}\cdot\sin(L\cdot\sqrt{K}) & \cos(L\cdot\sqrt{K}) \end{pmatrix}$$

- **Thick to thin: make L smaller and smaller, this permits:**

  Taylor/power expansion (of sin and cos) in "small" length L:

$$\mathcal{M} = \underbrace{L^0\cdot\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + L^1\cdot\begin{pmatrix} 0 & 1 \\ -K & 0 \end{pmatrix}}_{\text{thin lens in ''linear lectures''}} + L^2\cdot\begin{pmatrix} -\frac{K}{2} & 0 \\ 0 & -\frac{K}{2} \end{pmatrix} + \ldots$$

[*)] **.. it isn't**

- **Keep up to <u>first order</u> term in L (contribution with L^2 is small)**

$$\mathcal{M} = L^0 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + L^1 \cdot \begin{pmatrix} 0 & 1 \\ -K & 0 \end{pmatrix}$$

$$\mathcal{M} = \begin{pmatrix} 1 & L \\ -K \cdot L & 1 \end{pmatrix} + \mathcal{O}(L^2)$$

- **Precise to first order $\mathcal{O}(L^1)$**
- **det $\mathcal{M} = (1 + KL^2) \neq$ 1, non-symplectic !**

A possible remedy out of the blue:

If we add a term $-KL^2$ the matrix becomes symplectic:

$$\Rightarrow \quad \mathcal{M} = \begin{pmatrix} 1 & L \\ -K \cdot L & 1-KL^2 \end{pmatrix}$$

$\Rightarrow$ **det** $\mathcal{M} = (1 - KL^2 + KL^2) =$ **1**

- **The model is <u>exact</u> and symplecticity is recovered, the (magnet) model is slightly inaccurate (approximate, remember yesterday)**

- **We have not damaged the accuracy too much, the original truncated matrix is inaccurate to order $\mathcal{O}(L^2)$ anyway ...**

Carry on:

Keep up to <u>second order</u> term in L

$$\mathcal{M} = \begin{pmatrix} 1 - \frac{1}{2}KL^2 & L \\ -K \cdot L & 1 - \frac{1}{2}KL^2 \end{pmatrix} + \mathcal{O}(L^3)$$

- **Precise to second order $\mathcal{O}(L^2)$**
- **More accurate than before, but again not symplectic**

Make it symplectic by adding $-\frac{1}{4}KL^3$

$$\mathcal{M} = \begin{pmatrix} 1 - \frac{1}{2}KL^2 & L - \frac{1}{4}KL^3 \\ -K \cdot L & 1 - \frac{1}{2}KL^2 \end{pmatrix} + \mathcal{O}(L^3)$$

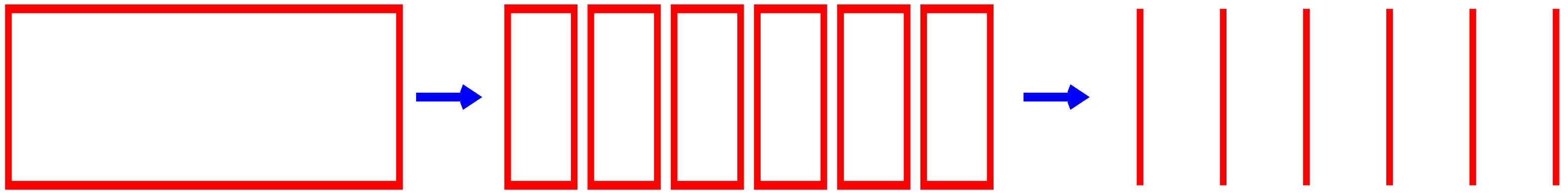
This model is more accurate and symplectic, error is of higher order than before

For many kicks: every "kick" is symplectic

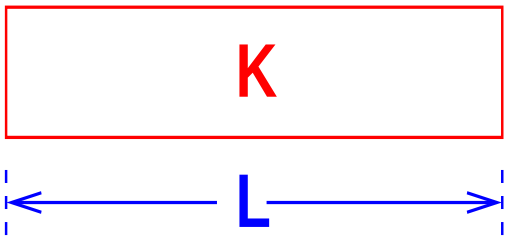
Is there a physical picture behind the approximations ?

Yes → geometry of thin lens kicks ...

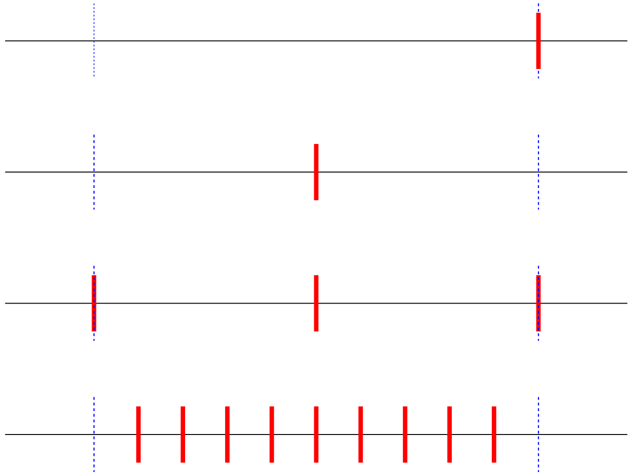
A thick element we should split into one <u>or more</u> thin elements with drifts between them, (cut and shrink) e.g.:



Thick quadrupole $\Rightarrow$ thin quadrupoles $\Rightarrow$ "kicks"



some options:



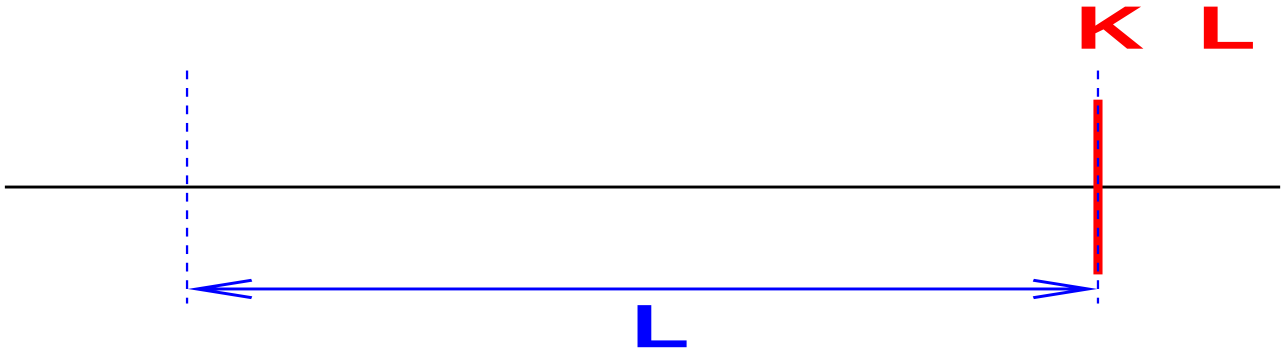
Represented by one or more "thin" lenses (kicks)

$\boxed{\textbf{How many and where ?}}$

Which is a good strategy ? → accuracy and simplicity

Be reasonable, no need to have a longitudinal position precise to 10^{-15} m .. (it is a small step for a man but a giant leap to nonsense)

Option 1



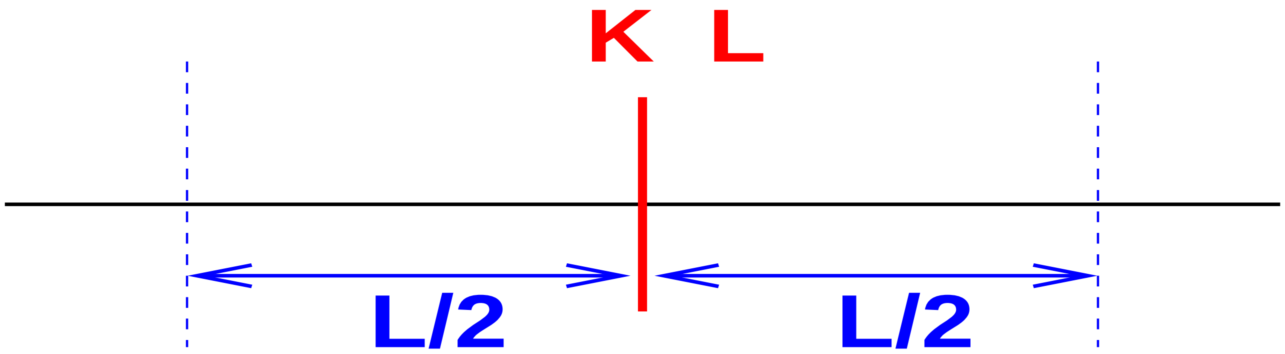
One thin quadrupole "kick" and one drift combined

$$\mathcal{M}_{drift\,+\,kick} = \overbrace{\begin{pmatrix} 1 & 0 \\ -K \cdot L & 1 \end{pmatrix}}^{M_{kick}} \overbrace{\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}}^{M_{drift}} = \begin{pmatrix} 1 & L \\ -K \cdot L & 1 - KL^2 \end{pmatrix}$$

Reminder: product of symplectic matrices is symplectic

Resembles our "symplectification" of order O(1)

Option 2



One thin quadrupole "kick" between two drifts of half length

$$\mathcal{M} = \begin{pmatrix} 1 & \frac{1}{2}L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -K \cdot L & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{2}L \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 - \frac{1}{2}KL^2 & L - \frac{1}{4}KL^3 \\ -K \cdot L & 1 - \frac{1}{2}KL^2 \end{pmatrix}$$

Resembles more accurate "symplectification" of order O(2)

Accuracy of thin lenses

- **One kick at the end (or beginning):**
  - → **Error (inaccuracy) of second order** $\mathcal{O}(L^2)$
- **One kick in the centre:**
  - → **Error (inaccuracy) of third order** $\mathcal{O}(L^3)$

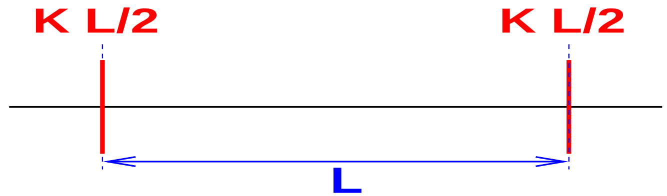
It is very relevant how to apply thin lenses !

If you describe a quadrupole like : $\begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}$

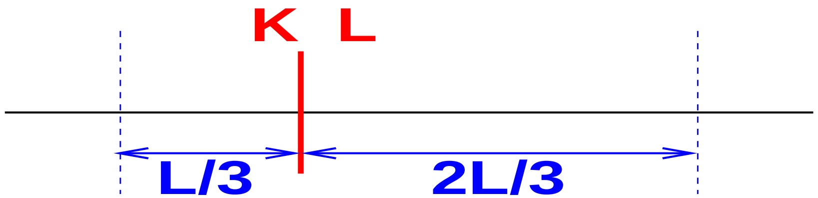
The aim should be to be precise and fast (and simple to implement)

Check whether the approximation is valid for you ...

Increase the number of "kicks": What about these options ?

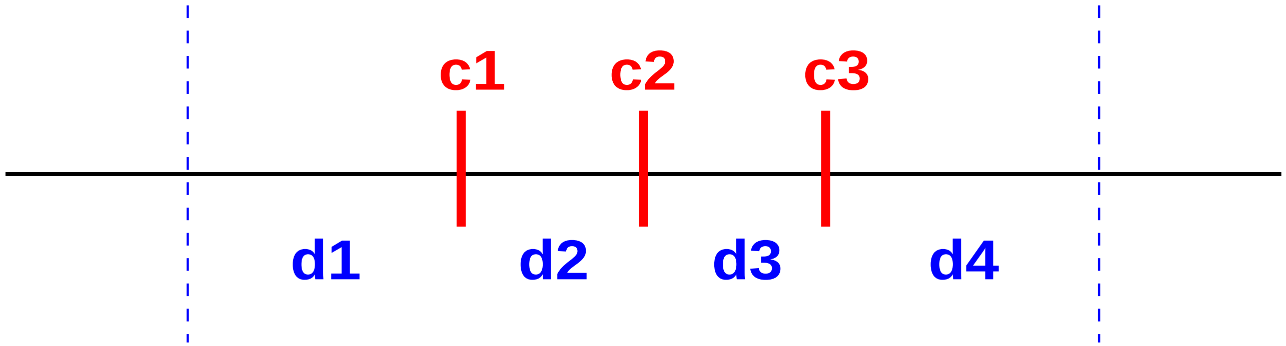


or:



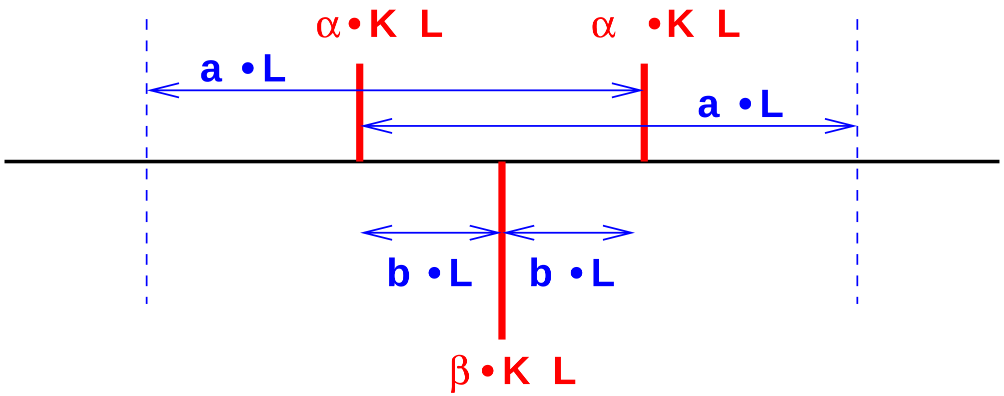
➡ Home exercises: Are they symplectic ?

Can one do better ? Try a model with 3 kicks:



- To get best accuracy (i.e. deviation from exact solution):
- You have 7 free parameters to minimize deviation:
  - Kicks c1, c2, c3 (allow different strength)
  - Drifts d1, d2, d3, d4 (allow any position of the kicks, but NO need to be accurate to attometres)

The optimization gives us: (for the derivation, e.g. [AC1])



with:

$$a = \frac{1}{2} \cdot \frac{1}{2 - 2^{1/3}}, \qquad b = \frac{1 - 2^{1/3}}{2} \cdot \frac{1}{2 - 2^{1/3}}$$

$$\alpha = \frac{1}{2 - 2^{1/3}}, \qquad \beta = -\,2^{1/3} \cdot \frac{1}{2 - 2^{1/3}}$$

We have a $\mathcal{O}(4)$ integrator ... (without proof)

Watch out: it is $-(2.0^{1/3})$ **and not** $(-2.0^{1/3})$ **(remember yesterday !)**

Resulting matrix $\mathcal{M}$ (from the 7 matrices: 4 drifts, 3 kicks) becomes:

$$\mathcal{M}(\mathcal{O}4) = \begin{pmatrix} 1 - \frac{1}{2}k^2L^2 + \frac{1}{24}k^4L^4 + \frac{2^{1/3}}{48(2-2^{1/3})^3}k^6L^6 & L - \frac{1}{6}k^2L^3 + \frac{1-2^{1/3}}{24(2-2^{1/3})^2}k^4L^5 + \frac{2^{1/3}}{96(2-2^{1/3})^4}k^6L^7 \\ -k^2L + \frac{1}{6}k^4L^3 + \frac{2^{1/3}}{24(2-2^{1/3})^2}k^6L^5 & 1 - \frac{1}{2}k^2L^2 + \frac{1}{24}k^4L^4 + \frac{2^{1/3}}{48(2-2^{1/3})^3}k^6L^6 \end{pmatrix}$$

For the ambitious - Prove that it is symplectic

(MATHEMATICA® is really a good friend ...)

Why all that ? (answer in a few minutes)

- **What we do is a Symplectic Integration**
- **From a lower order integration scheme (1 kick), construct higher order scheme:**

$$\overbrace{O(2)}^{1\ kick} \Rightarrow \overbrace{O(4)}^{3\ kicks} \Rightarrow \overbrace{O(6)}^{?\ kicks} \Rightarrow \ldots$$

Formally (for the formulation of S_k see later):

From a 2nd order scheme (1 kick) S_2 we construct a 4th order scheme (3 kicks = 3 x 1 kick) like:

$S_4 = S_2(x_1) \circ S_2(x_0) \circ S_2(x_1)$ **with scaling coefficients:**

$$x_0 = \frac{-2^{1/3}}{2 - 2^{1/3}} \qquad x_1 = \frac{1}{2 - 2^{1/3}}$$

- **Can be considered an iterative scheme (see e.g. H. Yoshida, 1990, E. Forest, 1998):**

If S_{2k} is a symmetric integrator of order $2k$, then:

$$S_{2k+2} = S_{2k}(x_1) \circ S_{2k}(x_0) \circ S_{2k}(x_1)$$

$$\text{with :} \quad x_0 = \frac{-\sqrt[2k+1]{2}}{2 - \sqrt[2k+1]{2}} \qquad x_1 = \frac{1}{2 - \sqrt[2k+1]{2}}$$

- **Higher order integrators can be obtained in a similar way:**

$$S_{2k} \Rightarrow S_{2k+2} \Rightarrow S_{2k+4} \Rightarrow S_{2k+6} \Rightarrow \ldots$$

- **Stop at the desired order, rather simple to implement on a computer (with paper and pencil makes you a lunatic)**

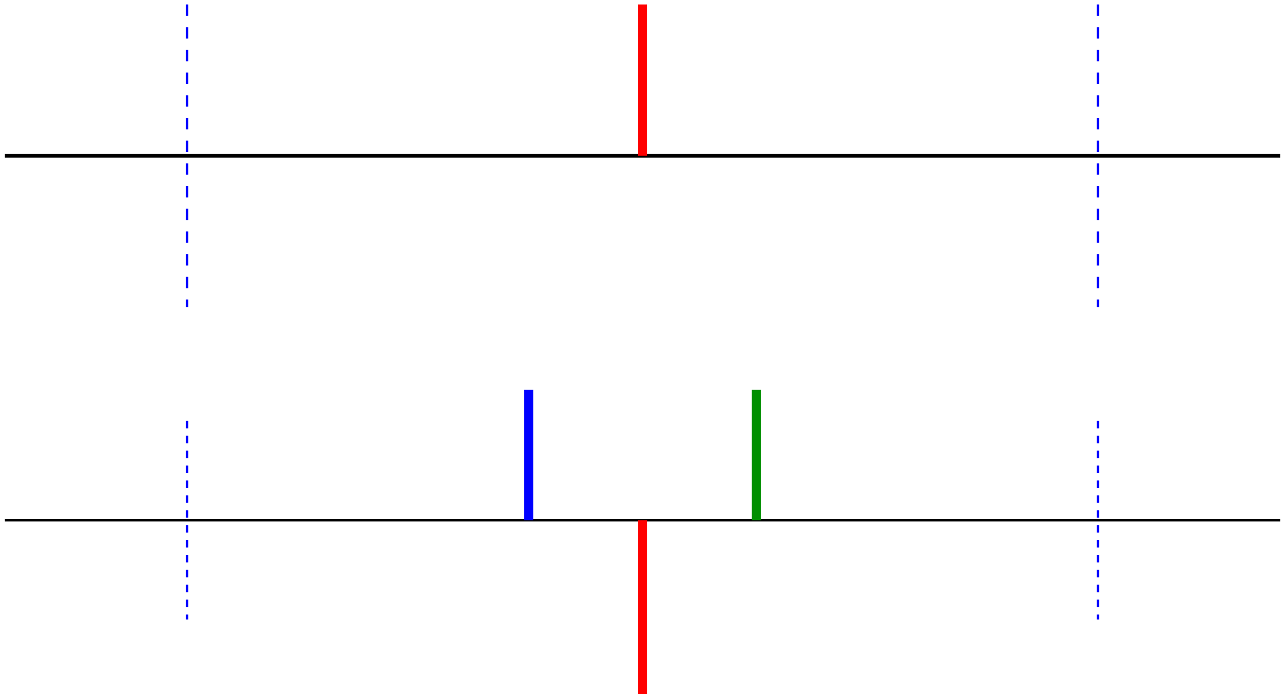
Example: From a 4th order to 6th order

$$S_6 = S_4(x_1) \circ S_4(x_0) \circ S_4(x_1)$$

Replace <u>each kick</u> of a 4th order integrator by a 4th order integrator, using the same scaling factors

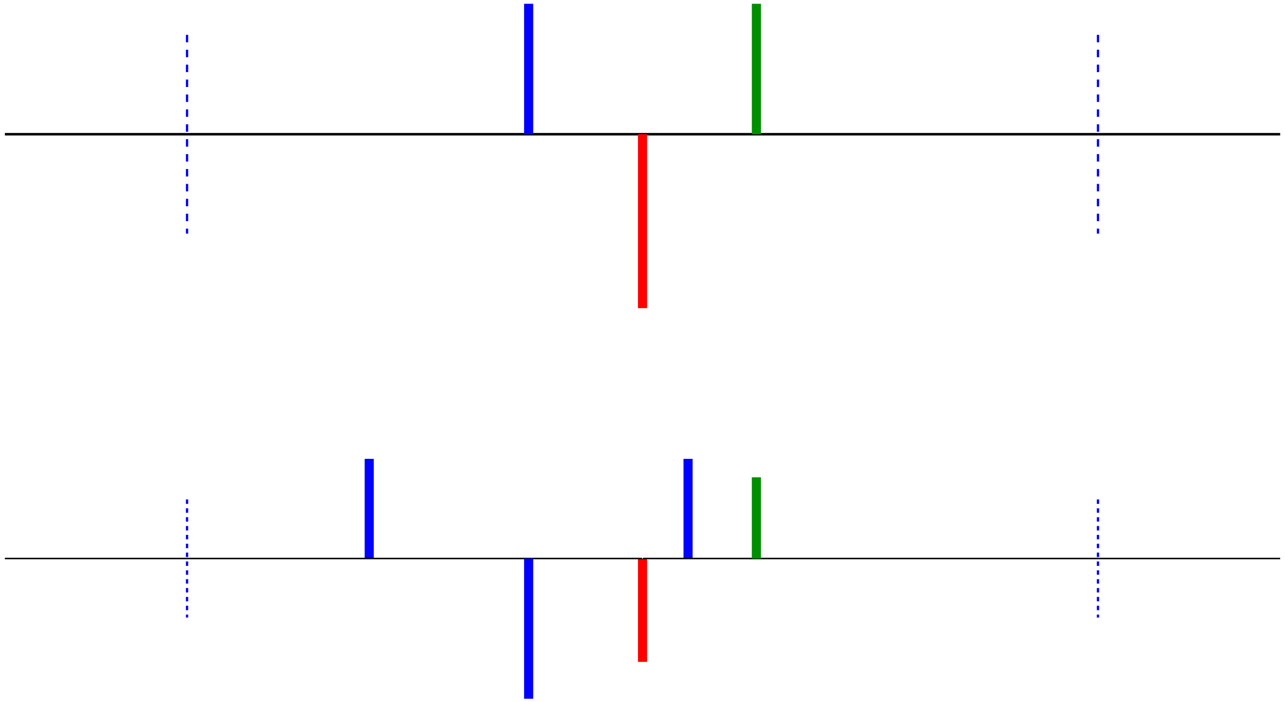
We get 3 times 4th order with 3 kicks each, we have the 9 kick, 6th order integrator mentioned earlier

Integrator of order 2 $\Longrightarrow$ 4



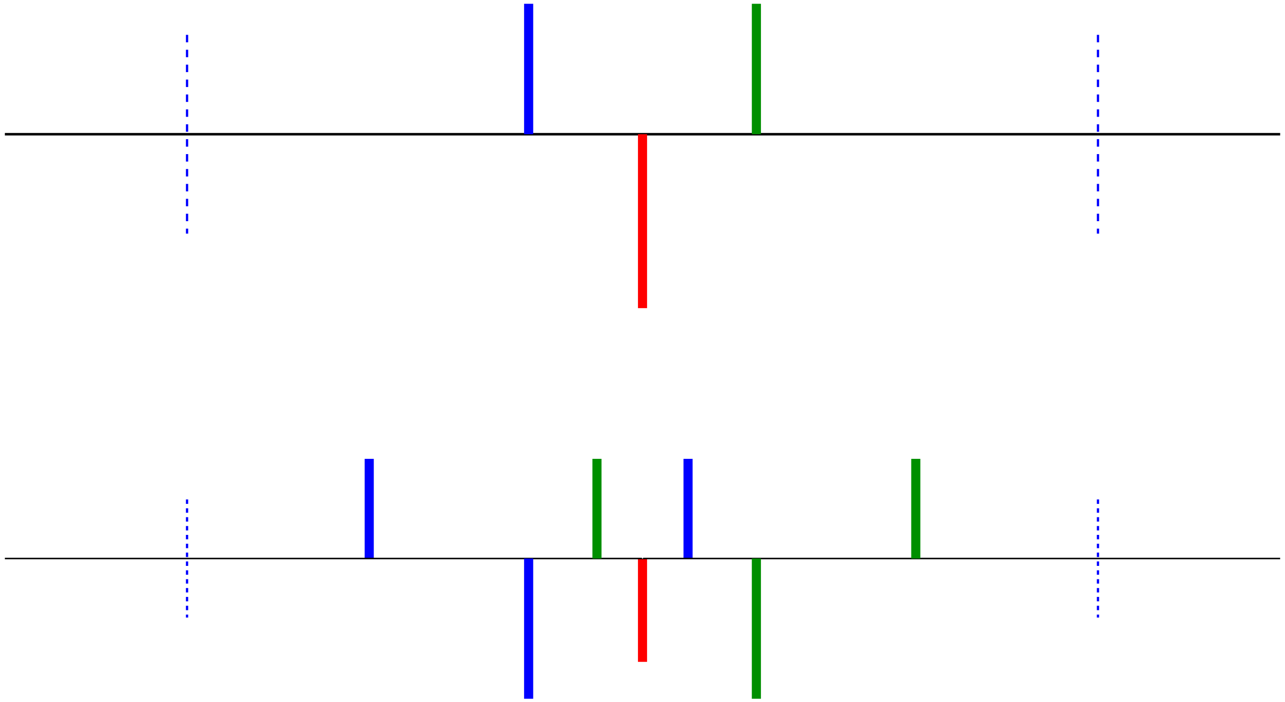
- Replace <u>kick</u> by 4th order integrator

Integrator of order 4 $\Longrightarrow$ 6 - step by step



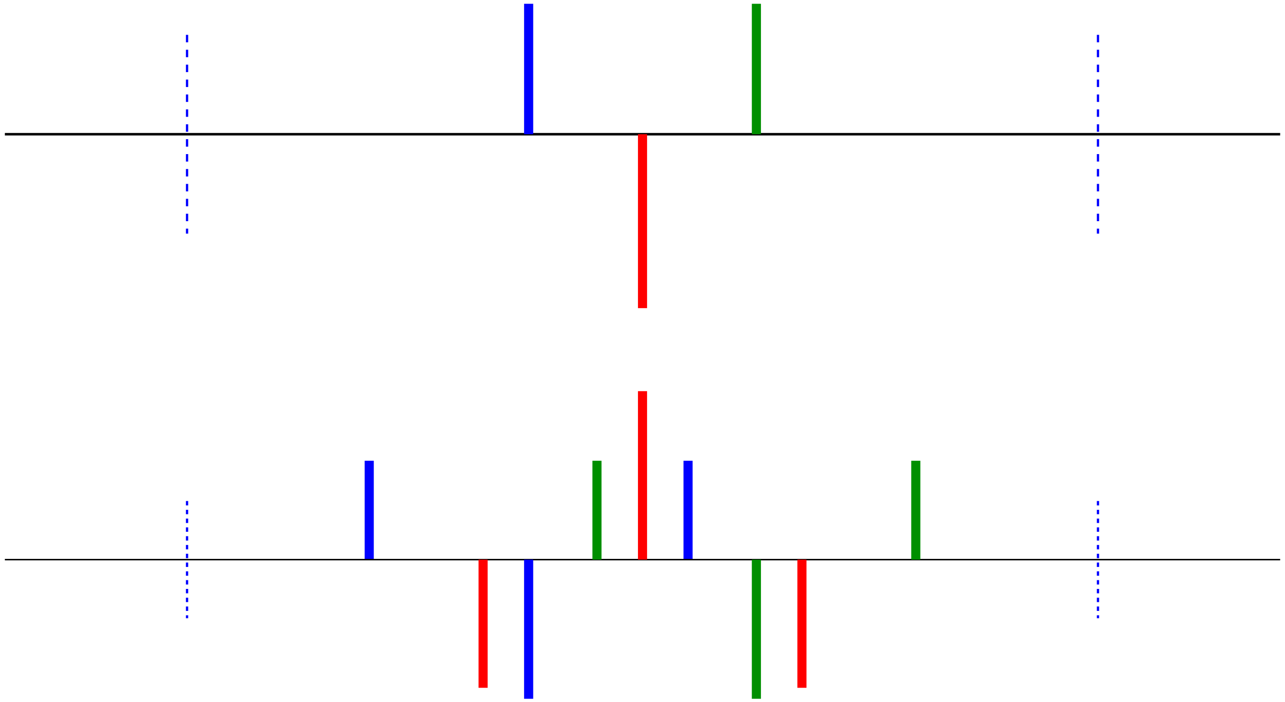
- Replace each <u>kick</u> by 4th order integrator

Integrator of order 4 $\Longrightarrow$ 6 - step by step



- Replace each <u>kick</u> by 4th order integrator

Integrator of order 4 $\Longrightarrow$ 6 - step by step



- Replace each kick by 4th order integrator, requires 9 kicks
- We have 3 interleaved 4th order integrators (compute $\mathcal{M}(\mathcal{O}6)$), repeat the procedure to go to higher orders

- We have used a linear map (quadrupole) to demonstrate the integration
- Can that be applied for other maps (solenoids, higher order, nonlinear maps) ?
  - Yes !!
  - One gets the same coefficients !
  - Proof and systematic (and easy) extension in the form of Lie operators (see tomorrow)

→ Without proof: best possible accuracy for thin lenses (be smart: a scheme with more thin lenses may be less precise, even if the position is as accurate as attometres !)

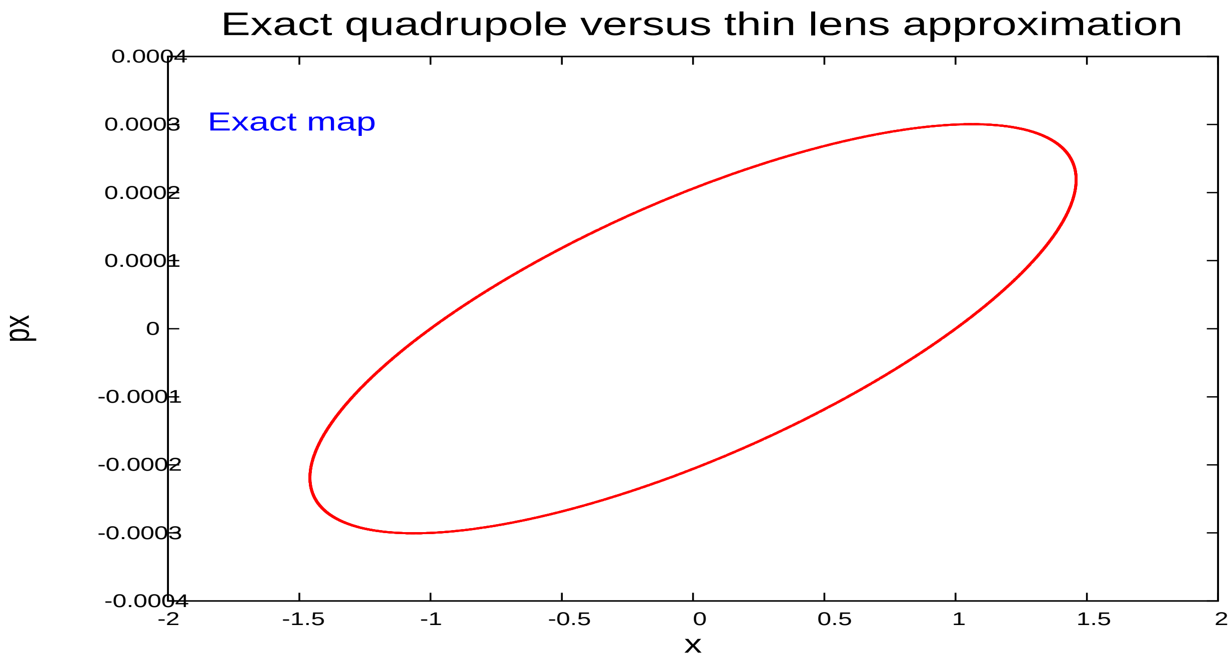
To remember:

Given a truncated Power map we construct a symplectic map whose lower order terms agree with the exact non-symplectic Power expansion and whose higher order (neglected) terms are small.

Key question:

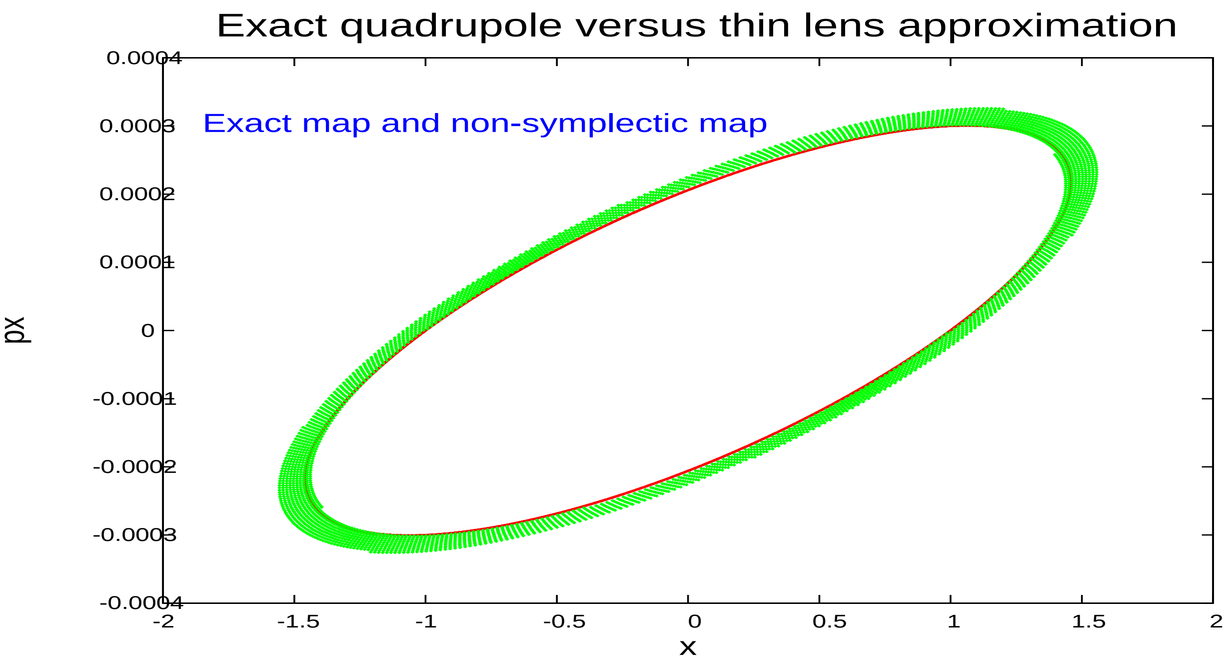
How can we say that the neglected terms do not exceed a tolerable limit ?

What is the point ???



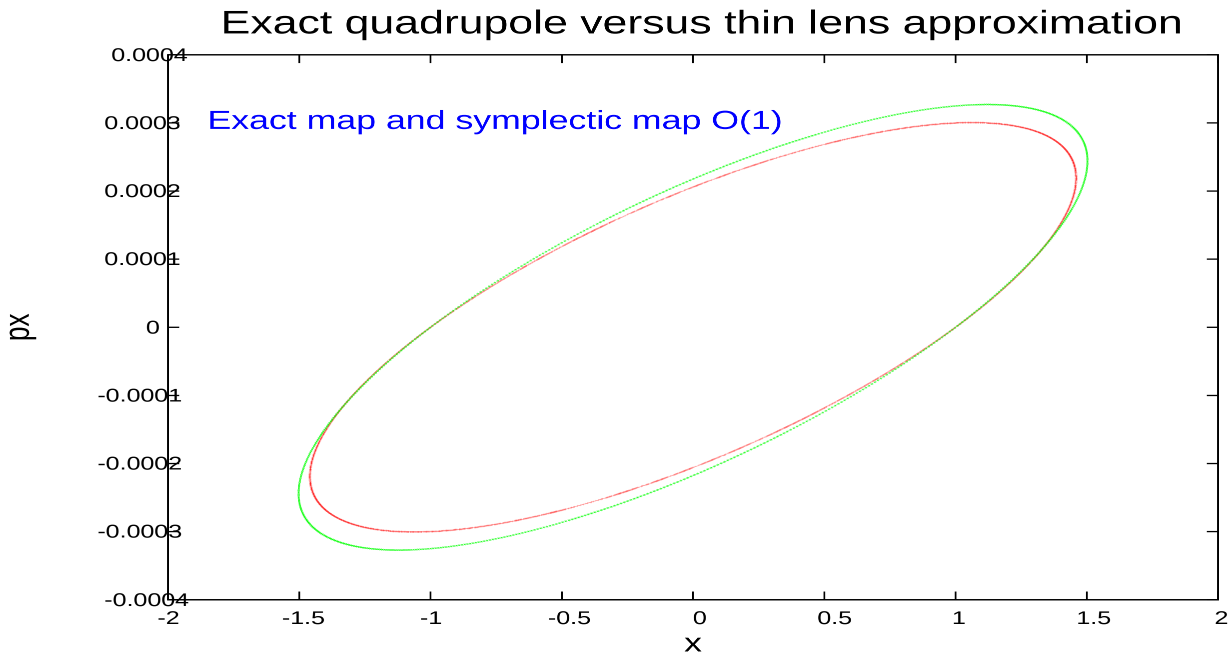
Phase space ellipse - quadrupole exact solution

Quadrupole non-symplectic solution L^1



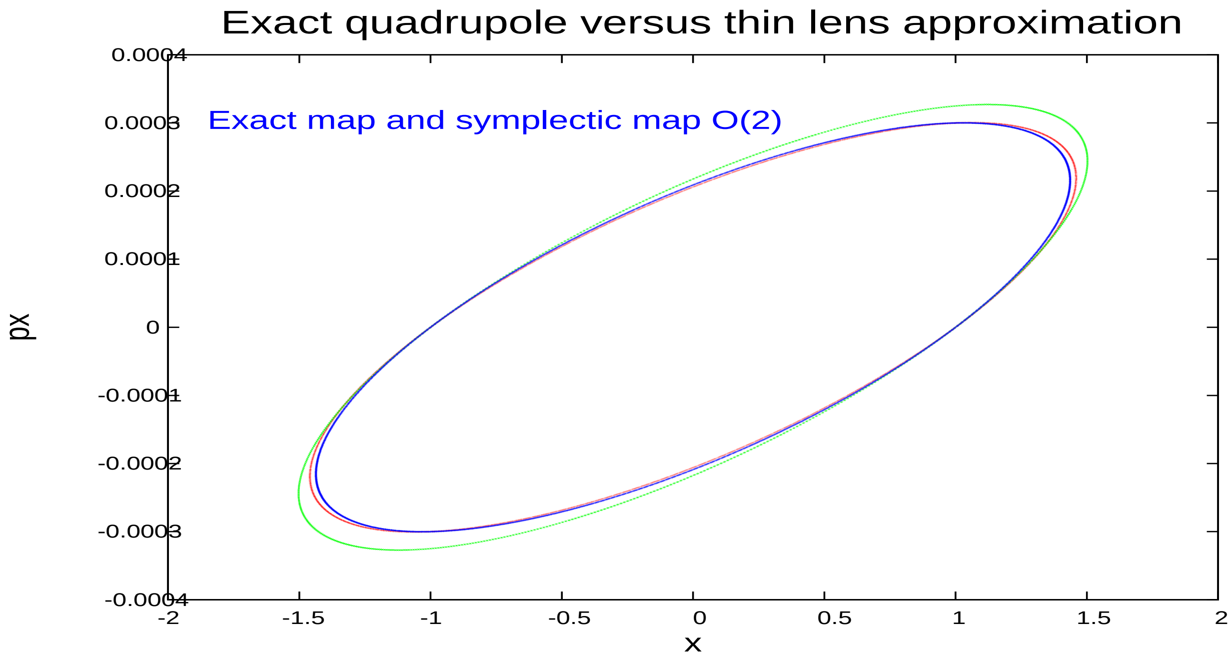
Non-symplecticity: particles spiral towards outside, artifact of approximation (of the algorithm)

Quadrupole symplectic $\mathcal{O}(L^1)$ solution



symplectic, solution order $\mathcal{O}(L^1)$, but visible inaccuracy

Quadrupole symplectic $\mathcal{O}(L^2)$ solution



symplectic, solution order $\mathcal{O}(L^2)$, but good accuracy

Quantitatively: Accuracy of (nonlinear) thin lenses

Nonlinear elements are usually thin (thinner than dipoles, quadrupoles ...)

- **Dipoles:** $\approx$ **14.3 m**
- **Quadrupole:** $\approx$ **2 - 5 m**
- **Sextupoles, Octupoles:** $\approx$ **0.30 m**
- **Decapole:** $\approx$ **0.07 m**

Assume a kick from a general function of x**:**

deflection : $\Delta x' = f(x)$

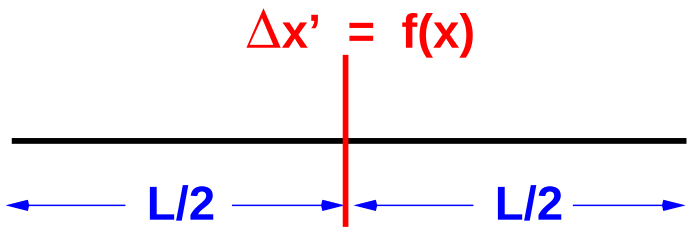
e.g. quadrupole $f(x) = k \cdot x^1$

e.g. sextupole $f(x) = k \cdot x^2$

e.g. octupole $f(x) = k \cdot x^3$

→ **Can try our simplest thin lens approximation O(2) first ...**

Drift - Kick - Drift



$$1.\,Step \quad \begin{pmatrix} x \\ x' \end{pmatrix}_{s_1+L/2} = \begin{pmatrix} 1 & \frac{L}{2} \\ 0 & 1 \end{pmatrix} \circ \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}_{s_1}$$

$$2.\,Step \quad \begin{pmatrix} x \\ x' \end{pmatrix}_{s_1+L/2} = \begin{pmatrix} x \\ x' + \Delta x' \end{pmatrix}_{s_1+L/2} = \begin{pmatrix} x \\ x' + f(x) \end{pmatrix}_{s_1+L/2}$$

$$3.\,Step \quad \begin{pmatrix} x \\ x' \end{pmatrix}_{s_1+L} = \begin{pmatrix} 1 & \frac{L}{2} \\ 0 & 1 \end{pmatrix} \circ \begin{pmatrix} x \\ x' \end{pmatrix}_{s_1+L/2}$$

Putting it together and written in explicit form:

$$\begin{pmatrix} x(L) \\ x'(L) \end{pmatrix} = \begin{pmatrix} x_0 + \frac{L}{2} \cdot (x_0' + x'(L)) \\ x_0' + L \cdot f(x_0 + \frac{L}{2} x_0') \end{pmatrix}$$

$$x(L) \approx x_0 + L \cdot x_0' \quad + \frac{L^2}{2} \cdot \underbrace{f(x_0 + \frac{L}{2} x_0')}$$

(using: $f(z + \Delta z) \approx f(z) + f'(z) \cdot \Delta z$ **for small** Δz**)**

$$\boxed{x(L) \approx x_0 + L \cdot x_0' \quad + \frac{L^2}{2} \cdot f(x_0) + \frac{L^3}{4} \cdot f'(x_0) x_0'}$$

It is symplectic !!

Comparison:

the (exact, but non-symplectic) Taylor expansion of $f(x)$ gives:

$$x(L) \quad = \quad x_0 + x_0' L + \frac{L^2}{2} f(x_0) + \frac{L^3}{6} f'(x_0) x_0' + ...$$

the (approximate, but symplectic) algorithm gives:

$$x(L) \quad = \quad x_0 + x_0' L + \frac{L^2}{2} f(x_0) + \frac{L^3}{4} f'(x_0) x_0' + ...$$

- **Errors are $\mathcal{O}(L^3)$ (is correct to $\mathcal{O}(L^2)$ by construction)**

  Errors are $\mathcal{O}(L^5)$ for the $\mathcal{O}(L^4)$ (3 kicks) scheme

- **For small L acceptable, and symplectic**

just for illustration : $\dfrac{L^3_{decapole}}{L^3_{dipole}} \approx 10^{-7}$

An application, a (1D) sextupole with:

$$f(x) \quad = \quad k \cdot x^2$$

using the thin lens approximation gives:

$$\begin{aligned} x(L) \quad &= \quad x_0 + x_0' L + \frac{1}{2} k x_0^2 L^2 + \overbrace{\frac{1}{2} k x_0 x_0' L^3}^{\frac{1}{3} k x_0 x_0' L^3} + \ldots \\ x'(L) \quad &= \quad x_0' + k x_0^2 L + k x_0 x_0' L^2 + \frac{1}{4} k x_0'^2 L^3 + \ldots \end{aligned}$$

Map for thick sextupole of length L in thin lens approximation, accurate to $\mathcal{O}(L^2)$

Short summary: thin lens computations

- **Are exactly symplectic**
- **Simulations based on thin lenses fast and efficient**

  successfully applied to large (storage) rings (e.g. SPS, Tevatron, LHC, LEP, ...)

But they do not represent an exact <u>model</u> of the accelerator

If used blindly: .. an exact solution to a wrong problem

For (large) accelerators the thin lenses are usually a good approximation and tool (because we do not have to go to very high-order integrators to get proper results).

An extension and more accurate treatment in particular for non-linear elements in terms of Hamiltonian dynamics