(Special) Relativity

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(http://cern.ch/Werner.Herr/CAS2014/lectures/Relativity.pdf)

Why Special Relativity ?

- **Most beams are relativistic (or at least fast)**
- **Strong implications for beam dynamics:**
  - **Transverse dynamics (e.g. momentum compaction, radiation, ...)**
  - **Longitudinal dynamics (e.g. transition, ...)**
  - **Collective effects (e.g. space charge, beam-beam, ...)**
  - **Luminosity in colliders**
  - **Particle lifetime and decay (e.g. μ, π, Z_0, Higgs, ...)**

OUTLINE

- **Principle of Relativity (Newton, Galilei)**
  - **Motivation, Ideas and Terminology**
  - **Formalism, Examples**
- **Principle of Special Relativity (Einstein)**
  - **Why ?**
  - **Formalism and Consequences**
  - **Four-vectors and applications (accelerators)**

Mathematical derivations and proofs are mostly avoided ...
(see bibliography)

Reading Material

(1) A. Einstein, Zur Elektrodynamik bewegter Körper, Ann. Phys. 17, (1905).

(2) R.P. Feynman, Feynman lectures on Physics, Vol. 1 + 2, (Basic Books, 2011).

(3) R.P. Feynman, Six not-so-easy pieces, (Basic Books, 2011).

(4) J. Freund, Special Relativity, (World Scientific, 2008).

(5) J.D. Jackson, *Classical Electrodynamics* **(Wiley, 1998 ..)**

(6) J. Hafele and R. Keating, Science 177, (1972) 166.

Principles of Relativity

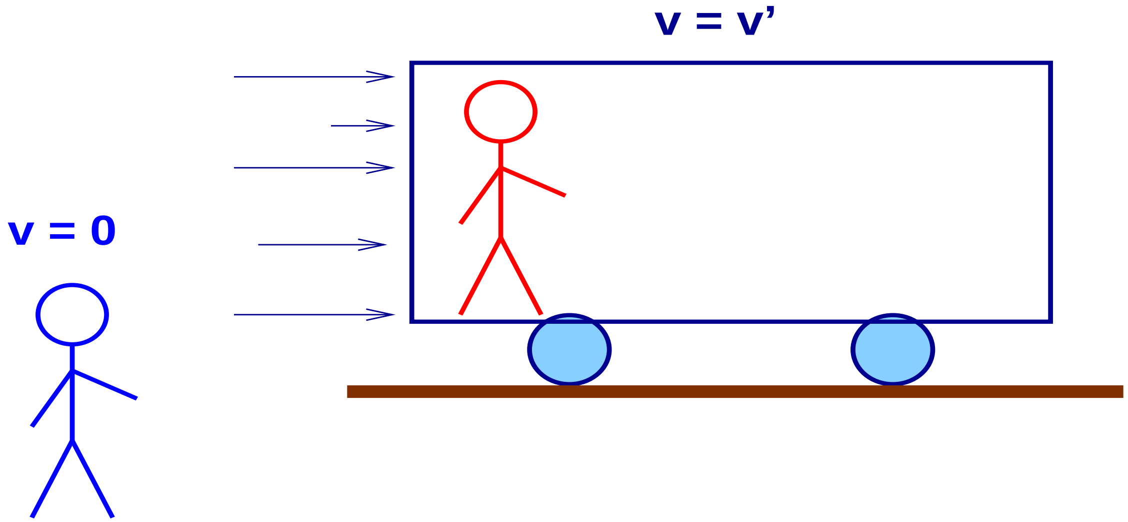
- **Relativity tells us how to relate observations in one frame of reference to the observation of the same thing in another frame of reference**
- **Looking at the same event may be perceived differently from different frames**
- **For example:**
  - **Two people observing from different locations**
  - **Two people moving with constant velocity with respect to one another.**
- **But: the laws of physics must be the same !**

Setting the scene (terminology) ..

- **To describe an observation and physics laws we use:**
  - **Space coordinates:** $\vec{x} = (x, y, z)$
  - **Time:** t
- **What is a "Frame":**
  - **Where we observe physical phenomena and properties as function of their position** $\vec{x}$ **and time** t**.**
  - **In different frames** $\vec{x}$ **and** t **are usually different.**
- **What is an "Event":**
  - **Something happening at** $\vec{x}$ **at time** t **is an "event", given by four numbers** $(x, y, z),\ t$

Example: two frames ...

Assume a frame at rest (F) and another frame (F') moving in x-direction with velocity $\vec{v} = (v', 0, 0)$



- **Passenger performs an experiment and measures the results <u>within</u> his frame**

- **Observer measures the results <u>from</u> the rest frame**

(e.g.: Passenger = Particle, Observer = Accelerator engineer)

Principles of Relativity (Newton, Galilei)

Definition:

A frame moving at constant velocity is an (Inertial System)

Postulate: "Physical laws are invariant in all inertial systems"

invariant:

➡ **the mathematical equations keep the same form**

Example:

we would like to have

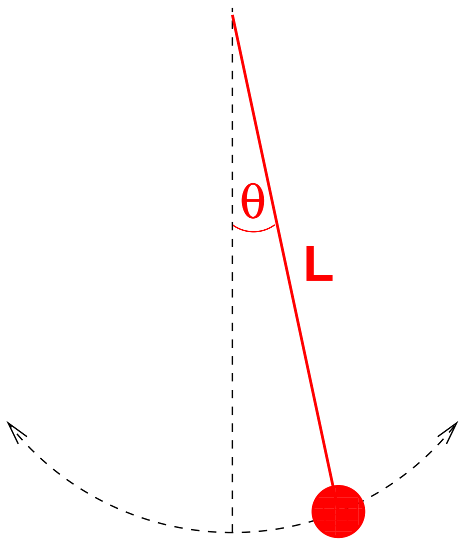
$Force = m \cdot a$ **and** $Force' = m \cdot a'$

Consequence: no system is privileged

In an Inertial Frame:

cannot tell whether system is moving or not (Galilei's ship)

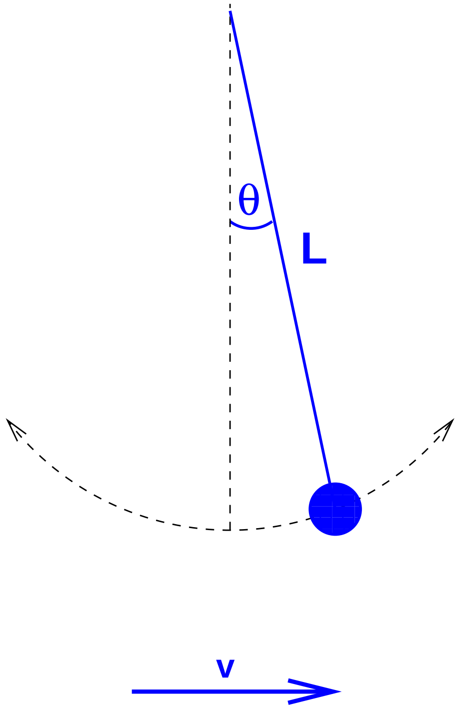
Example 2: Pendulum



$$f = \frac{1}{2\pi}\sqrt{\frac{g}{L}}$$

Frequency of the pendulum is f

Example: Pendulum



$$f' = \frac{1}{2\pi}\sqrt{\frac{g}{L}}$$

Frequency of the pendulum the same in all inertial systems: f = f'

Relativity: so how to we relate observations ?

- **We have observed an event in rest frame F**

  using coordinates (x, y, z) and time t for the description

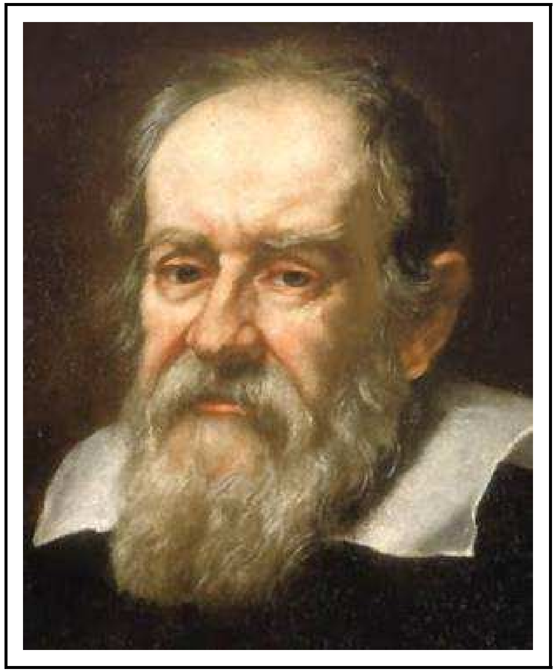
- **How can we describe it seen from a moving frame F' ?**

  - **Need to transform coordinates to the moving system (x', y', z') and t'.**

  - **For Galilei/Newton's principle of relativity need a transformation for:**
    (x, y, z) and t ➡ (x', y', z') and t'.

- **Then laws <u>should</u> look the same, have the same form**

Galilei transformation



$$x' = x - v_x t$$
$$y' = y$$
$$z' = z$$
$$t' = t$$

Galilei transformations relate observations in two frames moving relative to each other (here with constant velocity v_x in x-direction).

Only the position is changing with time

Consequences of Galilei transformation

Frame moves in x-direction with velocity v_x:

Space coordinates are changed, time is not changed !

Space and time are <u>independent</u> quantities

There exists an absolute space where physics laws are the same

There exists an absolute time when physics laws are the same

If you want to try yourself:

F' (in your - moving - frame) F (seen by observer)

- In frame F' (moving horizontally with velocity v'_x):

  Ball starts with vertical velocity $v'_y = 0$

  Ball goes straight down and is accelerated: $v'_y = -g \cdot t'$

- In rest frame for observer F: ball describes a curve (parabola ?)

Equation of motion in moving frame $x'(t')$ and $y'(t')$:

$$x'(t') = 0, \quad v'_y(t') = -g \cdot t'$$
$$y'(t') = \int v'_y(t')\mathrm{d}t' = -\frac{1}{2}gt'^2$$

To get equation of motion in rest frame $x(t)$ and $y(t)$:
Galilei transform: $y(t) \equiv y'(t')$**,** $t \equiv t'$**,** $x(t) = x' + v_x \cdot t = v_x \cdot t$
and get for the trajectories $y(t)$ and $y(x)$ in the rest frame:

$$y(t) = -\frac{1}{2}gt^2 \qquad\qquad y(x) = -\frac{1}{2}g\frac{x^2}{v_x^2}$$

For $y(x)$ we get a parabola, observed from the rest frame.

Another consequence ...

Velocities can be added

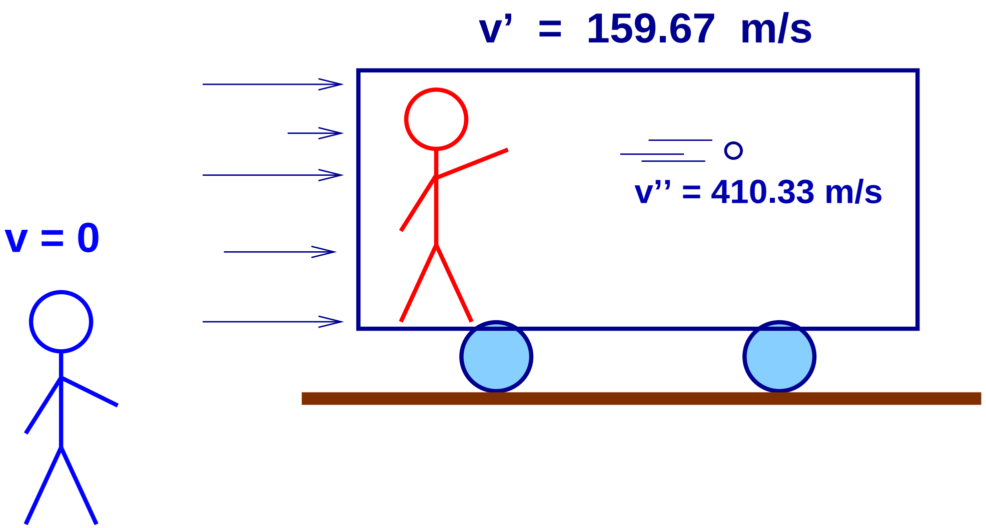
- **From Galilei transformation, take derivative:**

$$x' = x - v_x t$$

$$\dot{x}' = \dot{x} - v_x \quad \rightarrow \quad v' = v - v_x$$

- **For an object moving with velocity v' in a frame moving with velocity v_x we have in rest frame** $v = v' + v_x$

Adding velocities (Galilei)



Fling a ball with 410.33 m/s in a frame moving with 159.67 m/s:

Observed from a non-moving frame:
speed of ping-pong ball: 570 m/s

Where the trouble starts:

A charge moving with a (constant) velocity $\vec{v}(x,t)$:

We get a current density $\vec{j}(x,t)$:

$$\rho(x,t) \longrightarrow \rho(x,t)\cdot\vec{v}(x,t) = \vec{j}(x,t)$$

Induces a magnetic field (see lecture before):

$$\nabla\times\vec{B} = \mu_0\vec{j}$$

- **Observation in moving frame of the charge: only electric field**
- **Observation in rest frame: electric and magnetic fields**

Enter Einstein:

Zur Elektrodynamik bewegter Körper **leading to** *Special Relativity*

Problems with Galilei transformation

- **Maxwell's equations are different when Galilei transformations are applied**
- **Could exceed speed of light when velocities are added:**

  $0.8 \cdot c + 0.5 \cdot c = 1.3 \cdot c$ **?**

  c = 299792458.000 m/s
- **From experiments: Speed of light <u>in vacuum</u> is upper limit and the same in <u>all</u> frames and <u>all</u> directions (not affected by the motion of the source)**

Speed of light ...

- **Speed of light in vacuum c is the maximum speed for:**
  - **Propagation of matter**
  - **Propagation of information**
  - **Moving as group velocity or signal velocity**
- **Phase velocity can exceed speed of light !**
  - **Phase velocity of matter waves**

Principle(s) of Special Relativity (Einstein)

All physical laws (e.g. Maxwell's) in inertial frames must have equivalent forms

Speed of light in vacuum c must be the same in all frames

Elementary charge e must be the same in all frames

- **Cannot determine absolute speed of an inertial frame measuring speed of light**
- **Need Transformations (not Galileaen) which make the physics laws (Maxwell !) look the same !**

Coordinates must be transformed differently

- **Transformation must keep speed of light constant**

Constant speed of light requires:

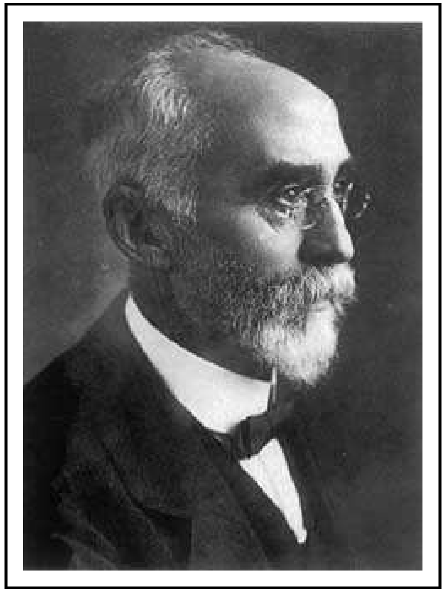
$$x^2 + y^2 + z^2 - c^2t^2 = 0 \;\Rightarrow\; x'^2 + y'^2 + z'^2 - c^2t'^2 = 0$$

(front of a light wave moving at speed c)

- **To fulfill this condition (c = c'): Time must be changed by transformation as well as space coordinates**
- **Transform** $(x, y, z),\ t \quad \rightarrow \quad (x', y', z'),\ t'$

➡ **Defines the Lorentz transformation**

Lorentz transformation



$$x' = \frac{x - vt}{\sqrt{(1 - \frac{v^2}{c^2})}} = \gamma \cdot (x - vt)$$

$$y' = y$$

$$z' = z$$

$$t' = \frac{t - \frac{v \cdot x}{c^2}}{\sqrt{(1 - \frac{v^2}{c^2})}} = \gamma \cdot (t - \frac{v \cdot x}{c^2})$$

Transformation for constant velocity v along x-axis

Time is now also transformed

Note: for $v \ll c$ it reduces to a Galilei transformation !

Definitions: relativistic factors

$$\beta_r = \frac{v}{c}$$

$$\gamma = \frac{1}{\sqrt{(1 - \frac{v^2}{c^2})}} = \frac{1}{\sqrt{(1 - \beta_r^2)}}$$

β_r **relativistic speed:** $\beta_r = [\mathbf{0},\ \mathbf{1}]$

γ **relativistic factor:** $\gamma = [\mathbf{1},\ \infty]$

(unfortunately, you will also see other β **and** γ **... !)**

Consequences of Einstein's interpretation

- **Space and time and <u>NOT</u> independent quantities**
- **There are no absolute time and space, no absolute motion**
- **Relativistic phenomena:**
  - **No speed of moving objects can exceed speed of light**
  - **(Non-) Simultaneity of events in independent frames**
  - **Lorentz contraction**
  - **Time dilation**
- **Formalism with four-vectors introduced (see later)**

Addition of velocities

Galilei: $v = v_1 + v_2$

With Lorentz transform we have:

$$v = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}}$$

for $v_1, v_2, v_3, ... = 0.5c$ **we get:**

$\mathbf{0.5c + 0.5c = 0.8c}$

$\mathbf{0.5c + 0.5c + 0.5c = 0.93c}$

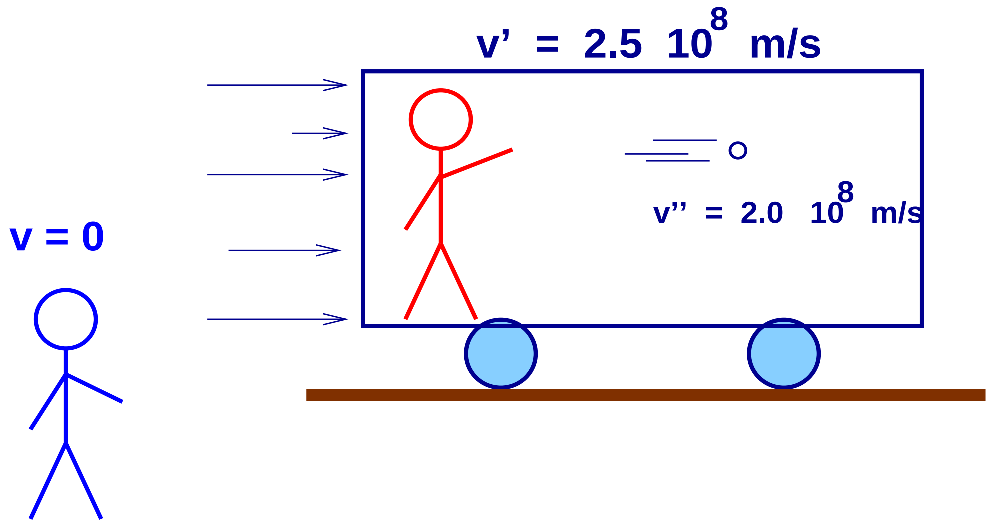
$\mathbf{0.5c + 0.5c + 0.5c + 0.5c = 0.976c}$

$\mathbf{0.5c + 0.5c + 0.5c + 0.5c + 0.5c = 0.992c}$

➡ **Speed of light can never be exceeded by adding velocities !**

Special case: $\mathbf{0.5c + 1.0c = 1.0c}$

Adding velocities (Lorentz)



Fling a ping-pong ball with 2.0$\cdot\ 10^8$ **m/s in a frame moving with 2.5**$\cdot\ 10^8$ **m/s:**

Observed from a non-moving frame:
speed of ping-pong ball: 2.89$\cdot\ 10^8$ **m/s** $\approx$ **c (and not 4.5**$\cdot\ 10^8$ **m/s)**

- Simultaneity -

Simultaneity between moving frames

Assume two events in frame F at positions x_1 and x_2 happen simultaneously at times $t_1 = t_2$

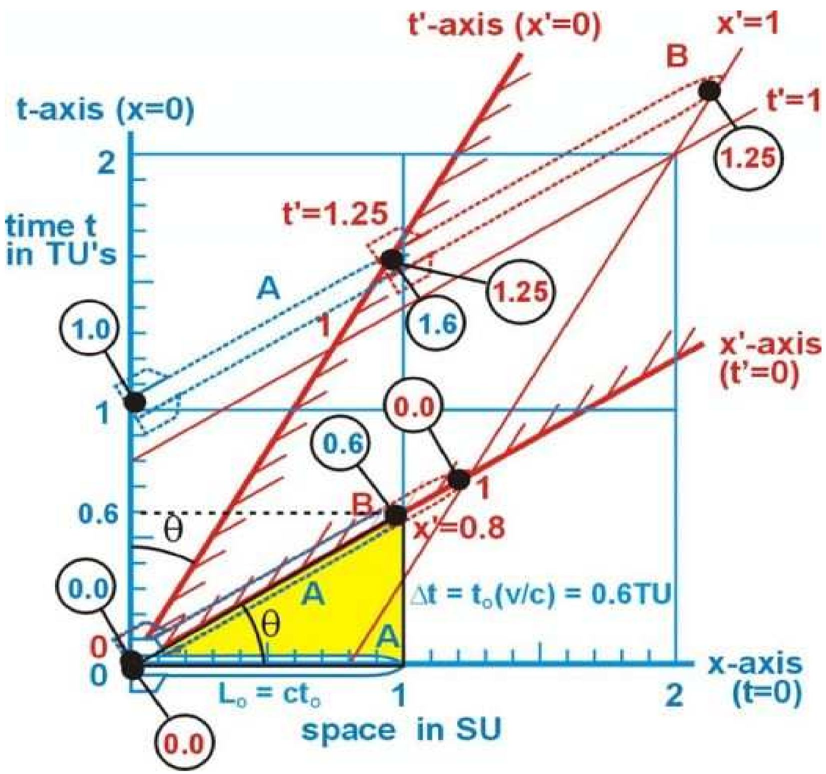
The times t_1' and t_2' in F' we get from:

$$t_1' = \frac{t_1 - \frac{v \cdot x_1}{c^2}}{\sqrt{(1 - \frac{v^2}{c^2})}} \quad \textbf{and} \quad t_2' = \frac{t_2 - \frac{v \cdot x_2}{c^2}}{\sqrt{(1 - \frac{v^2}{c^2})}}$$

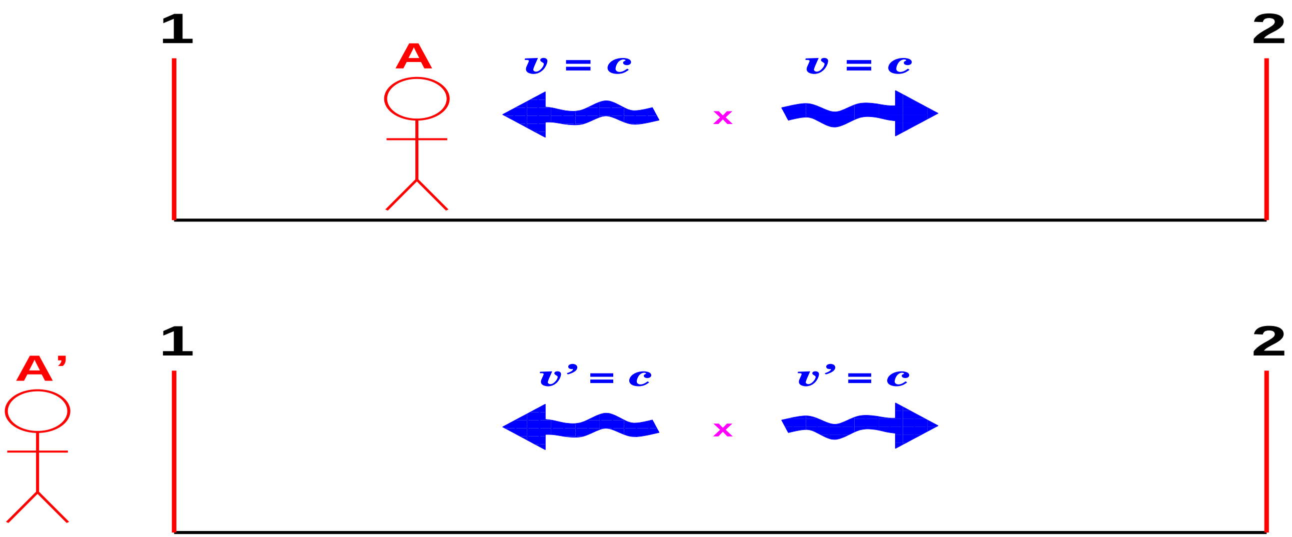
$x_1 \neq x_2$ **in F implies that $t_1' \neq t_2'$ in frame F' !!**

- **Two events simultaneous at positions x_1 and x_2 in F are not simultaneous in F'**

Lack of Simultaneity - explanation:



Simultaneity between moving frames

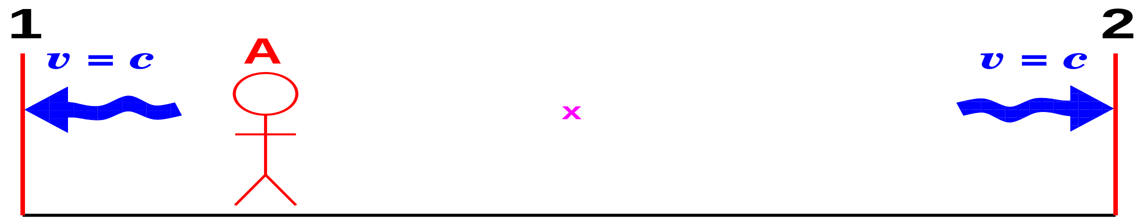


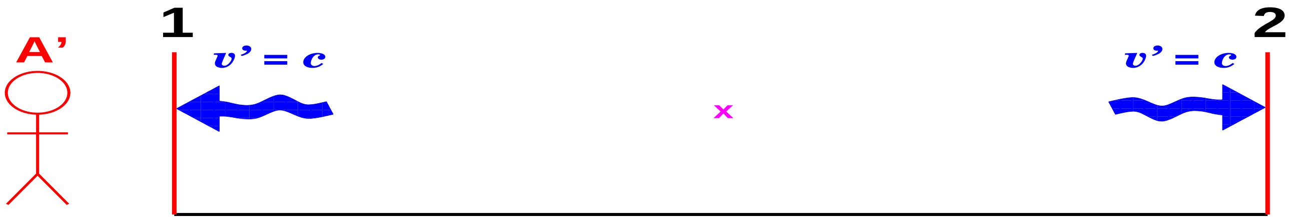
- System with a light source (x) and detectors (1, 2) and
- flashes moving from light source towards detectors

Observer (A) inside this frame

Observer (A') outside

After some time:



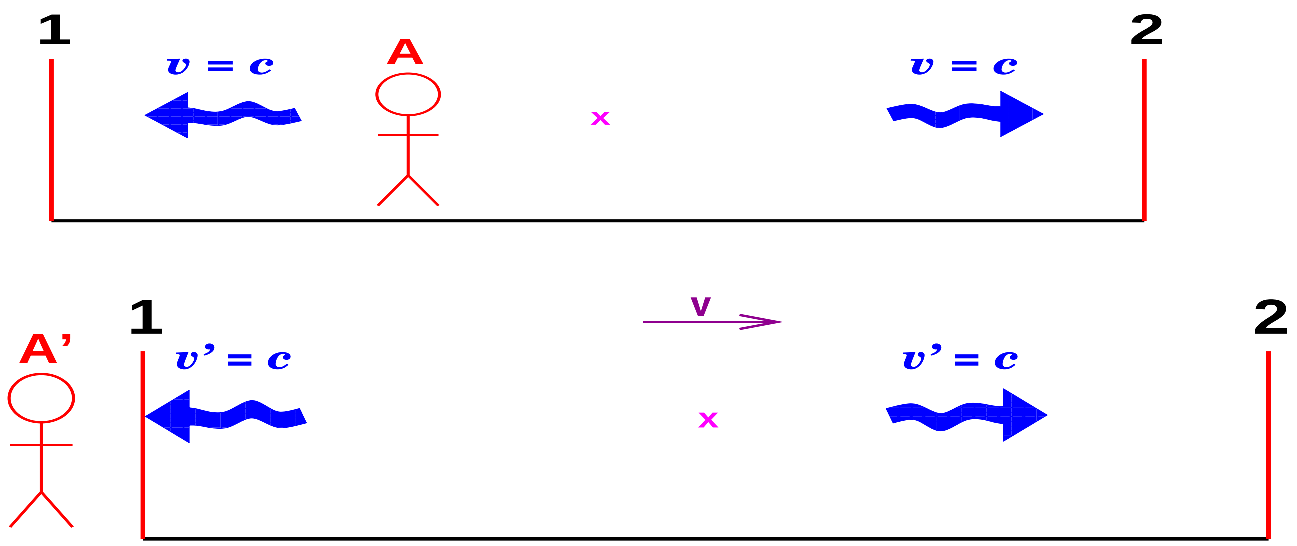


A: both flashes arrive simultaneously at 1 and 2

A': both flashes arrive simultaneously at 1 and 2

What if the frame is moving relative to observer A' ?

Now one frame is moving with speed v:



A: both flashes arrive simultaneously in 1,2

A': flash arrives first in 1, later in 2

A simultaneous event in F is not simultaneous in F'

Why do we care ??

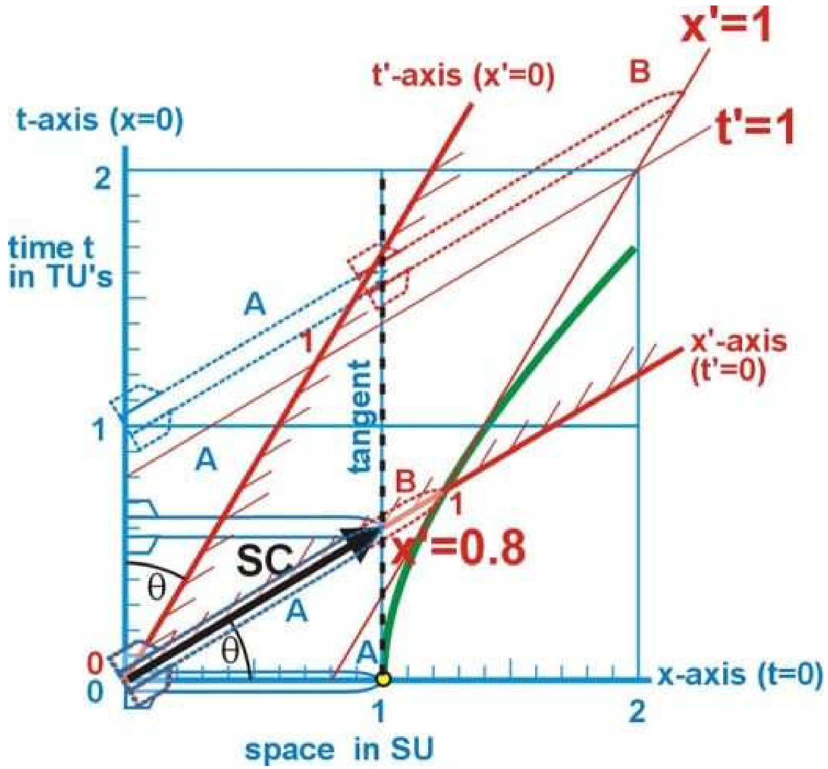
Why care about simultaneity ?

- **Simultaneity is not frame independent**
- **This is a key in special relativity**
- **Most paradoxes are explained by that (although not the twin paradox) !**
- **Different observers see a different reality, in particular the sequence of events can change !**
  - **For $t_1 < t_2$ we may find (not always*) !) a frame where $t_1 > t_2$ (concept of before and after depends on the observer)**

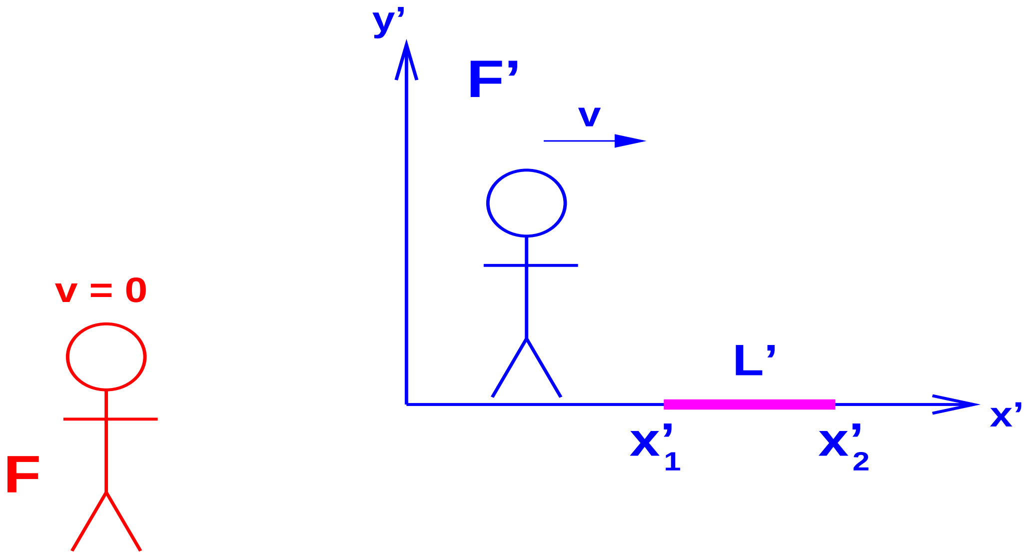
*) **ask later ...**

- Lorentz contraction -

Lorentz contraction - explanation:



How to measure the length of an object ?



Have to measure position of both ends simultaneously !

Length of a rod in F' is $L' = x'_2 - x'_1$, measured simultaneously at a fixed time t' in frame F', what is the length L measured from F ??

Consequences: length measurement

We have to measure simultaneously (!) the ends of the rod at a fixed time t in frame F, i.e.: $L = x_2 - x_1$

Lorentz transformation of "rod coordinates" into rest frame:

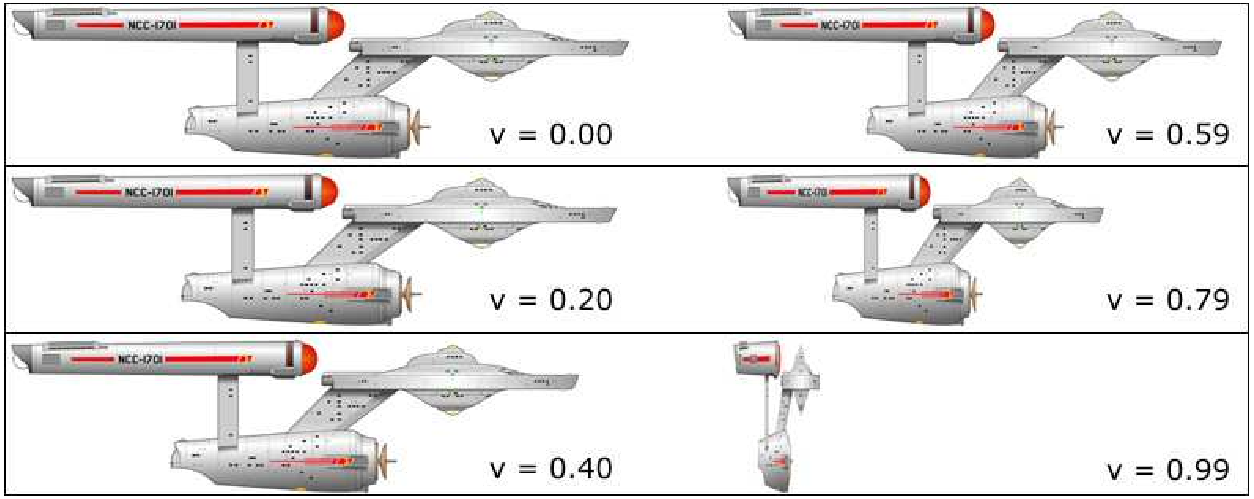
$$x'_1 = \gamma \cdot (x_1 - vt) \quad \text{and} \quad x'_2 = \gamma \cdot (x_2 - vt)$$

$$L' = x'_2 - x'_1 = \gamma \cdot (x_2 - x_1) = \gamma \cdot L$$

$$\rightarrow \quad L = L'/\gamma$$

In accelerators: bunch length, electromagnetic fields, ...

Lorentz contraction - schematic



Length of spaceship measured from earth

→ Appears shorter at higher speed

Question for bar discussion:

What do you see on a photograph taken from the earth ??

Lorentz contraction - schematic

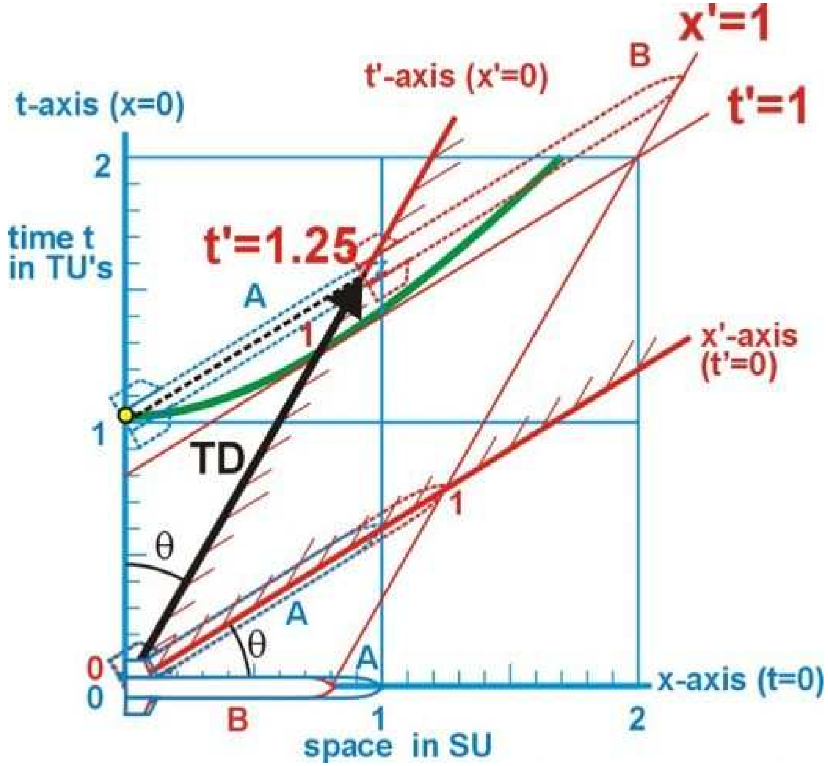
Earth measured from spaceship

- **Both observers mesure the other object contracted**
- **No inertial frame is privileged**

(Ping-pong ball at 2.0 10^8 m/s appears shorter by 25%)

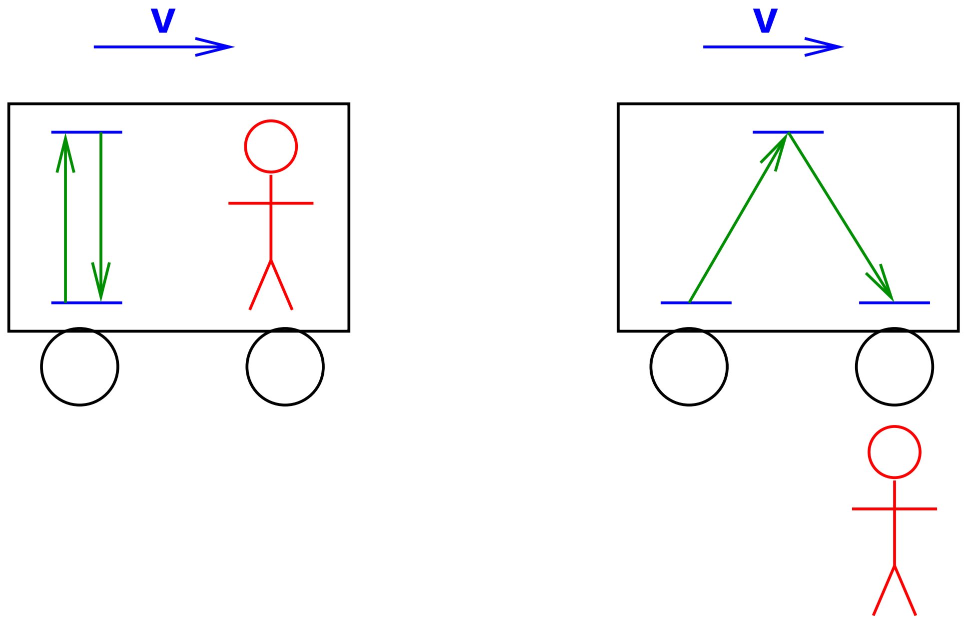
- Time dilation -

Time dilation - explanation:



Time dilation - schematic

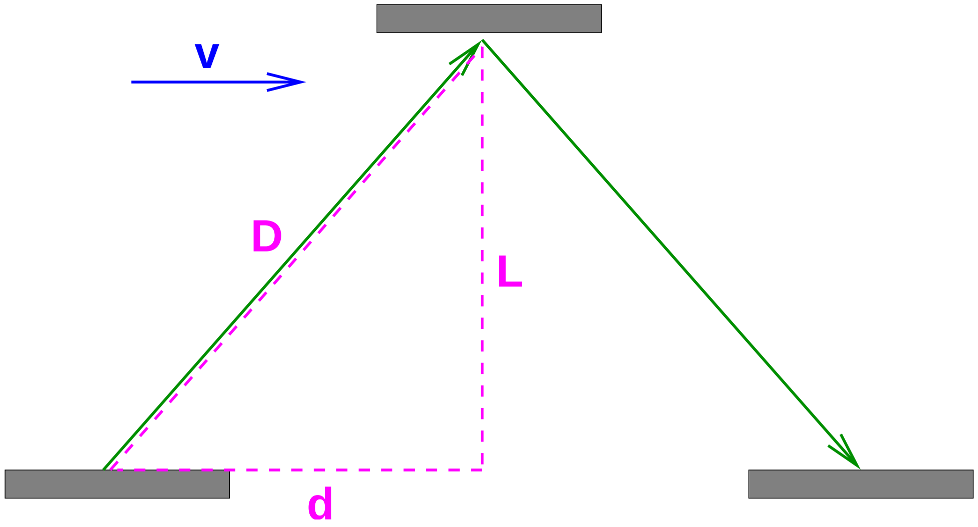
Reflection of light between 2 mirrors seen inside moving frame and from outside



Frame moving with velocity v

Seen from outside the path is longer, but c must be the same ..

Time dilation - derivation



In frame F': **light travels L in time $\Delta t'$**

In frame F: **light travels D in time Δt**

system moves d in time Δt

$$L = c \cdot \Delta t' \qquad D = c \cdot \Delta t \qquad d = v \cdot \Delta t$$

$$(c \cdot \Delta t)^2 = (c \cdot \Delta t')^2 + (v \cdot \Delta t)^2$$

$$\rightarrow \quad \Delta t^2 = \; \Delta t'^2 \quad \rightarrow \quad \boxed{\Delta t = \gamma \cdot \Delta t'}$$

Time dilation - the headache

One can find different derivations:

The car is moving: $\Delta t = \gamma \cdot \Delta t'$

The observer is moving: $\Delta t' = \gamma \cdot \Delta t$

Seems like a contradiction ...

The meaning:

The time variable on the right is always the proper time of the process

i.e. the time measured by the observer at rest relative to the process

Nota bene: ditto for Lorentz contraction ...

Proper Length and Proper Time

Time and distances are relative :

- **τ is a fundamental time: proper time τ**
- **The time measured by an observer in its own frame**
- **From frames moving relative to it, time appears longer**

- **$\mathcal{L}$ is a fundamental length: proper length $\mathcal{L}$**
- **The length measured by an observer in its own frame**
- **From frames moving relative to it, it appears shorter**

The importance of "proper time"

$\Delta\tau$ is the time interval measured <u>inside</u> the moving frame

Consider decay of cosmic μ: $\mu \rightarrow$ e $+ \bar{\nu_e} + \nu_u$

- μ **lifetime is** $\approx$ **2** μ**s**
- **decay in** $\approx$ **2** μ**s in their frame, i.e. their "proper time"**
- **decay in** $\approx$ $\gamma\cdot$ **2** μ**s in the laboratory frame, i.e. earth**
- μ **appear to exist longer than 2** μ**s in the laboratory frame, i.e. earth**
- $\gamma \geq$ **150, they survive 100 km to reach earth from upper atmosphere**

Moving clocks appear to go slower

Travel by airplane:

Assume regular airplane, cruising at $\approx$ 900 km/h

On a flight from Montreal to Geneva, the time is slower by 25 - 30 ns (considering only special relativity) !

That was tested experimentally with atomic clocks (1971 and 1977).

➡ **No strong effect flying in the airplane ...**

(but driving in a car !)

Every day example (GPS satellite):

- 20000 km above ground, (unlike popular believe: not on geostationary orbits)
- Orbital speed 14000 km/h (i.e. relative to observer on earth)
- On-board clock accuracy 1 ns
- Relative precision of satellite orbit $\leq 10^{-8}$

  (scaled to LHC: $\leq$ 40 μm !)
- At GPS receiver, for 5 m need clock accuracy $\approx$ 10 ns

Do we correct for relativistic effects (strongly contradictory opinions !) ?

Do the math:

Orbital speed 14000 km/h $\approx$ 3.9 km/s

➡ $\beta \approx 1.3 \cdot 10^{-5}$, $\gamma \approx 1.0000000084$

Small, but accumulates 7 μs during one day compared to reference time on earth !

After one day: your position wrong by $\approx$ 2 km !!

(nota bene: including general relativity error would be 10 km per day, not treated here, for the interested: coffee break or after dinner discussions)

➡ **Countermeasures (are these <u>corrections</u> ?):**

(1) Minimum 4 satellites (avoid reference time on earth)

(2) Detune data transmission frequency from 1.023 MHz to 1.02299999543 MHz prior to launch

To make it clear:

Key to understand relativity

- **Lorentz contraction:**
  - **It is not the matter that is changed**
  - **It is the space that is distorted**
- **Time dilation:**
  - **It is not the clock that is changed**
  - **It is the time that is distorted**

What about the mass m **?**

Lorentz transformation of velocities

As usual: frame F' moves with constant speed of $\vec{v} = (v, 0, 0)$ **relative to frame F**

Object inside moving frame moves with $\vec{u}' = (u'_x, u'_y, u'_z)$

What is the velocity $\vec{u} = (u_x, u_y, u_z)$ **of the object in the frame F (homework*)) ?**

$$u_x = \frac{u'_x + v}{1 + \frac{u'_x v}{c^2}} \qquad u_y = \frac{u'_y}{\gamma(1 + \frac{u'_x v}{c^2})} \qquad u_z = \frac{u'_z}{\gamma(1 + \frac{u'_x v}{c^2})}$$

*) **hint:** $u_x = \frac{dx}{dt} = \frac{dx}{dt'}\frac{dt'}{dt}$
and we know the transformation of x and t

Momentum conservation

To simplify the computation:
Object inside moving frame F' moves with $\vec{u}' = (0, u'_y, 0)$

We want the expression:

$$\vec{F} = m \cdot \vec{a} = m \cdot \frac{d\vec{v}}{dt}$$

in the same form in all frames, transverse momentum must be conserved:

$$p_y = p'_y$$
$$mu_y = m'u'_y$$
$$mu'_y/\gamma = m'u'_y$$

implies:

$$m = \gamma m'$$

In a frame with $v = 0$ **we call the mass the <u>rest mass</u>** m_0

Relativistic mass

For momentum conservation: <u>mass must also be transformed</u> !

Using the expression for the mass m:

$$m \;=\; m_0\sqrt{1-\left(\frac{v}{c}\right)^2} \;=\; \gamma \cdot m_0$$

and expand it for small speeds:

$$m \;\cong\; m_0 \;+\; \frac{1}{2}m_0 v^2\left(\frac{1}{c^2}\right)$$

and multiplied by c^2:

$$mc^2 \;\cong\; m_0c^2 \;+\; \frac{1}{2}m_0v^2 \;=\; m_0c^2 \;+\; T$$

The second term is the kinetic energy T

Relativistic energy

Interpretation:

$$E = mc^2 = m_0c^2 + T$$

- **Total energy** E **is** $E = mc^2$
- **Sum of kinetic energy plus rest energy**
- **Energy of particle at rest is** $E_0 = m_0c^2$

$$\boxed{E = m \cdot c^2 = \gamma m_0 \cdot c^2}$$

using the definition of relativistic mass again: $m = \gamma m_0$

Interpretation of relativistic energy

- **For any object, $m \cdot c^2$ is the total energy**
- **Follows directly from momentum conservations**
  - **Object can be composite, like proton ..**
  - **m is the mass (energy) of the object "in motion"**
  - **m_0 is the mass (energy) of the object "at rest"**
- **The mass m is not the same in all inertial systems, the rest mass m_0 is !**

  For a proof: see later !

Relativistic momentum

Classically:

$$p \;=\; m\; v$$

with $m = \gamma m_0$**:**

$$p \;=\; \gamma \cdot m_0\; v \;=\; \gamma \cdot \beta \cdot c \cdot m_0$$

we re-write:

$$E^2 \;=\; m^2 c^4 \;=\; \gamma^2 m_0^2 c^4 \;=\; (1 + \gamma^2 \beta^2) m_0^2 c^4$$

and finally get:

$$E^2 \;=\; (m_0^2)^2 + (pc)^2 \quad \rightarrow \quad \frac{E}{c} \;=\; \sqrt{(m_0 c)^2 \;+\; p^2}$$

First summary

- **Physics laws the same in all inertial frames ...**
- **Speed of light in vacuum c is the same in all frames and requires Lorentz transformation**
- **Moving objects appear shorter**
- **Moving clocks appear to go slower**
- **Mass is not independent of motion ($m = \gamma \cdot m_0$) and total energy is $E = m \cdot c^2$**
- **No absolute space or time: where it happens and when it happens is not independent**
- **Next: how to calculate something and applications ...**

Introducing four-vectors

Since space and time are not independent, must reformulate physics taking both into account:

Separated time and space (Euclidean space):

$$t, \quad \vec{a} \;=\; (x, y, z)$$

Replace by vector including the time (Minkowski space):

$$A \;=\; (ct, x, y, z)$$

(time t multiplied by c to get the same units)
This is the position four-vector, you also find a^μ instead of A

Scalar products revisited

Scalar product for (usual) vectors like: $\vec{a} \cdot \vec{b}$**,**

$\vec{a} = (x_a, y_a, z_a) \qquad \vec{b} = (x_b, y_b, z_b)$

Standard definition (Euclidean geometry):

$$\vec{a} \cdot \vec{b} = (x_a, y_a, z_a) \cdot (x_b, y_b, z_b) = (x_a \cdot x_b + y_a \cdot y_b + z_a \cdot z_b)$$

Scalar product for four-vectors like: $A \odot B$

$A = (ct_a, x_a, y_a, z_a) \qquad B = (ct_b, x_b, y_b, z_b)$

$$A \odot B = ct_a \cdot ct_b - \vec{a} \cdot \vec{b} = (ct_a \cdot ct_b - x_a \cdot x_b - y_a \cdot y_b - z_a \cdot z_b)$$

Note the – sign !!

Distance between events in space-time

We can describe a distance in the space-time between two points A_1 and A_2:

$$\Delta X = A_2 - A_1 = (ct_2 - ct_1,\ x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1)$$

Scalar product of the difference is the distance^2 $=$ D^2:

$$D^2 \ = \ \Delta A^2 \ = \ c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$$

D^2 can be positive (time-like) or negative (space-like)

Important for calculation of particle scattering and causality, see also back-up slides or ask a lecturer

Using four-vectors: $X = (ct, \vec{x})$**,** $X' = (ct', \vec{x'})$

$$X \odot X = c^2 t^2 - x^2 - y^2 - z^2$$

and

$$X' \odot X' = c^2 t'^2 - x'^2 - y'^2 - z'^2$$

we have:

$$X \odot X = X' \odot X'$$

because this is our condition for constant speed of light c **!**

This product is an Invariant Quantity

Invariant Quantities have the same values in all inertial frames

Why bother about four-vectors ?

- **We want invariant laws of physics in different frames**
- **The solution: write the laws of physics in terms of four vectors**
- **Without proof: any four-vector (scalar) product $F \odot F$ has the same value in all inertial frames:**

$$F \odot F = F' \odot F'$$

All scalar products of four-vectors are invariant !

We have important four-vectors:

$$\begin{aligned}
&\text{Coordinates}: && X = (ct, x, y, z) = (ct, \vec{x}) \\
&\text{Velocities}: && V = \tfrac{dX}{dt} \cdot \tfrac{dt}{d\tau} \; = \; \gamma(c, \dot{\vec{x}}) = \gamma(c, \vec{v}) \\
&\text{Momenta}: && P = mV = m\gamma(c, \vec{v}) = \gamma(mc, \vec{p}) \\
&\text{Force}: && F = \tfrac{dP}{dt} = \gamma \tfrac{d}{dt}(mc, \vec{p})
\end{aligned}$$

A special invariant

From the velocity four-vector V:

$$V \;=\; \gamma(c, \vec{v})$$

we get the scalar product:

$$V \odot V = \gamma^2(c^2 - \vec{v}^2) = c^2 \;\;!!$$

➡ c **is an invariant, has the same value in all inertial frames**

$$V \;\odot\; V \;\;=\;\; V' \;\odot\; V' \;\;=\;\; c^2$$

➡ **The invariant of the velocity four-vector** V **is the speed of light** c**, i.e. it is the same in ALL frames !**

Another important invariant

Momentum four-vector P:

$$P = m_0 V = m_0 \gamma (c, \vec{v}) = (\mathbf{m}c, \vec{p}) \;\; = \;\; (\frac{E}{c}, \vec{p})$$

$$P' = m_0 V' = m_0 \gamma (c, \vec{v}') = (\mathbf{m}c, \vec{p}') \;\; = \;\; (\frac{E}{c}, \vec{p}')$$

We can get another invariant:

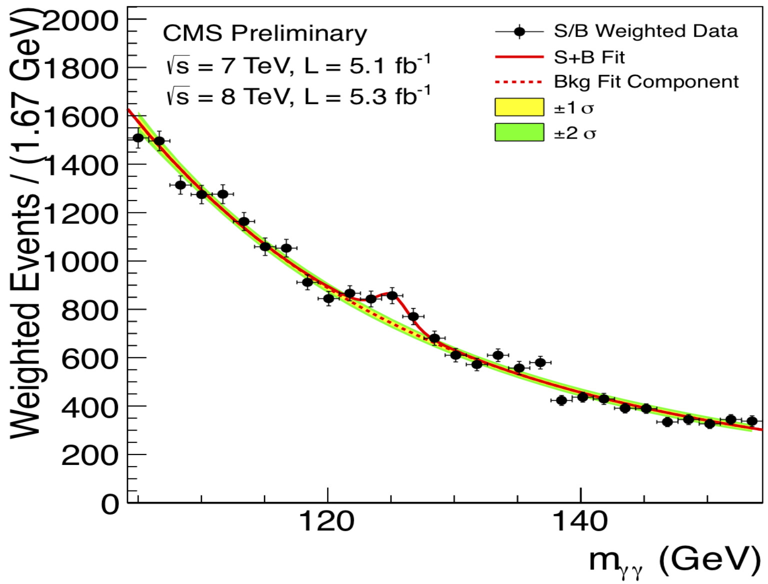
$$P \odot P \;\; = \;\; P' \odot P' \;\; = \;\; m_0^2 c^2$$

Invariant of the four-momentum vector is the mass m_0

➡ **The rest mass is the same in all frames !**

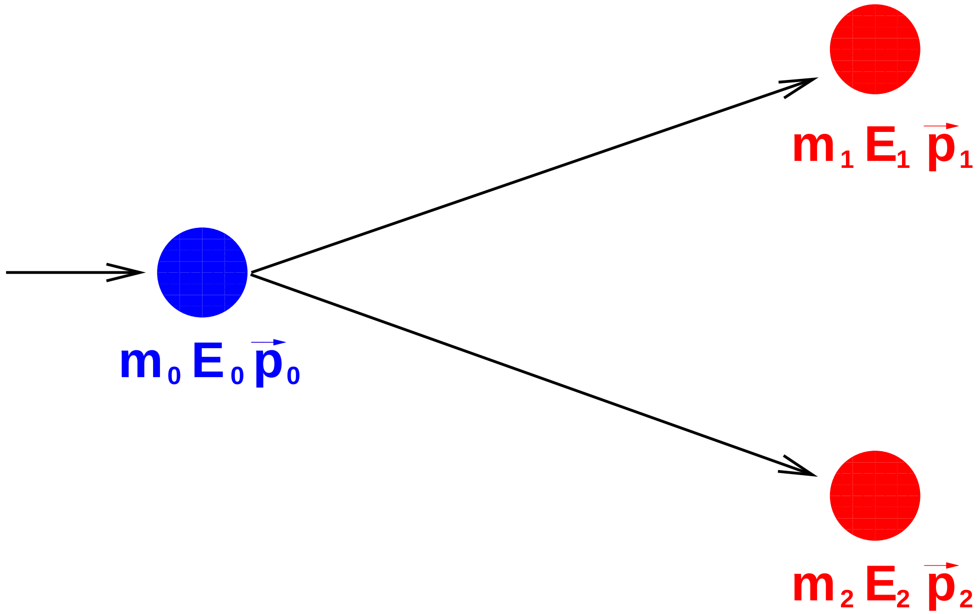
(otherwise we could tell whether we are moving or not !!)

Application: reconstruction of a decay



- **Here: Higgs** → $\gamma\gamma$
- **How to get the mass of the original particle ?**

Particle P_0 decaying into two particles: $P_0 \rightarrow P_1 + P_2$



P_1 and P_2 we can measure (i.e. $\vec{p_2}, \vec{p_1}, m_1, m_2, E_1, E_2$)

P_0 (i.e. $\vec{p_0}, m_0, E_0$) are unknown

$$P_1 = (E_1, \vec{p_1}), \quad E_1 = \sqrt{m_1^2 + \vec{p_1^2}}$$

$$P_2 = (E_2, \vec{p_2}), \quad E_2 = \sqrt{m_2^2 + \vec{p_2^2}}$$

- **Get <u>sum</u> of four momenta (four momentum of original particle):**

$$P_0 = (P_1 + P_2) = (E_1 + E_2, \vec{p_1} + \vec{p_2})$$

- **We take the scalar product and know:** $P_0^2 = m_0^2 c^4$

$$m_0 c^2 = \sqrt{(E_1 + E_2)^2 - (\vec{p_1} c + \vec{p_2} c)^2}$$

We plot: m_0 **for every decay we observe and have the histogram above !**

Practical units

Standard units are not very convenient, easier to use:

$$[\mathbf{E}] = \mathbf{eV} \quad [\mathbf{p}] = \mathbf{eV}/c \quad [\mathbf{m}] = \mathbf{eV}/c^2$$

Mass of a proton: $m_p = 1.672 \cdot 10^{-27}$ **Kg**
Energy(at rest): $m_p c^2 = 938$ **MeV = 0.15 nJ**

- **Other example, 2.7 gram of material (ping-pong ball) equivalent to:**

  $\approx$ **700000 times the full LHC beam** ($2.4 \cdot 10^{14}$ **J**)

  $\approx$ **60 kilotons of TNT**

Masses in accelerators

The mass of a fast moving particle is increasing like:

$$m = \gamma m_0 = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

and the kinetic energy T:

$$T \;=\; m_0(\gamma - 1)c^2$$

- **Why do we care ?**
  - **Particles cannot go faster than c !**
  - **What happens when we accelerate ?**

Masses in accelerators

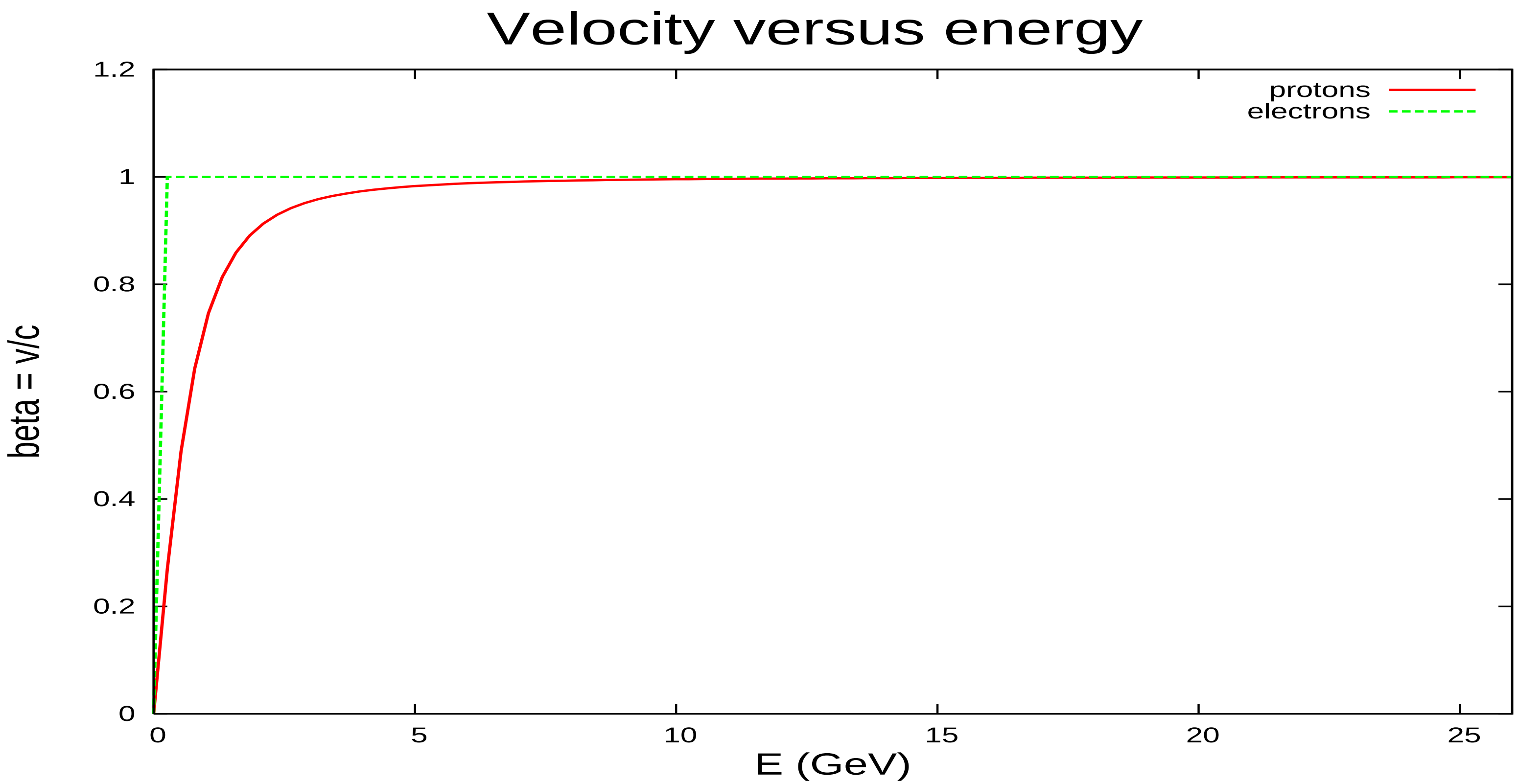
When we accelerate:

- **For v $\ll$ c:**
  - **- E, m, p, v increase ...**
- **For v $\approx$ c:**
  - **- E, m, p increase, but v does not !**

$$\beta = \frac{v}{c} \approx \sqrt{1 - \frac{m_0^2 c^4}{T^2}}$$

Rest mass m_0 makes the difference ...

Speed versus energy (protons)



Why do we care ??

E (GeV)	v (km/s)	γ	β	T (LHC)
450	299791.82	479.74	0.99999787	88.92465 μs
7000	299792.455	7462.7	0.99999999	88.92446 μs

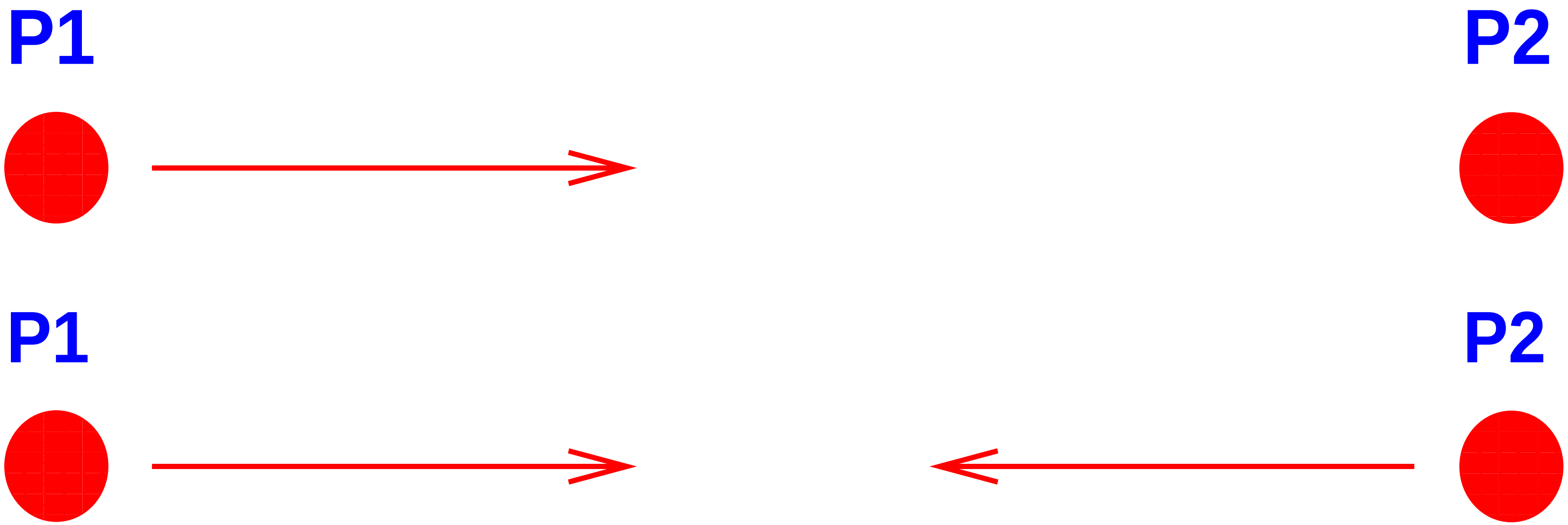
- For identical circumference very small change in revolution time
- If path for faster particle slightly longer, the faster particle arrives later !

➤ Concept of transition (see later lecture)

Four vectors

- Use of four-vectors simplify calculations significantly
- Follow the rules and look for invariants
- In particular kinematic relationships, e.g.
  - Particle decay (find mass of parent particle)
  - Particle collisions →

Particle collisions



What is the available collision energy ?

Particle collisions - collider

Assume identical particles and beam energies, colliding head-on

The four momentum vectors are:

$$P1 = (E, \vec{p}) \qquad P2 = (E, -\vec{p})$$

The four momentum vector in centre of mass system is:

$$P^{*} = P1 + P2 = (E + E, \vec{p} - \vec{p}) = (2E, \vec{0})$$

The four momentum vector in centre of mass system is:

$$P^* = P1 + P2 = (E + E, \vec{p} - \vec{p}) = (2E, \vec{0})$$

The square of the total available energy s in the centre of mass system is the momentum invariant:

$$s = P^* \odot P^* = 4E^2$$

$$E_{cm} = \sqrt{P^* \odot P^*} = 2E$$

i.e. in a (symmetric) collider the total energy is twice the beam energy

Particle collisions - fixed target

The four momentum vectors are:

$$P1 = (E, \vec{p}) \qquad P2 = (m_0, \vec{0})$$

The four momentum vector in centre of mass system is:

$$P^* = P1 + P2 = (E + m_0, \vec{p})$$

Particle collisions - fixed target

With the above it follows:

$$P^* \odot P^* = E^2 + 2m_0E + m_0^2 - \vec{p}^2$$

since $E^2 - \vec{p}^2 = m_0^2$ **we get:**

$$s = 2m_0E + m_0^2 + m_0^2$$

if E **much larger than** m_0 **we find:**

$$E_{cm} = \sqrt{s} = \sqrt{2m_0E}$$

Particle collisions - fixed target

Homework: try for $E1 \neq E2$ **and** $m1 \neq m2$

Examples:

collision	beam energy	$\sqrt{s}$ (collider)	$\sqrt{s}$ (fixed target)
pp	315 (GeV)	630 (GeV)	24.3 (GeV)
pp	7000 (GeV)	14000 (GeV)	114.6 (GeV)
e+e−	100 (GeV)	200 (GeV)	0.320 (GeV)

Kinematic invariant

We need to make cross sections (and therefore luminosity) invariant !
This is done by a calibration factor which is (without derivation):

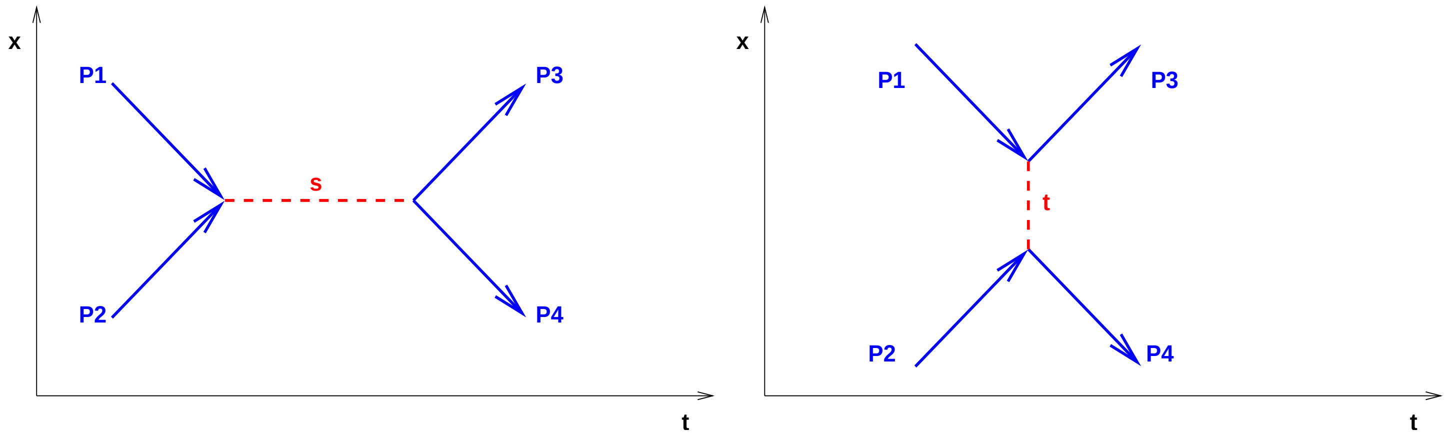
$$K = \sqrt{(\vec{v_1} - \vec{v_2})^2 - (\vec{v_1} \times \vec{v_2})^2/c^2}$$

Here $\vec{v_1}$ **and** $\vec{v_2}$ **are the velocities of the two (relativistic) beams.**
For a (symmetric) collider, e.g. LHC, we have:

$$\vec{v_1} = -\vec{v_2}, \qquad \vec{v_1} \times \vec{v_2} \;=\; 0 \quad \mathrm{head-on!}$$

→ $K \;=\; 2 \;\cdot\; c$ **!**

For completeness ...



Squared centre of mass energy:

$$s = (P1 + P2)^2 = (P3 + P4)^2$$

Squared momentum transfer in particle scattering (small t - small angle, see again lecture on Luminosity):

$$t = (P1 - P3)^2 = (P2 - P4)^2$$

Kinematic relations

How are kinematic variables related to each other, e.g.:

β**,** v**,** T**,** E**,** γ**,** p**, ...**

Important: relate <u>changes</u> between the variables

Examples for relative changes (classically):

$$\frac{dp}{p} = \frac{dv}{v}, \qquad \frac{dv}{v} = \frac{1}{2} \cdot \frac{dT}{T}, .. \qquad \text{etc.}$$

What changes with special relativity ?

Kinematic relations - logarithmic derivatives

	$\frac{d\beta}{\beta}$	$\frac{dp}{p}$	$\frac{dT}{T}$	$\frac{dE}{E} = \frac{d\gamma}{\gamma}$
$\frac{d\beta}{\beta} =$	$\frac{d\beta}{\beta}$	$\frac{1}{\gamma^2}\frac{dp}{p}$	$\frac{1}{\gamma(\gamma+1)}\frac{dT}{T}$	$\frac{1}{(\beta\gamma)^2}\frac{d\gamma}{\gamma}$
$\frac{dp}{p} =$	$\gamma^2\frac{d\beta}{\beta}$	$\frac{dp}{p}$	$[\gamma/(\gamma+1)]\frac{dT}{T}$	$\frac{1}{\beta^2}\frac{d\gamma}{\gamma}$
$\frac{dT}{T} =$	$\gamma(\gamma+1)\frac{d\beta}{\beta}$	$\left(1+\frac{1}{\gamma}\right)\frac{dp}{p}$	$\frac{dT}{T}$	$\frac{\gamma}{(\gamma-1)}\frac{d\gamma}{\gamma}$
$\frac{dE}{E} =$	$(\beta\gamma)^2\frac{d\beta}{\beta}$	$\beta^2\frac{dp}{p}$	$\left(1-\frac{1}{\gamma}\right)\frac{dT}{T}$	$\frac{d\gamma}{\gamma}$
$\frac{d\gamma}{\gamma} =$	$(\gamma^2-1)\frac{d\beta}{\beta}$	$\frac{dp}{p} - \frac{d\beta}{\beta}$	$\left(1-\frac{1}{\gamma}\right)\frac{dT}{T}$	$\frac{d\gamma}{\gamma}$

Example LHC (7 TeV): $\frac{\Delta p}{p} \approx 10^{-4}$ **implies:** $\frac{\Delta v}{v} \approx 2 \cdot 10^{-12}$

With four-vectors, Lorentz transformation can be written in a compact form with matrix multiplication:

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

$$A' = \Lambda \, A$$

Nota bene: this matrix Λ is also used for Lorentz transformation of fields, forces, derivatives, etc., if they are written as four-vectors ...

Relativity and electrodynamics

- **Life made easy with four-vectors ..**
- **Back to the start: electrodynamics and Maxwell equations**

Write potentials and currents as four-vectors:

$$A \;=\; (\frac{\Phi}{c}, \vec{A}) \qquad \text{and} \qquad J \;=\; (\rho \cdot c, \vec{j})$$

What about the fields ?

Lorentz transformation of fields - field tensor

Electromagnetic field described by field-tensor $F^{\mu\nu}$:

$$F = \begin{pmatrix} 0 & \frac{-E_x}{c} & \frac{-E_y}{c} & \frac{-E_z}{c} \\ \frac{E_x}{c} & 0 & -B_z & B_y \\ \frac{E_y}{c} & B_z & 0 & -B_x \\ \frac{E_z}{c} & -B_y & B_x & 0 \end{pmatrix}$$

It transforms via: $F \; = \; \Lambda \; F' \; \Lambda^T$ **(same Λ as before)**

Transform the four-current like:

$$\begin{pmatrix} \rho c \\ j_x \\ j_y \\ j_z \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \rho' c \\ j'_x \\ j'_y \\ j'_z \end{pmatrix}$$

It transforms via: $J = \Lambda J'$ **(always the same** Λ**)**

We can define a four-gradient now:

$$\vec{\nabla} \quad \Rightarrow \quad \partial \; = \; (\frac{1}{c}\frac{\partial}{\partial t}, \vec{\nabla})$$

and transform like:

$$\begin{pmatrix} \frac{\partial}{c\partial t} \\ \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial}{c\partial t'} \\ \frac{\partial}{\partial x'} \\ \frac{\partial}{\partial y'} \\ \frac{\partial}{\partial z'} \end{pmatrix}$$

It transforms via: $\partial \; = \; \Lambda \; \partial'$ **(always the same** Λ**)**

Start from Faraday's law in S-frame:

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \;=\; -\frac{\partial B_x}{\partial t}$$

applying the appropriate transformations from above to the S'-frame

$$\frac{\partial E'_z}{\partial y'} - \frac{\partial E'_y}{\partial z'} \;=\; -\frac{\partial B'_x}{\partial t'}$$

Now Maxwell's equation have the same form (try for the others ..)

Lorentz transformation $F = \Lambda \ F' \ \Lambda^T$ **for components:**

$$
\begin{aligned}
\vec{E}'_\perp &= \gamma(\vec{E}_\perp + \vec{v} \times \vec{B}) \\
\vec{B}'_\perp &= \gamma\left(\vec{B}_\perp - \frac{\vec{v}\times\vec{E}}{c^2}\right) \\
\vec{E}'_\parallel &= \vec{E}_\parallel \\
\vec{B}'_\parallel &= \vec{B}_\parallel
\end{aligned}
$$

- **Fields perpendicular to movement are transformed**
- **Coulomb force is turned into the Lorentz force**

Of course there is also a four-vector for the Lorentz force:

$$F = \gamma q(\frac{\vec{E}\cdot\vec{v}}{c}, \ \vec{E} + \vec{v}\times\vec{B})$$

Summary I (things to understand)

- Special Relativity is very simple, few basic principles
- Physics laws the same in inertial systems, speed of light in vacuum the same in all systems
- Everyday phenomena lose their meaning:
  - ➡ Only union of space and time preserve an independent reality: **space-time**
  - ➡ Only union of electric and magnetic fields preserve an independent reality:

    electromagnetic field tensor and potential

Life is invariant and very easy:

Four-vectors and space-time

Summary II (things to remember)

- **Relativistic effects in accelerators (used in later lectures)**
- **The formulation with four-vectors gives you automatically what is needed:**
  - **Lorentz contraction and Time dilation**
  - **Invariants !**
  - **Relativistic mass effects and dynamics**
  - **Modification of electromagnetic fields**

Summary III (not treated here)

- **Principles of Special Relativity apply to inertial (non-accelerated) systems**
- **Is it conceivable that the principle applies to accelerated systems ?**
- **Introduces General Relativity, with consequences:**
  - **Space and time are dynamical entities:**

    → **space and time change in the presence of matter**
  - **Explanation of gravity (sort of ..)**
  - **Black holes, worm holes, ...**
  - **Time depends on gravitational potential, different at different heights (GPS !)**

A last word ...

- **If you do not have enough or cannot sleep, look up some of the popular paradoxes:**
  - **Ladder-garage paradox (*)**
  - **Submarine paradox (**)**
  - **Twin paradox (**)**
  - **J. Bell's rocket-rope paradox (***)**
  - **...**

- BACKUP SLIDES -

Small history

- **1678 (Römer, Huygens): Speed of light c is finite ($c \approx 3 \cdot 10^8$ m/s)**
- **1630-1687 (Galilei,Newton): Principles of Relativity**
- **1863 (Maxwell): Electromagnetic theory, light are waves moving through static ether with speed c**
- **1887 (Michelson, Morley): Speed c independent of direction, → no ether**
- **1892 (Lorentz, FitzGerald, Poincaré): Lorentz transformations, Lorentz contraction**
- **1897 (Larmor): Time dilation**
- **1905 (Einstein): Principles of Special Relativity**
- **1907 (Einstein, Minkowski): Concepts of Spacetime**

Relativistic Principles

- **Relativity in (classical) inertial systems:**
  - Classical relativity
  - Newton, Galilei
- **Relativity in (all) inertial systems:**
  - Special relativity
  - Lorentz, Einstein, Minkowski
- **Relativity in accelerated systems:**
  - General relativity
  - Einstein

Lorentz contraction

- **In moving frame an object has always the same length (it is invariant, our principle !)**
- **From stationary frame moving objects appear contracted by a factor γ (Lorentz contraction)**
- **Why do we care ?**
- **Turn the argument around: assume length of a proton bunch appears always at 0.1 m in laboratory frame (e.g. in the RF bucket), what is the length in its own (moving) frame ?**
  - **At 5 GeV ($\gamma \approx 5.3$) $\rightarrow$ L' = 0.53 m**
  - **At 450 GeV ($\gamma \approx 480$) $\rightarrow$ L' = 48.0 m**

Relations to remember

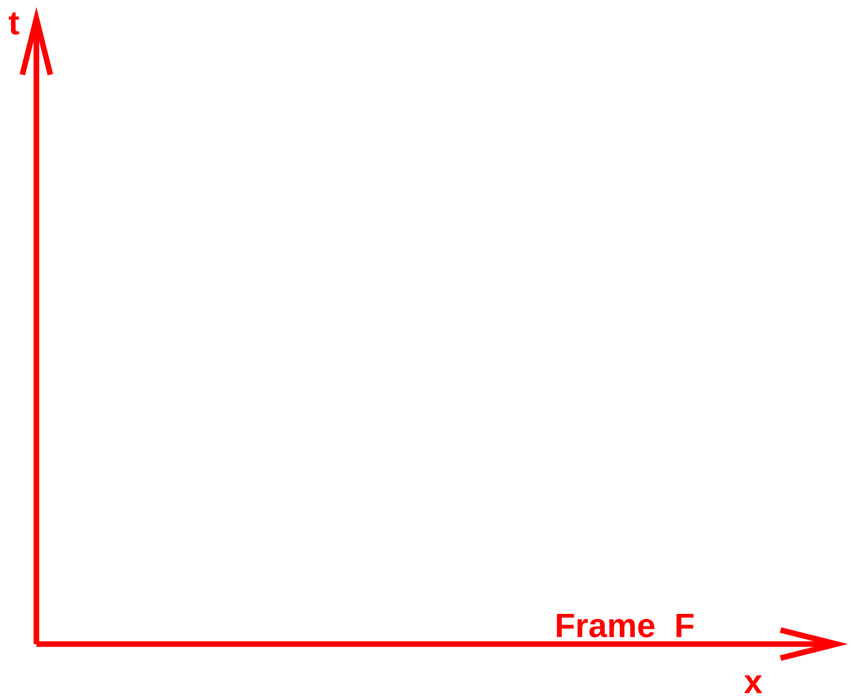
Note:

$$E = mc^2 = \gamma \cdot m_0 c^2 \quad \rightarrow \quad \boxed{E = \gamma m_0}$$

$$p = m_0 \gamma v = \gamma m_0 \cdot \beta c \quad \rightarrow \quad \boxed{p = \gamma m_0 \cdot \beta}$$

$$T = m_0(\gamma - 1) \cdot c^2 \quad \rightarrow \quad \boxed{T = \gamma m_0 - m_0}$$

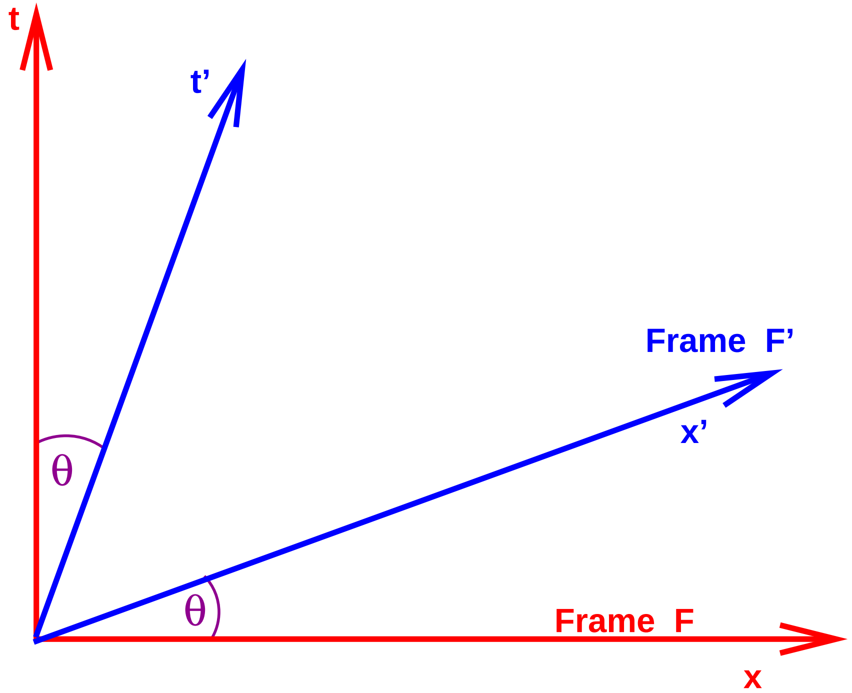
Lorentz transformation - schematic



- **Rest frame (x only, difficult to draw many dimensions)**

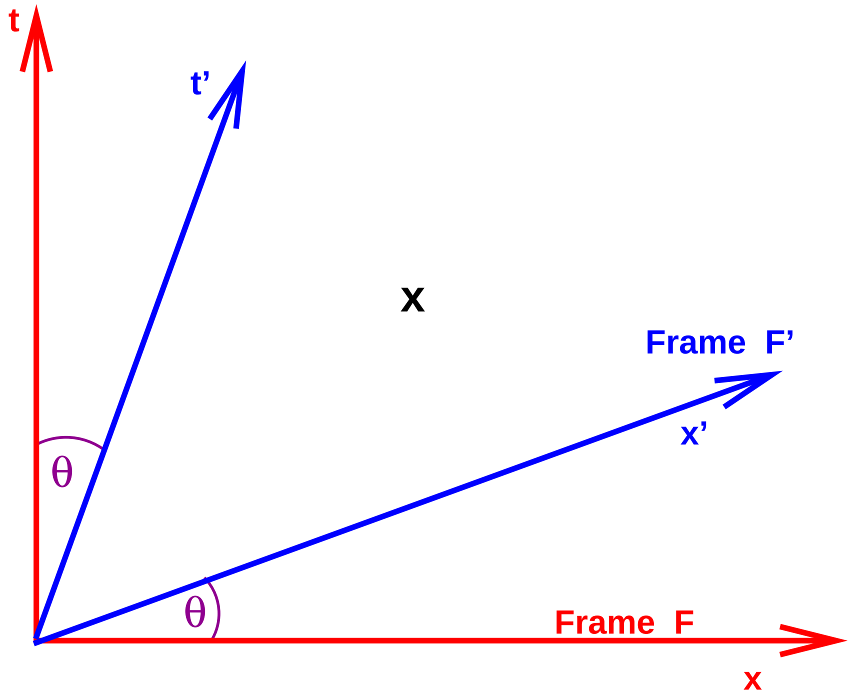
  y and z coordinates are not changed (transformed)

Lorentz transformation - schematic



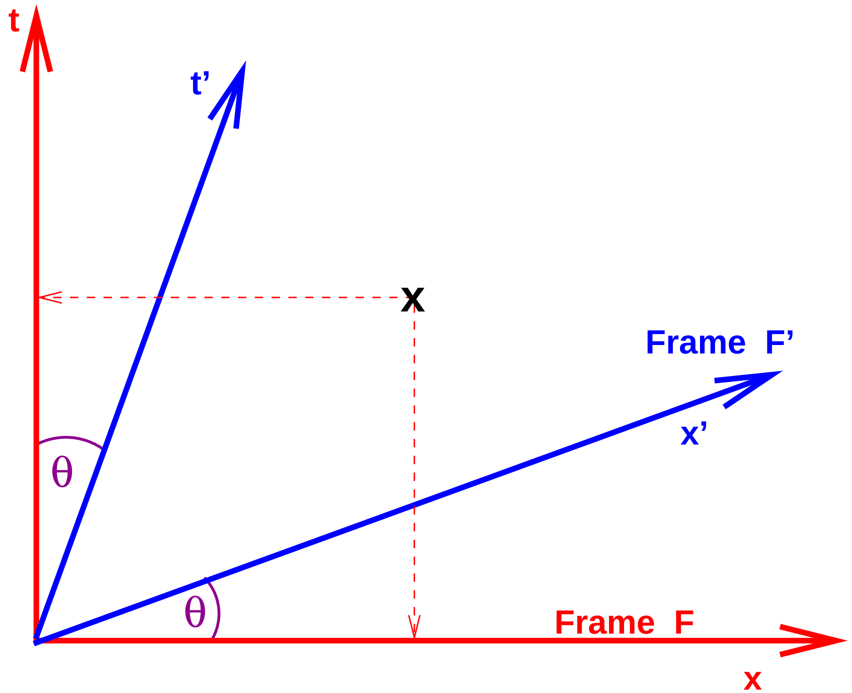
- **Rest frame and moving frame**
- $\tan(\theta) = \frac{v}{c}$

Lorentz transformation - schematic



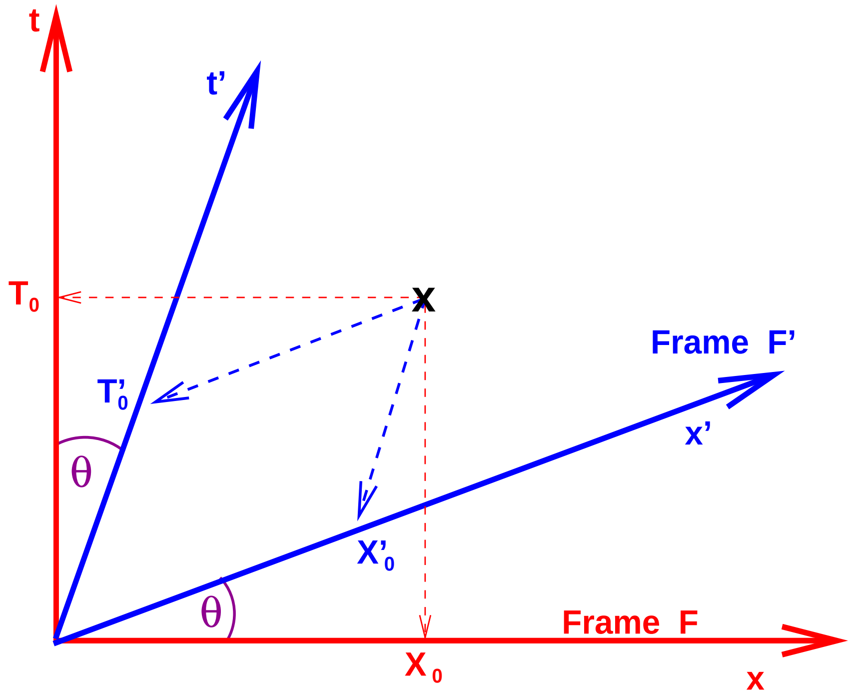
- **An event X**

Lorentz transformation - schematic



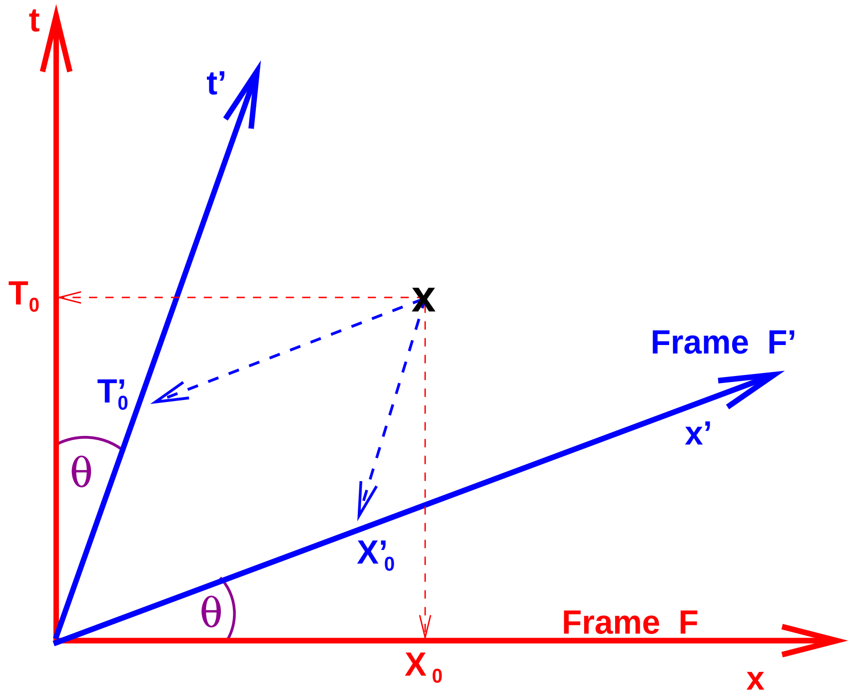
- **Event X as seen from rest frame, projected on F-axes**

Lorentz transformation - schematic



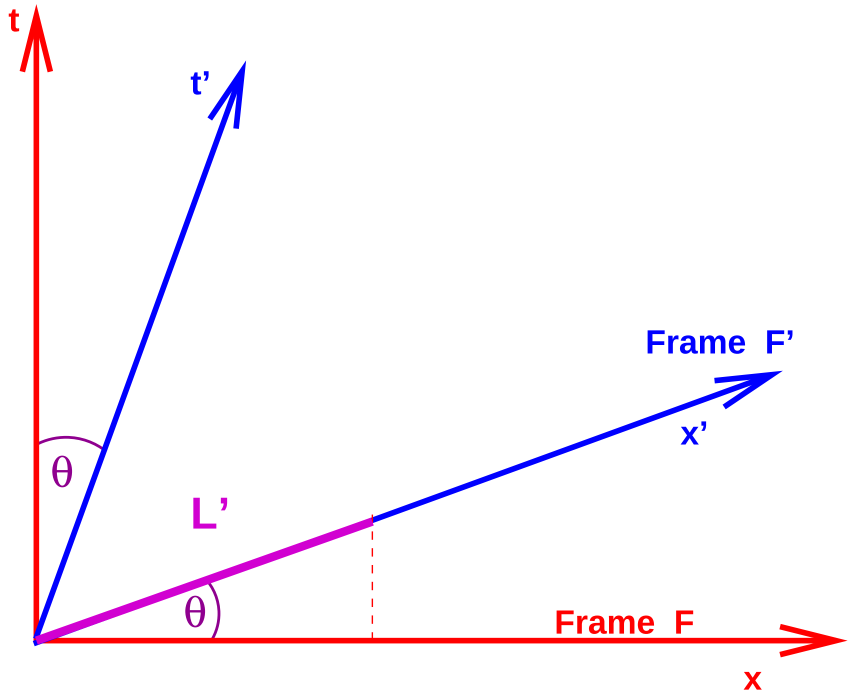
- **Event X seen at different time and location in the two frames, projected on axes of F and F'**

Lorentz transformation - schematic



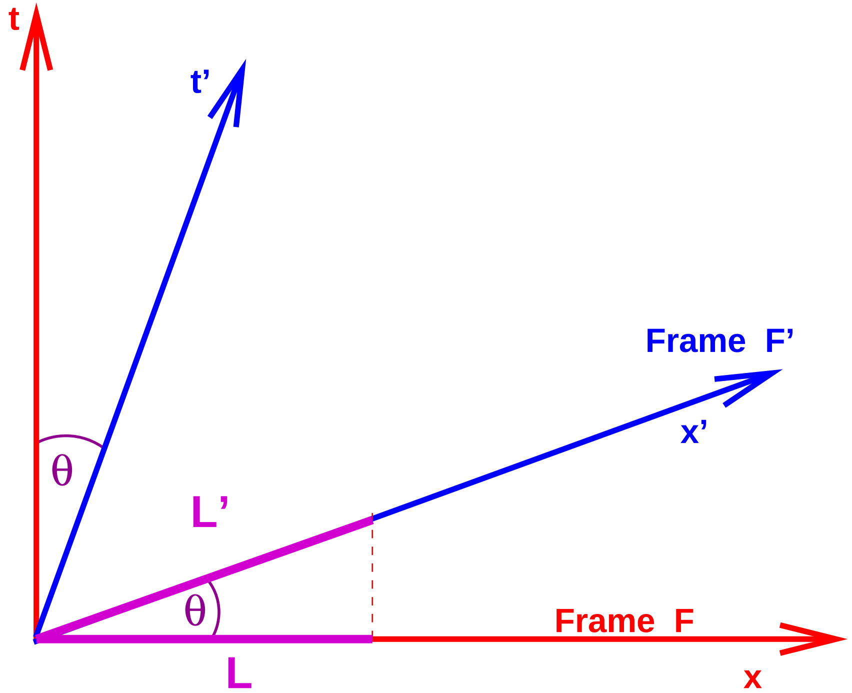
- **Q: How would a Galilei-transformation look like ??**

Lorentz contraction - schematic



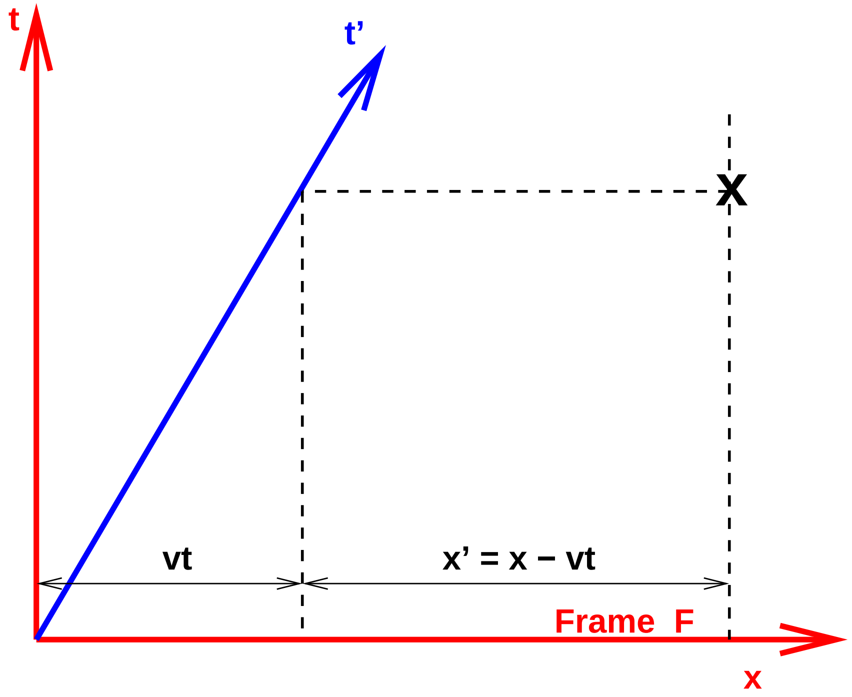
- Length L' as measured in moving frame

Lorentz contraction - schematic



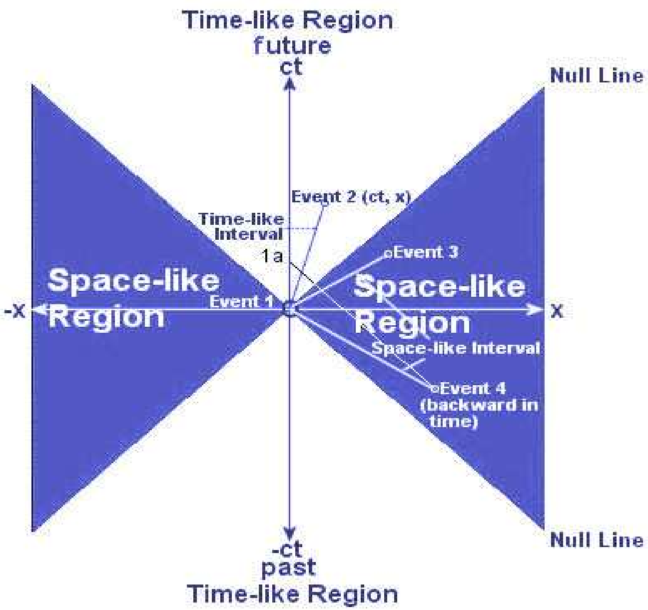
- From moving frame: L appears shorter in rest frame
- Length is maximum in frame (F') where object is at rest

Galilei transformation - schematic



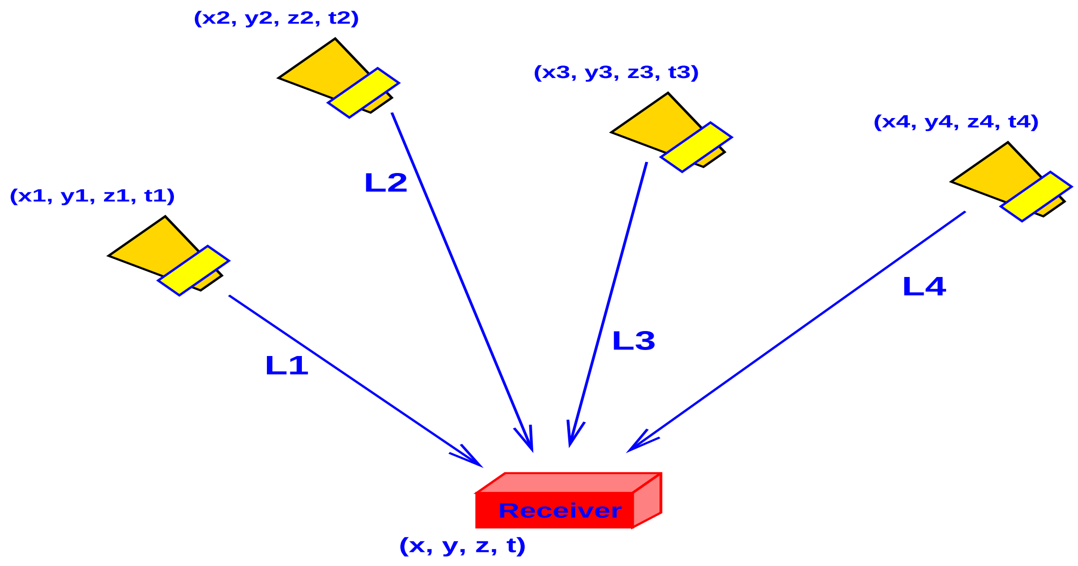
- **Rest frame and Galilei transformation ...**

Time-like and Space-like events



- **Event 1 can communicate with event 2**
- **Event 1 cannot communicate with event 3, would require to travel faster than the speed of light**

GPS principle ...



$$L_1 = \quad c(t - t_1) = \quad \sqrt{((x - x_1)^2 \; + \; (y - y_1)^2 \; + \; (z - z_1)^2}$$
$$L_2 = \quad c(t - t_2) = \quad \sqrt{((x - x_2)^2 \; + \; (y - y_2)^2 \; + \; (z - z_2)^2}$$
$$L_3 = \quad c(t - t_3) = \quad \sqrt{((x - x_3)^2 \; + \; (y - y_3)^2 \; + \; (z - z_3)^2}$$
$$L_4 = \quad c(t - t_4) = \quad \sqrt{((x - x_4)^2 \; + \; (y - y_4)^2 \; + \; (z - z_4)^2}$$

$t_1, t_2, t_3, t_4,$ **need relativistic correction !**

4 equations and 4 variables → (x, y, z, t) **of the receiver !**

Kinematic relations

	cp	**T**	**E**	γ
$\beta =$	$\frac{1}{\sqrt{(\frac{E_0}{cp})^2+1}}$	$\sqrt{1-\frac{1}{(1+\frac{T}{E_0})^2}}$	$\sqrt{1-(\frac{E_0}{E})^2}$	$\sqrt{1-\gamma^{-2}}$
cp $=$	cp	$\sqrt{T(2E_0+T)}$	$\sqrt{E^2-E_0^2}$	$E_0\sqrt{\gamma^2-1}$
$E_0 =$	$\frac{cp}{\sqrt{\gamma^2-1}}$	$T/(\gamma-1)$	$\sqrt{E^2-c^2p^2}$	E/γ
T $=$	$cp\sqrt{\frac{\gamma-1}{\gamma+1}}$	**T**	$E-E_0$	$E_0(\gamma-1)$
$\gamma =$	$cp/E_0\beta$	$1+T/E_0$	E/E_0	γ

Kinematic relations

Example: CERN Booster

At injection: T = 50 MeV

➡ **E = 0.988 GeV, p = 0.311 GeV/c**

➡ γ **= 1.0533,** β **= 0.314**

At extraction: T = 1.4 GeV

➡ **E = 2.338 GeV, p = 2.141 GeV/c**

➡ γ **= 2.4925,** β **= 0.916**

Lorentz transformation of fields

Assuming a motion in x-direction, we write for the components:

$$
\begin{aligned}
E'_x &= E_x & B'_x &= B_x \\
E'_y &= \gamma(E_y - v \cdot B_z) & B'_y &= \gamma(B_y + \tfrac{v}{c^2} \cdot E_z) \\
E'_z &= \gamma(E_z + v \cdot B_y) & B'_z &= \gamma(B_z - \tfrac{v}{c^2} \cdot E_y)
\end{aligned}
$$

An interesting consequence:
Since $\vec{B}' = 0$ and plugging it into the Lorentz force, we get for the <u>transverse</u> forces:

$$\vec{F_{mag}} = -\beta^2 \cdot \vec{F_{el}}$$

➡ **for very relativistic motion $\beta \approx 1$ the transverse <u>forces</u> cancel ...**