Basic Mathematics for Accelerators

Rende Steerenberg - CERN - Beams Department

CERN **A**ccelerator **S**chool
Introduction to Accelerator Physics
31 August – 12 September 2014
Prague – Czech Republic

Marathon

~~PROGRAMME~~ FOR INTRODUCTION TO ACCELERATOR PHYSICS
31 August – 12 September, 2014, Prague, Czech Republic

Time	Sunday 31 Aug	Monday 1 September	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday	Monday	Tuesday	Wednesday	Thursday 11 September	Friday 12 September
08:30 – 09:30		Opening Talks V. Petracek J. Pechlat R. Bailey										FELs A. Wolski	Cyclotrons M. Seidel
09:30 – 10:30		Mathematics for Accelerators R. Steerenberg										Collective Effects II G. Franchetti	Applications of Accelerators S. Sheehy
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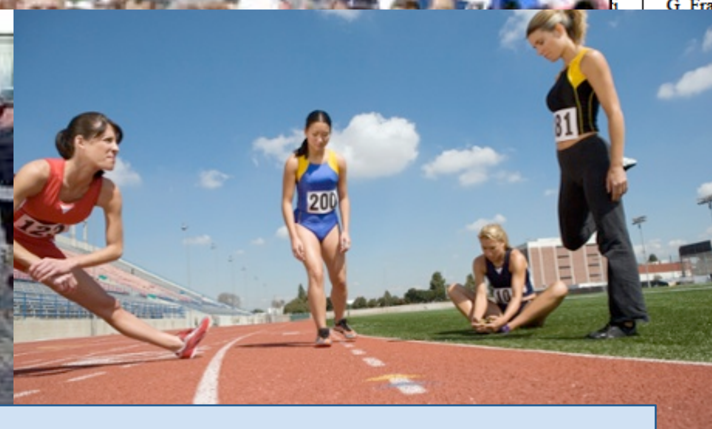
11:00 – 12:00	A R R I V A L											…ial …ral	Putting It All Together W. Herr
												…CH	LUNCH
14:00 – 15:00												…tion …and Its …ences …gger	D E P A R T U R E
15:00 – 16:00		W. Herr										…Gs S. Sheehy	
		TEA	TEA	TEA					TEA	TEA	N	TEA	
16:30 – 17:30	Regis-tration	1 Slide 1 Minute R. Bailey	Private Study	Guided Study on TBD					17:00 Seminar 60 Years of Science for Peace R. Heuer	Seminar Laser Driven Plasma Acceleration D. Margarone	O O N	Seminar Proton Therapy V. Vondracek	
19:30	Buffet Dinner	Dinner	Dinner	Dinner				…er	Dinner	Dinner	Special Dinner	Dinner	



This overview or review of the basic mathematics used in accelerator physics is a part of the warming up

Contents

- What Maths are needed and Why ?
- Differential Equations
- Vector Basics
- Matrices

- **What Maths are needed and Why ?**
- Differential Equations
- Vector Basics
- Matrices

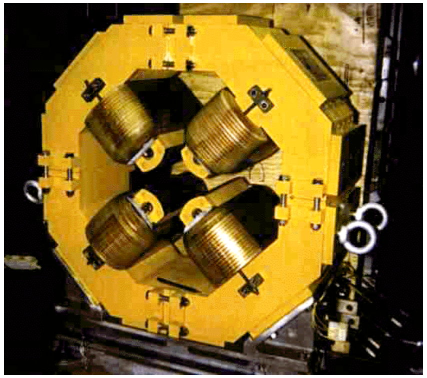
Forces in Accelerators

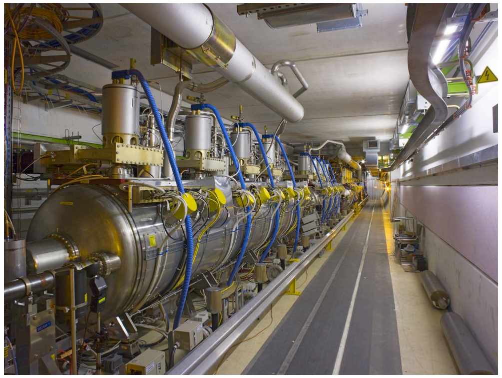
Particles in our accelerator will oscillate around the circumference of the machine under the influence of external forces

Mainly

Magnetic forces in the transverse plane

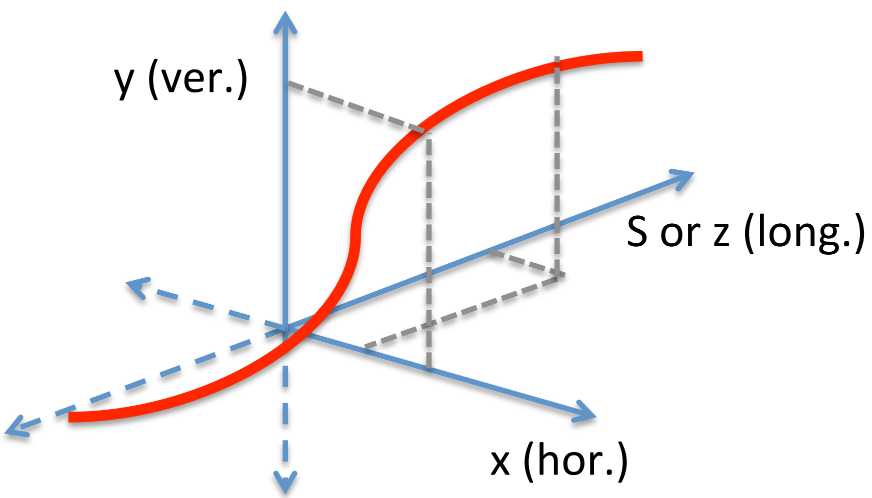
Electric forces in the longitudinal plane



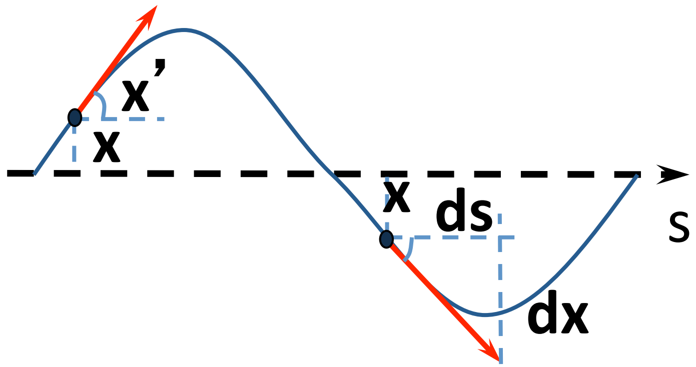


Particle Motion

These external forces result in oscillatory motion of the particles in our accelerator



Decompose x and y motion



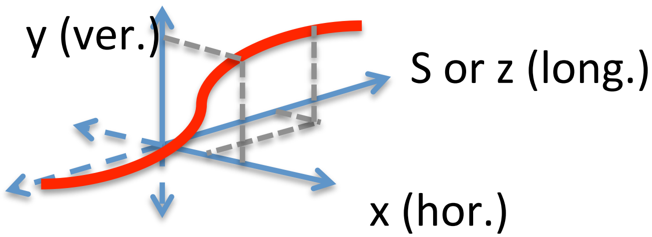
Transverse Motion
&
Longitudinal Motion

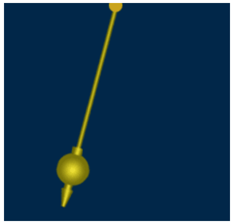
x = hor. Displacement

x' = dx/ds = hor. angle

What maths & Why ?

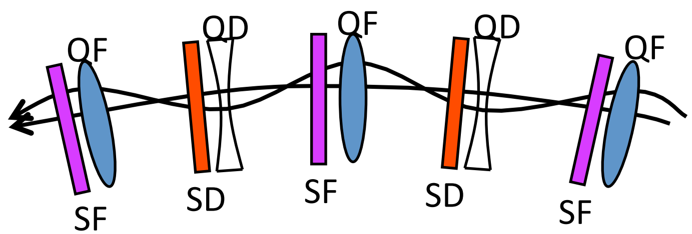
Oscillations of particle in accelerator by **differential equations**





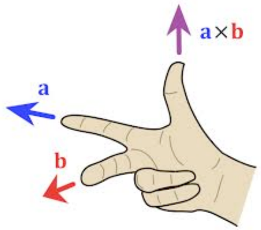
$$\frac{d^2(x)}{dt^2} + (K)x = 0$$

Optics described using **matrices** (also oscillations)

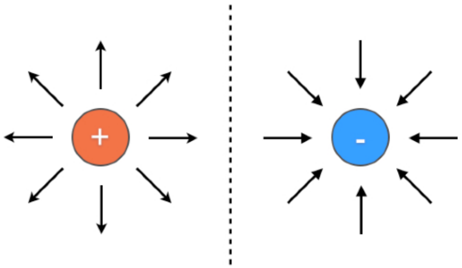


$$\begin{pmatrix} x_2 \\ x_2' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{LK}{(B\rho)} & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_1' \end{pmatrix}$$

Electro – Magnetic fields described by **vectors**



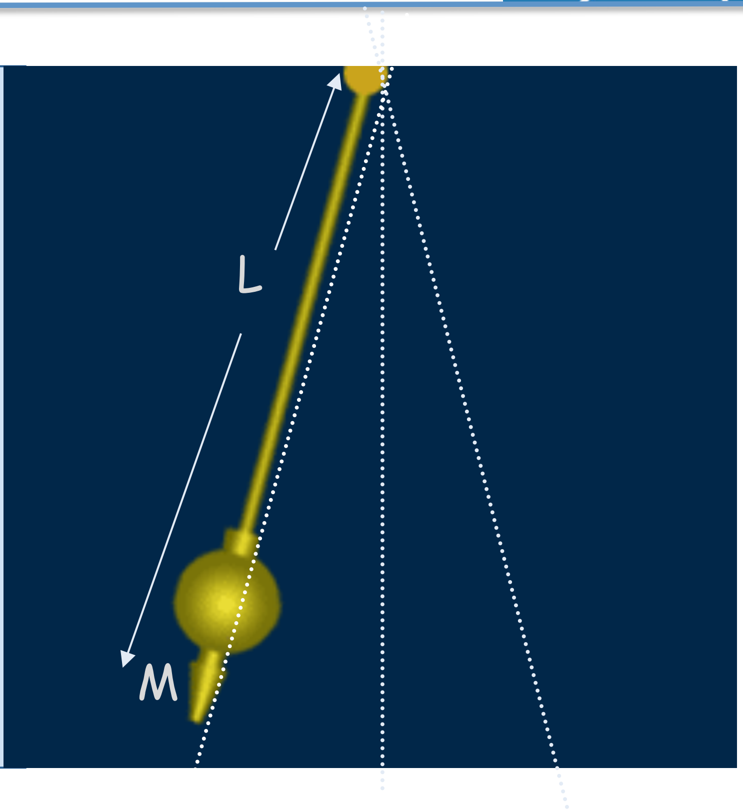
$$F = e(\vec{v} \times \vec{B})$$



- What Maths are needed and Why ?
- **Differential Equations**
- Vector Basics
- Matrices

The Pendulum

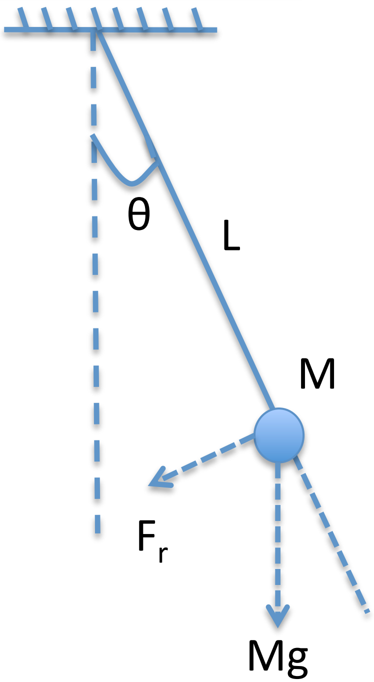
- This motion in accelerators is similar to the motion of a pendulum
- The **length** of the pendulum is **L**
- It has a **mass m** attached to it
- It moves back and forth under the **influence of gravity**



The motion of the pendulum is described by a

Second Order Differential Equation

Simple Harmonic Motion



- Position: $x = L\theta$ (provided θ is small)
- Velocity: $v = \dfrac{dx}{dt} = \dfrac{d(L\theta)}{dt}$
- Acceleration: $a = \dfrac{dv}{dt} = \dfrac{d^2(x)}{dt^2} = \dfrac{d^2(L\theta)}{dt^2}$
- Restoring force: $F_r = -MgSin\theta$

$$MgSin\theta = M\frac{d^2(L\theta)}{dt^2} \quad \Rightarrow \quad \frac{d^2(\theta)}{dt^2} + \frac{g}{L}\theta = 0$$

Hill's equation

Second order differential equation describing a the **Simple Harmonic Motion** of the pendulum

$$\frac{d^2(\theta)}{dt^2} + \frac{g}{L}\theta = 0 \quad \Rightarrow \quad \theta = A\cos\sqrt{\frac{g}{L}}t$$

The motion of the particles in our accelerators can also be described by a **2nd order differential equation**

Hill's equation

$$\frac{d^2(x)}{dt^2} + K(s)x = 0$$

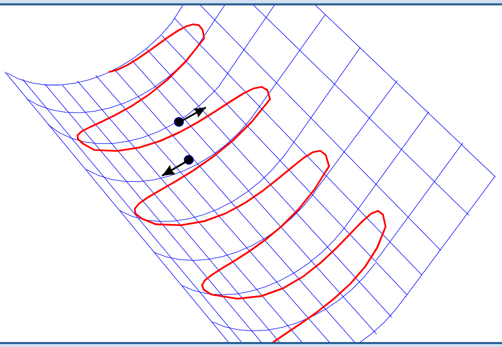
Where:

- restoring forces are magnetic fields
- K(s) is related to the magnetic gradients

Position & Velocity

The differential equation that describes the transverse motion of the particles as they move around our accelerator.

$$\frac{d^2(x)}{dt^2} + (K)x = 0$$



The solution of this second order differential equation describes a Simple Harmonic Motion and needs to be solved for different values of K

For any system, performing simple harmonic motion, where the restoring force is proportional to the displacement, the solution for the displacement will be of the form:

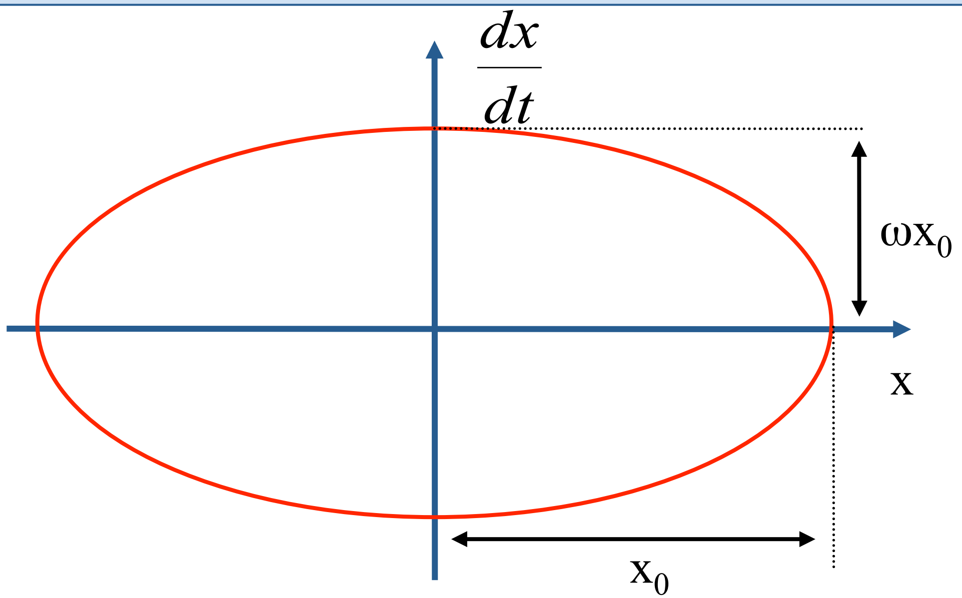
$$x = x_0 \cos(\omega t) \qquad \frac{dx}{dt} = -x_0 \omega \sin(\omega t)$$

Phase Space Plot

Plot the **velocity** as a function of **displacement**:

$$x = x_0 \cos(\omega t)$$

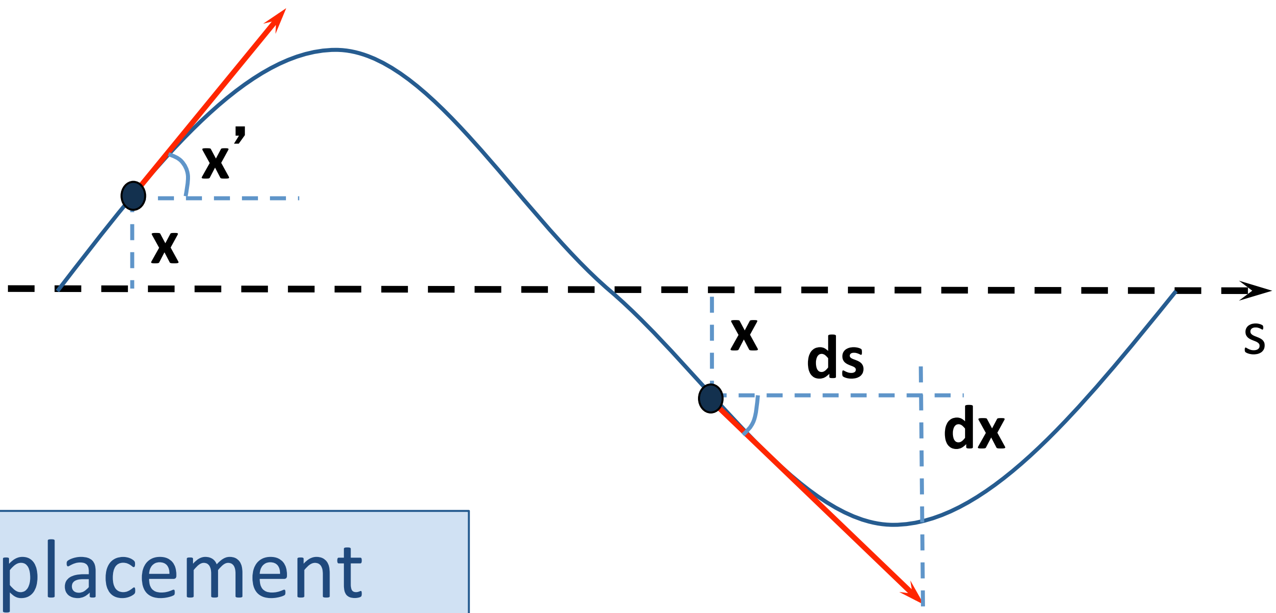
$$\frac{dx}{dt} = -x_0 \omega \sin(\omega t)$$



- It is an ellipse.
- As ωt advances by 2 π it repeats itself.
- This continues for (ω t + k 2π), with k=0,±1, ±2,..,..etc

Oscillations in Accelerators

Under the influence of the **magnetic fields** the **particle oscillate**



x = displacement

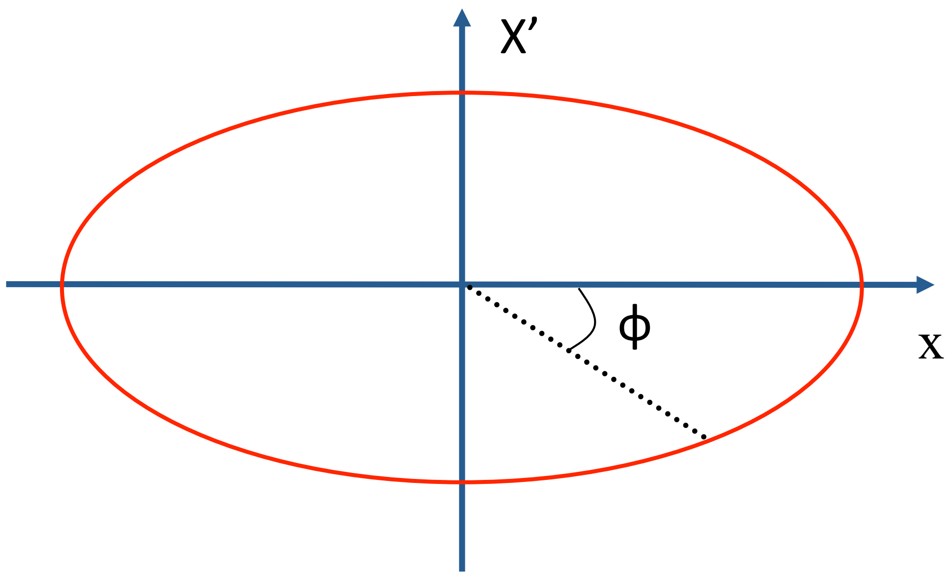
x' = angle = dx/ds

Transverse Phase Space Plot

This changes slightly the Phase Space plot

Position: x

Angle: $x' = \dfrac{dx}{ds}$



- ϕ = ωt is called the **phase angle**
- X-axis is the horizontal or vertical position (or time in longitudinal case).
- Y-axis is the horizontal or vertical phase angle (or energy in longitudinal case)

- What Maths are needed and Why ?
- Differential Equations
- **Vector Basics**
- Matrices

Scalars & Vectors

Scalar:

Simplest physical quantity that can be completely specified by its magnitude, a single number together with the units in which it is measured

Age

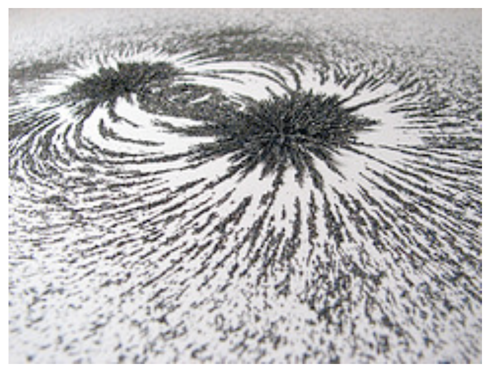
Weight

Temperature

Scalars & Vectors

Vector:

A quantity that requires both a magnitude (≥ 0) and a direction in space to specify it



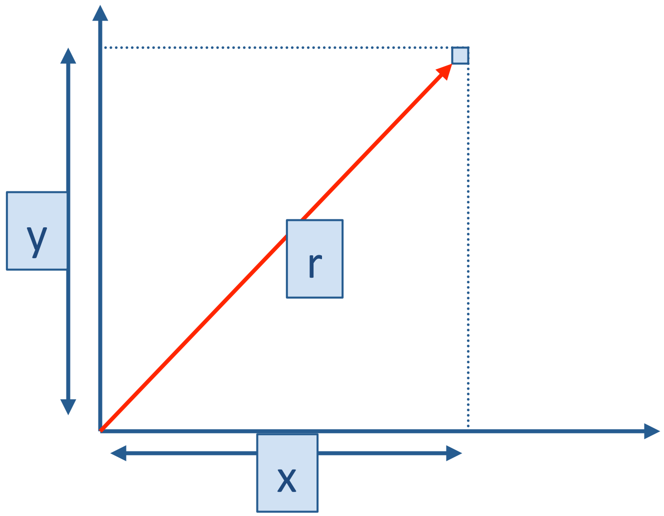
Velocity

(magnetic) fields

Force

Vectors

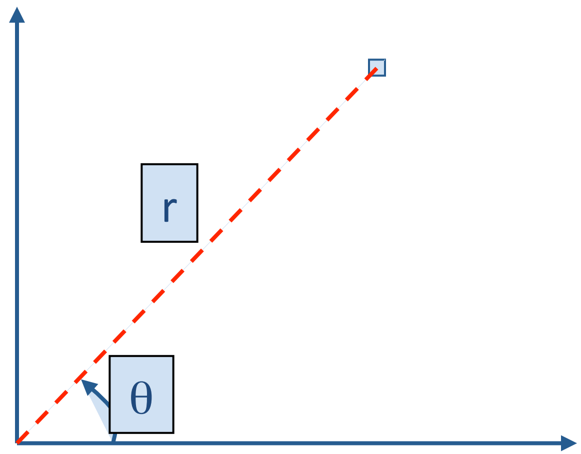
Cartesian coordinates (x,y)



r is the length of the vector

$$r = \sqrt{x^2 + y^2}$$

Polar coordinates (r,θ)

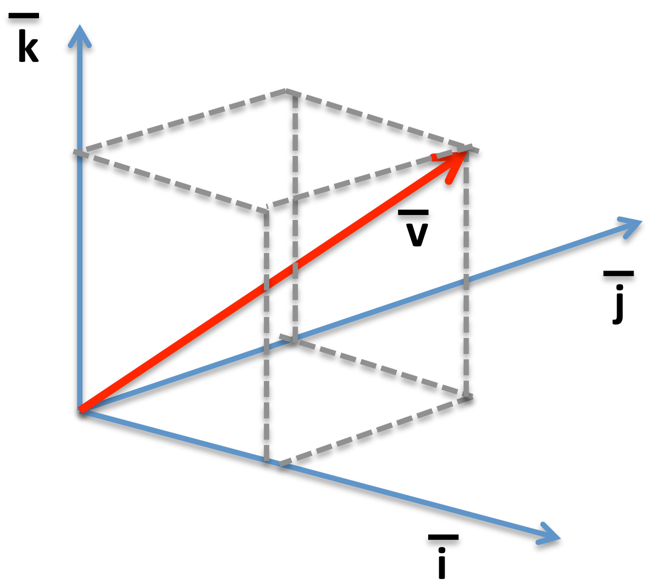


θ gives the direction of the vector

$$\tan(\theta) = \frac{y}{x} \Rightarrow \theta = \arctan\left(\frac{y}{x}\right)$$

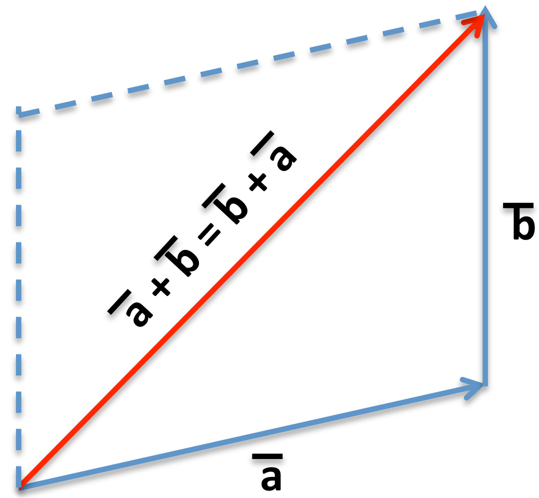
Vector Addition - Subtraction

Vector components



$$\bar{v} = a_x \bar{i} + a_y \bar{j} + a_z \bar{k}$$

addition



Subtraction

$$\bar{a} + (-\bar{b}) = (-\bar{b}) + \bar{a}$$

Vector Multiplication

Two products are commonly defined:

- Vector product → vector
- Scalar product → just a number

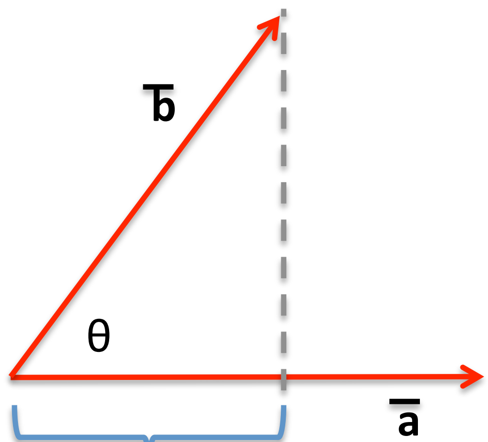
A third is multiplication of a vector by a scalar:

- Same direction as the original one but proportional magnitude
- The scalar can be positive, negative or zero

$$\overline{v} = \alpha \overline{s}$$

Scalar Product

$\bar{a}$ and $\bar{b}$ are two vectors in the in a plane separated by angle θ



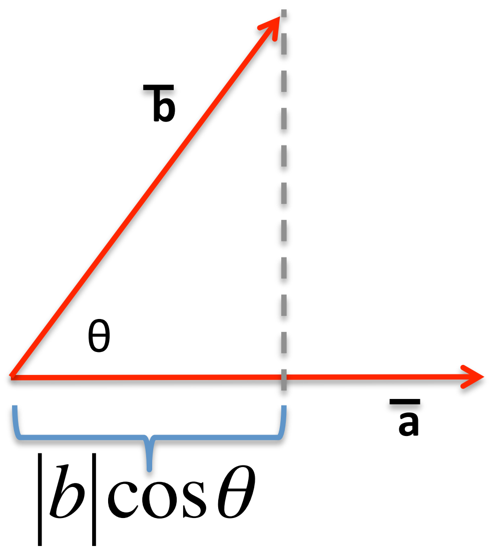
$$\left|b\right|\cos\theta$$

The scalar product of two vectors $\bar{a}$ and $\bar{b}$ is the magnitude vector $\bar{a}$ multiplied by the projection of vector $\bar{b}$ onto vector $\bar{a}$

$$\left|\vec{a}\bullet\vec{b}\right| = \left|\vec{a}\right|\left|\vec{b}\right|\cos(\theta) \quad \text{with} \quad 0 \le \theta \le \pi$$

The scalar product is also called dot product

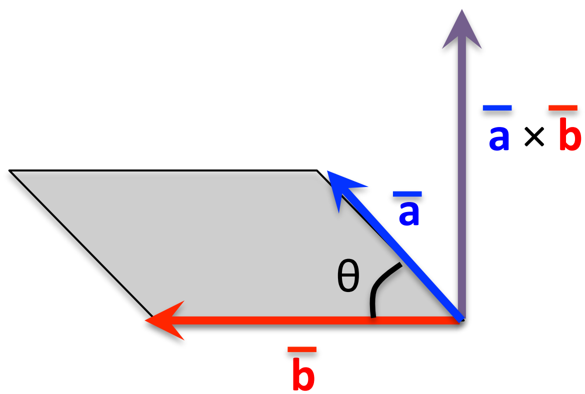
Use of Scalar Products in Physics

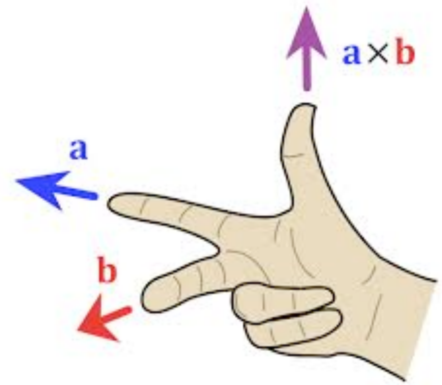


- Test if an angle between vectors is perpendicular
- Determine the angle between two vectors, when expressed in Cartesian form
- Find the component of a vector in the direction of another

Vector Product

$\bar{a}$ and $\bar{b}$ are two vectors in the in a plane separated by angle θ





The cross product $\bar{a} \times \bar{b}$ is defined by:

- **Direction**: $\bar{a} \times \bar{b}$ is perpendicular (normal) on the plane through $\bar{a}$ and $\bar{b}$
- The **length** of $\bar{a} \times \bar{b}$ is the surface of the parallelogram formed by $\bar{a}$ and $\bar{b}$

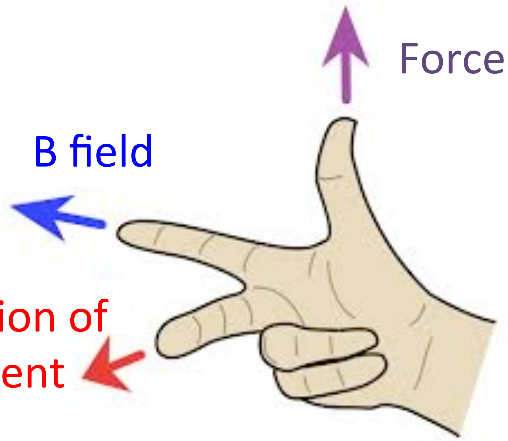
$$\left|\vec{a} \times \vec{b}\right| = \left|\vec{a}\right| \cdot \left|\vec{b}\right| \cdot \sin(\theta)$$

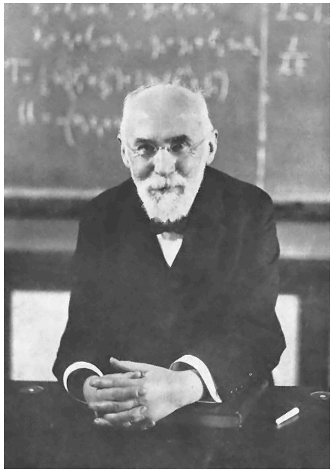
The vector product is also called cross product

Use of Vector Products in Physics

The Lorentz force is a magnetic field

$$F = e(\vec{v} \times \vec{B})$$





The reason why our particles move around our "circular" accelerators under the influence of the magnetic fields

- What Maths are needed and Why ?
- Differential Equations
- Vector Basics
- **Matrices**

Matrices

In physics applications we often encounter sets of simultaneous linear equations. In general we may have M equations with N unknowns, of which some may be expressed by a single matrix equation.

$$Ax = b$$

$$\begin{bmatrix} A_{11} & A_{12} & \rightarrow & A_{1N} \\ A_{21} & A_{22} & \rightarrow & A_{2N} \\ \downarrow & \downarrow & & \downarrow \\ A_{M1} & A_{M2} & \rightarrow & A_{MN} \end{bmatrix} \begin{bmatrix} x \\ x_2 \\ \downarrow \\ x_N \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \downarrow \\ b_M \end{bmatrix}$$

Addition & Subtraction

Adding matrices means simply adding all corresponding individual cells of both matrices an putting the result at the same cell in the sum matrix

$$\begin{bmatrix} s_{11} & \boxed{s_{12}} & s_{13} \\ s_{21} & s_{22} & s_{23} \end{bmatrix} = \begin{bmatrix} a_{11} & \boxed{a_{12}} & a_{13} \\ a_{21} & a_{21} & a_{23} \end{bmatrix} + \begin{bmatrix} b_{11} & \boxed{b_{12}} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}+b_{11} & \boxed{a_{12}+b_{12}} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{21}+b_{22} & a_{23}+b_{23} \end{bmatrix}$$

Subtraction is similar to addition e.g.: $S_{12} = a_{12} + (-b_{12})$

The matrices must be of the same dimension (i.e. both M × N)

Matrix multiplication

Multiplication by a scalar:

$$\lambda \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \end{bmatrix} = \begin{bmatrix} \lambda s_{11} & \lambda s_{12} & \lambda s_{13} \\ \lambda s_{21} & \lambda s_{22} & \lambda s_{23} \end{bmatrix}$$

Multiplication of a matrix and a column vector

$$\begin{bmatrix} y_1 \\ y_2 \\ \downarrow \\ y_M \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & \rightarrow & A_{1N} \\ A_{21} & A_{22} & \rightarrow & A_{2N} \\ \downarrow & \downarrow & & \downarrow \\ A_{M1} & A_{M2} & \rightarrow & A_{MN} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \downarrow \\ x_N \end{bmatrix}$$

$$y_2 = A_{12}x_1 + A_{22}x_2 + \cdots + A_{2N}x_N$$

Matrix multiplication

Multiplication of two matrices

$$\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$$

$$p_{11} = a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} \qquad p_{12} = a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31}$$

Matrix multiplication is associative: $A(BC) = (AB)C$

Multiplication is not commutative: $AB \neq BA$

Multiplication is distributive over addition:

$(A+B)C = AC + BC$ and $C(A+B) = CA + CB$

Null & Identity Matrix

Null Matrix (exceptional case)

$$A0 = 0 = 0A \qquad A + 0 = 0 + A = A$$

Identity Matrix (exceptional case)

$$AI = IA = A$$

$$I_N \begin{bmatrix} 1 & 0 & \rightarrow & 0 \\ 0 & 1 & \rightarrow & 0 \\ \downarrow & \downarrow & & \downarrow \\ 0 & 0 & \rightarrow & 1 \end{bmatrix}$$

Transpose

The *transpose* of a matrix A, often written as A^T is simply the matrix whose columns are the rows of matrix A

$$A = \begin{bmatrix} A_{11} & A_{12} & \rightarrow & A_{1N} \\ A_{21} & A_{22} & \rightarrow & A_{2N} \\ \downarrow & \downarrow & & \downarrow \\ A_{M1} & A_{M2} & \rightarrow & A_{MN} \end{bmatrix} \qquad A^T = \begin{bmatrix} A_{12} & A_{21} & \rightarrow & A_{M1} \\ A_{12} & A_{22} & \rightarrow & A_{M2} \\ \downarrow & \downarrow & & \downarrow \\ A_{1N} & A_{2N} & \rightarrow & A_{MN} \end{bmatrix}$$

If A is an M × N matrix then A^T is an N × M matrix

$$(AB)^T = A^T B^T$$

Trace of a Matrix

Sometime one wishes to derive a single number from a matrix which is denoted by Tr A. This *trace* of A quantity is defined as the sum of the diagonal elements of the matrix

$$A = \begin{bmatrix} A_{11} & A_{12} & \rightarrow & A_{1N} \\ A_{21} & A_{22} & \rightarrow & A_{2N} \\ \downarrow & \downarrow & & \downarrow \\ A_{M1} & A_{M2} & \rightarrow & A_{MN} \end{bmatrix}$$

$$Tr(A) = A_{12} + A_{22} + \cdots + A_{MN}$$

$$Tr(A) = \sum_{i=1}^{N} A_{ii}$$

The *trace* is only defined for square matrices

$$Tr(A \pm B) = Tr(A) \pm Tr(B) \qquad Tr(ABC) = Tr(BCA) = Tr(CAB)$$

Determinant of a Matrix

For a given matrix the *determinant* Det(A) is a single number that depends upon the elements of A

If a matrix is N × N then its *determinant* is denoted by:

$$\det(A) = |A| = \begin{vmatrix} A_{11} & A_{12} & \rightarrow & A_{1N} \\ A_{21} & A_{22} & \rightarrow & A_{2N} \\ \downarrow & \downarrow & & \downarrow \\ A_{N1} & A_{N2} & \rightarrow & A_{NN} \end{vmatrix}$$

The *determinant* is only defined for square matrices

Cofactor and Minor

In order to define the *determinant* of an N × N matrix we will need the *cofactor* and the *minor*

The *minor* of M_{ij} of the element A_{ij} of an N × N matrix A is the determinant of the (N-1) × (N-1) matrix obtained by removing all the elements of the i^{th} row and j^{th} column of A.

The associated *cofactor* is found by multiplying the *minor* by the result of $(-1)^{i+j}$

Lets look at an example using:

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

Cofactor of a Matrix

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Removing all the elements of the 2nd row and the 3rd column of matrix A and forming the *determinant* of the remainder term gives the *minor*

$$M_{23} = \begin{vmatrix} A_{11} & A_{12} \\ A_{31} & A_{32} \end{vmatrix}$$

Multiplying the *minor* by $(-1)^{2+3} = (-1)^5 = -1$ then gives

$$C_{23} = -\begin{vmatrix} A_{11} & A_{12} \\ A_{31} & A_{32} \end{vmatrix}$$

The *determinant* is the sum of the products of the elements of any row or column and their *cofactor* (Laplace expansion).

Determinant of a Matrix

As an example the first of these expansions, using the elements of the 2nd row of the *determinant* and their corresponding *cofactors* we can write the *Laplace expansion*

$$|A| = A_{21}(-1)^{(2+1)}M_{21} + A_{22}(-1)^{(2+2)}M_{22} + A_{23}(-1)^{(2+3)}M_{23}$$

$$= -A_{21}\begin{vmatrix} A_{12} & A_{13} \\ A_{32} & A_{33} \end{vmatrix} + A_{22}\begin{vmatrix} A_{11} & A_{13} \\ A_{31} & A_{33} \end{vmatrix} - A_{23}\begin{vmatrix} A_{11} & A_{12} \\ A_{31} & A_{32} \end{vmatrix}$$

The *determinant* is independent of the row or column chosen

Determinant of a Matrix

We now need to find the order-2 *determinants* of the 2 × 2 *minors* in the *Laplace expansion*

$$\begin{vmatrix} A_{12} & A_{13} \\ A_{32} & A_{33} \end{vmatrix} = A_{12}(-1)^{(1+1)}\left|A_{33}\right| + A_{13}(-1)^{(1+2)}\left|A_{32}\right| = A_{12}A_{33} - A_{13}A_{32}$$

Now repeat the same for the other 2 *minors*

$$\begin{vmatrix} A_{11} & A_{13} \\ A_{31} & A_{33} \end{vmatrix} = A_{11}A_{33} - A_{13}A_{31} \qquad \begin{vmatrix} A_{11} & A_{12} \\ A_{31} & A_{32} \end{vmatrix} = A_{11}A_{32} - A_{12}A_{31}$$

Determinant of a Matrix

Combing the previous:

$$|A| = -A_{12}(A_{12}A_{33} - A_{13}A_{32}) + A_{22}(A_{11}A_{33} - A_{13}A_{13}) - A_{23}(A_{11}A_{32} - A_{12}A_{31})$$

Instead of taking the 2nd row we could have taken the first row, which would have resulted in:

$$|A| = A_{11}(A_{22}A_{33} - A_{23}A_{32}) - A_{12}(A_{23}A_{31} - A_{21}A_{33}) + A_{13}(A_{21}A_{32} - A_{22}A_{31})$$

Repeating this with a concrete example would result in the same scalar for both cases

Determinant: Concrete Example

$$A = \begin{bmatrix} 1 & 4 & 2 \\ 6 & 2 & 4 \\ 2 & 8 & 6 \end{bmatrix}$$

Use row 1:

$$|A| = -44$$

$$|A| = \begin{bmatrix} 1 & & \\ & 2 & 4 \\ & 8 & 6 \end{bmatrix} \Rightarrow +1(2 \times 6 - 4 \times 8)$$

$$|A| = \begin{bmatrix} & 4 & \\ 6 & & 4 \\ 2 & & 6 \end{bmatrix} \Rightarrow -4(6 \times 6 - 4 \times 2)$$

$$|A| = \begin{bmatrix} & & 2 \\ 6 & 2 & \\ 2 & 8 & \end{bmatrix} \Rightarrow +2(6 \times 8 - 2 \times 2)$$

Inverse Matrix

If a matrix A describes the transformation of an initial vector into a final vector then one could define a matrix that performs the inverse transformation, obtaining the initial vector from the final vector. This inverse matrix is denoted as A^{-1}

$$y = Ax \Leftrightarrow x = A^{-1} y$$

Matrix times Inverse Matrix gives the Identity matrix

$$AA^{-1} = I = \begin{bmatrix} 1 & 0 & \rightarrow & 0 \\ 0 & 1 & \rightarrow & 0 \\ \downarrow & \downarrow & & \downarrow \\ 0 & 0 & \rightarrow & 1 \end{bmatrix}$$

Inverse Matrix

If a matrix A has a *determinant* which is zero, then matrix A is called *singular*, otherwise it is *non-singular*
If a matrix is *non-singular* and then matrix A will have an inverse matrix A^{-1}

Finding the inverse matrix A^{-1} can be done in several ways. One method is to construct the matrix C containing the *cofactors* of the elements of A.
The inverse matrix can then be found by taking the *transpose* of C and divide by the determinant of A

$$(A^{-1})_{ij} = \frac{(C)^T_{ij}}{|A|} = \frac{C_{ji}}{|A|}$$

Inverse Matrix: Concrete Example

$$A = \begin{bmatrix} 1 & 4 & 2 \\ 6 & 2 & 4 \\ 2 & 8 & 6 \end{bmatrix}$$

We previously found: $|A| = -44$

Note: this is non-zero, hence A is non-singular

Some *cofactors*:

$$C_{11} = \begin{vmatrix} 2 & 4 \\ 8 & 6 \end{vmatrix} \qquad C_{23} = \begin{vmatrix} 1 & 4 \\ 2 & 8 \end{vmatrix}$$

$$C = \begin{bmatrix} +C_{11} & -C_{12} & +C_{13} \\ -C_{21} & +C_{22} & -C_{23} \\ +C_{31} & -C_{32} & +C_{33} \end{bmatrix} = \begin{bmatrix} -20 & -24 & 44 \\ -8 & 2 & 0 \\ 12 & 8 & -22 \end{bmatrix}$$

Inverse Matrix: Concrete Example

Transposing the *cofactor* matrix:

$$C = \begin{bmatrix} -20 & -24 & 44 \\ -8 & 2 & 0 \\ 12 & 8 & -22 \end{bmatrix} \Leftrightarrow C^T = \begin{bmatrix} -20 & -8 & 12 \\ -24 & 2 & 8 \\ 44 & 0 & -22 \end{bmatrix}$$

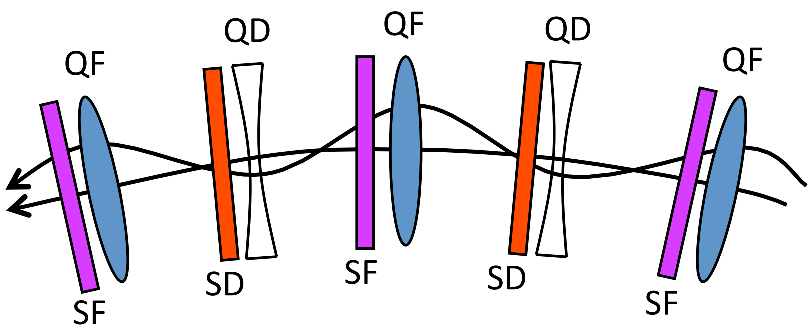
Hence the inverse matrix of A is:

$$A^{-1} = \frac{C^T}{|A|} = \frac{1}{-44}\begin{bmatrix} -20 & -8 & 12 \\ -24 & 2 & 8 \\ 44 & 0 & -22 \end{bmatrix}$$

Use of Inverse Matrix

Each element in an accelerator can be describe by a transfer matrix, describing the change of horizontal and vertical position and angle of our particle(s)

- Modelling the accelerator with these transfer matrices requires matrix multiplication
- Inverse matrices will allow reconstructing initial conditions of the beam, knowing final conditions and the transformation matrices



$$y = Ax \Leftrightarrow x = A^{-1}y$$

Another Practical example

- Changing the current in two sets of quadrupole magnets (F & D) changes the horizontal and vertical tunes (Q_h & Q_v).
- This can be expressed by the following matrix relationship:

$$\begin{pmatrix} \Delta Q_h \\ \Delta Q_v \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \Delta I_F \\ \Delta I_D \end{pmatrix} \quad \text{or} \quad \overline{\Delta Q} = M \, \overline{\Delta I}$$

- Change I_F then I_D independently and measure the changes in Q_h and Q_v
- Calculate the matrix M
- Calculate the inverse matrix M^{-1}
- Use now M^{-1} to calculate the current changes (ΔI_F and ΔI_D) needed for any required change in tune (ΔQ_h and ΔQ_v).

$$\overline{\Delta I} = M^{-1} \overline{\Delta Q}$$

Eigenvalue & Eigenvector

An *eigenvalue* is a number that is derived from a square matrix and is usually represented by λ
We say that a number is the *eigenvalue* for square matrix A if and only if there exists a non-zero vector $\bar{x}$ such that:

$$A\bar{x} = \lambda\bar{x}$$

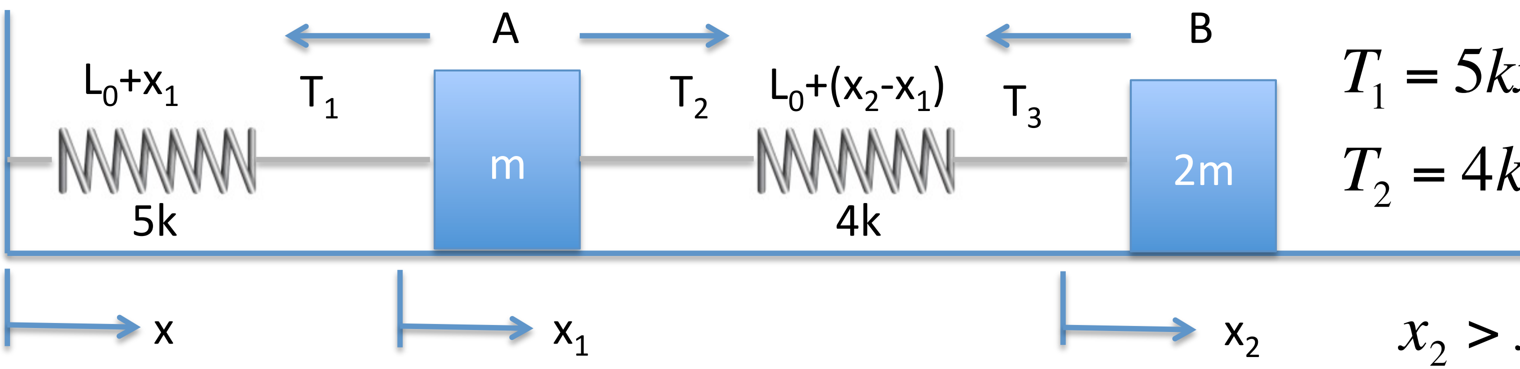
where A is a square matrix, x is the non-zero vector and λ is a non-zero value

In that case:

- λ is the *eigenvalue*
- $\bar{x}$ is the *eigenvector*

Example (Normal modes)

Eigenvalues and *eigenvectors* are particularly useful in simple harmonic systems to find normal modes an displacements



$$T_1 = 5kx_1$$

$$T_2 = 4k(x_2 - x_1)$$

$$x_2 > x_1 > 0$$

Equations of motion for A

$$m\frac{d^2x_1}{dt^2} = T_2 - T_1$$

$$= 4k(x_2 - x_1) - 5kx_1$$

$$= -9kx_1 + 4kx_2$$

Equations of motion for A

$$2m\frac{d^2x_2}{dt^2} = -T_3 = -4k(x_2 - x_1)$$

$$m\frac{d^2x_2}{dt^2} = -9kx_1 + 4kx_2$$

Eigenvalue & Eigenvector

A normal mode of a mechanical system is a motion of the system in which all the masses execute simple harmonic motion with the same angular frequency called normal mode angular frequency

Let: $\frac{d^2 x_1}{dt^2} = -\omega^2 x_1$ and $\frac{d^2 x_2}{dt^2} = -\omega^2 x_2$

The equations of motion become then

$$-9kx_1 + 4kx_2 = -m\omega^2 x_1$$
$$2kx_1 - 2kx_2 = -m\omega^2 x_2$$

In matrix form

$$\begin{bmatrix} -9k & 4k \\ 2k & -2k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -m\omega^2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$A\bar{x} = \lambda \bar{x}$$

Eigenvalue & Eigenvector

Eigenvalue problem to be solved using: $|A-\lambda I|=0 \qquad \lambda=-m\omega^2$

$$|A-\lambda I| = \left|\begin{bmatrix} -9k & 4k \\ 2k & -2k \end{bmatrix} + m\omega^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right| = \begin{vmatrix} -9k+m\omega^2 & 4k \\ 2k & -2k+m\omega^2 \end{vmatrix} = 0$$

$$(m\omega^2)^2 - 11km\omega^2 + 10k = 0$$

$$(m\omega^2 - 10k)(m\omega^2 - k) = 0$$

$\Rightarrow$ $\omega^2 = \dfrac{10k}{m}$ or $\omega^2 = \dfrac{k}{m}$

The normal mode angular frequencies are:

$$\omega_1 = \sqrt{\frac{10k}{m}} \qquad \omega_2 = \sqrt{\frac{k}{m}}$$

Eigenvalue & Eigenvector

Now *eigenvectors* can be calculated to find displacement

$$\omega_1 = \sqrt{\frac{10k}{m}} : \quad (A - \lambda I) = 0 \Rightarrow \begin{bmatrix} -9k & 4k \\ 2k & -2k \end{bmatrix} + m\omega^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -8k & 4k \\ 2k & -k \end{bmatrix}$$

$$\lambda = -m\omega^2$$

$$\Rightarrow \begin{bmatrix} -8k & 4k \\ 2k & -k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \frac{x_1}{x_2} = \frac{1}{2}$$

$$\omega_1 = \sqrt{\frac{10k}{m}} : \quad \begin{bmatrix} k & 4k \\ 2k & 8k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \frac{x_1}{x_2} = \frac{1}{-\frac{1}{4}}$$

Since the final displacements will depend on the initial conditions we can only calculate the displacement ratio between A and B

Eigenvalue & Eigenvector

The example treated a simple harmonic oscillator that was coupled through springs. Using eigenvalues and eigenvectors we could conclude something about:

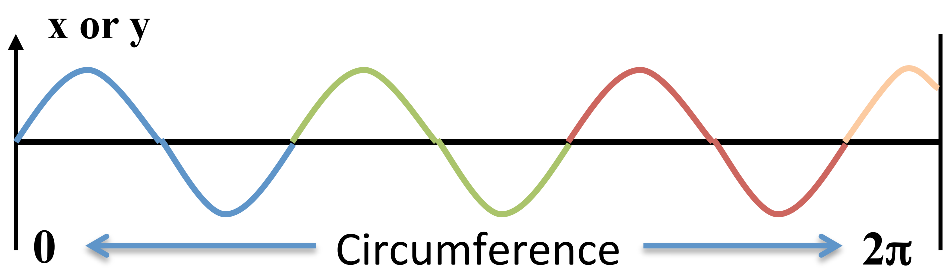
- Oscillation frequencies
- Displacements

The particles in our accelerators make simple harmonic oscillation under the influence of magnetic fields in the horizontal and vertical plane.

Through magnets, but also collective effect, the particle oscillations in the horizontal plane and vertical plane can become coupled. Eigenvalues and Eigenvector can be used to characterise this coupling

The use of Eigenvalues & Eigenvectors

Under the influence of the quadrupoles the particles make oscillations that can be decomposed in horizontal and vertical oscillations:



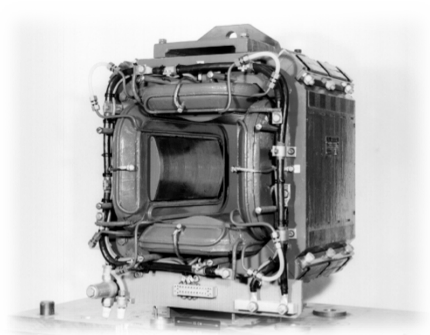
The number of oscillations a particle makes for one turn around the accelerator is called the betatron tune:

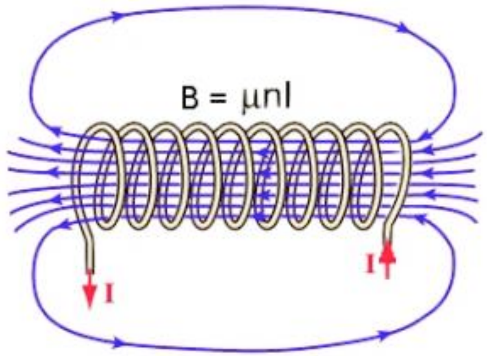
- Q_h or Q_x for the horizontal betatron tune
- Q_v or Q_y for the vertical betatron tune

Eigenvalue and eigenvectors will provide directly information on the tunes, optics and the beam stability

(de-)Coupling through magnets

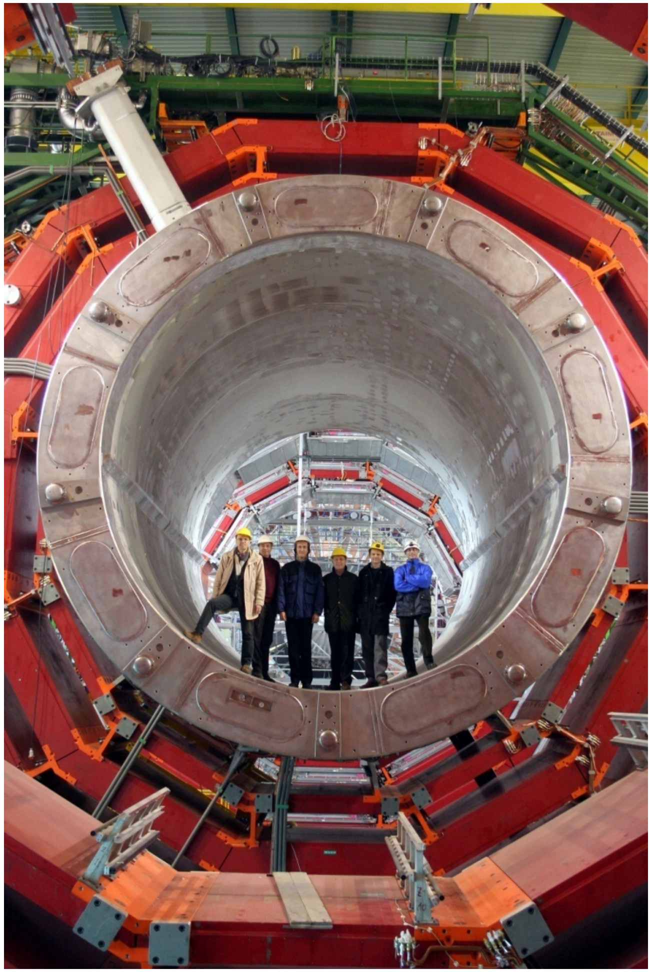
Solenoid fields from the LHC Experiments cause coupling of the oscillations in the horizontal and vertical plane. This coupling needs to be compensated



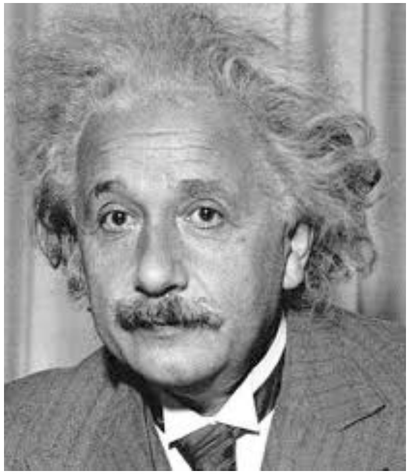


Skew quadrupoles are often used to compensate for coupling introduced by magnetic errors

Eigenvalues and eigenvectors help us providing theoretical insight in this coupling phenomena



Everything must be made as simple as possible. But not simpler….



Albert Einstein

-- Spare Slides --

Solving a Differential Equation

$$\frac{d^2(\theta)}{dt^2} + \left(\frac{g}{L}\right)\theta = 0$$

Differential equation describing the motion of a pendulum at small amplitudes.

Find a solution……Try a good "guess"……

$$\theta = A\cos(\omega t)$$

Differentiate our guess (twice)

$$\frac{d(\theta)}{dt} = -A\omega\sin(\omega t) \quad \text{and} \quad \frac{d^2(\theta)}{dt^2} = -A\omega^2\cos(\omega t)$$

Put this and our "guess" back in the original Differential equation.

$$-\omega^2\cos(\omega t) + \left(\frac{g}{L}\right)\cos(\omega t) = 0$$

Solving a Differential Equation

Now we have to find the solution for the following equation:

$$-\omega^2 \cos(\omega t) + \left(\frac{g}{L}\right)\cos(\omega t) = 0$$

Solving this equation gives:

$$\omega = \sqrt{\frac{g}{L}}$$

The final solution of our differential equation, describing the motion of a pendulum is as we expected :

$$\theta = A\cos\sqrt{\left(\frac{g}{L}\right)}\,t$$

Oscillation amplitude (A)

Oscillation frequency