Direct Vlasov solvers – part I

Nicolas Mounet, CERN/BE-ABP-HSC

Acknowledgements: Sergey Arsenyev, Xavier Buffat, Giovanni Iadarola, Kevin Li, Elias Métral, Adrian Oeftiger, Giovanni Rumolo

Direct Vlasov solvers

Part I

- Introduction: collective effects
- Motivation for Vlasov solvers
- Vlasov equation historically, and in the context of accelerators
- Transverse impedance and instabilities
- Building of a simple Vlasov solver for impedance instabilities

Collective effects

- Collective effects: phenomena in which the evolution of the particle beam cannot be studied as if the beam was a collection of single particles behaving independently, but rather as an ensemble of interacting particles.

- Examples (with the potential effect on the beam):
  - ✓ Impedance & wake fields, i.e. interaction with the beam's own self-generated electromagnetic (EM) fields (instabilities, heat load),
  - ✓ Intra-beam scattering & Touschek effect (emittance growth, intensity loss),
  - ✓ Interactions with trapped ions (coherent instabilities),
  - ✓ Build up of an electron cloud and interaction with it (heat load, coherent instabilities),
  - ✓ Interaction with another counter-rotating beam – so-called beam-beam effects (emittance growth, intensity loss, possibly coherent instabilities).

Collective effects - modeling instabilities

- Coherent instability: self-enhanced, typically exponentially growing, oscillation of the full beam (or a significant part of it, e.g. one bunch).
- A first approach is simply to perform multi-particle tracking (see previous CAS lectures), including the collective effect under study (e.g. collision between particles, EM fields from ensemble of particles, etc.).
- This approach is, in principle:
  - ✓ simple and efficient, especially if a model is available for the self-interaction fields (e.g. a wake function),
  - ✓ easy to extend to complex situations,
  - ✓ potentially very realistic.

So why should we do anything else than this?

Motivation for another kind of modeling

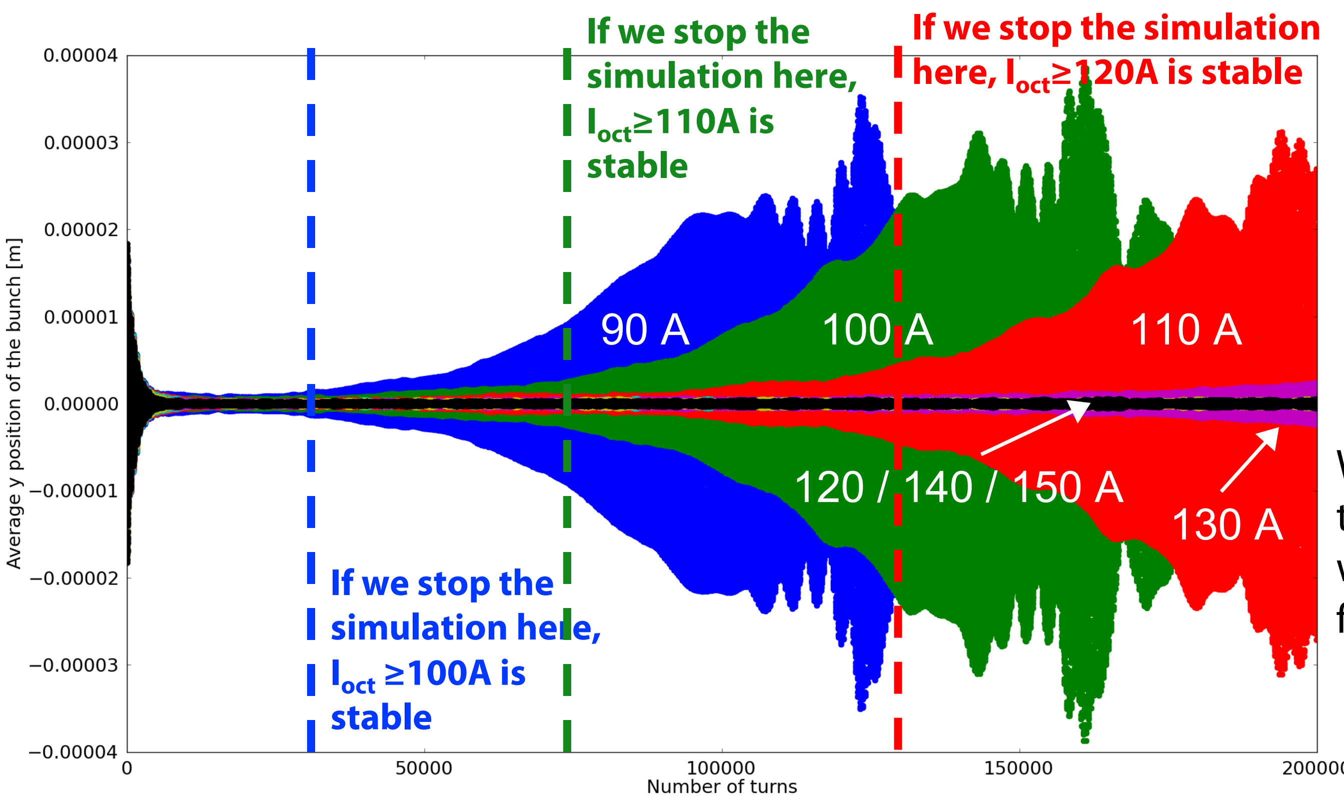
- Multi-particle tracking still exhibit a number of drawbacks:
  - ✗ It can be slow: one needs to track thousands to millions of macroparticles, sometimes with a complex interaction mechanism (PIC solver, bunch slicing for wake fields, etc.).
  - ✗ Most importantly, it does not always help for an understanding of what's happening.

$\Rightarrow$ It's not always easy to understand what parameters are the important ones to e.g. stabilize an unstable beam.

Motivation for another kind of modeling

- Multi-particle tracking can also be misleading: as a time domain technique, a beam that looks stable might actually be unstable if we track more turns.

  Example: average vertical position in the LHC vs octupole current I_{oct} (i.e. with increasing damping from transverse non-linearities):



What is the real threshold? Are we really stable for I_{oct}≥140A?

Alternative for instability computation

- Multi-particles is one way to discretize the phase space – very close to reality as the beams are indeed made of distinct particles, albeit much more numerous than in typical simulations.

- *A contrario*, one can also consider the whole phase space distribution as a continuum, and look for modes arising from collective interactions, that could develop and lead to instabilities.

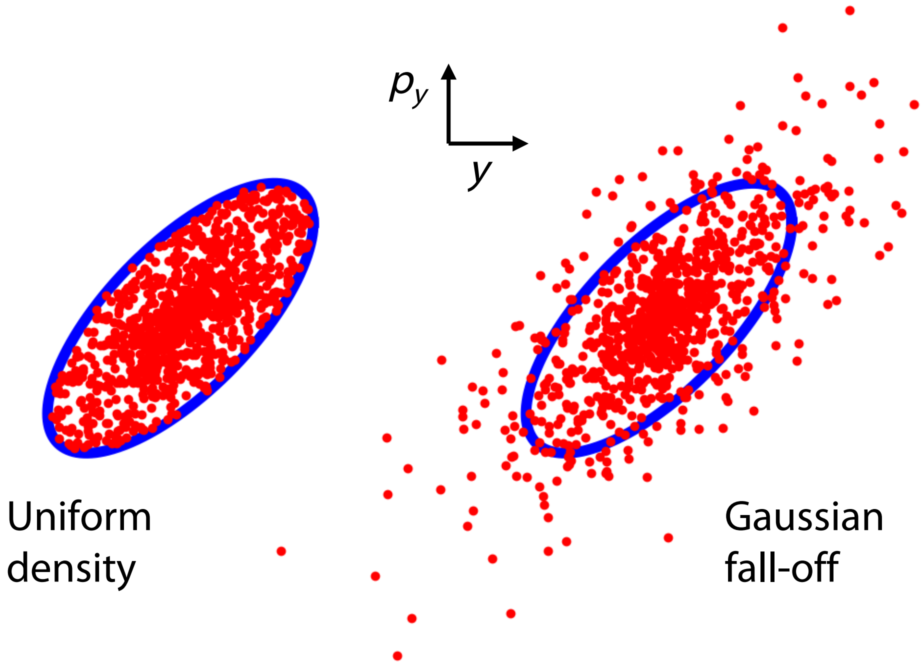
  $\Rightarrow$ Vlasov solvers – named after the equation to be solved.

  $\Rightarrow$ Switch from time to mode domain, the stability of each mode being predictable irrespectively of its rapidity to develop.

- Historically, this was the first approach adopted to try to understand instabilities in particle accelerators [L. J. Laslett, V. K. Neil, and A. M. Sessler *(1965)*, F. J. Sacherer *(1972)*].

Distribution of particles in phase space

- In a classical (i.e. not quantum-mechanical) picture, each beam particles has a certain position and momentum for each of the three coordinates *(x, y, z)*.
- For a 2D distribution, in e.g. vertical, such a distribution of particles can be easily pictured in phase space (y,p_y):

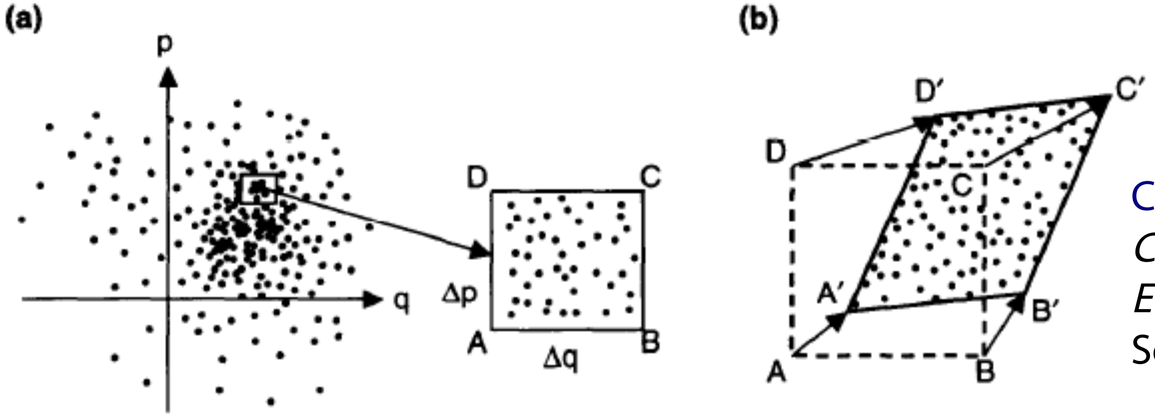


$\Rightarrow$ the distribution function ψ represents the density of particles in phase space

Total number of particles $N = \iiint_{position} \iiint_{momenta} \psi(x, p_x, y, p_y, z, p_z; t)\, dx\, dp_x\, dy\, dp_y\, dz\, dp_z$

Liouville theorem

- Vlasov equation is based on Liouville theorem (or equivalently, on the collisionless Boltzmann transport equation), which expresses that the local phase space density does not change when one follows the flow (i.e. the trajectory) of particles.
- In other words: local phase space area is conserved in time: $\boxed{\frac{d\psi}{dt} = 0}$



Courtesy A. W. Chao, *Physics of Collective Beam Instabilities in High Energy Accelerators*, John Wiley & Sons (1993), chap. 6.

Figure 6.3. (a) Phase space distribution of particles at time t. A rectangular box $ABCD$ with area $\Delta q\,\Delta p$ is drawn and magnified. (b) At a later time, $t+dt$, the box moves and deforms into a parellelogram with the same area as $ABCD$. All particles inside the box move with the box.

Vlasov equation [**A. A. Vlasov**, *J. Phys. USSR 9, 25 (1945)*]

- Vlasov equation was first written in the context of plasma physics, where the standard collision-based Boltzmann approach, with Coulomb collisions, was failing.
- As Coulomb interactions have a long-range character, the idea of Vlasov was to integrate the collective, self-interaction EM fields into the Hamiltonian, instead of writing them as a collision term.
- Assumptions:
  - conservative & deterministic system (governed by Hamiltonian) – no damping or diffusion from external sources (no synchrotron radiation),
  - particles are interacting only through the collective EM fields (no short-range collision).

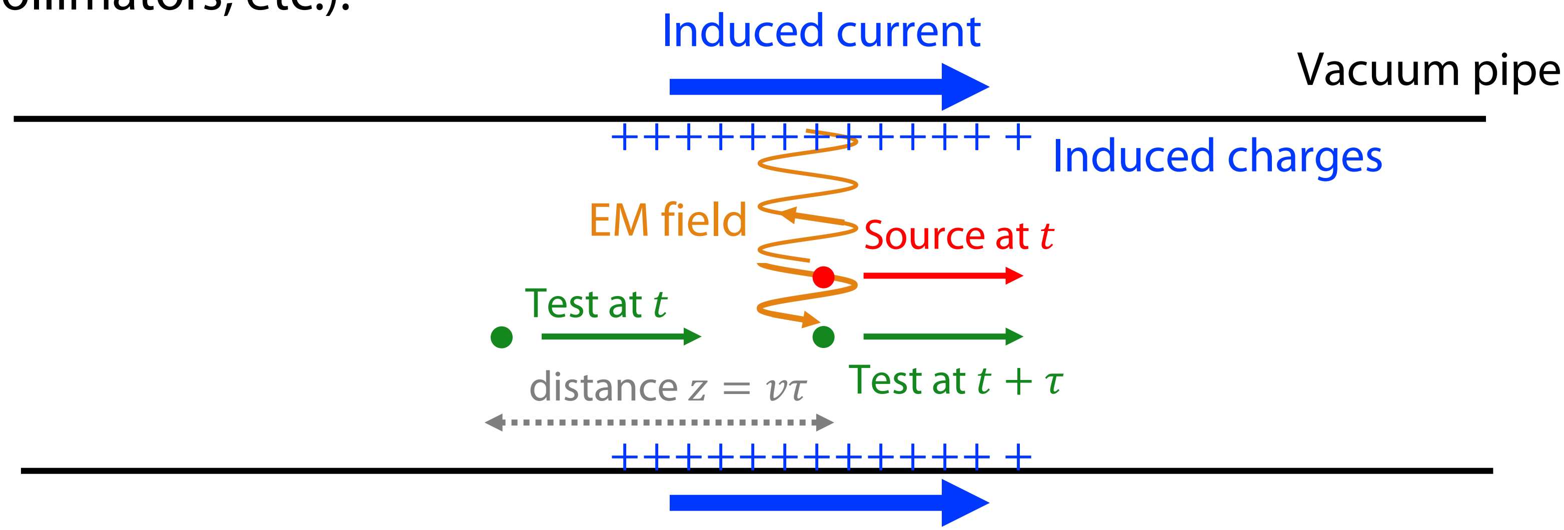
Vlasov solvers for synchrotrons

- Vlasov solvers can be used in principle for various kinds of collective effects involving self-generated EM fields, e.g.:
  - Transverse impedance effects (see later for references),
  - Longitudinal impedance effects [e.g. **M. Venturini** et al, *Phys. Rev. ST Accel. Beams 10 (2007), 054403*],
  - Beam-beam effects [e.g. **Y. Alexahin**, *Nucl. Instr. Meth. in Phys. Res. A 480 (2002) pp. 253–288*],
  - Electron-cloud, or more generally two stream effects [e.g. **E. A. Perevedentsev**, *Proc. workshop on e-cloud simul. for proton & positron beams, Geneva, Switzerland, CERN-2002-001* (*2002) pp. 171-194*],
  - Space-charge (& impedance) [e.g. **M. Blaskiewicz**, *Phys. Rev. ST Accel. Beams 1 (1998), 044201*].
- In this lecture we will rather focus on transverse impedance effects without space-charge, in circular machines.

Still, the approach adopted here can be applied to other collective effects.

Impedance & wake function

- Impedance is a quantity that characterizes the electromagnetic (EM) fields generated by a single particle ("source") on another particle ("test") through interaction with the beam surroundings (vacuum pipe, cavities, collimators, etc.):



- The force felt by the test, averaged over the device length and normalized by source and test charges, is the wake function (here in vertical, length=$2\pi R$ for a vacuum pipe all round the ring):

Imaginary unit

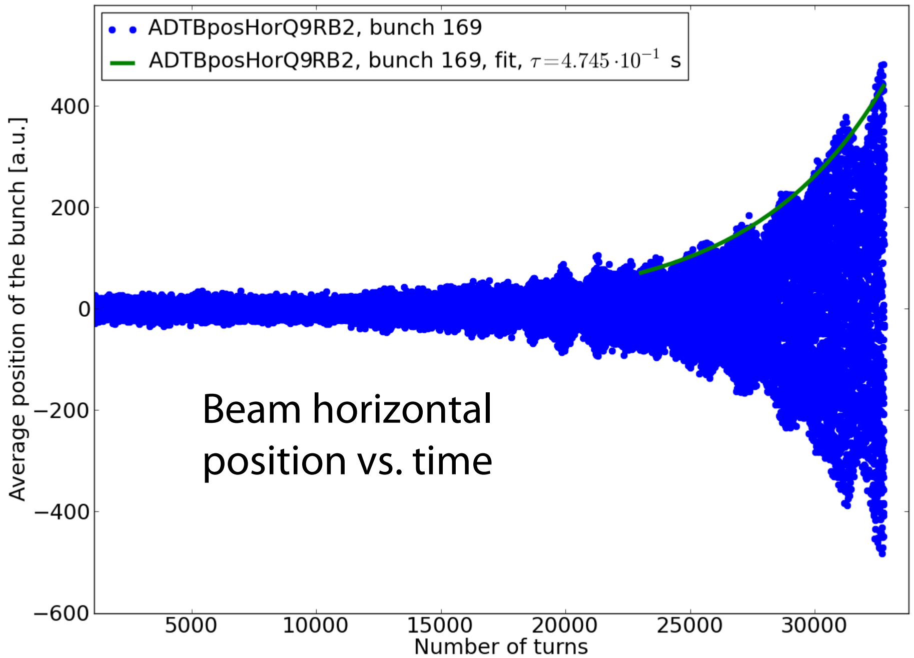
$$W_y(z) = \frac{2\pi R}{e^2} F_y(x_{test}, y_{test}, z) = -\frac{j}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{j\omega\frac{z}{v}} Z_y(\omega)$$

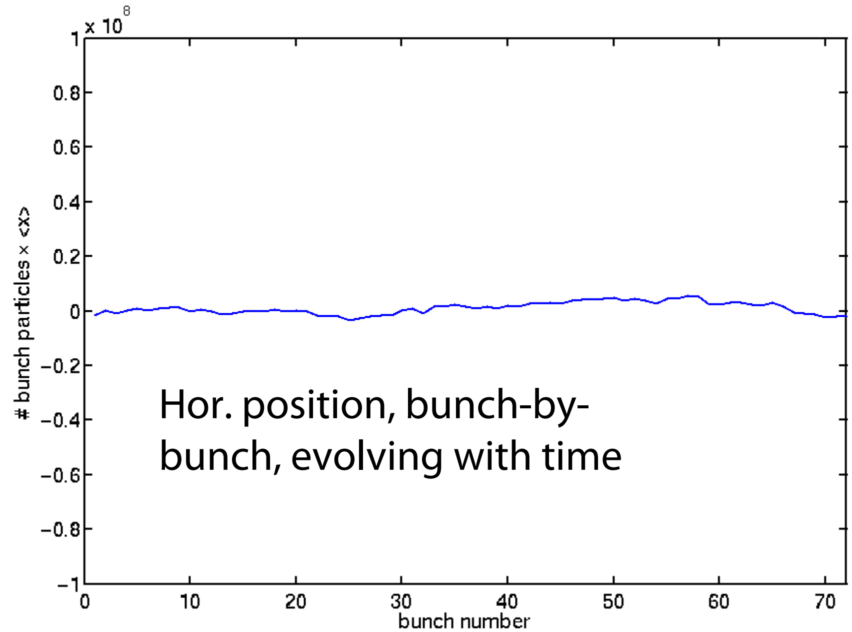
It's the inverse Fourier transform of the impedance

Transverse instability modes

- Coherent instabilities are self-enhanced modes, characterized by a beam position growing with time (typically exponentially) :

Measurements in the LHC

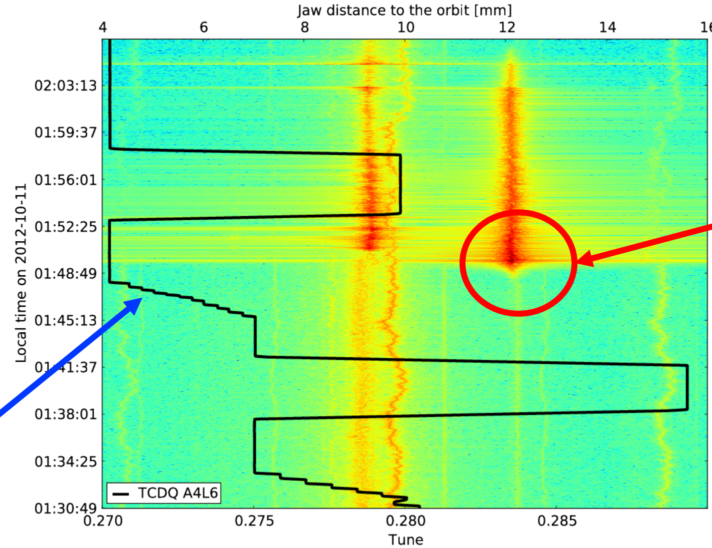




Transverse instability modes

➢ Typically, instabilities happen at a certain frequency, close to the tune

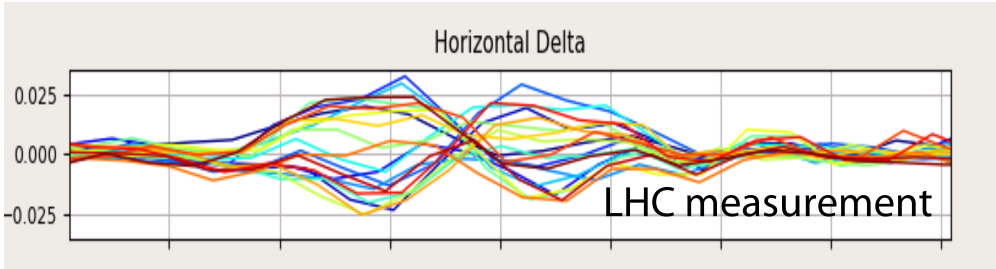
Frequency spectrum over time for the LHC beam hor. position while moving a collimator jaw closer to the beam

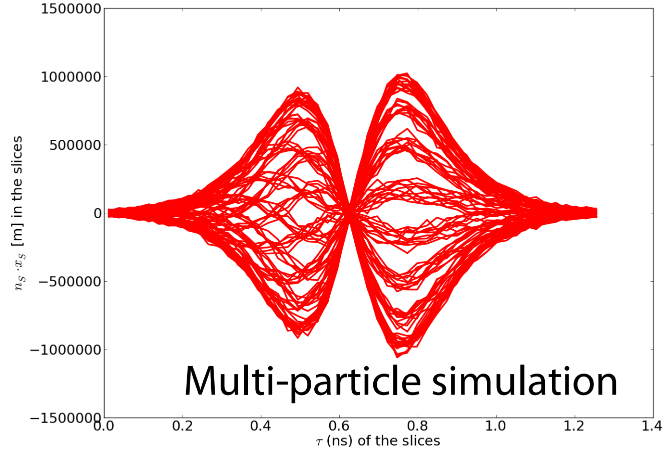


Instability

Movement of collimator jaw

➢ … and an intra-bunch pattern:





Vlasov solvers for transverse impedance

- Vlasov equation was first used to compute stability conditions for a given excitation, obtaining dispersion relations, by Laslett et al (1965) [1].
- The seminal Sacherer integral equation was derived (1972) [2], and a simple formula for instability growth rates obtained from it (1974) [3].
- Besnier devised a method to solve Sacherer Integral eq. using orthogonal polynomials (1979) [4], and Laclare developped an equivalent approach in frequency domain (1985) [5].
- Several codes were implemented over the years, e.g. MOSES (1985) [6], NHTVS (2014) [7], DELPHI [8] (2014) and GALACTIC (2018) [9].
- Extension to include synchrotron radiation for lepton machines do exist, solving Vlasov-Fokker-Planck equation, see e.g. Ref. [10].
- Reviews, courses and books can be found, in e.g. Refs. [3,5] and Chao's book [11].

[1] **L. J. Laslett, V. K. Neil, and A. M. Sessler**, *Rev. Sci. Instrum. 36, 4 (1965) pp. 436–448.*
[2] **F. J. Sacherer**, *CERN/SI-BR/72-5 (1972).*
[3] **B. Zotter & F. J. Sacherer**, *Proc. 1st Int. School Part. Acc., Erice, Italy (1976) pp. 175– 218.*
[4] **G. Besnier, D. Brandt, and B. Zotter**, *CERN LEP-TH/84-11, LHC Note 17 (1985).*
[5] **J. L. Laclare**, *Proc. CERN Accelerator School, Oxford, UK (1985) pp. 264–326.*
[6] **Y.-H. Chin**, *CERN/SPS/85-2 (1985) and CERN/LEP-TH/88-05 (1988).*
[7] **A. V. Burov**, *Phys. Rev. ST Accel. Beams, 17 (2014) 021007.*
[8] **N. Mounet**, *CERN Yellow Reports: Conference Proceedings, 1 (2018) p. 77.*
[9] **E. Métral** et al, *Proc. IPAC'18, Vancouver, Canada (2018) pp. 3076–3079.*
[10] **R. L. Warnock**, *Nucl. Instr. Meth. in Phys. Res. A 561 (2006) pp. 186–194.*
[11] **A. W. Chao**, *Physics of Collective Beams Instabilities in High Energy Accelerators. John Wiley and Sons (1993), chap. 6.*

How to build a Vlasov solver

- It would be numerically very difficult to solve Vlasov equation with "brute force", as a partial differential equation of 7 variables:

$$\frac{d\psi}{dt} = \frac{\partial\psi}{\partial t} + \frac{\partial\psi}{\partial x}\frac{dx}{dt} + \frac{\partial\psi}{\partial p_x}\frac{dp_x}{dt} + \frac{\partial\psi}{\partial y}\frac{dy}{dt} + \frac{\partial\psi}{\partial p_y}\frac{dp_y}{dt} + \frac{\partial\psi}{\partial z}\frac{dz}{dt} + \frac{\partial\psi}{\partial p_z}\frac{dp_z}{dt} = 0$$

  Moreover, we would lose any asset with respect to tracking:
  - no particular insight or understanding,
  - solution in time domain → no identification of modes.

- To build a useful (i.e. fast and simple enough) Vlasov solver, one rather needs to do some analytical work first, essentially aiming at reducing the number of variables.

- Typical end results of this "pencil and paper" work is either a fully analytical formula (e.g. Sacherer formula), an eigenvalue problem, or a non-linear equation to solve against a single parameter.

- Now we will first focus on the initial analytical work, on an example.

Building a simple Vlasov solver

- Let's consider a simple case, to understand how it works:
  - Impedance $Z_y(\omega)$ is the only source of instability considered, and gives the EM force arising from the interaction of the beam with the resistive or geometric elements around it,
  - only vertical plane, with position and "momentum" $\left(y, y' = \frac{dy}{ds}\right)$ (using for convenience y' rather than p_y)
  - purely linear, uncoupled optics in transverse, within smooth approximation,
  - no longitudinal motion, i.e. essentially rigid bunches in *z*,
  - chromaticity $Q_y' = \frac{dQ_y}{d\delta} = 0$,
  - Phase space distribution function is then

$$\psi = \psi(y, y'; t)$$

Longitudinal coordinate along the accelerator

Building a Vlasov solver: method outline

1. Write the stationary distribution
2. Introduce a perturbation to the distribution function
3. Get the time derivatives through the equations of motion
4. Simplify and linearize Vlasov equation
5. Transform the system of coordinates
6. Decompose appropriately the perturbation
7. Reduce the number of variables
8. Write the impedance force
9. Get the final equation

Stationary distribution

Stationary distribution
Perturbation
Equations of motion
Simplification - Linearization
Coordinate transform
Perturbation decomposition
Reduction variables
Impedance force
Final equation

Let's say there is no impedance, and only the optics plays a role (perfect quadrupoles, focusing the beam around the orbit):

$$\frac{d\psi}{dt} = 0$$

is satisfied by $\psi = \psi(\text{invariants of motion})$

This is a general rule: in the absence of time dependent perturbation, stationary solutions of Vlasov equation are simply ANY phase space distribution function which depends ONLY on the invariants of motion.

The stationary distribution is the starting point of our Vlasov solver.

Stationary distribution

Stationary distribution
Perturbation
Equations of motion
Simplification - Linearization
Coordinate transform
Perturbation decomposition
Reduction variables
Impedance force
Final equation

In vertical, for linear optics, the invariant is the action defined as (see appendix for a derivation)

Single particle tune

Machine physical radius (=circum/2π)

$$J_y = \frac{1}{2}\left[y^2 \frac{Q_y}{R} + y'^2 \frac{R}{Q_y}\right]$$

such that the unperturbed distribution function is

$$\psi(y, y'; t) = \psi_0(J_y)$$

From the expression of the invariant J_y it is easy to show the existence of the angle variable θ_y such that

$$y = \sqrt{\frac{2J_y R}{Q_y}} \cos\theta_y \qquad \text{and} \qquad y' = \sqrt{\frac{2J_y Q_y}{R}} \sin\theta_y$$

Perturbation theory

Stationary distribution
Perturbation
Equations of motion
Simplification - Linearization
Coordinate transform
Perturbation decomposition
Reduction variables
Impedance force
Final equation

It's rather difficult to solve Vlasov equation without making any assumption on the distribution function.

→ instead one typically solves it using linear perturbation theory, i.e. from the knowledge of a stationary distribution, that we slightly perturb to include the (collective) effect under study:

$$\psi = \psi_0(J_y) + \Delta\psi(y, y'; t)$$
$$= \psi_0(J_y) + \Delta\psi(J_y, \theta_y; t)$$

Stationary distribution

Perturbation, assumed infinitesimally small, that we can express indifferently in (y, y') or (J_y, θ_y) variables

Perturbation theory

Stationary distribution
Perturbation
Equations of motion
Simplification - Linearization
Coordinate transform
Perturbation decomposition
Reduction variables
Impedance force
Final equation

$$\psi = \psi_0(J_y) + \Delta\psi(y, y'; t)$$

Vlasov equation becomes:

$$\frac{d\psi}{dt} = 0$$

$$\Leftrightarrow \frac{\partial \Delta\psi}{\partial t} + \frac{\partial \psi}{\partial y}\frac{dy}{dt} + \frac{\partial \psi}{\partial y'}\frac{dy'}{dt} = 0 \qquad \text{(chain rule)}$$

First, how do we get these?

Equations of motion

Stationary distribution
Perturbation
Equations of motion
Simplification - Linearization
Coordinate transform
Perturbation decomposition
Reduction variables
Impedance force
Final equation

Beam velocity = βc

$$\frac{dy}{dt} = \frac{dy}{ds} \cdot \frac{ds}{dt} = v \cdot y'$$

$$\frac{dy'}{dt} = \left(\frac{dy'}{dt}\right)^{optics} + \left(\frac{dy'}{dt}\right)^{impedance}$$

Next step is to express these as a function of $(y, y'; t)$.

Equations of motion

- Stationary distribution
- Perturbation
- Equations of motion
- Simplification - Linearization
- Coordinate transform
- Perturbation decomposition
- Reduction variables
- Impedance force
- Final equation

$$\left(\frac{dy'}{dt}\right)^{optics} = \frac{d}{dt}\left(\frac{dy}{ds}\right) = \frac{d^2y}{ds^2}\cdot v = -vy\left(\frac{Q_y}{R}\right)^2$$

Using Hill's equation in the smooth approximation

$$\frac{d^2y}{ds^2} + \left(\frac{Q_y}{R}\right)^2 y = 0$$

$$\left(\frac{dy'}{dt}\right)^{impedance} = \frac{d}{dt}\left(\frac{dy}{dt}\cdot\frac{dt}{ds}\right) = \frac{d}{dt}\left(\frac{v_y}{v}\right) = \frac{1}{m_0\gamma v}\frac{dp_y}{dt}$$

$$= \frac{F_y^{impedance}}{m_0\gamma v}$$

Particle rest mass

Relativistic mass factor $\gamma = \frac{1}{\sqrt{1-\beta^2}}$

Simplifying and linearizing Vlasov equation

Stationary distribution

Perturbation

Equations of motion

Simplification - Linearization

Coordinate transform

Perturbation decomposition

Reduction variables

Impedance force

Final equation

$$\psi = \psi_0 + \Delta\psi$$

$$\frac{\partial \Delta\psi}{\partial t} + \frac{\partial \psi}{\partial y}\frac{dy}{dt} + \frac{\partial \psi}{\partial y'}\frac{dy'}{dt} = 0$$

$$\Leftrightarrow \frac{\partial \Delta\psi}{\partial t} + \frac{\partial \psi}{\partial y} v y' + \frac{\partial \psi}{\partial y'}\left(\frac{F_y^{impedance}}{m_0\gamma v} - v y \left(\frac{Q_y}{R}\right)^2\right) = 0$$

$$\Leftrightarrow \frac{\partial \Delta\psi}{\partial t} + \left(\frac{\partial \psi_0}{\partial y} + \frac{\partial \Delta\psi}{\partial y}\right) v y' + \left(\frac{\partial \psi_0}{\partial y'} + \frac{\partial \Delta\psi}{\partial y'}\right)\left(\frac{F_y^{imp}}{m_0\gamma v} - v y \left(\frac{Q_y}{R}\right)^2\right) = 0$$

$$\Leftrightarrow \frac{\partial \Delta\psi}{\partial t} + \cancel{\left(\frac{\partial \psi_0}{\partial y} v y' - \frac{\partial \psi_0}{\partial y'} v y \left(\frac{Q_y}{R}\right)^2\right)}$$

Identically zero from Vlasov eq. on ψ_0

$$+\left(\frac{\partial \Delta\psi}{\partial y} v y' - \frac{\partial \Delta\psi}{\partial y'} v y \left(\frac{Q_y}{R}\right)^2 + \frac{\partial \psi_0}{\partial y'}\frac{F_y^{imp}}{m_0\gamma v}\right) + \cancel{\frac{\partial \Delta\psi}{\partial y'}\frac{F_y^{imp}}{m_0\gamma v}} = 0$$

2nd order

Transformation of coordinates

Since the unperturbed distribution is a function of the action J_y alone, it's natural to switch to action-angle variables:

$$y = \sqrt{\frac{2J_y R}{Q_y}} \cos\theta_y\,, \qquad y' = \sqrt{\frac{2J_y Q_y}{R}} \sin\theta_y$$

$$J_y = \frac{1}{2}\left[y^2 \frac{Q_y}{R} + y'^2 \frac{R}{Q_y}\right], \qquad \theta_y = \operatorname{atan}\left(\frac{R}{Q_y}\frac{y'}{y}\right)$$

and for the partial derivatives:

$$\frac{\partial J_y}{\partial y} = \frac{y\, Q_y}{R}, \qquad \frac{\partial J_y}{\partial y'} = \frac{y' R}{Q_y}$$

$$\frac{\partial \theta_y}{\partial y} = -\sqrt{\frac{Q_y}{2J_y R}} \sin\theta_y\,, \qquad \frac{\partial \theta_y}{\partial y'} = \sqrt{\frac{R}{2J_y Q_y}} \cos\theta_y$$

Transformation of coordinates

Stationary distribution
Perturbation
Equations of motion
Simplification - Linearization
Coordinate transform
Perturbation decomposition
Reduction variables
Impedance force
Final equation

Using the partial derivatives computed previously:

$$\frac{\partial \psi_0}{\partial y'} = \frac{d\psi_0}{dJ_y} \cdot \frac{\partial J_y}{\partial y'} = \psi_0'(J_y)\frac{y'R}{Q_y}$$

$$\frac{\partial \Delta\psi}{\partial y} = \frac{\partial \Delta\psi}{\partial J_y} \cdot \frac{\partial J_y}{\partial y} + \frac{\partial \Delta\psi}{\partial \theta_y} \cdot \frac{\partial \theta_y}{\partial y} = \frac{\partial \Delta\psi}{\partial J_y} \cdot \frac{y\,Q_y}{R} + \frac{\partial \Delta\psi}{\partial \theta_y} \cdot \left(-\sqrt{\frac{Q_y}{2J_yR}}\sin\theta_y\right) \quad \times vy'$$

$$\frac{\partial \Delta\psi}{\partial y'} = \frac{\partial \Delta\psi}{\partial J_y} \cdot \frac{\partial J_y}{\partial y'} + \frac{\partial \Delta\psi}{\partial \theta_y} \cdot \frac{\partial \theta_y}{\partial y'} = \frac{\partial \Delta\psi}{\partial J_y} \cdot \frac{y'R}{Q_y} + \frac{\partial \Delta\psi}{\partial \theta_y} \cdot \left(\sqrt{\frac{R}{2J_yQ_y}}\cos\theta_y\right) \quad \times vy\left(\frac{Q_y}{R}\right)^2$$

such that

$$\frac{\partial \Delta\psi}{\partial y} vy' - \frac{\partial \Delta\psi}{\partial p_y} vy\left(\frac{Q_y}{R}\right)^2 = -\frac{\partial \Delta\psi}{\partial \theta_y} Q_y \frac{v}{R}$$

Angular revolution frequency $\omega_0 = \frac{v}{R}$

An already simpler Vlasov equation

Stationary distribution

Perturbation

Equations of motion

Simplification - Linearization

Coordinate transform

Perturbation decomposition

Reduction variables

Impedance force

Final equation

$$\frac{d\psi}{dt} = 0$$

$$\Leftrightarrow \frac{\partial \Delta\psi}{\partial t} - \frac{\partial \Delta\psi}{\partial \theta_y} Q_y \omega_0 + \psi_0'(J_y) \frac{1}{m_0 \gamma v} \sqrt{\frac{2 J_y R}{Q_y}} \sin\theta_y \, F_y^{imp} = 0$$

→ Only one partial derivative of the coordinates is left.

Writing the perturbation

- Stationary distribution
- Perturbation
- Equations of motion
- Simplification - Linearization
- Coordinate transform
- **Perturbation decomposition**
- Reduction variables
- Impedance force
- Final equation

Now it's time to take a closer look at $\Delta\psi$:

➢ We first make just one assumption: its time dependence is that of a single mode of coherent angular frequency Ω, close to $\omega_0 Q_y$ (with $\omega_0 \equiv \frac{v}{R}$ the angular revolution frequency) – well justified when one computes a growing instability mode, which supersedes exponentially any other mode:

$$\Delta\psi(J_y, \theta_y; t) = \Delta\psi_1(J_y, \theta_y) e^{j\Omega t}$$

➢ Then we decompose this mode using a Fourier series of the angle θ_y:

$$\Delta\psi(J_y, \theta_y; t) = e^{j\Omega t} \sum_{p=-\infty}^{+\infty} f_p(J_y) e^{jp\theta_y}$$

Reducing the number of variables

Stationary distribution
Perturbation
Equations of motion
Simplification - Linearization
Coordinate transform
Perturbation decomposition
Reduction variables
Impedance force
Final equation

Injecting the perturbation into Vlasov equation, we can simplify it even more:

$$\frac{\partial \Delta\psi}{\partial t} - \frac{\partial \Delta\psi}{\partial \theta_y} Q_y \omega_0 + \psi_0'(J_y) \sqrt{\frac{2J_y R}{Q_y}} \sin\theta_y \frac{F_y^{imp}}{m_0 \gamma v} = 0$$

$$\Leftrightarrow e^{j\Omega t} \sum_{p=-\infty}^{+\infty} f_p(J_y) e^{jp\theta_y} (j\Omega - jpQ_y\omega_0) = -\psi_0'(J_y) \sqrt{\frac{2J_y R}{Q_y}} \sin\theta_y \frac{F_y^{imp}}{m_0 \gamma v}$$

$$\Leftrightarrow e^{j\Omega t} \sum_{p=-\infty}^{+\infty} f_p(J_y) e^{jp\theta_y} (j\Omega - jpQ_y\omega_0) = -\psi_0'(J_y) \sqrt{\frac{2J_y R}{Q_y}} \frac{e^{j\theta_y} - \cancel{e^{-j\theta_y}}}{2j} \frac{F_y^{imp}}{m_0 \gamma v}$$

Term by term identification leads to

$$f_p(J_y) = 0 \text{ for any } p \neq \pm 1$$

Then, the assumption $\Omega \approx Q_y \omega_0$, gives

$$f_{-1}(J_y) \approx 0$$

Reducing the number of variables

- Stationary distribution
- Perturbation
- Equations of motion
- Simplification - Linearization
- Coordinate transform
- Perturbation decomposition
- Reduction variables
- Impedance force
- Final equation

We end-up with (taking away the $e^{j\theta_y}$ on both sides):

$$e^{j\Omega t} f_1(J_y)(\Omega - Q_y\omega_0) = \psi_0'(J_y)\sqrt{\frac{J_y R}{2Q_y}}\,\frac{F_y^{imp}(t)}{m_0\gamma v}$$

This already gives us the J_y dependency of the perturbative distribution!

$$f_1(J_y) \propto \psi_0'(J_y)\sqrt{\frac{J_y R}{2Q_y}}$$

$$\Rightarrow \Delta\psi(J_y, \theta_y; t) = D e^{j\Omega t} e^{j\theta_y} \psi_0'(J_y)\sqrt{\frac{J_y R}{2Q_y}}$$

Constant

Force from impedance

Stationary distribution
Perturbation
Equations of motion
Simplification - Linearization
Coordinate transform
Perturbation decomposition
Reduction variables
Impedance force
Final equation

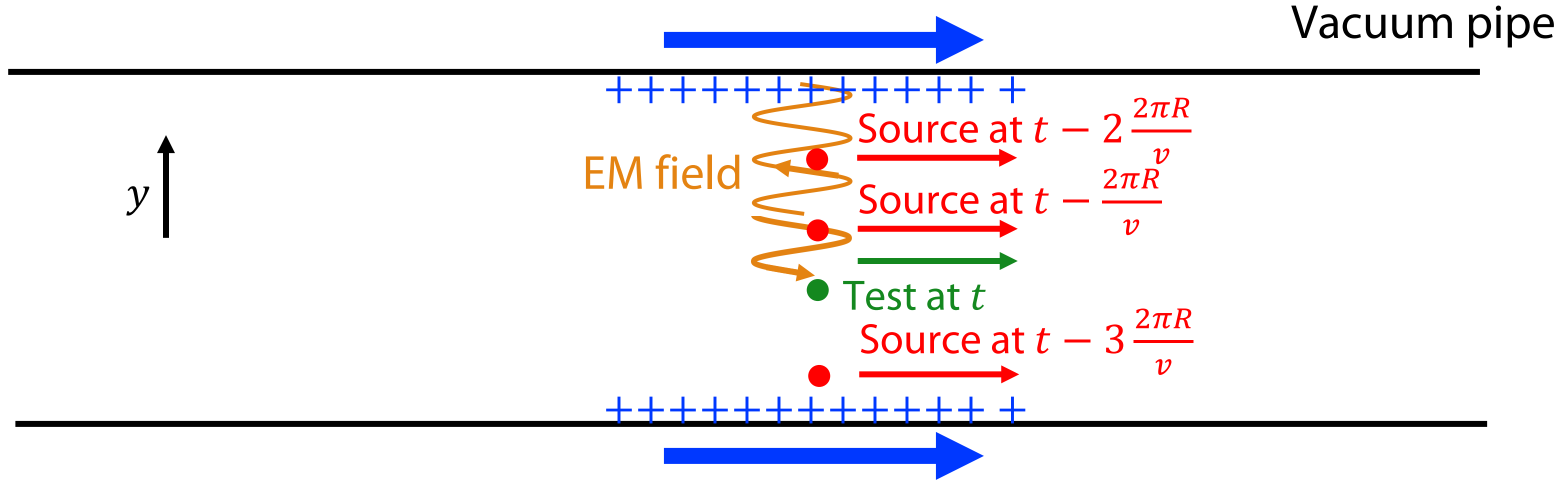
Summing the wakes from the bunch passage at all previous (and subsequent) turns

Wake function, assumed constant within a single-bunch

$$F_y^{imp} = \frac{e^2}{2\pi R} \sum_{k=-\infty}^{+\infty} \iint dy\, dy'\, \psi\left(y, y'; t - k\frac{2\pi R}{v}\right) y W_y(2\pi k R)$$

Integration over phase space

Revolution time



Force from impedance

Stationary distribution
Perturbation
Equations of motion
Simplification - Linearization
Coordinate transform
Perturbation decomposition
Reduction variables
Impedance force
Final equation

$$F_y^{imp} = \frac{e^2}{2\pi R} \sum_{k=-\infty}^{+\infty} \iint dy\, dy'\, \psi\left(y, y'; t - k\frac{2\pi R}{v}\right) y W_y(2\pi k R)$$

$$= \frac{e^2}{2\pi R} \sum_{k=-\infty}^{+\infty} \iint dy\, dy'\, \Delta\psi\left(y, y'; t - k\frac{2\pi R}{v}\right) y W_y(2\pi k R)$$

F_y^{imp} only depends on the perturbation $\Delta\psi$ because the stationary distribution is centered around the orbit ($y = 0$):

$$\iint dy\, dy'\, \psi_0(y, y') y = 0$$

Force from impedance

Stationary distribution

Perturbation

Equations of motion

Simplification - Linearization

Coordinate transform

Perturbation decomposition

Reduction variables

Impedance force

Final equation

After the usual change of variables $(y, y') \rightarrow (J_y, \theta_y)$:

$$F_y^{imp} = \frac{e^2}{2\pi R} \sum_{k=-\infty}^{+\infty} W_y(2\pi k R)$$

$$\times \iint dJ_y \, d\theta_y \, \Delta\psi\left(J_y, \theta_y; t - k\frac{2\pi R}{v}\right) \underbrace{\sqrt{\frac{2 J_y R}{Q_y}} \cos\theta_y}_{y}$$

Force from impedance

Stationary distribution
Perturbation
Equations of motion
Simplification - Linearization
Coordinate transform
Perturbation decomposition
Reduction variables
Impedance force
Final equation

Using what we know from the perturbation

$$\Delta\psi(J_y,\theta_y;t) = De^{j\Omega t}e^{j\theta_y}\psi_0'(J_y)\sqrt{\frac{J_yR}{2Q_y}}$$

we get

$$F_y^{imp} = \frac{e^2De^{j\Omega t}}{2\pi Q_y}\sum_{k=-\infty}^{+\infty} e^{\frac{-j2\pi k\Omega R}{v}}W_y(2\pi kR)\iint dJ_y\, d\theta_y J_y\psi_0'(J_y)\cos\theta_y e^{j\theta_y}$$

$$= -\frac{Ne^2D}{4\pi Q_y}e^{j\Omega t}\sum_{k=-\infty}^{+\infty} e^{\frac{-j2\pi k\Omega R}{v}}W_y(2\pi kR)$$

Can we simplify this?

Number of particles

from

$$\int_0^\infty dJ_y\, J_y\psi_0'(J_y) = \cancel{[J_y\psi_0(J_y)]_0^\infty} - \int_0^\infty dJ_y\psi_0(J_y) = -\frac{N}{2\pi}$$

and $$\int_0^{2\pi} d\theta_y\, \mathrm{e}^{\mathrm{j}\theta_y}\cos\theta_y = \pi$$

Force from impedance

Stationary distribution

Perturbation

Equations of motion

Simplification - Linearization

Coordinate transform

Perturbation decomposition

Reduction variables

Impedance force

Final equation

Recall the definition of a wake function as a Fourier transform of the impedance:

$$W_y(z) = -\frac{j}{2\pi}\int_{-\infty}^{\infty} d\omega e^{j\omega\frac{z}{v}} Z_y(\omega)$$

We get

$$\sum_{k=-\infty}^{+\infty} e^{\frac{-j2\pi k\Omega R}{v}} W_y(2\pi kR) = \frac{-j}{2\pi}\int_{-\infty}^{+\infty} d\omega Z_y(\omega) \sum_{k=-\infty}^{+\infty} e^{\frac{-j2\pi kR}{v}(\Omega-\omega)}$$

$$= \frac{-j}{2\pi}\int_{-\infty}^{+\infty} d\omega Z_y(\omega) \sum_{k=-\infty}^{+\infty} \delta\left(\frac{\Omega R}{v} + k - \frac{\omega R}{v}\right)$$

Dirac comb

$$= \frac{-j\omega_0}{2\pi}\sum_{k=-\infty}^{+\infty} Z_y\left(\Omega + k\omega_0\right)$$

$$\omega_0 = \frac{v}{R}$$

Final expression of Vlasov equation…

- Stationary distribution
- Perturbation
- Equations of motion
- Simplification - Linearization
- Coordinate transform
- Perturbation decomposition
- Reduction variables
- Impedance force
- Final equation

Dropping $D, e^{j\Omega t}, \psi_0'(J_y), \sqrt{\frac{J_y R}{2Q_y}}$ on both sides:

$$\Omega - Q_y\omega_0 = \frac{j\omega_0 N e^2}{8\pi^2 m_0 \gamma v Q_y} \sum_{k=-\infty}^{+\infty} Z_y\left(\Omega + k\omega_0\right)$$

In principle, this is a non-linear equation of Ω.

Still $Z_y(\omega)$ is typically is very smooth (at the level of the tune shifts we are looking for) such that in the right-hand side one can make the approximation:

$$\Omega \approx Q_y\omega_0$$

and we get finally

$$\boxed{\Omega - Q_y\omega_0 = \frac{j\omega_0 N e^2}{8\pi^2 m_0 \gamma v Q_y} \sum_{k=-\infty}^{+\infty} Z_y\left(Q_y\omega_0 + k\omega_0\right)}$$

which is a fully analytical formula giving the frequency shift of the mode → that's our Vlasov solver!

Direct Vlasov solvers – Summary part I

- We introduced the topic of collective effects, and more specifically transverse instabilities from impedance.
- We provided some motivation for an alternative to multi-particle simulations.
- We sketched a brief overview of the underlying principles of Vlasov equation, and its historical uses.
- We built our first “naive” Vlasov solver for longitudinally rigid bunches, providing a general outline of the method.

$\Rightarrow$ Some algebra is required, but not much advanced knowledge is needed, in order to build a Vlasov solver.

$\Rightarrow$ But with a few more tools, we can do it more efficiently and elegantly – this is part II.

Appendix

Another alternative for instability computation

➢ It is also possible to adopt an approach "in-between" multi-particle simulations and Vlasov solvers, still computing instability modes:

- assume a single "macro-particle" in transverse
- discretize the longitudinal phase space using a 2D mesh, in polar coordinates
  → transfer map in matrix form
  → diagonalization
  → modes

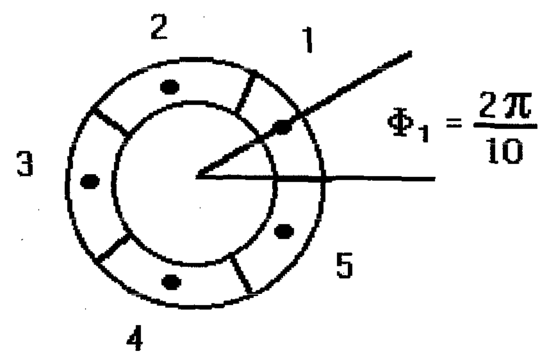


Fig. 2. Division of the longitudinal phase space into mesh elements for the hollow beam model.

Courtesy V. V. Danilov & E. A. Perevedentsev [1]

⇒ circulant matrix model [1], later extended by S. White and X. Buffat [2].

[1] **V. V. Danilov & E. A. Perevedentsev**, *Nucl. Instr. Meth. in Phys. Res. A 391 (1997) pp. 77-92.*
[2] **S. White** et al, *Phys. Rev. ST Accel. Beams 17 (2014), 041002.*

Invariant of motion: linear optics

Starting from Hill's equation (in the smooth approximation):

$$\frac{d^2y}{ds^2} + \left(\frac{Q_y}{R}\right)^2 y = 0$$

$$\times\left(\frac{dy}{ds}\right) \qquad \Rightarrow \frac{d^2y}{ds^2}\cdot\frac{dy}{ds} + \left(\frac{Q_y}{R}\right)^2 y\,\frac{dy}{ds} = 0$$

$$\Rightarrow \frac{1}{2}\left\{\frac{d}{ds}\left[\left(\frac{dy}{ds}\right)^2\right] + \left(\frac{Q_y}{R}\right)^2 \frac{d}{ds}(y^2)\right\} = 0$$

$$\times\left(\frac{R}{Q_y}\right)\int ds \qquad \Rightarrow \frac{1}{2}\frac{R}{Q_y}\left[\left(\frac{dy}{ds}\right)^2 + \left(\frac{Q_y}{R}\right)^2 y^2\right] = \text{constant}$$

$$\Rightarrow \frac{1}{2}\left[\frac{R}{Q_y}\left(\frac{p_y}{m_0\gamma v}\right)^2 + \frac{Q_y}{R}y^2\right] = \text{constant}$$

using $\frac{dy}{ds} = \frac{dy}{dt}\frac{dt}{ds} = \frac{v_y}{v} = \frac{p_y}{m_0\gamma v}$