# *Field Solvers*

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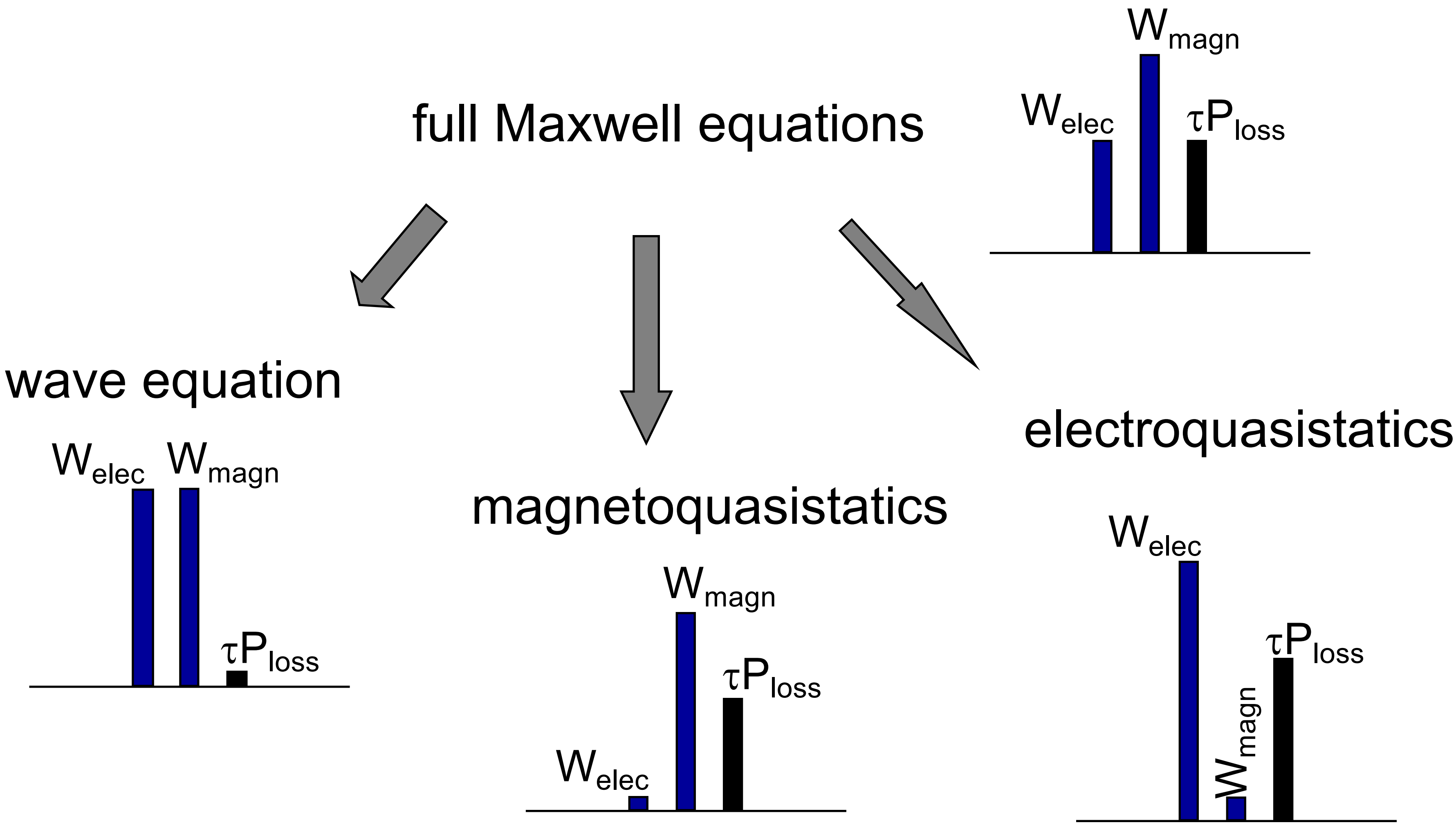
## Lecture 2 : Magnetoquasistatic Formulation and Discretisation

# Overview

- magnetoquasistatic formulation
- discretisation in space
- boundary and symmetry conditions
- reduction to 2D models
- modelling of coils and permanent magnets

# EM Field Simulation



# Magnetoquasistatics (1)

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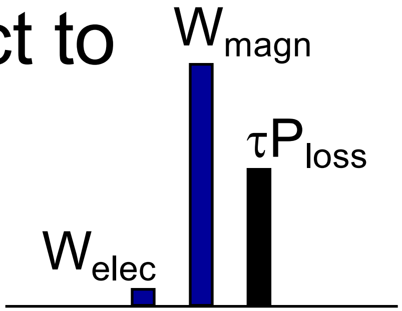
- neglect displacement currents with respect to conducting currents
  - Ampère-Maxwell

$$\nabla\times\vec{H} = \vec{J} + \cancel{\frac{\partial\vec{D}}{\partial t}}$$




- magnetic vector potential $\vec{A}$
  – conservation of magnetic flux

$$\nabla\cdot\vec{B} = 0 \quad \Rightarrow \quad \vec{B} = 0 + \nabla\times\vec{A}$$

- electric scalar potential $\varphi$ (voltage)
  –Faraday-Lenz

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} = -\nabla\times\frac{\partial\vec{A}}{\partial t} \quad \Rightarrow \quad \vec{E} = -\frac{\partial\vec{A}}{\partial t} - \nabla\varphi$$

# Magnetoquasistatics (2)

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## Ampère

$$\nabla \times \vec{H} = \vec{J}$$

$$\vec{B} = \mu \vec{H} = \frac{1}{\nu} \vec{H}$$

permeability

reluctivity

$$\vec{J} = \sigma \vec{E}$$

conductivity

⬇

$$\nabla \times \left( \nu \vec{B} \right) = \sigma \vec{E}$$

⬇

$$\nabla \times \left( \nu \nabla \times \vec{A} \right) + \sigma \frac{\partial \vec{A}}{\partial t} = \underbrace{-\sigma \nabla \varphi}_{\vec{J}_s}$$

source current density

***parabolic*** **partial differential equation**

**↔ elliptic PDEs (e.g. electrostatics, magnetostatics)**

**↔ hyperbolic PDEs (e.g. wave equation)**

## Physical Meaning (1)

**flux**

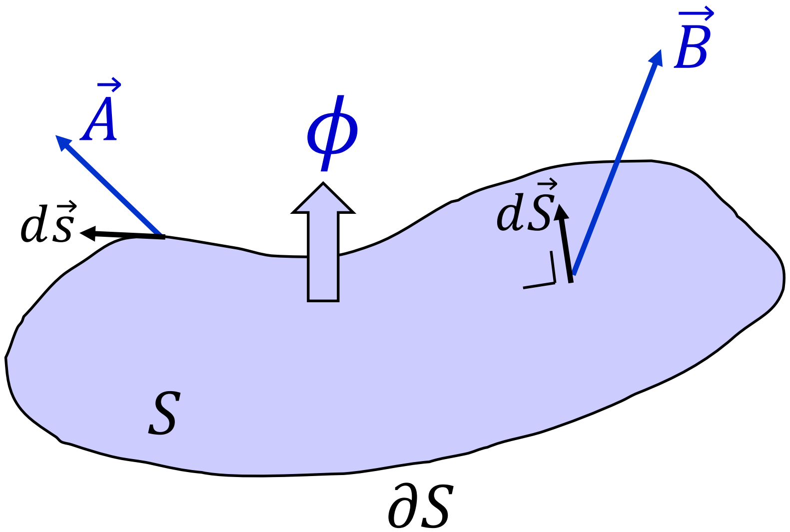
$$\phi = \int_S \vec{B} \cdot d\vec{S}$$

definition magnetic vector potential

$$\phi = \int_S \nabla \times \vec{A} \cdot d\vec{S}$$

Stokes

$$\phi = \oint_{\partial S} \vec{A} \cdot d\vec{s}$$




**induced voltage**

$$u_{\mathrm{ind}} = -\frac{d}{dt} \oint_{\partial S} \vec{A} \cdot d\vec{s}$$

# Physical Meaning (2)

**flux**

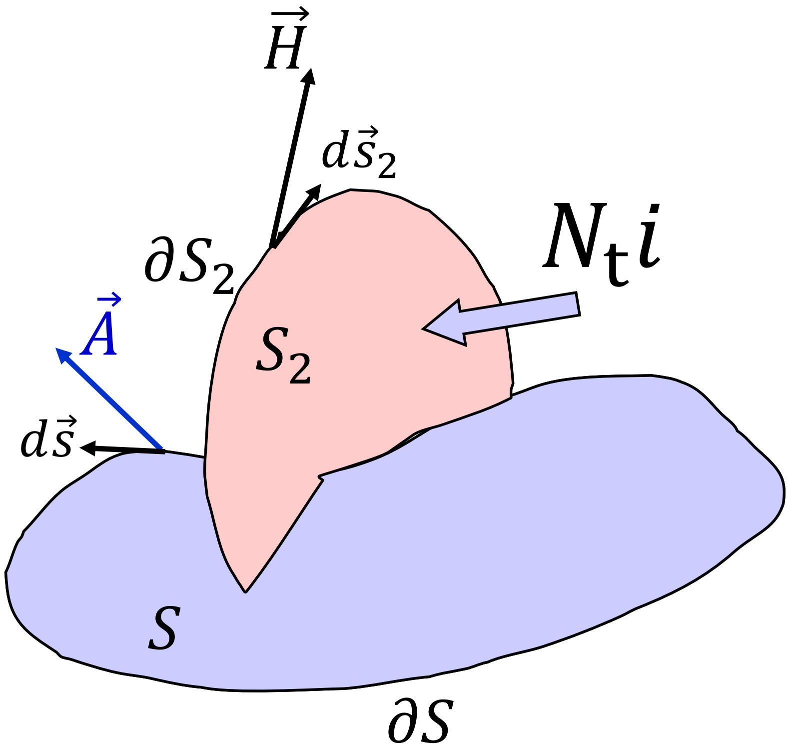
$$\phi = \oint_{\partial S} \vec{A} \cdot d\vec{s}$$

**induced voltage**

$$u_{\text{ind}} = -\frac{d}{dt}\oint_{\partial S} \vec{A} \cdot d\vec{s}$$

**Ampère**

$$N_{\text{t}} i = \oint_{\partial S_2} \vec{H} \cdot d\vec{s}_2$$




make (a lot of) drawings (also if they look stupid)

# Overview

- magnetoquasistatic formulation
- discretisation in space
- boundary and symmetry conditions
- reduction to 2D models
- modelling of coils and permanent magnets

# Spatial Discretisation (1)

- weighted residual approach

$$\nabla\times\left(\nu\nabla\times\vec{A}\right)+\sigma\frac{\partial\vec{A}}{\partial t}=\vec{J}_s \qquad \text{in} \quad \Omega$$

$$\int_V\left(\nabla\times\left(\nu\nabla\times\vec{A}\right)+\sigma\frac{\partial\vec{A}}{\partial t}\right)\cdot\vec{w}_i\,\mathrm{d}V=\int_V\vec{J}_s\cdot\vec{w}_i\,\mathrm{d}V \qquad \forall\,\vec{w}_i(x,y,z)$$

$\vec{w}_i(x,y,z)$ vectorial „weighting functions“
vectorial „test functions“

- scalar product : $\left(\vec{\alpha},\vec{\beta}\right)=\int_V\vec{\alpha}\cdot\vec{\beta}\,\mathrm{d}V$

# Spatial Discretisation (2)

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$\rightarrow$ weak formulation

do not study derivations by heart,
know how to get from one to the next line!

$$\int_V \left( \nabla \times \left( \nu \nabla \times \vec{A} \right) + \sigma \frac{\partial \vec{A}}{\partial t} \right) \cdot \vec{w}_i \, \mathrm{d}V = \int_V \vec{J}_s \cdot \vec{w}_i \, \mathrm{d}V \qquad \forall \vec{w}_i(x,y,z)$$

$$(\nabla \times \vec{v}) \cdot \vec{w} = \nabla \cdot (\vec{v} \times \vec{w}) + \vec{v} \cdot \nabla \times \vec{w}$$

$$\int_V \left( \nabla \cdot \left( \nu \nabla \times \vec{A} \times \vec{w}_i \right) + \nu \nabla \times \vec{A} \cdot \nabla \times \vec{w}_i + \sigma \frac{\partial \vec{A}}{\partial t} \cdot \vec{w}_i \right) \mathrm{d}V = \int_V \vec{J}_s \cdot \vec{w}_i \, \mathrm{d}V$$

Gauss

$$\oint_{\partial V} \underbrace{\nu \nabla \times \vec{A}}_{\vec{H}} \times \vec{w}_i \cdot \mathrm{d}\vec{S} + \int_V \left( \underbrace{\nu \nabla \times \vec{A} \cdot \nabla \times \vec{w}_i}_{} + \sigma \frac{\partial \vec{A}}{\partial t} \cdot \vec{w}_i \right) \mathrm{d}V = \int_V \vec{J}_s \cdot \vec{w}_i \, \mathrm{d}V$$

only first derivative required $\Longrightarrow$ „weak" formulation

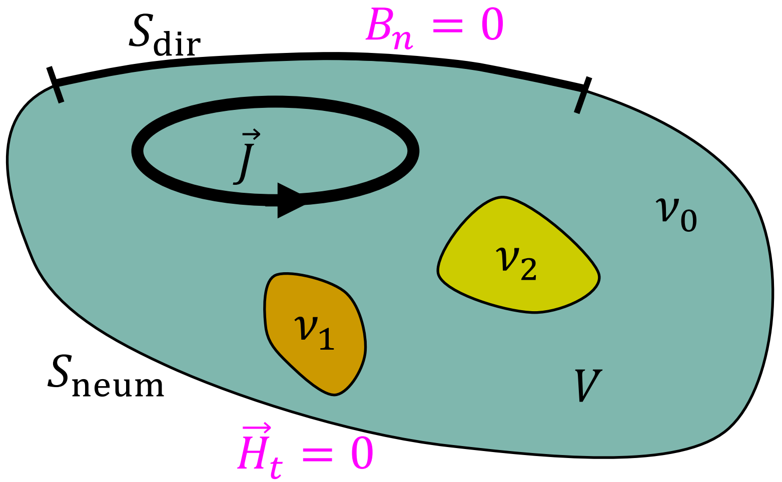
# Spatial Discretisation (3)

Dirichlet BC at $S_{\mathrm{dir}}$

$$\vec{A}\times\vec{n} = \vec{A}_{\mathrm{dir}}\times\vec{n} \;\Leftrightarrow\; \vec{B}\cdot\vec{n} = B_n$$

homogeneous Neumann BC at $S_{\mathrm{neum}}$

$$\vec{H}_t = (\nu\nabla\times\vec{A})\times\vec{n} = 0$$




$$\underbrace{\int_{S_{\mathrm{neum}}} \overbrace{(\nu\nabla\times\vec{A})}^{\vec{H}}\times\vec{w}_i\cdot\mathrm{d}\vec{S}}_{=0} + \underbrace{\int_{S_{\mathrm{dir}}} (\nu\nabla\times\vec{A})\times\vec{w}_i\cdot\mathrm{d}\vec{S}}_{=0} \qquad \forall\, \vec{w}_i(x,y,z)$$

$\forall\, \vec{w}_i : \vec{w}_i\times\vec{n} = 0$ at $S_{\mathrm{dir}}$

„natural" boundary condition

„essential" boundary condition

# Spatial Discretisation (4)

- discretisation

$$\vec{A}(x,y,z) = \sum_j u_j \, \vec{v}_j(x,y,z) \qquad \vec{v}_j(x,y,z) \times \vec{n} = 0 \ \text{ at } S_{\mathrm{dir}}$$

$$\begin{cases} \vec{v}_j(x,y,z) & \text{„shape/form functions", „trial functions"} \\ u_j & \text{unknowns, degrees of freedom} \end{cases}$$

- Ritz-Galerkin method

$$\vec{v}_j(x,y,z) = \vec{w}_j(x,y,z)$$

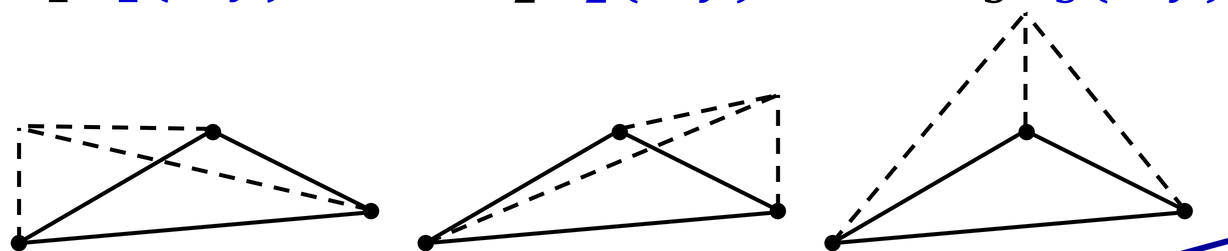
- Petrov-Galerkin method

$$\vec{v}_j(x,y,z) \neq \vec{w}_j(x,y,z)$$

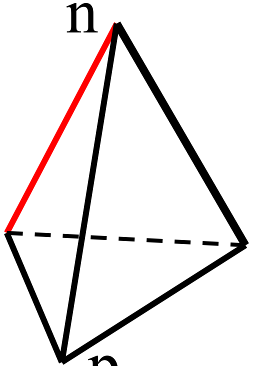
# Spatial Discretisation (5)

- nodal shape functions

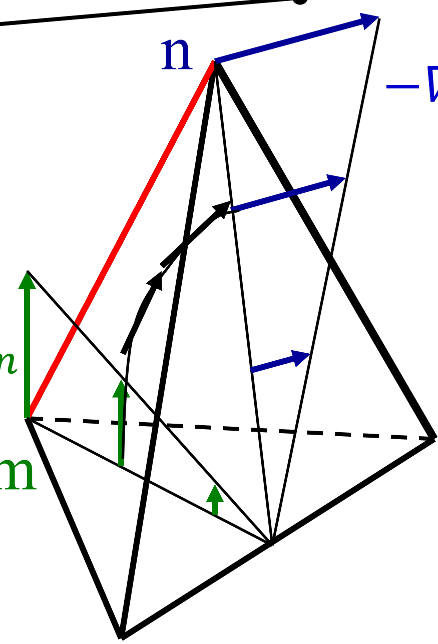
$$\varphi(x,y) = \quad u_1 N_1(x,y) \quad + u_2 N_2(x,y) \quad + u_3 N_3(x,y)$$



- edge shape functions




$$\vec{v}(\vec{x}) = N_m \nabla N_n - N_n \nabla N_m$$



# Spatial Discretisation (6)

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- discretization

$$\int_V \left( \nu \nabla \times \vec{A} \cdot \nabla \times \vec{w}_i + \sigma \frac{\partial \vec{A}}{\partial t} \cdot \vec{w}_i \right) \mathrm{d}V = \int_V \vec{J}_s \cdot \vec{w}_i \, \mathrm{d}V \qquad \forall\, \vec{w}_i(x,y,z)$$

$$\vec{A}(x,y,z) = \sum_j u_j \vec{w}_j(x,y,z)$$

curlcurl_l1.m

edgemass_l1.m

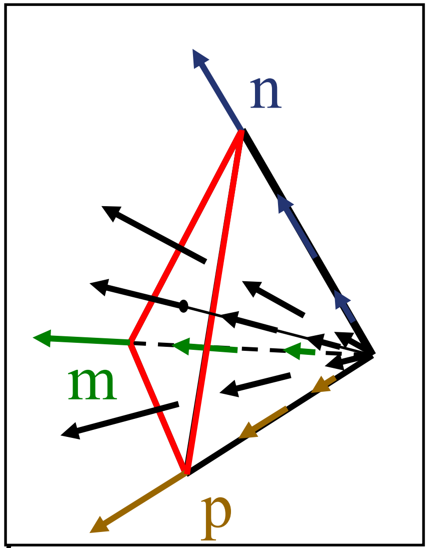
$$\sum_j \left( u_j \underbrace{\int_V \nu \nabla \times \vec{w}_j \cdot \nabla \times \vec{w}_i \, \mathrm{d}V}_{= k_{ij}} + \frac{\mathrm{d}u_j}{\mathrm{d}t} \underbrace{\int_V \sigma \vec{w}_j \cdot \vec{w}_i \, \mathrm{d}V}_{= m_{ij}} \right) = \underbrace{\int_V \vec{J}_s \cdot \vec{w}_i \, \mathrm{d}V}_{= f_i}$$

$$[k_{ij}][u_j] + [m_{ij}]\left[\frac{\mathrm{d}u_j}{\mathrm{d}t}\right] = [f_i]$$

K and M symmetric, semi-positive-definite

# Spatial Discretisation (7)

$$\sum_j \left( u_j \int_V \nu \nabla\times\vec{w}_j \cdot \nabla\times\vec{w}_i \,\mathrm{d}V + \frac{\mathrm{d}u_j}{\mathrm{d}t} \int_V \sigma \vec{w}_j \cdot \vec{w}_i \,\mathrm{d}V \right)$$

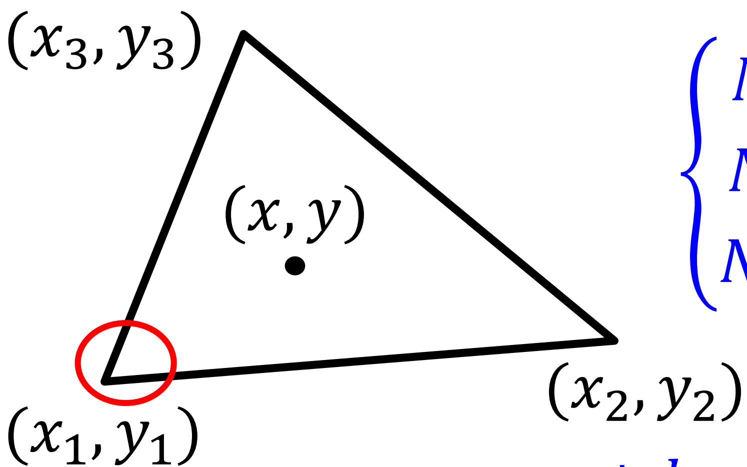



$$\nabla\times\vec{w}_j = \sum_q c_{jq} \vec{z}_q$$

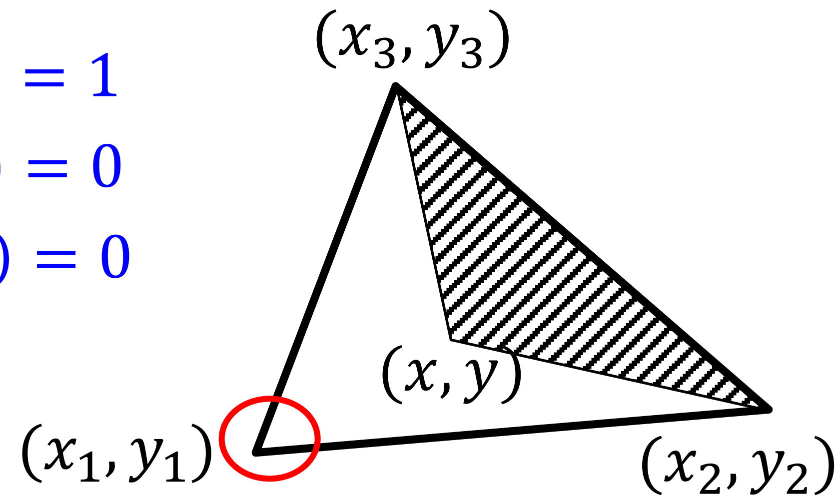
$$\sum_j \left( \underbrace{u_j}_{\hat{\mathbf{a}}_j} \sum_p \sum_q c_{ip} c_{jq} \underbrace{\int_V \nu\, \vec{z}_q \cdot \vec{z}_p \,\mathrm{d}V}_{\mathbf{M}^{\mathrm{FE}}_{\nu,p,q}} + \frac{\mathrm{d}u_j}{\mathrm{d}t} \underbrace{\int_V \sigma \vec{w}_j \cdot \vec{w}_i \,\mathrm{d}V}_{\mathbf{M}^{\mathrm{FE}}_{\sigma,i,j}} \right) = \underbrace{\int_V}_{\hat{\hat{\mathbf{j}}}_{s,i}}$$

$$\tilde{\mathbf{C}} \mathbf{M}^{\mathrm{FE}}_{\nu} \mathbf{C} \hat{\mathbf{a}} + \mathbf{M}^{\mathrm{FE}}_{\sigma} \frac{\mathrm{d}\hat{\mathbf{a}}}{\mathrm{d}t} = \hat{\hat{\mathbf{j}}}_{\mathrm{s}}$$

# 2D Nodal Shape Functions




$$\begin{cases} N_i(x_i, y_i) = 1 \\ N_i(x_j, y_j) = 0 \\ N_i(x_k, y_k) = 0 \end{cases}$$




$$N_i(x, y) = \frac{a_i + b_i x + c_i y}{2A_{ijk}}$$

$$\begin{cases} a_i = x_j y_k - x_k y_j \\ b_i = y_j - y_k \\ c_i = x_k - x_j \\ A_{ijk} = \text{element area} \end{cases}$$

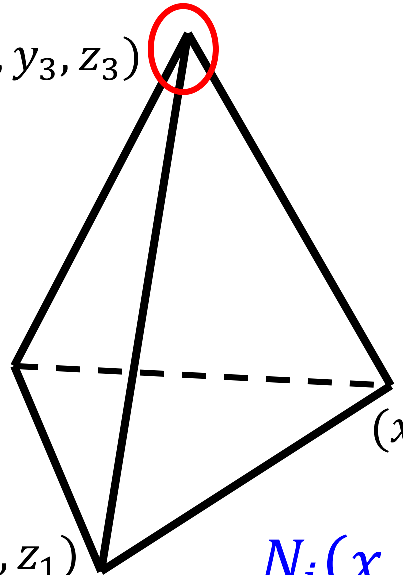
surface coordinates

$$N_i(x, y) = \frac{A_{pjk}}{A_{ijk}}$$

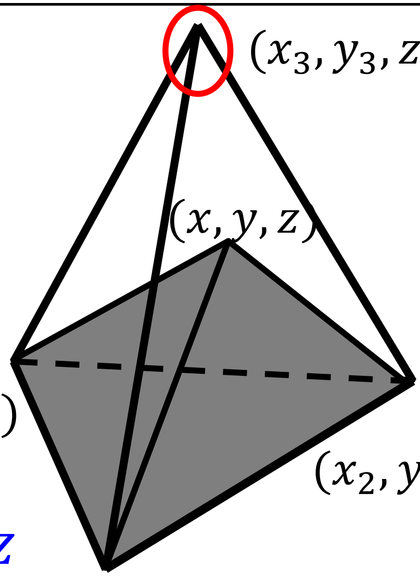
partition of unity

$$\sum_{i=1}^{3} N_i(x, y) = 1, \qquad \forall (x, y)$$

# 3D Nodal Shape Functions




$$\begin{cases} N_i(x_i, y_i, z_i) = 1 \\ N_i(x_j, y_j, z_j) = 0 \\ N_i(x_k, y_k, z_k) = 0 \\ N_i(x_\ell, y_\ell, z_\ell) = 0 \end{cases}$$




$$N_i(x, y, z) = \frac{a_i + b_i x + c_i y + d_i z}{6V_{ijk\ell}}$$

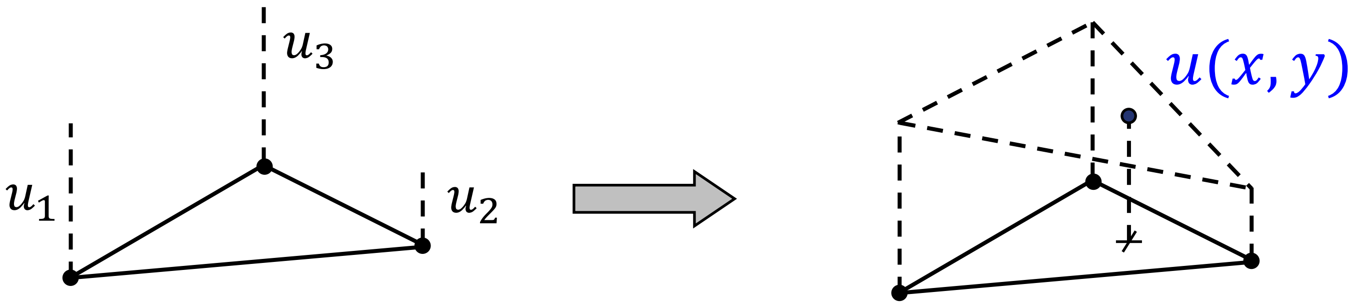
$$\begin{cases} a_i = \\ b_i = \\ c_i = \\ d_i = \\ V_{ijk\ell} = \text{element volume} \end{cases}$$

volume coordinates $N_i(x, y, z) = \dfrac{V_{pjk\ell}}{V_{ijk\ell}}$

partition of unity $\displaystyle\sum_{i=1}^{4} N_i(x, y, z) = 1, \forall (x, y, z)$

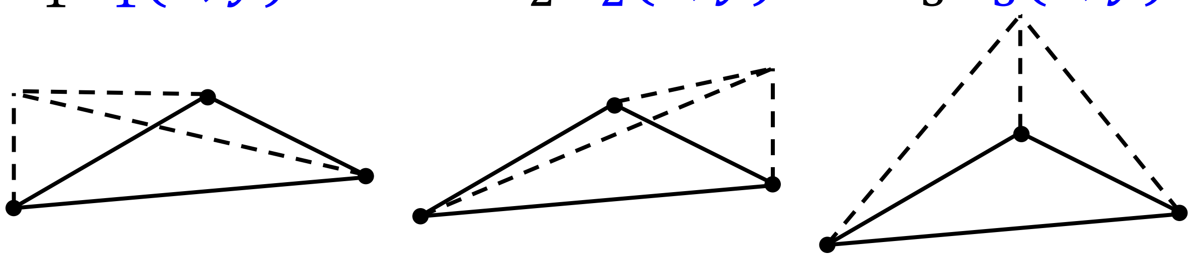
# Dofs or Nodal Values ?

- solution $\varphi(x,y,z) = \varphi_{\mathrm{dir}}(x,y,z) + \sum_j u_j N_j(x,y,z)$
  - interpolation :
    nodal values & interpolating functions




  - series development :
    degrees of freedom & shape functions

$$u_1, u_2, u_3 \quad \Rightarrow \quad u_1 N_1(x,y) \quad + u_2 N_2(x,y) \quad + u_3 N_3(x,y)$$

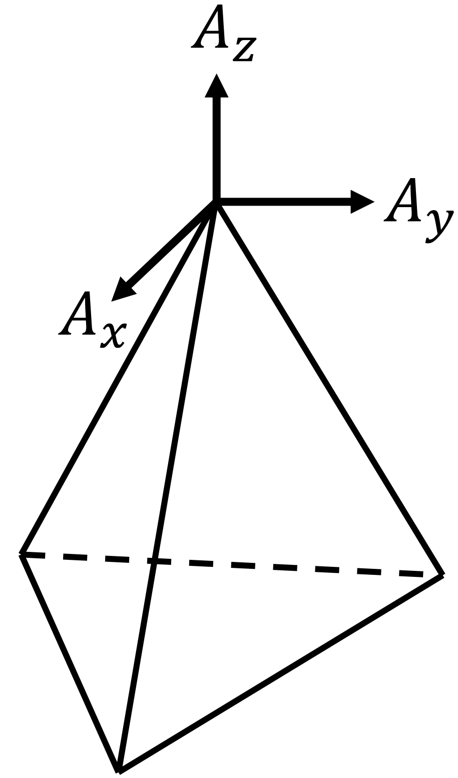


# Vectorial Shape Functions

- vector field $\vec{A} = A_x \vec{e}_x + A_y \vec{e}_y + A_z \vec{e}_z$
  - interpolate each component separately by *scalar* shape functions

$$\vec{A} = \left(\sum_j u_{x,j} N_j\right) \vec{e}_x + \left(\sum_j u_{y,j} N_j\right) \vec{e}_y + \left(\sum_j u_{z,j} N_j\right) \vec{e}_z$$

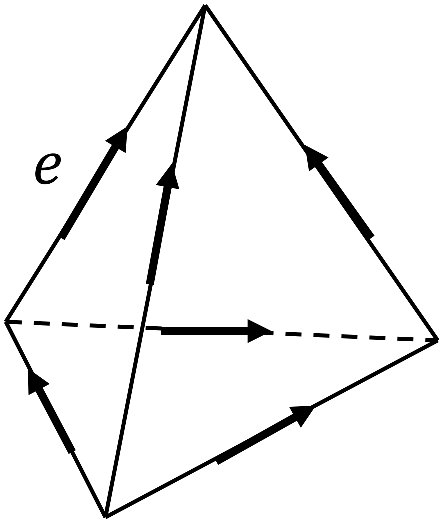
  - BUT
    - too much continuity (both normal and tangential components)
    - spurious modes



# Edge Elements

- shape functions defined by

$$\begin{cases} \int_e \vec{w}_e \cdot d\vec{s} = 1 \\ \int_{e'} \vec{w}_e \cdot d\vec{s} = 0, e \neq e' \end{cases}$$




–linear combination $\vec{E}(\vec{x}) = \sum_{j=1}^{E} u_j \vec{w}_{e_j}(\vec{x})$

- features tangential continuity (1-form)

–physical meaning of the degrees of freedom

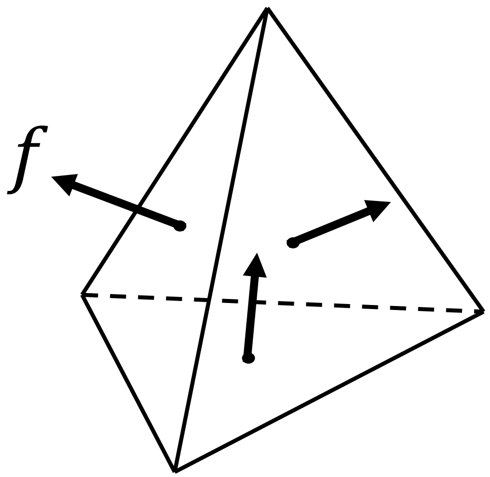
$$\int_e \vec{E} \cdot d\vec{s} = \sum_{j=1}^{E} u_j \int_e \vec{w}_{e_j} \cdot d\vec{s} = u_e \int_e \vec{w}_{e_j} \cdot d\vec{s} = u_e$$

- voltage drop along the edge

# Face(t) Elements

- shape functions defined by

$$\begin{cases} \int_f \vec{w}_f \cdot d\vec{A} = 1 \\ \int_{f'} \vec{w}_f \cdot d\vec{A} = 0, f' \neq f \end{cases}$$




– linear combination $\vec{B}(\vec{x}) = \sum_{j=1}^{F} u_j \, \vec{w}_{f_j}(\vec{x})$

  - features normal continuity (2-form)

– physical meaning of the degrees of freedom

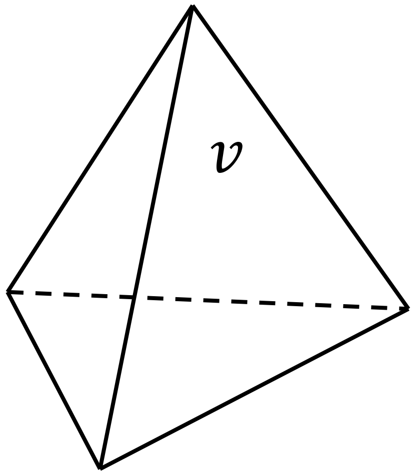
$$\int_f \vec{B} \cdot d\vec{A} = \sum_{j=1}^{F} u_j \int_f \vec{w}_{f_j} \cdot d\vec{A} = u_f \int_f \vec{w}_{f_j} \cdot d\vec{A} = u_f$$

  - flux through the facet

# Volume Elements

- shape functions defined by

$$\begin{cases} \int_v w_v dV = 1 \\ \int_{v'} w_v dV = 0, v' \neq v \end{cases}$$




–linear combination $$q(\vec{x}) = \sum_{j=1}^{V} u_j \, w_{v_j}(\vec{x})$$

- commonly discontinuous (3-form)

–physical meaning of the degrees of freedom

$$\int_v \rho dV = \sum_{j=1}^{V} u_j \int_v w_{v_j} dV = u_v \int_v w_{v_j} dV = u_v$$

- charge within the element

# Canonical Construction

- construction
  - nodal shape functions

$$N_m, N_n, N_p, N_q$$

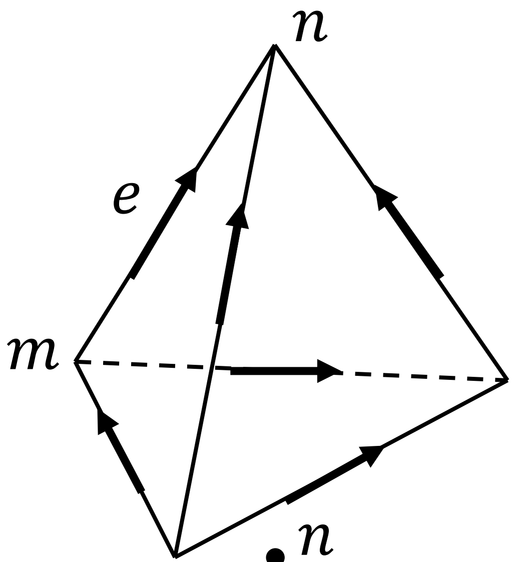
  - edge elements

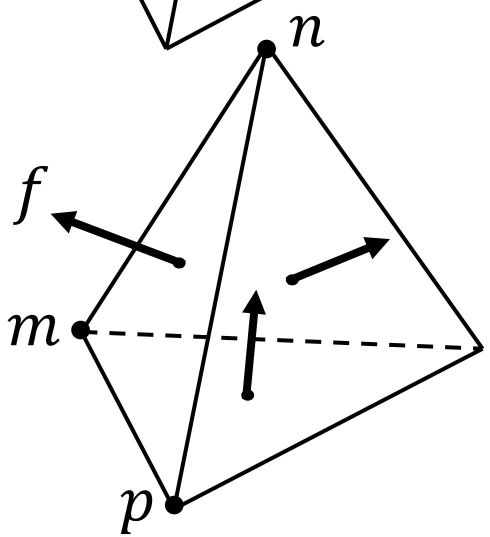
$$\vec{w}_e(\vec{x}) = N_m \nabla N_n - N_n \nabla N_m$$

  - facet elements

$$\vec{w}_f(\vec{x}) = 2(N_m \nabla N_n \times \nabla N_p + N_n \nabla N_p \times \nabla N_m + N_p \nabla N_m \times \nabla N_n)$$

  - volume elements $\quad w_v(\vec{x}) = \dfrac{1}{V}$



# Whitney Complex

- Whitney elements (for a given mesh)
  - $W^0$: nodal elements
  - $W^1$: edge elements
  - $W^2$: facet elements
  - $W^3$: volume elements
- Whitney complex

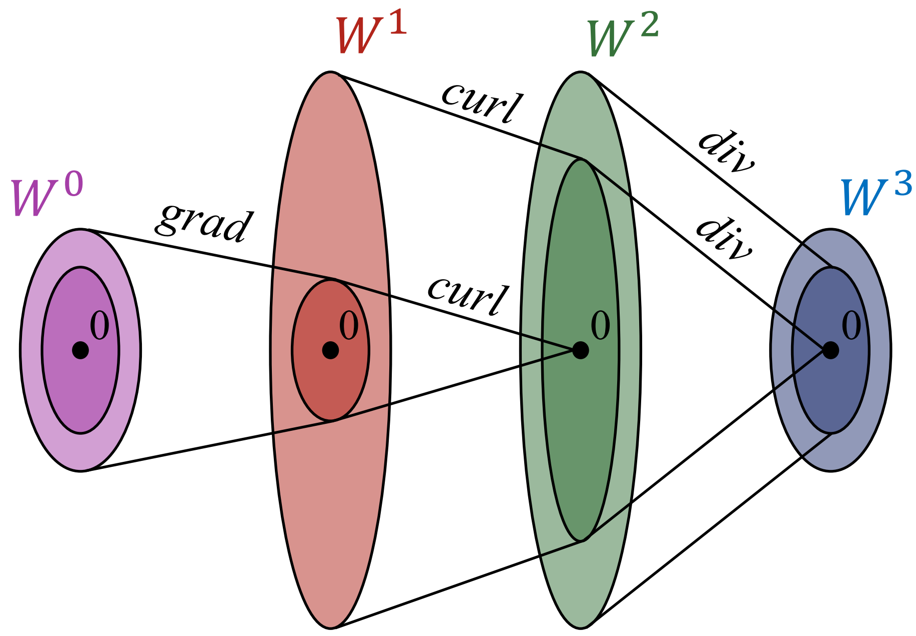
$$\operatorname{grad} W^0 \subset W^1$$

$$\operatorname{curl} W^1 \subset W^2$$

$$\operatorname{div} W^2 \subset W^3$$

$$\operatorname{curl} \operatorname{grad} W^0 = 0$$

$$\operatorname{div} \operatorname{curl} W^1 = 0$$



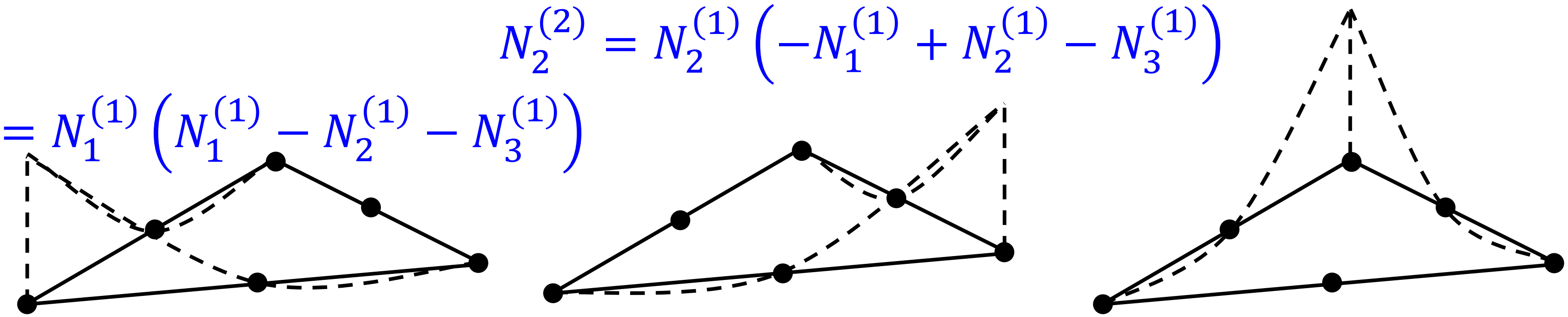
# 2D Higher-Order Elements

- nodal elements

$$N_3^{(2)} = N_3^{(1)}\left(-N_1^{(1)} - N_2^{(1)} + N_3^{(1)}\right)$$

$$N_2^{(2)} = N_2^{(1)}\left(-N_1^{(1)} + N_2^{(1)} - N_3^{(1)}\right)$$

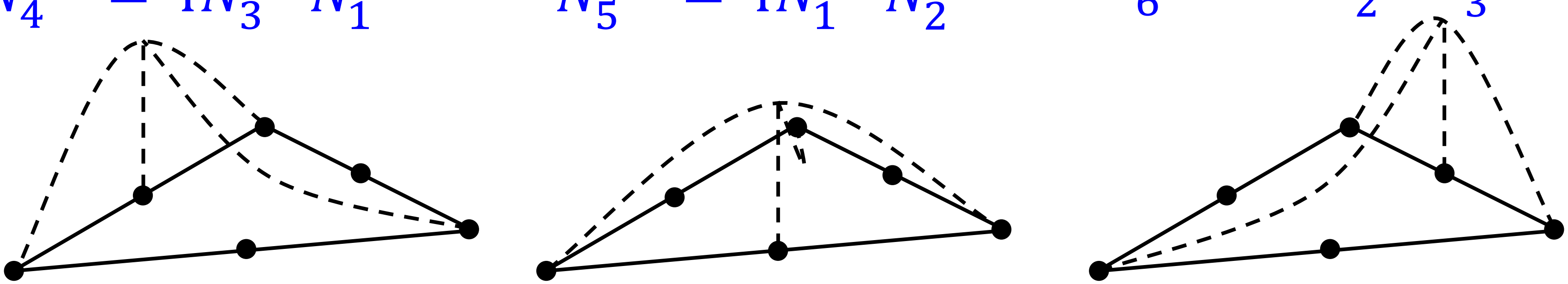
$$N_1^{(2)} = N_1^{(1)}\left(N_1^{(1)} - N_2^{(1)} - N_3^{(1)}\right)$$



$$N_4^{(2)} = 4N_3^{(1)}N_1^{(1)}$$

$$N_5^{(2)} = 4N_1^{(1)}N_2^{(1)}$$

$$N_6^{(2)} = 4N_2^{(1)}N_3^{(1)}$$



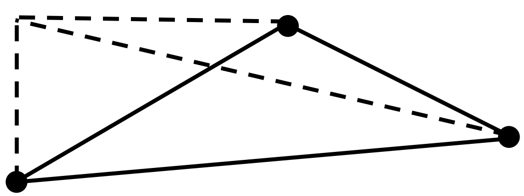
degrees of freedom == nodal values

# Hierarchical Finite Elements

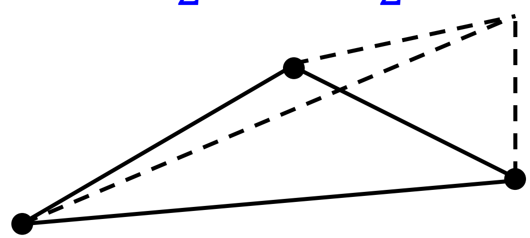
- nodal elements

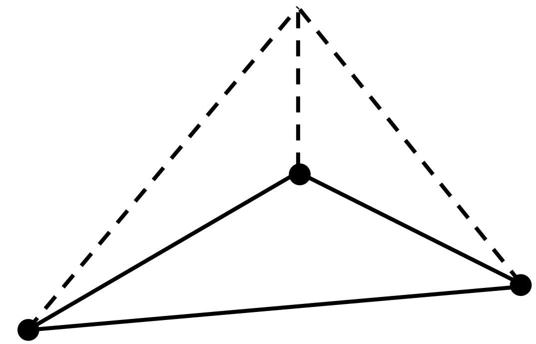
$$N_3^{(2)} = N_3^{(1)}$$

$$N_1^{(2)} = N_1^{(1)}$$

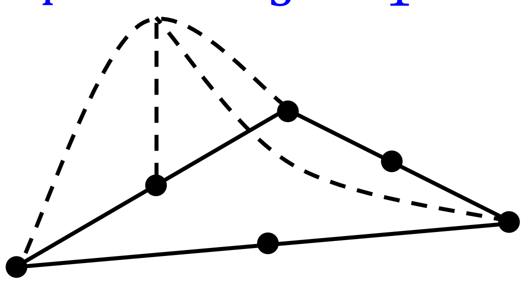


$$N_2^{(2)} = N_2^{(1)}$$

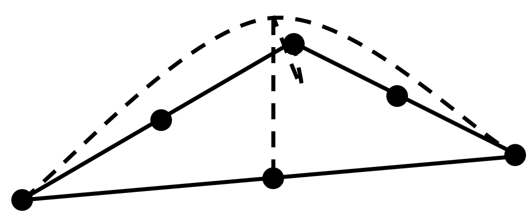




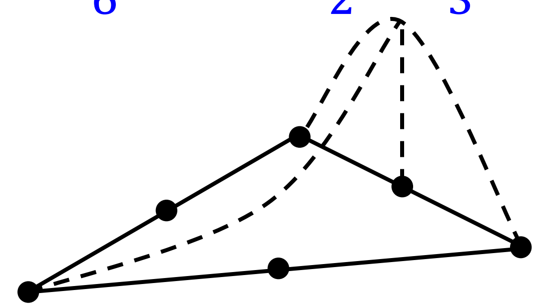
$$N_4^{(2)} = 4N_3^{(1)}N_1^{(1)}$$



$$N_5^{(2)} = 4N_1^{(1)}N_2^{(1)}$$



$$N_6^{(2)} = 4N_2^{(1)}N_3^{(1)}$$



degrees of freedom $\neq$ nodal values

# Hierarchical Finite Elements

- first-order system of equations

$$\left[K^{(11)}\right]\left[u^{(1)}\right] = \left[f^{(1)}\right]$$

- second-order system of equations

$$\begin{bmatrix} K^{(11)} & K^{(12)} \\ K^{(21)} & K^{(22)} \end{bmatrix} \begin{bmatrix} u^{(1)} \\ u^{(2)} \end{bmatrix} = \begin{bmatrix} f^{(1)} \\ f^{(2)} \end{bmatrix}$$

first-order system embedded in second-order system

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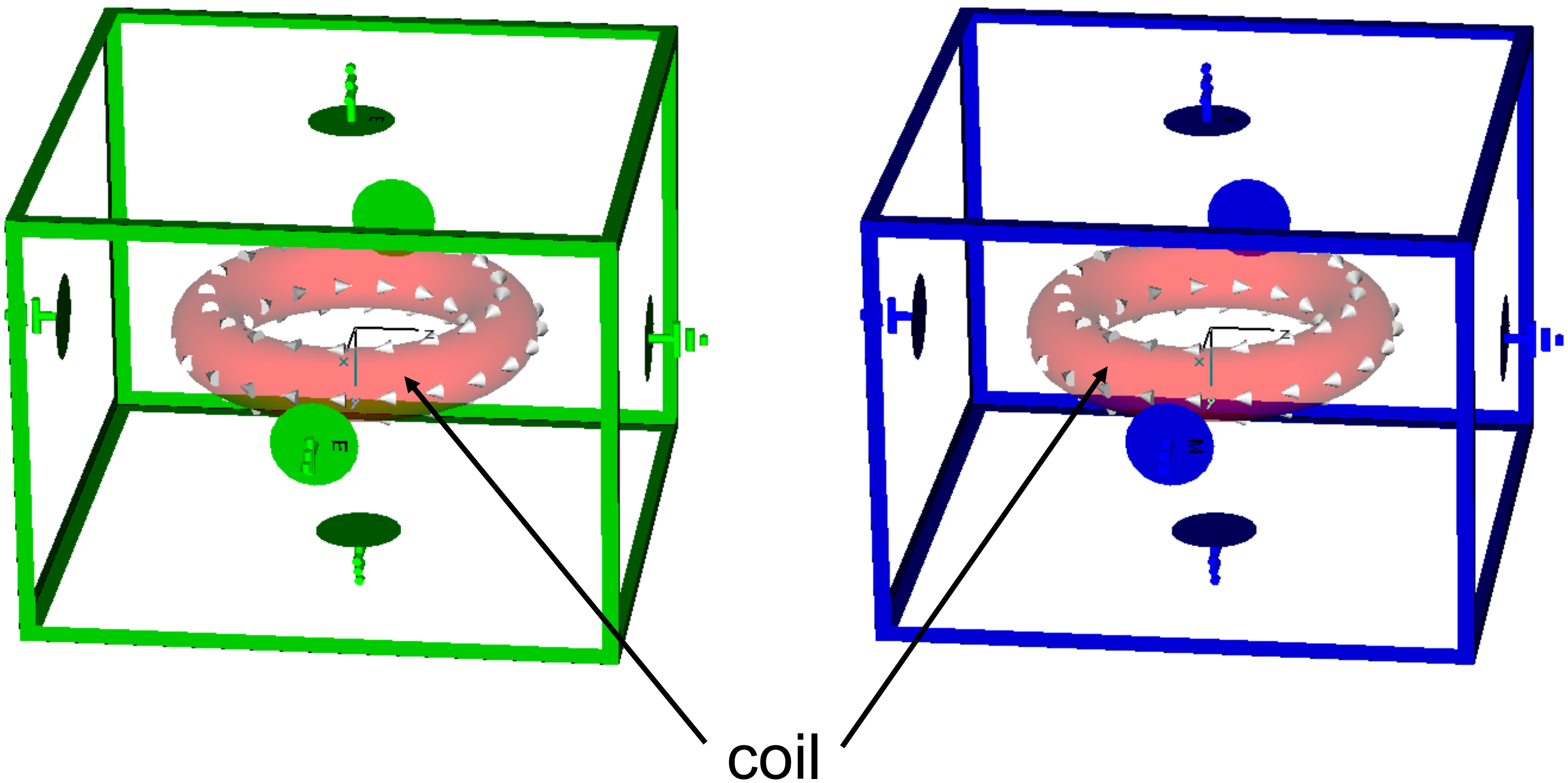
# Boundary Conditions (1)

| | electric BC<br>„flux wall“<br>„current gate“ | magnetic BC<br>„flux gate“<br>„current wall“ |
|---|---|---|
| definition | $\vec{E}_t = 0$ | $\vec{H}_t = 0$ |
| electric current | $\vec{J}_n \neq 0$ | $\vec{J}_n = 0$ |
| magnetic flux | $\vec{B}_n = 0$ | $\vec{B}_n \neq 0$ |
| magnetic vector potential formulation | Dirichlet BC | Neumann BC |
| magnetic scalar potential formulation | Neumann BC | Dirichlet BC |

# Boundary Conditions (2)

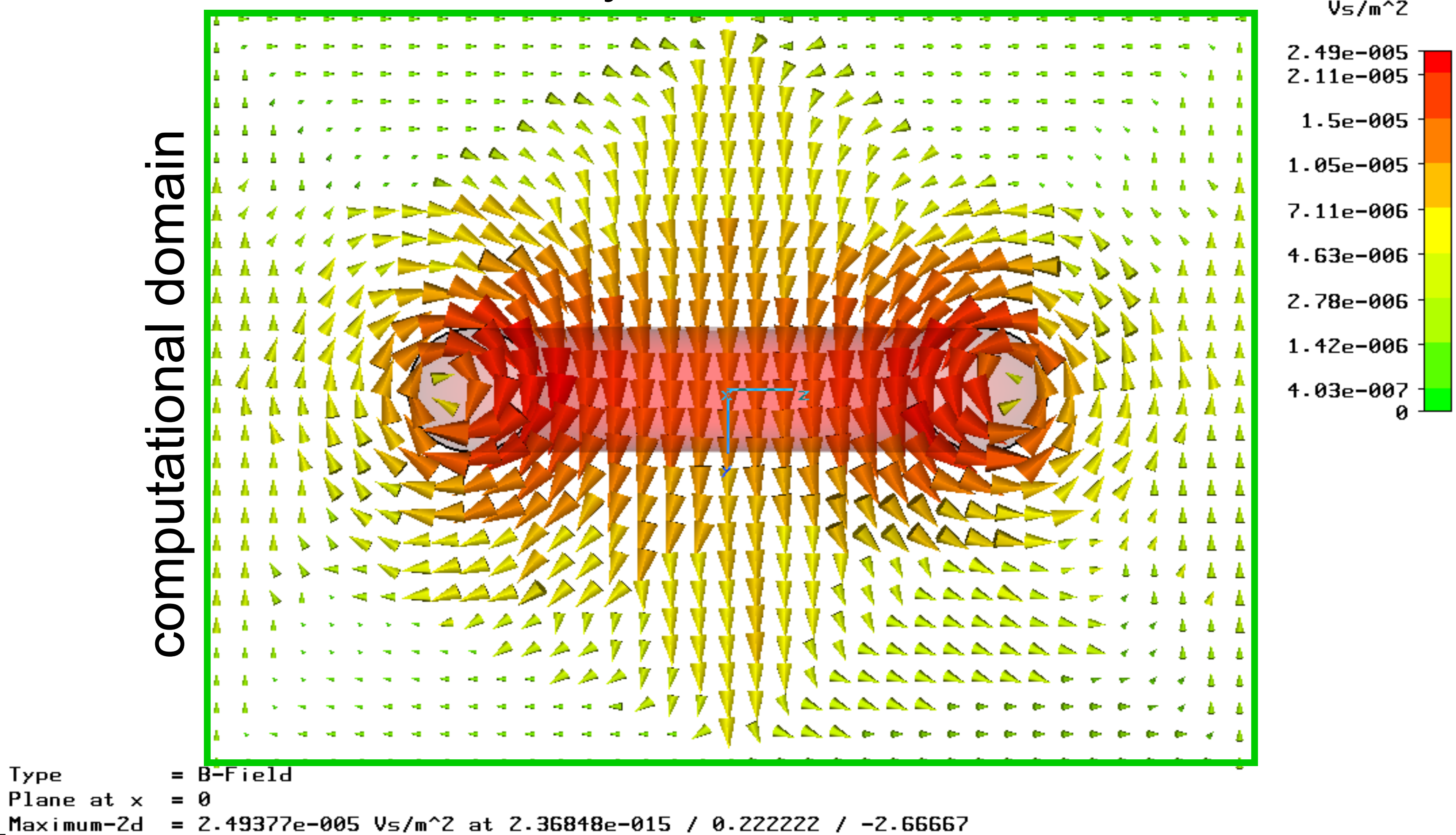
electric boundary conditions magnetic boundary conditions



# Boundary Conditions (3)

electric boundary conditions

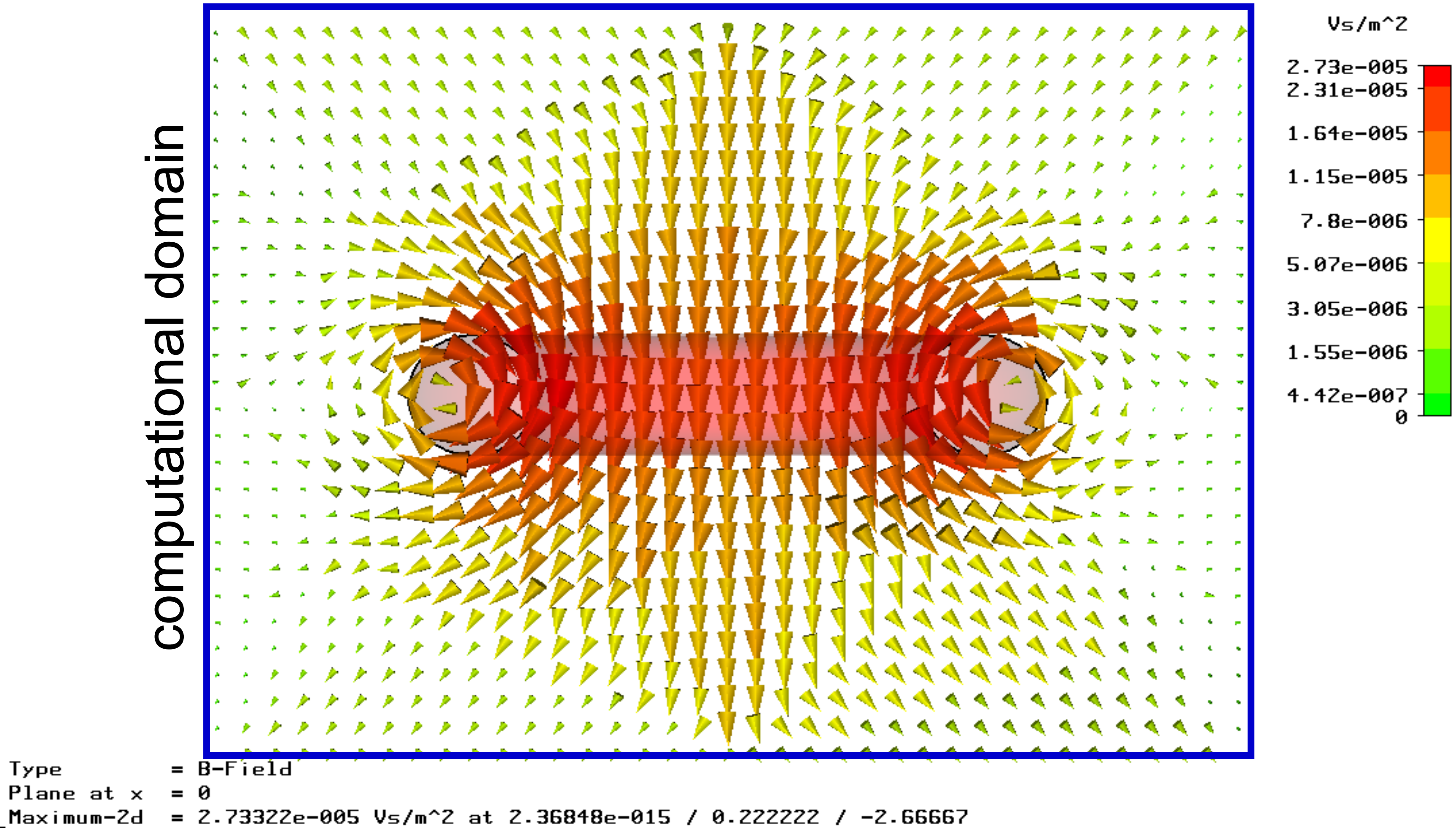
computational domain



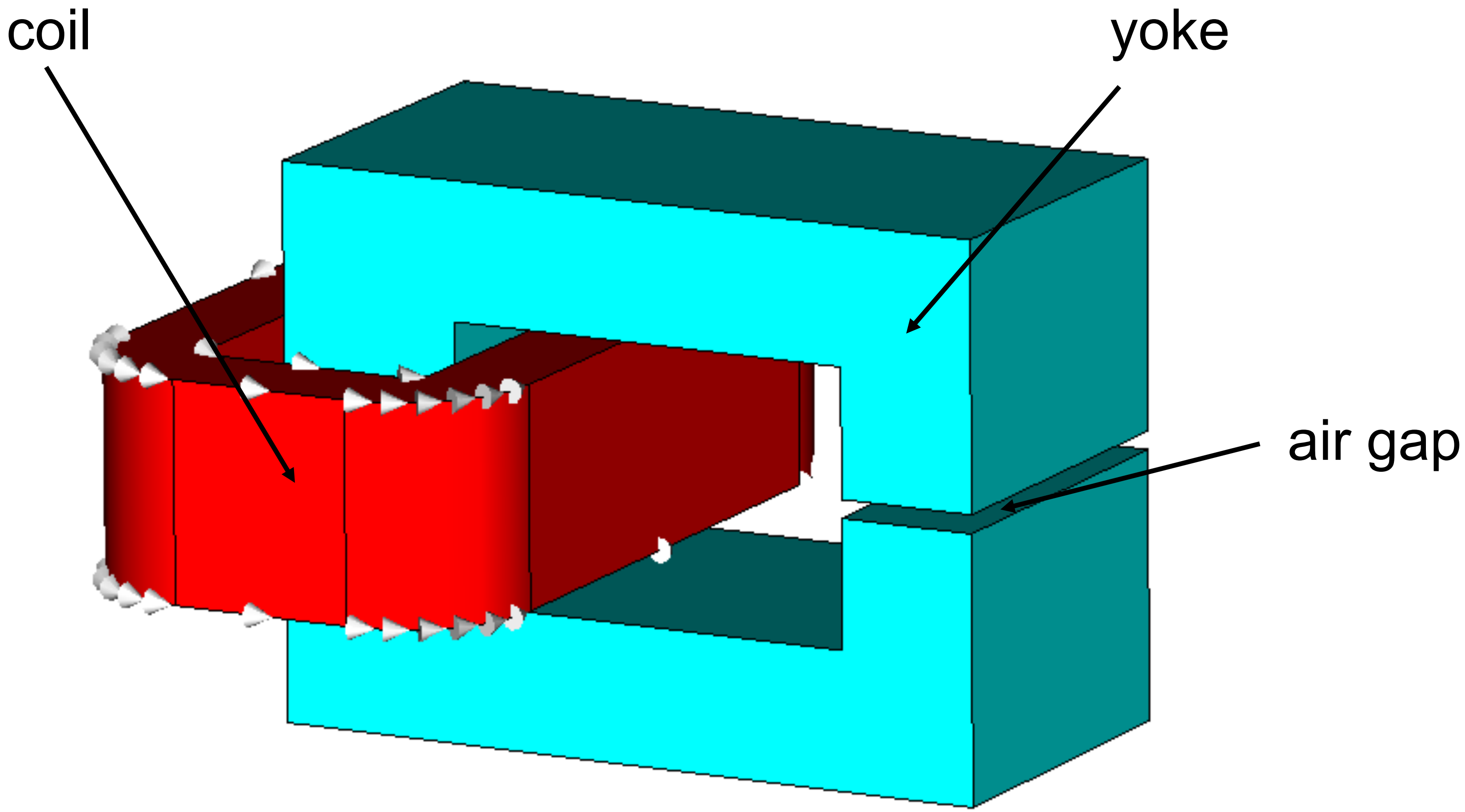
# Boundary Conditions (4)

magnetic boundary conditions

computational domain



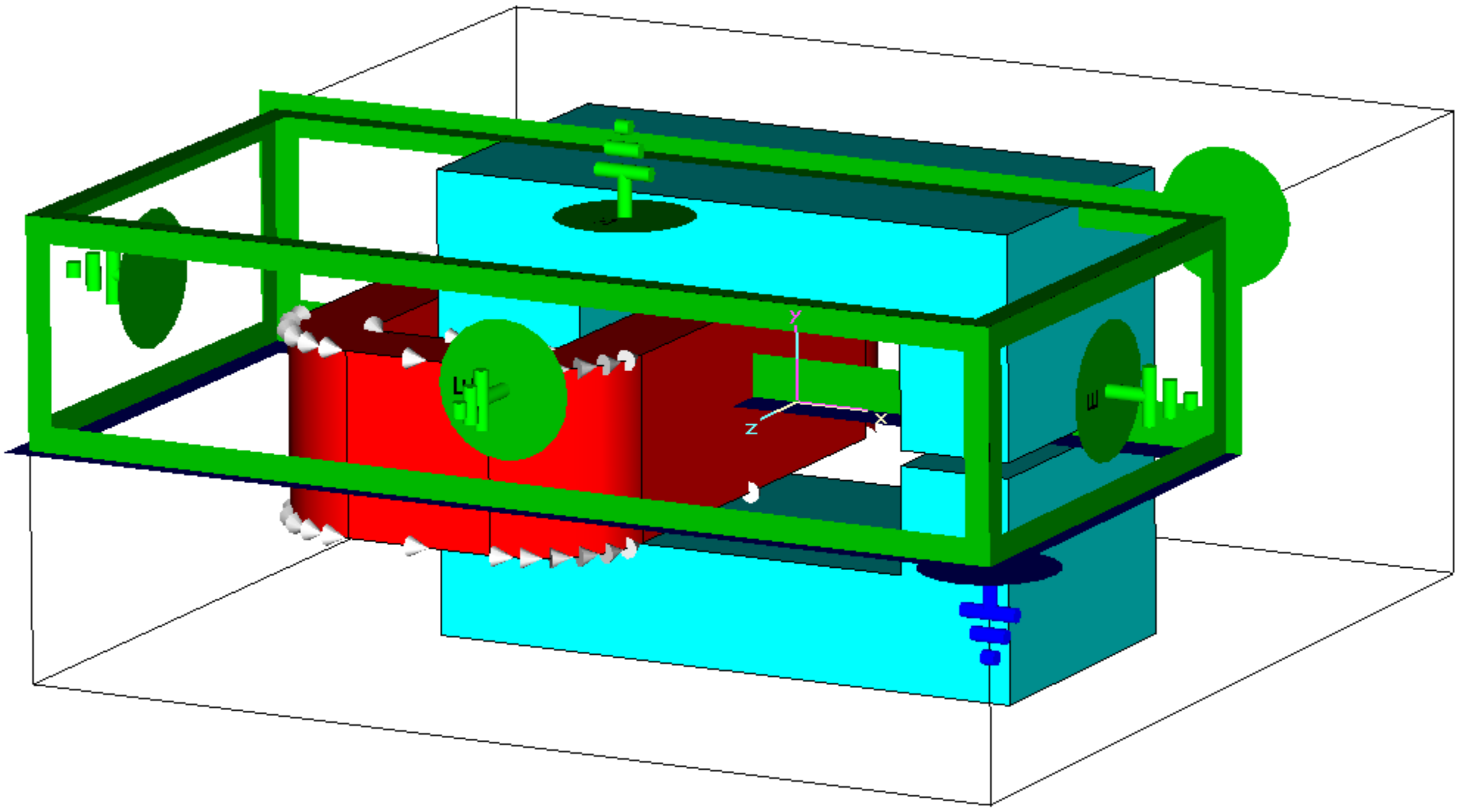
# Symmetries (1)



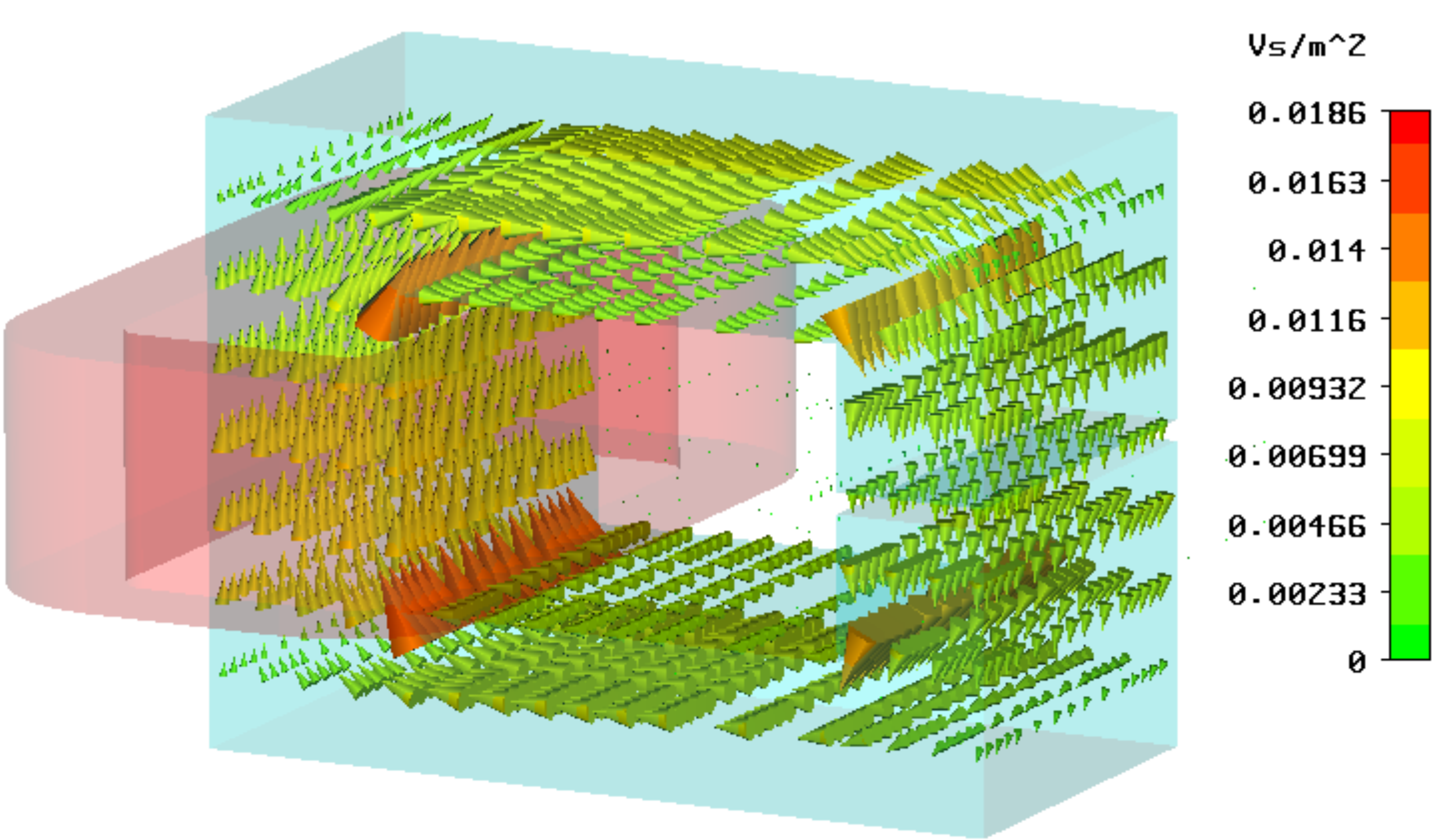
# Symmetries (2)

electric boundary conditions

magnetic boundary conditions



# Symmetries (3)



# Inserting Boundary Conditions

$$\begin{bmatrix} \boldsymbol{K}_{bb} & \boldsymbol{K}_{bc} \\ \boldsymbol{K}_{cb} & \boldsymbol{K}_{cc} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_b \\ \boldsymbol{u}_c \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_b \\ \boldsymbol{f}_c \end{bmatrix}$$

potentials living at Dirichlet boundaries

insert essential boundary conditions

$$\left[\begin{array}{cc:c} \boldsymbol{K}_{bb} & \boldsymbol{K}_{bc} & 0 \\ \boldsymbol{K}_{cb} & \boldsymbol{K}_{cc} & \boldsymbol{B}_{cq} \\ \hdashline 0 & \boldsymbol{B}_{qc} & 0 \end{array}\right] \left[\begin{array}{c} \boldsymbol{u}_b \\ \boldsymbol{u}_c \\ \hdashline \boldsymbol{y}_q \end{array}\right] = \left[\begin{array}{c} \boldsymbol{f}_b \\ \boldsymbol{f}_c \\ \hdashline 0 \end{array}\right]$$

eliminate known potentials

$$\boldsymbol{K}_{bb}\boldsymbol{u}_b = \boldsymbol{f}_b - \boldsymbol{K}_{bc}\boldsymbol{u}_c$$

$\boldsymbol{K}$ $\boldsymbol{f}$ → `bdrycond_shrink.m` → $\mathbf{K}_{bb}$ $\boldsymbol{f}_b - \boldsymbol{K}_{bc}\boldsymbol{u}_c$

`backslash`

$\boldsymbol{u}$ ← `bdrycond_inflate.m` ← $\boldsymbol{u}_b$

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# Reduction to 2D Models (1)

- 2D cartesian models

$$\vec{J} = \left(0{,}0, J_z(x,y)\right)$$

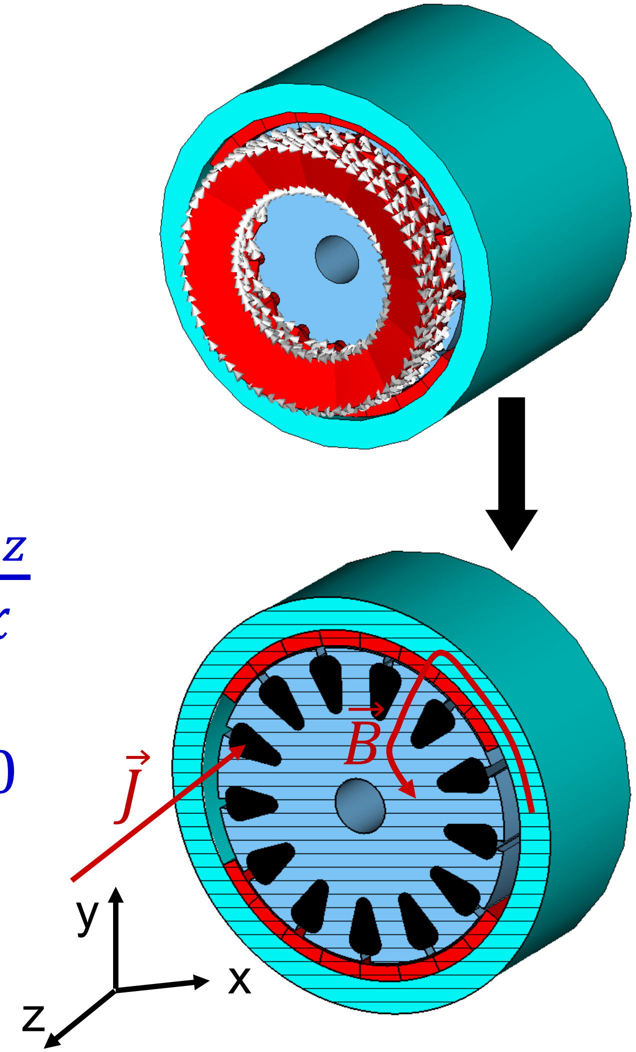
$$\vec{B} = \left(B_x(x,y), B_y(x,y), 0\right)$$

$$\vec{A} = \left(0{,}0, A_z(x,y)\right)$$

$$\vec{B} = \nabla\times\vec{A} \iff B_x = \frac{\partial A_z}{\partial y};\; B_y = -\frac{\partial A_z}{\partial x}$$

$$\nabla\cdot\vec{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = \frac{\partial^2 A_z}{\partial x\partial y} - \frac{\partial^2 A_z}{\partial x\partial y} = 0$$

$$\frac{\partial A_z}{\partial z}, \frac{\partial B_x}{\partial z}, \frac{\partial B_y}{\partial z}, \frac{\partial J_z}{\partial z} = 0 \quad \text{but} \quad \frac{\partial \varphi}{\partial z} \neq 0$$



# Reduction to 2D Models (2)

- 2D cartesian models

$$B_x = \frac{\partial A_z}{\partial y}; \quad B_y = -\frac{\partial A_z}{\partial x}$$

anisotropic material

$$\begin{bmatrix} H_x \\ H_y \end{bmatrix} = \begin{bmatrix} \nu_x & 0 \\ 0 & \nu_y \end{bmatrix} \begin{bmatrix} B_x \\ B_y \end{bmatrix}$$

$$H_x = \nu_x \frac{\partial A_z}{\partial y}; \quad H_y = -\nu_y \frac{\partial A_z}{\partial x}$$

$$J_z = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = J_{s,z} - \sigma \frac{\partial A_z}{\partial t}$$

Ampère
+ Faraday-Lenz

$$-\frac{\partial}{\partial x}\left(\nu_y \frac{\partial A_z}{\partial x}\right) - \frac{\partial}{\partial y}\left(\nu_x \frac{\partial A_z}{\partial y}\right) + \sigma \frac{\partial A_z}{\partial t} = J_{s,z}$$

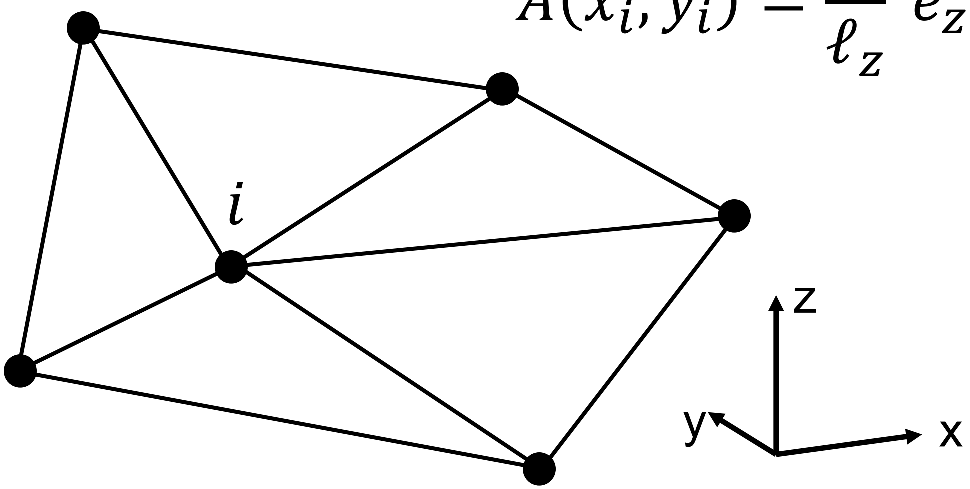
$$\Longleftrightarrow \quad \nabla \times \left(\nu \nabla \times \vec{A}\right) + \sigma \frac{\partial \vec{A}}{\partial t} = \vec{J}_s$$

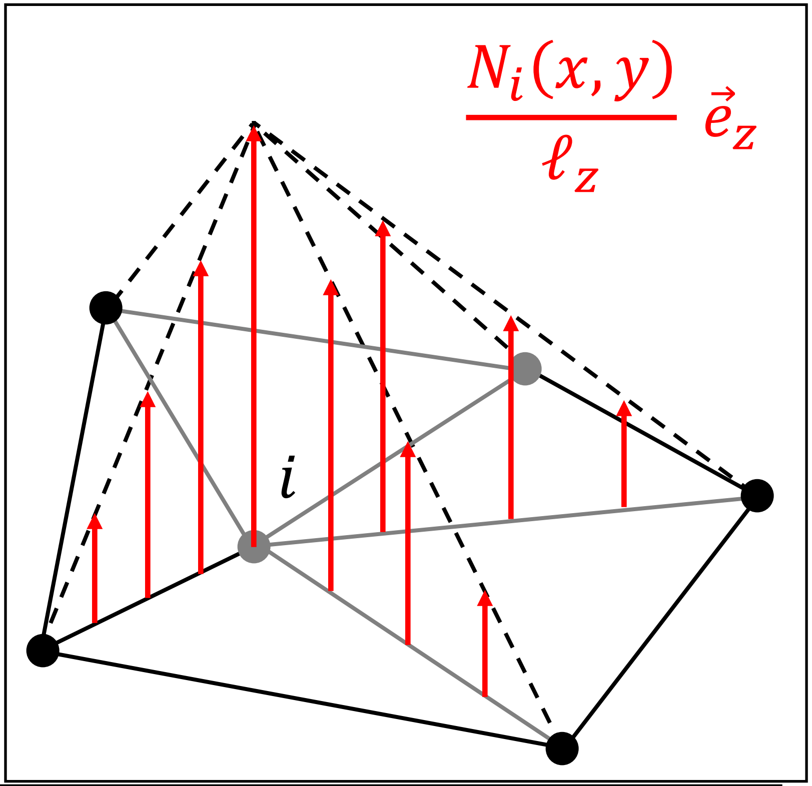
# 2D Discretisation (1)

$$\sum_j \left( u_j \int_V \nu \nabla\times\vec{w}_j \cdot \nabla\times\vec{w}_i \, \mathrm{d}V + \frac{\mathrm{d}u_j}{\mathrm{d}t} \int_V \sigma \vec{w}_j \cdot \vec{w}_i \, \mathrm{d}V \right) = \int_V \vec{J}_s \cdot \vec{w}_i \, \mathrm{d}V$$

$$\vec{A} = \sum_j u_j \vec{w}_j = \sum_j u_j \frac{N_j(x,y)}{\ell_z} \vec{e}_z$$

$$\vec{A}(x_i, y_i) = \frac{u_i}{\ell_z} \vec{e}_z$$



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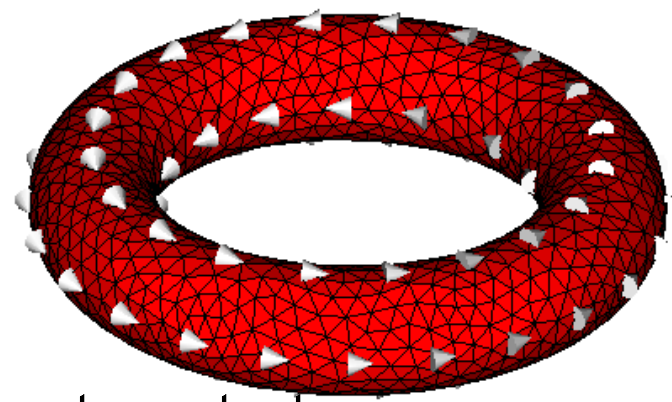
# Coil Model (1)

assumptions

- homogeneous current distribution
- no eddy currents

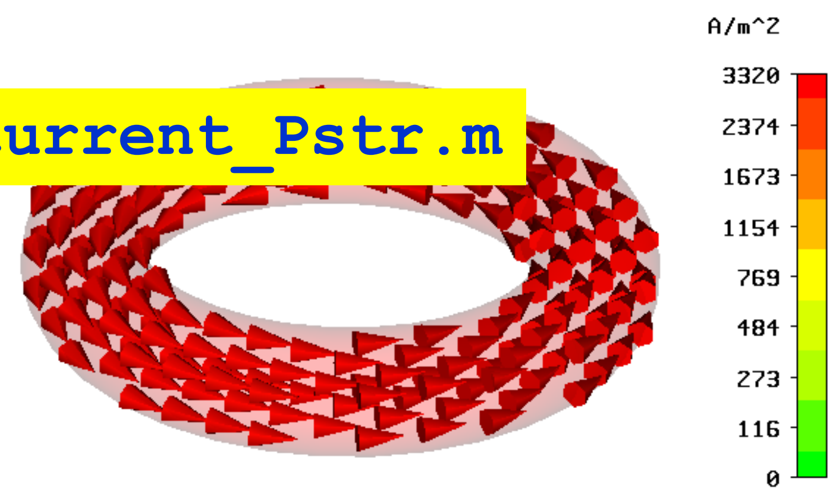
notice (*model*)

- there will be an induced voltage !!
- current density not ct when cross-section not constant



winding function $\vec{\chi}_{\text{coil},q}(x,y,z)$ [1/m²]

- computed geometrically
- by field solution (lecture V11)

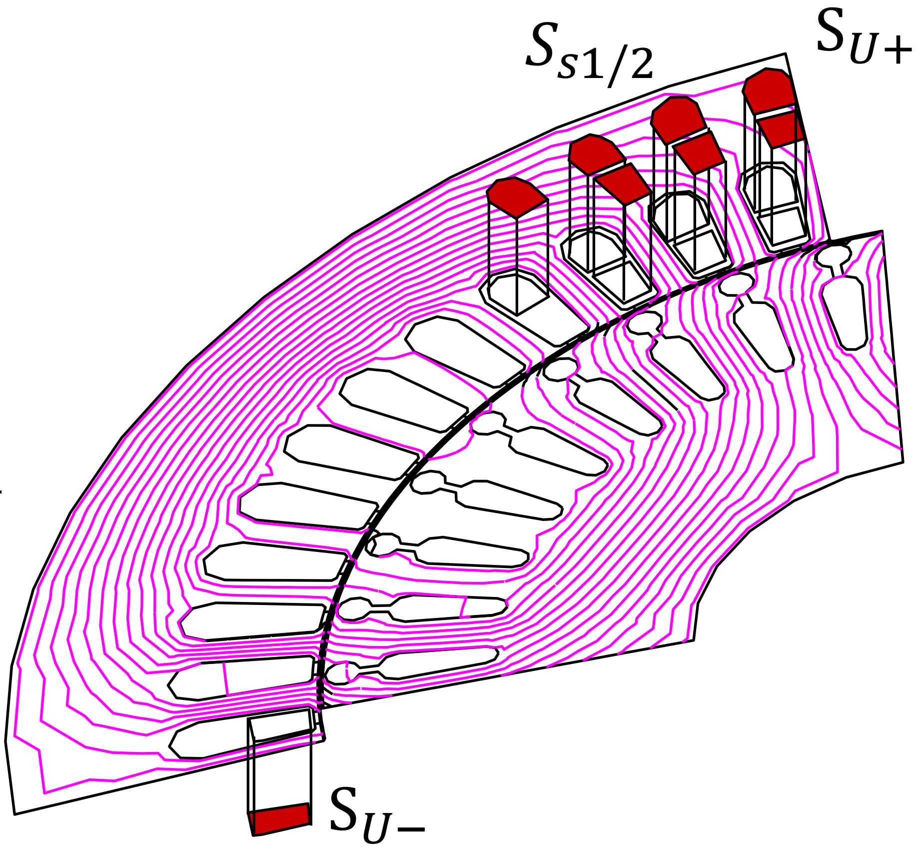



$$\vec{J}_s(x,y,z,t) = \sum_{q=1}^{n_{\text{coil}}} \underbrace{\vec{\chi}_{\text{coil},q}(x,y,z)\, i_q(t)}_{\vec{J}_{\text{coil},q}\,(x,y,z,t)}$$

# Coil Model (2)

- in 2D: $\vec{J}(x,y,t) = \big(0,0,J_z(x,y,t)\big)$

$$
\begin{cases}
\vec{\chi}_{\mathrm{coil},U}(x,y) = +\dfrac{N_\mathrm{t}}{S_{s1/2}}\vec{e}_z & \text{in } S_{U+} \\
\vec{\chi}_{\mathrm{coil},U}(x,y) = -\dfrac{N_\mathrm{t}}{S_{s1/2}}\vec{e}_z & \text{in } S_{U-} \\
\vec{\chi}_{\mathrm{coil},U}(x,y) = 0 & \text{in } S_{2\mathrm{D}} \backslash S_{U+} \backslash S_{U-}
\end{cases}
$$



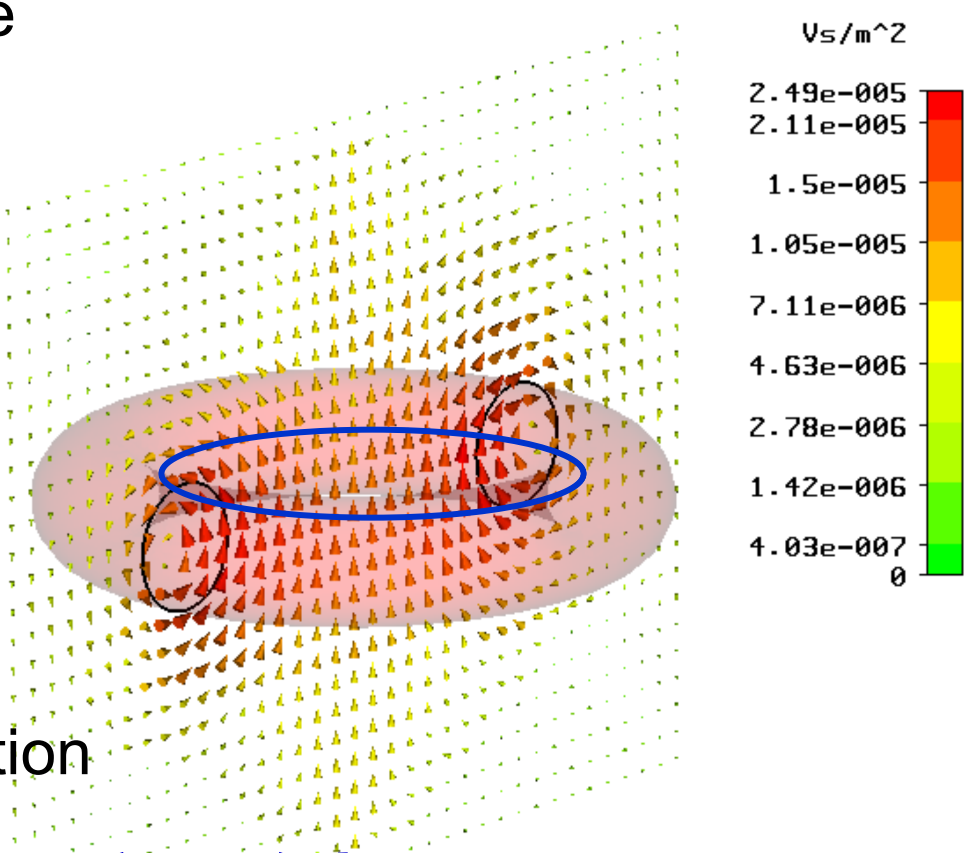
# Coil Model (3)

- induced voltage ~ flux linkage
- which flux is linked?

- for a single path

$$\phi = \oint_{\partial S} \vec{A} \cdot \mathrm{d}\vec{s}$$

- for a coil
- integrating along the coil
- average at the coil cross-section

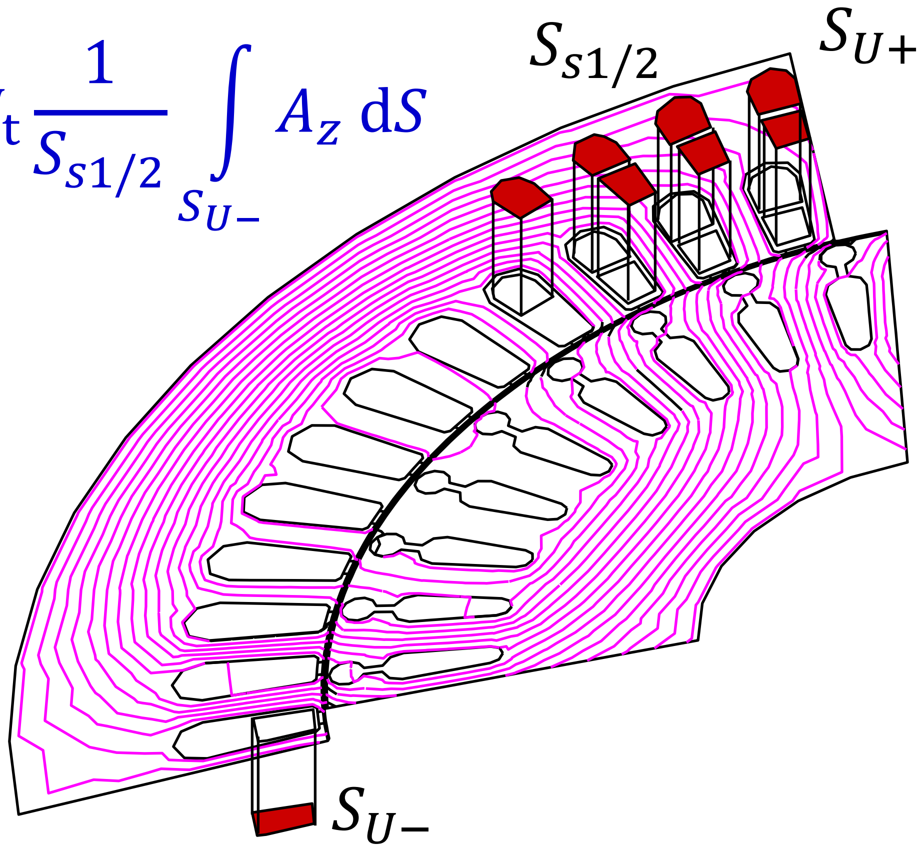



$$\psi_{\mathrm{coil},q}(t) = \int_V \vec{A}(x,y,z,t) \cdot \vec{\chi}_{\mathrm{coil},q}(x,y,z)\,\mathrm{d}V$$

# Coil Model (4)

- in 2D: $\psi_{\text{coil},q} = \int_V \vec{A} \cdot \vec{\chi}_{\text{coil},q} \, dV$

$$\psi_{\text{coil},U} = N_{\text{t}} \frac{1}{S_{s1/2}} \int_{S_{U+}} A_z \, \mathrm{d}S - N_{\text{t}} \frac{1}{S_{s1/2}} \int_{S_{U-}} A_z \, \mathrm{d}S$$



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# Permanent Magnet (1)

scalar and linear permanent-magnet model

$$B = B_\mathrm{r} + \mu H$$

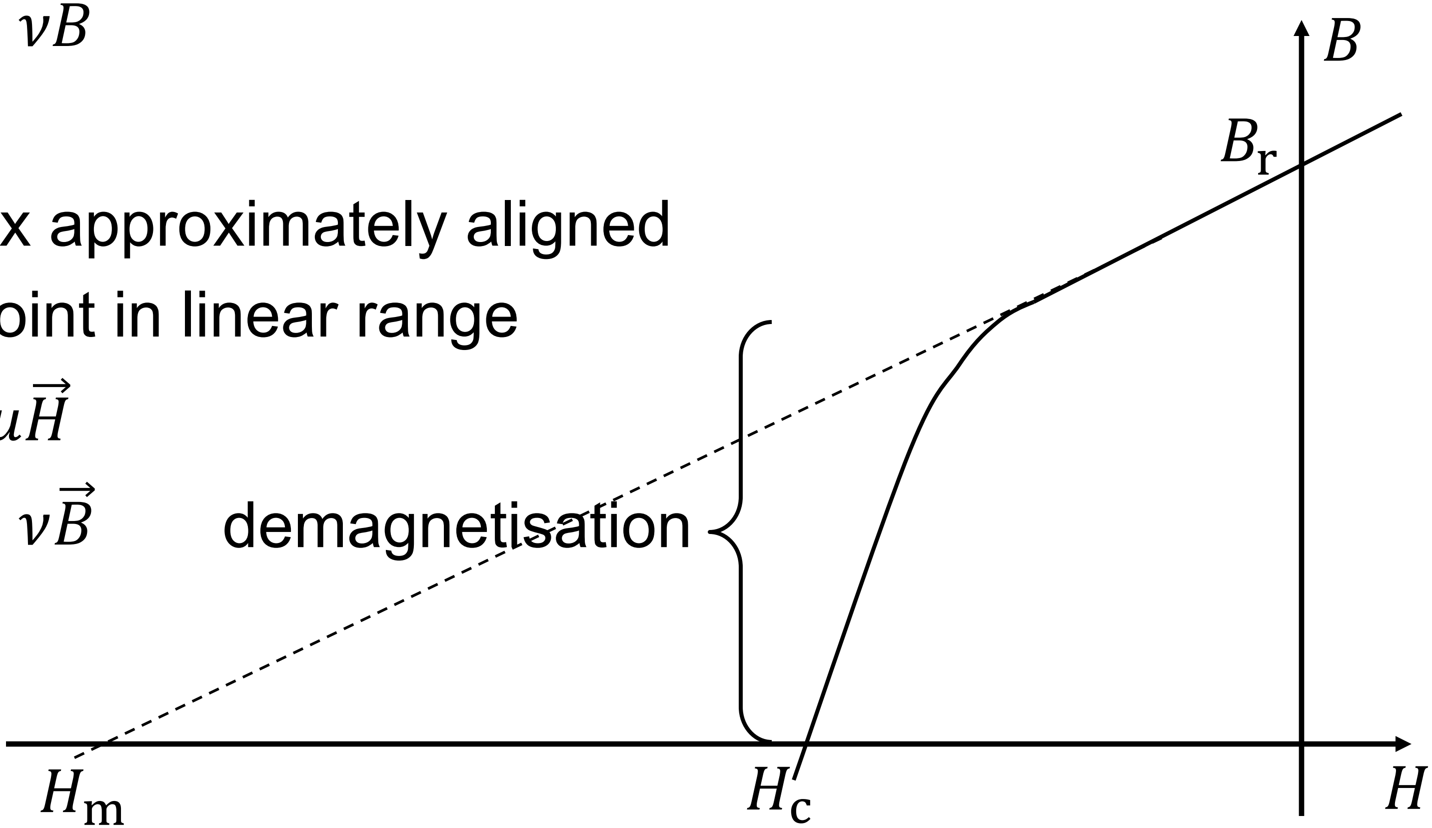
$$H = H_\mathrm{m} + \nu B$$

assumptions

- resulting flux approximately aligned
- operation point in linear range

$$\vec{B} = \vec{B}_\mathrm{r} + \mu \vec{H}$$

$$\vec{H} = \vec{H}_\mathrm{m} + \nu \vec{B}$$



# Permanent Magnet (2)

magnetoquasistatic formulation:

$$\vec{H} = \vec{H}_{\mathrm{m}} + \nu\vec{B}$$

$$\vec{H} = \vec{H}_{\mathrm{m}} + \nu\nabla\times\vec{A}$$

$$\nabla\times\left(\nu\nabla\times\vec{A}\right) = \vec{J} - \underbrace{\nabla\times\vec{H}_{\mathrm{m}}}_{\vec{J}_{\mathrm{m}}}$$

$\vec{J}_{\mathrm{m}}$ magnetisation current

in 2D:

$$-\frac{\partial}{\partial x}\left(\nu_y\frac{\partial A_z}{\partial x}\right) - \frac{\partial}{\partial y}\left(\nu_x\frac{\partial A_z}{\partial y}\right) = J_z - \frac{\partial H_{\mathrm{m},y}}{\partial x} + \frac{\partial H_{\mathrm{m},x}}{\partial y}$$

# Permanent Magnet (3)

discretisation:

$$\mathrm{RHS} = \int_V \vec{J}_\mathrm{m} \cdot \vec{w}_i \, \mathrm{d}V$$

$$\mathrm{RHS} = -\int_V \nabla \times \vec{H}_\mathrm{m} \cdot \vec{w}_i \, \mathrm{d}V$$

$$\mathrm{RHS} = \int_V \nabla \cdot \left( \vec{w}_i \times \vec{H}_\mathrm{m} \right) \mathrm{d}V - \int_V \vec{H}_\mathrm{m} \cdot \nabla \times \vec{w}_i \, \mathrm{d}V$$

$$\mathrm{RHS} = \underbrace{\oint_{\partial V} \vec{w}_i \times \vec{H}_\mathrm{m} \cdot \mathrm{d}\vec{S}}_{=0 \text{ when no PMs at the model boundary}} - \int_V \vec{H}_\mathrm{m} \cdot \nabla \times \vec{w}_i \, \mathrm{d}V$$

# Permanent Magnet (4)

discretisation (in 2D):

$$\mathrm{RHS} = -\int_V \vec{H}_\mathrm{m} \cdot \nabla\times\vec{w}_i \,\mathrm{d}V$$

$$\Downarrow \qquad \vec{w}_i = \frac{N_i(x,y)}{\ell_z}\,\vec{e}_z$$

$$\mathrm{RHS} = \int_V \frac{1}{\ell_z}\left(\frac{\partial N_i}{\partial y}H_{\mathrm{m}x} - \frac{\partial N_i}{\partial x}H_{\mathrm{m}y}\right)\mathrm{d}V$$

$$\mathrm{RHS} = \int_{\partial V}\left(\frac{\partial N_i}{\partial y}H_{\mathrm{m}x} - \frac{\partial N_i}{\partial x}H_{\mathrm{m}y}\right)\mathrm{d}S$$

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