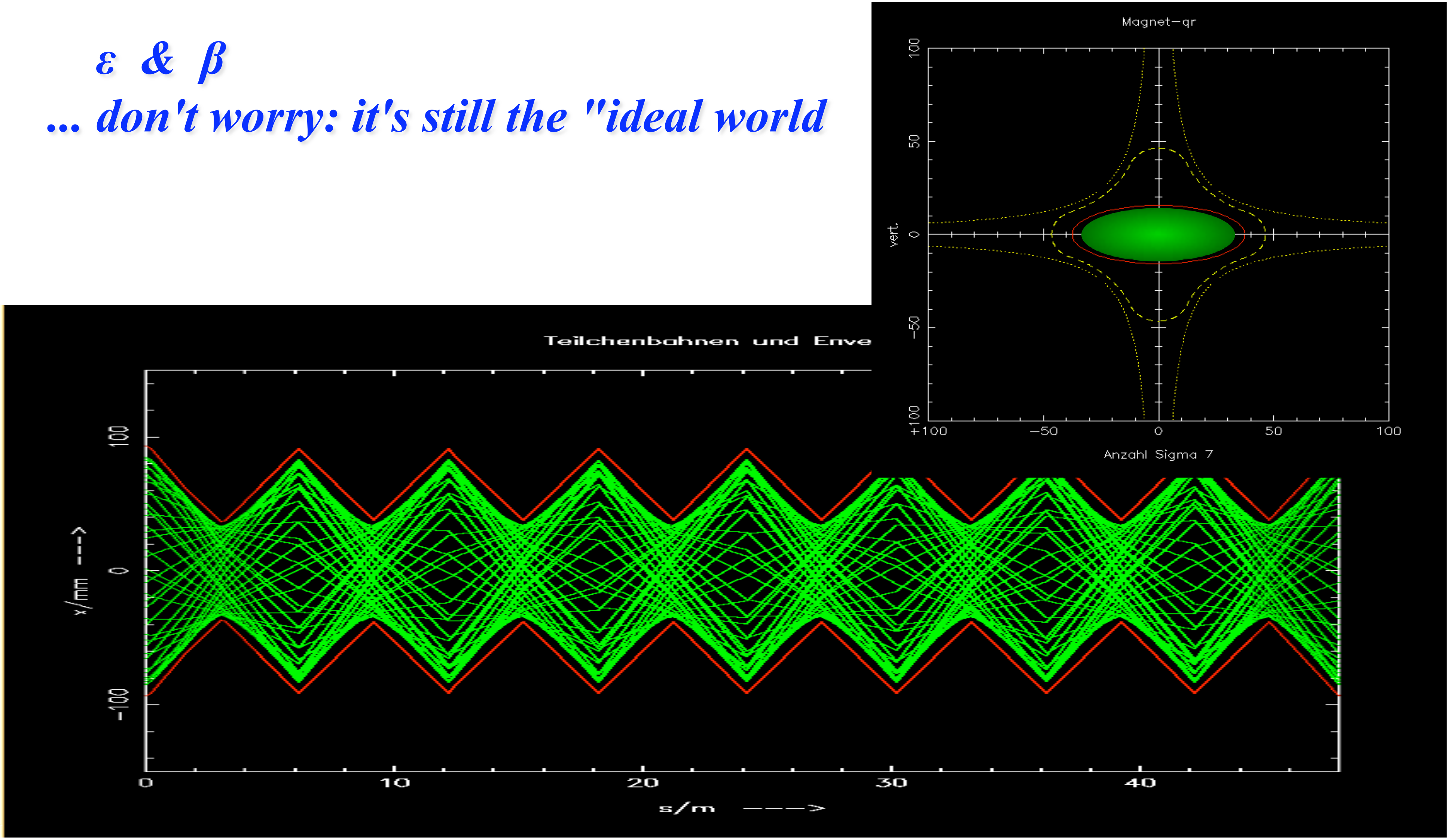
Introduction to Transverse Beam Optics

II.) Particle Trajectories, Beams & Bunch

ε & β

... don't worry: it's still the "ideal world



4.) Solution of Trajectory Equations

$$x'' + K\,x = 0$$

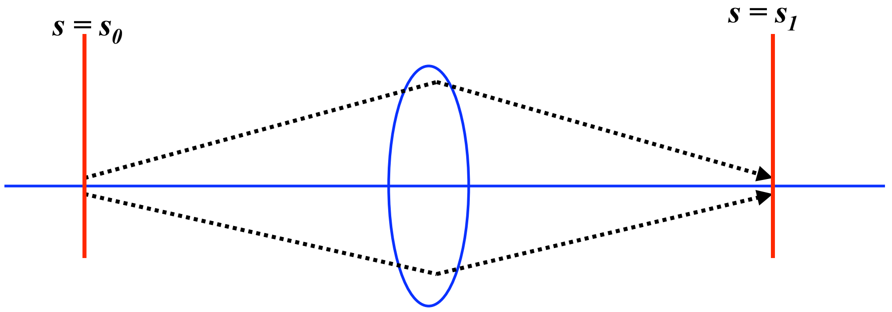
Hor. Focusing Quadrupole K > 0:

$$x(s) = x_0 \cdot \cos(\sqrt{|K|}s) + x'_0 \cdot \frac{1}{\sqrt{|K|}}\sin(\sqrt{|K|}s)$$

$$x'(s) = -x_0 \cdot \sqrt{|K|} \cdot \sin(\sqrt{|K|}s) + x'_0 \cdot \cos(\sqrt{|K|}s)$$

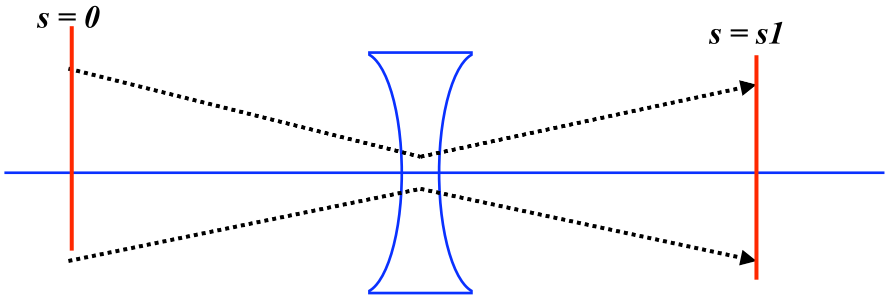
For convenience expressed in matrix formalism:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s1} = M_{foc} * \begin{pmatrix} x \\ x' \end{pmatrix}_{s0}$$



$$M_{foc} = \begin{pmatrix} \cos(\sqrt{|K|}s) & \frac{1}{\sqrt{|K|}}\sin(\sqrt{|K|}s \\ -\sqrt{|K|}\sin(\sqrt{|K|}s) & \cos(\sqrt{|K|}s) \end{pmatrix}_0$$

hor. defocusing quadrupole:



$$x'' - K\,x = 0$$

Remember from school:

$$f(s) = \cosh(s) \quad , \qquad f'(s) = \sinh(s)$$

Ansatz: $x(s) = a_1 \cdot \cosh(\omega s) + a_2 \cdot \sinh(\omega s)$

$$M_{defoc} = \begin{pmatrix} \cosh\sqrt{|K|}\,l & \frac{1}{\sqrt{|K|}}\sinh\sqrt{|K|}\,l \\ \sqrt{|K|}\sinh\sqrt{|K|}\,l & \cosh\sqrt{|K|}\,l \end{pmatrix}$$

drift space:

$K = 0$

$$M_{drift} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$$

! with the assumptions made, the motion in the horizontal and vertical planes are independent „ ... the particle motion in x & y is uncoupled"

Combining the two planes:

Clear enough (hopefully ... ?) : a quadrupole magnet that is focussing o-in one plane acts as defocusing lens in the other plane ... et vice versa.

hor foc. quadrupole lens

$$M_{foc} = \begin{pmatrix} \cos(\sqrt{|K|}s) & \frac{1}{\sqrt{|K|}}\sin(\sqrt{|K|}s \\ -\sqrt{|K|}\sin(\sqrt{|K|}s) & \cos(\sqrt{|K|}s) \end{pmatrix}$$

matrix of the same magnet in the vert. plane:

$$M_{defoc} = \begin{pmatrix} \cosh\sqrt{|K|}l & \frac{1}{\sqrt{|K|}}\sinh\sqrt{|K|}l \\ \sqrt{|K|}\sinh\sqrt{|K|}l & \cosh\sqrt{|K|}l \end{pmatrix}$$

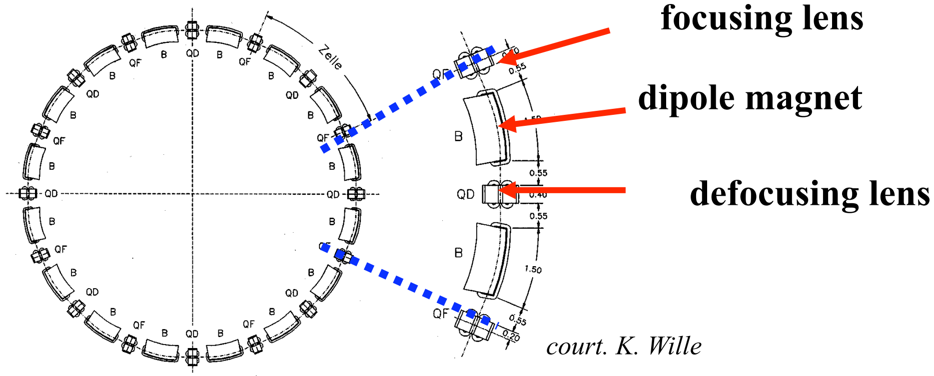
$$\begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_f = \begin{pmatrix} \cos(\sqrt{|k|}s) & \frac{1}{\sqrt{|k|}}\sin(\sqrt{|k|}s) & 0 & 0 \\ -\sqrt{|k|}\sin(\sqrt{|k|}s) & \cos(\sqrt{|k|}s) & 0 & 0 \\ 0 & 0 & \cosh(\sqrt{|k|}s) & \frac{1}{\sqrt{|k|}}\sinh(\sqrt{|k|}s) \\ 0 & 0 & \sqrt{|k|}\sinh(\sqrt{|k|}s) & \cosh(\sqrt{|k|}s) \end{pmatrix} * \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_i$$

Transformation through a system of lattice elements

combine the single element solutions by multiplication of the matrices

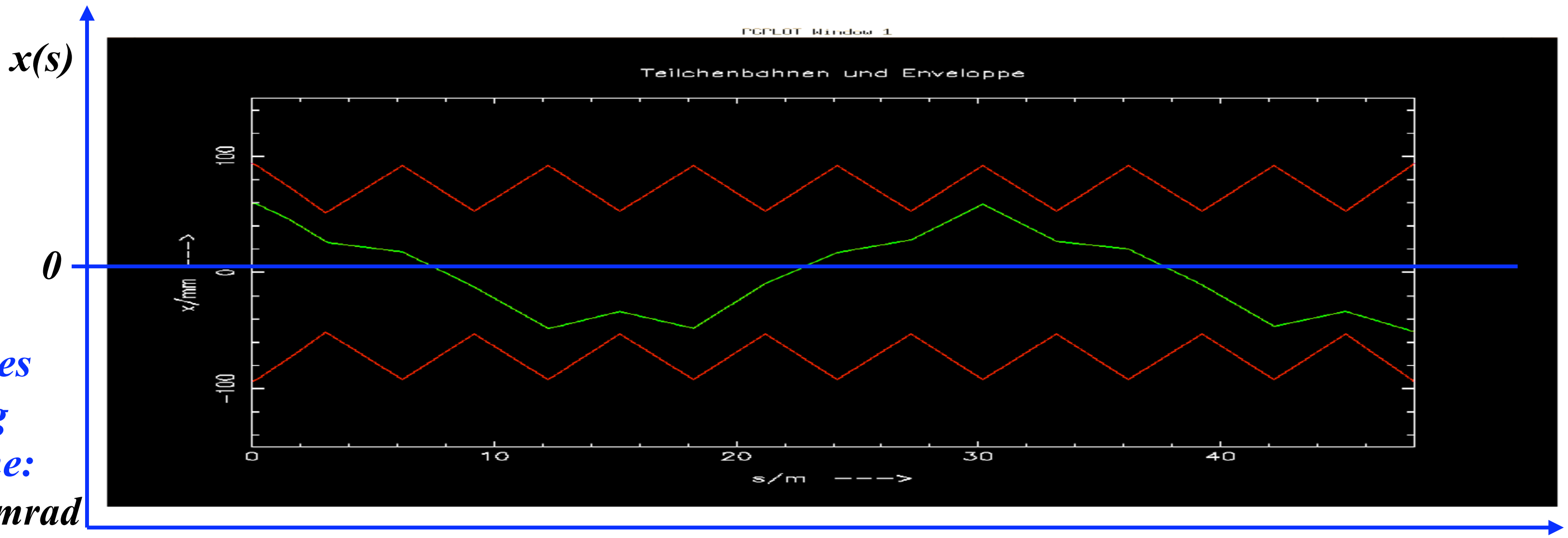
$$M_{total} = M_{QF} * M_D * M_{QD} * M_{Bend} * M_{D*.....}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s2} = M(s_2, s_1) * \begin{pmatrix} x \\ x' \end{pmatrix}_{s1}$$



court. K. Wille

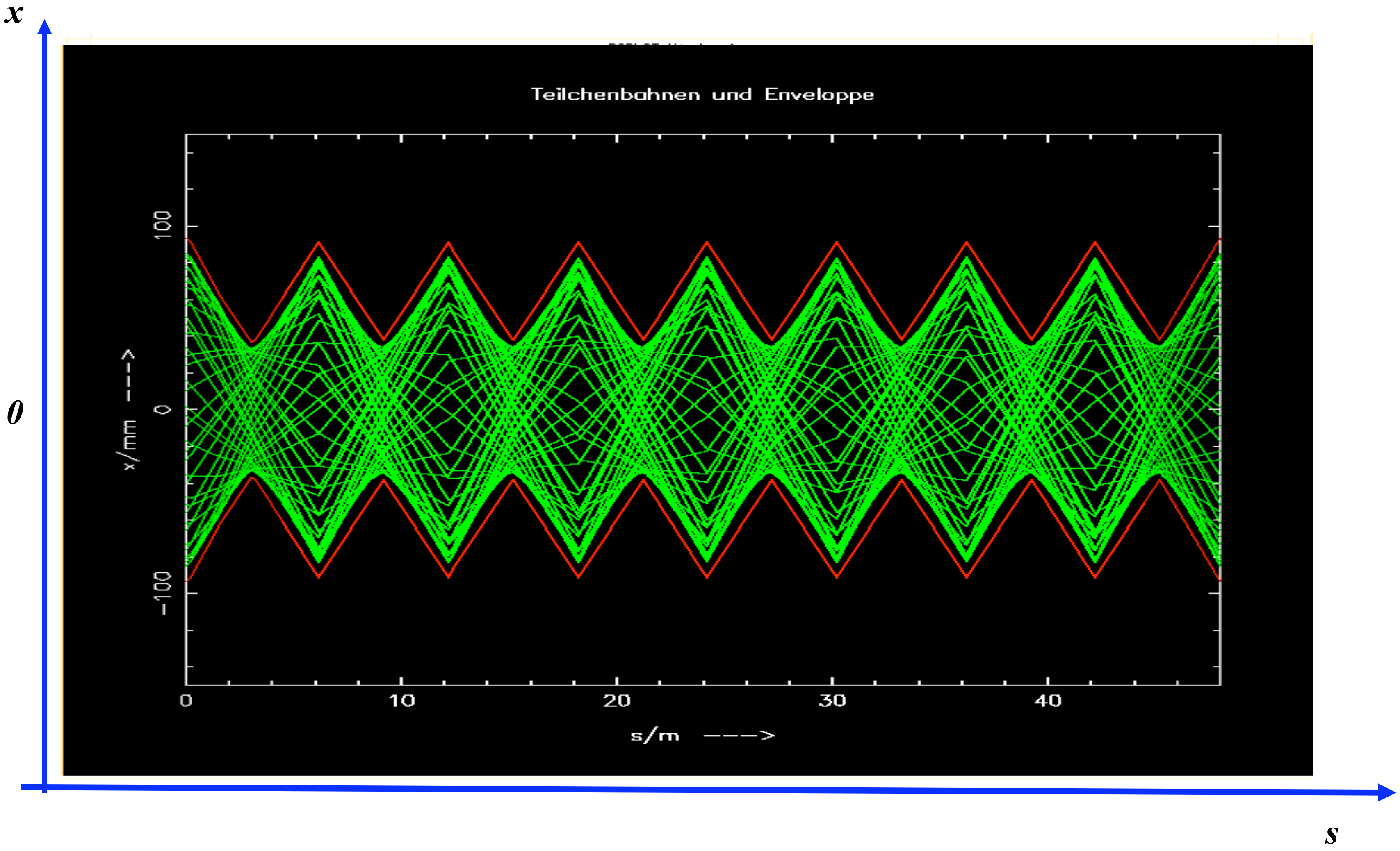
in each accelerator element the particle trajectory corresponds to the movement of a harmonic oscillator „



typical values in a strong foc. machine:
x ≈ mm, x´ ≤ mrad

Question: ***what will happen, if the particle performs a second turn ?***

... or a third one or ... 10^{10} turns



6.) The Beta Function

General solution of Hill´s equation:

(i) $$x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)} \cdot \cos(\psi(s) + \phi)$$

ε, Φ = integration constants determined by initial conditions

β(s) periodic function given by focusing properties of the lattice ↔ quadrupoles

$$\beta(s+L) = \beta(s)$$

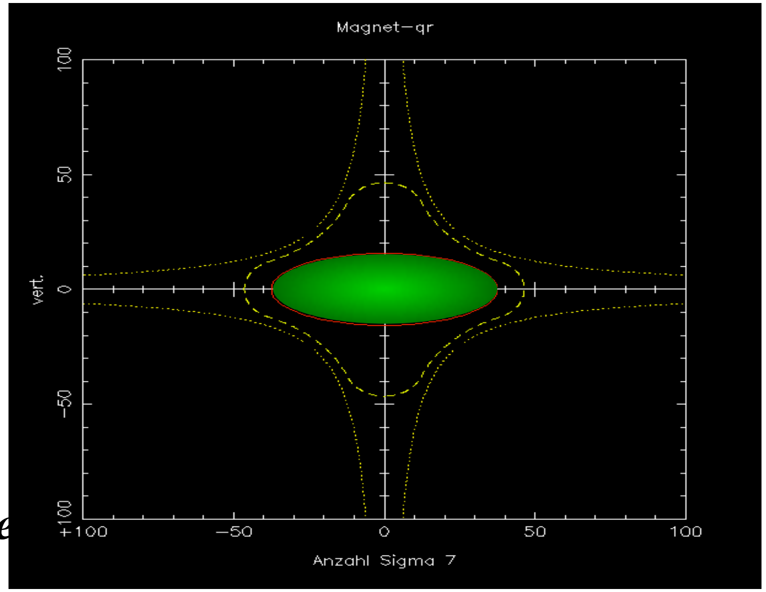
Inserting (i) into the equation of motion …

$$\psi(s) = \int_0^s \frac{ds}{\beta(s)}$$

Ψ(s) = „phase advance" of the oscillation between point „0" and „s" in the lattice.
For one complete revolution: number of oscillations per turn „Tune"

$$Q_y = \frac{1}{2\pi}\oint \frac{ds}{\beta(s)}$$

The Beta Function

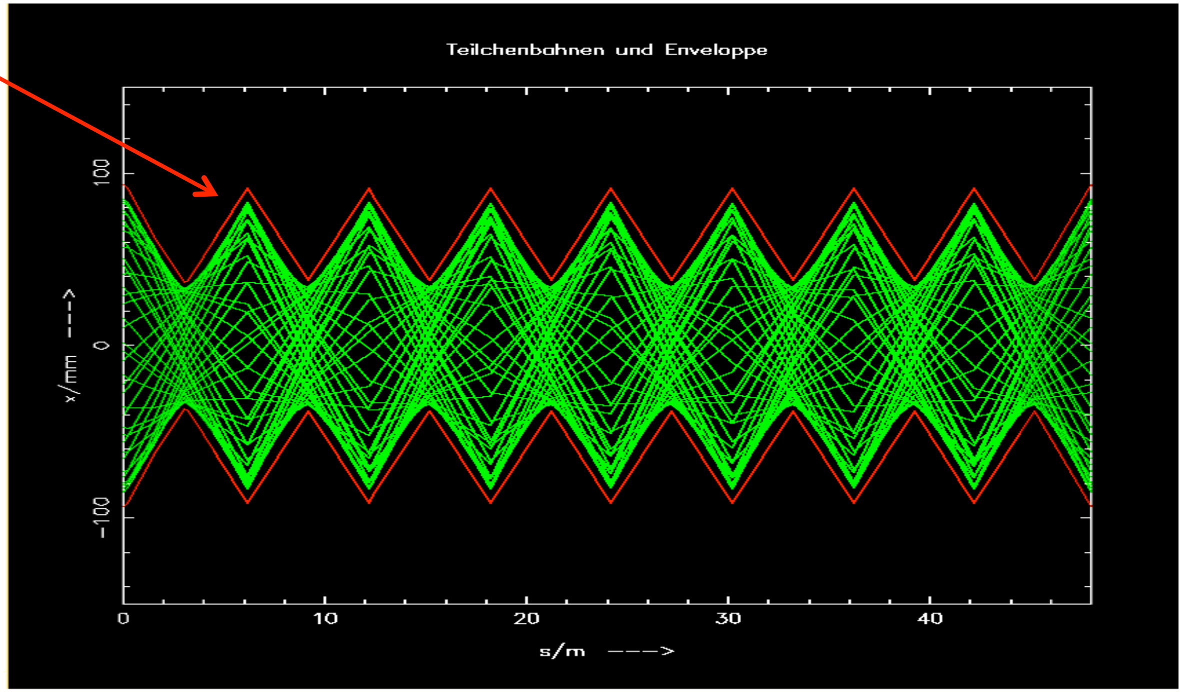


Amplitude of a particle trajectory:

$$x(s) = \sqrt{\varepsilon} * \sqrt{\beta(s)} * \cos(\psi(s) + \varphi)$$

Maximum size of a particle amplitude

$$\hat{x}(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}$$



β determines the beam size
(... the envelope of all particle trajectories at a given position "s" in the storage ring.

It reflects the periodicity of the magnet structure.

7.) Beam Emittance and Phase Space Ellipse

general solution of Hill equation

(1) $$x(s) = \sqrt{\varepsilon}\,\sqrt{\beta(s)}\,\cos(\psi(s)+\phi)$$

(2) $$x'(s) = -\frac{\sqrt{\varepsilon}}{\sqrt{\beta(s)}}\left\{\alpha(s)\cos(\psi(s)+\phi)+\sin(\psi(s)+\phi)\right\}$$

from (1) we get

$$\cos(\psi(s)+\phi) = \frac{x(s)}{\sqrt{\varepsilon}\,\sqrt{\beta(s)}}$$

$$\alpha(s) = \frac{-1}{2}\beta'(s)$$

$$\gamma(s) = \frac{1+\alpha(s)^2}{\beta(s)}$$

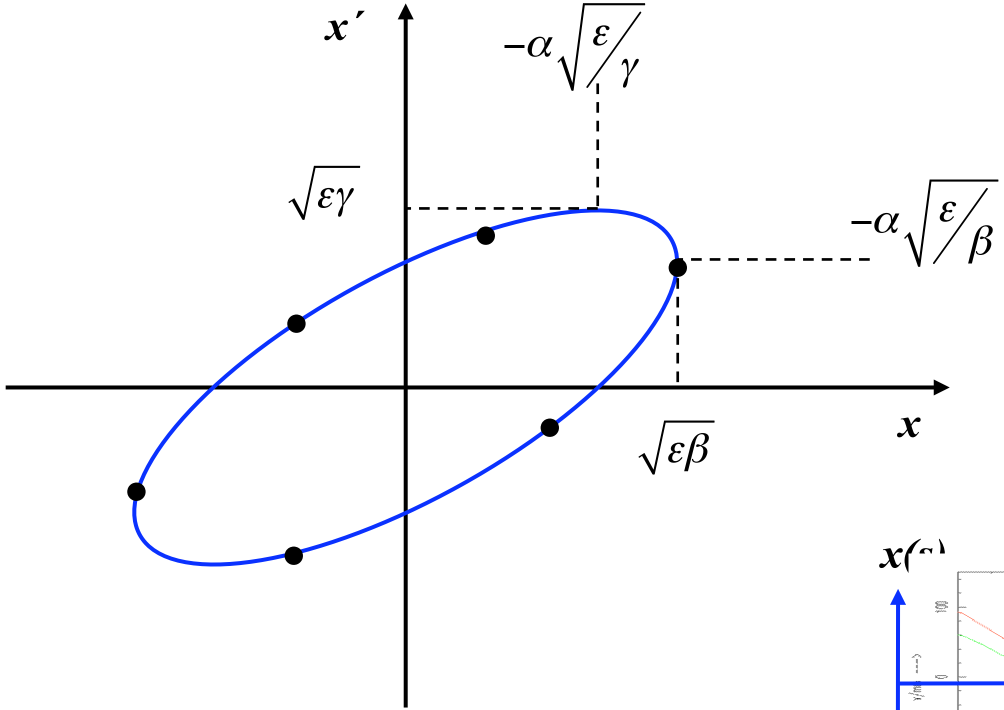
Insert into (2) and solve for ε

$$\varepsilon = \gamma(s)\,x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)\,x'^2(s)$$

***** ε is a constant of the motion … it is independent of „s“*
***** parametric representation of an ellipse in the x x‘ space*
***** shape and orientation of ellipse are given by α, β, γ*

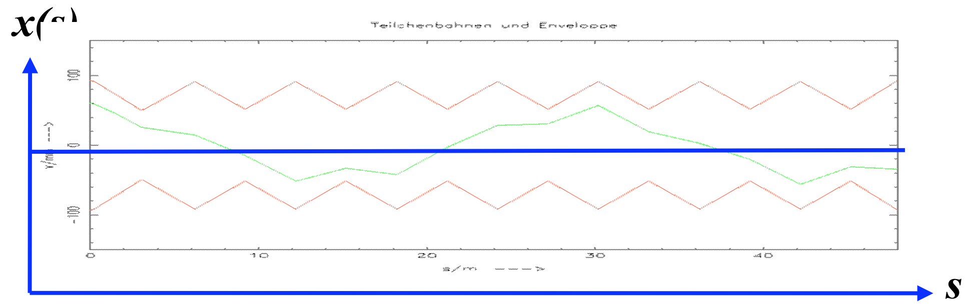
Beam Emittance and Phase Space Ellipse

$$\varepsilon = \gamma(s)\, x^2(s) + 2\alpha(s) x(s) x'(s) + \beta(s)\, x'^2(s)$$



***Liouville:* in reasonable storage rings**
area in phase space is constant.

A = π*ε=const



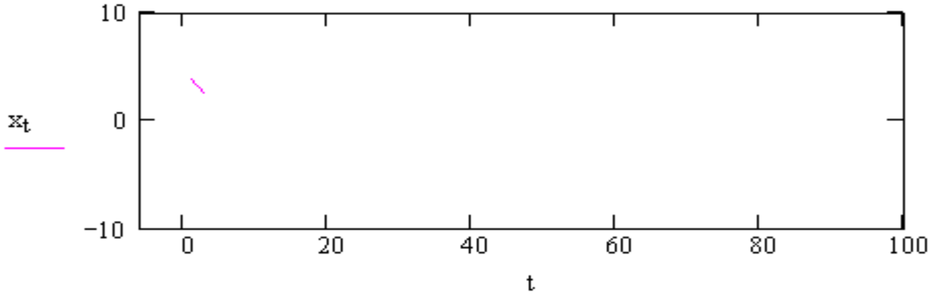
ε beam emittance = woozilycity of the particle ensemble, intrinsic beam parameter,
cannot be changed by the foc. properties.

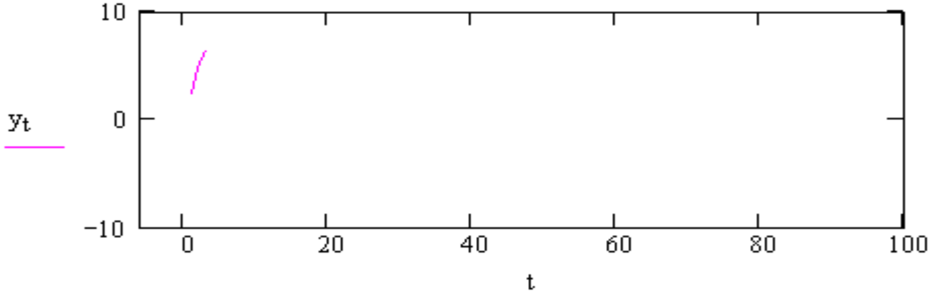
***Scientifiquely speaking:* area covered *in transverse x, x´ phase space* … and it is constant !!!**

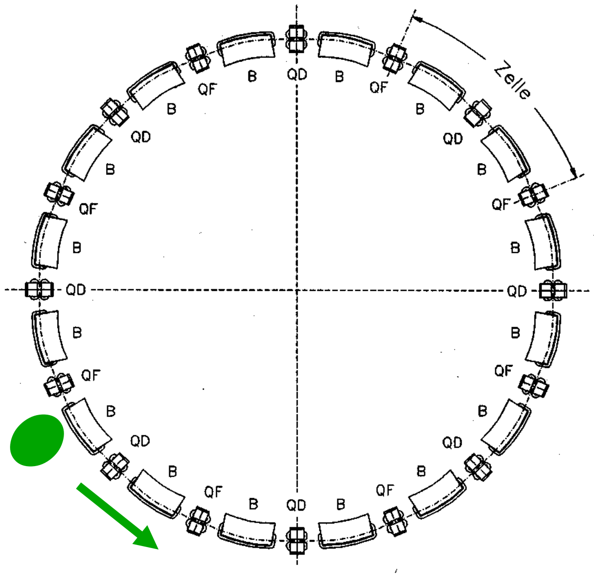
Particle Tracking in a Storage Ring

Calculate x, x´ for each linear accelerator element according to matrix formalism

plot x, x´as a function of „s"

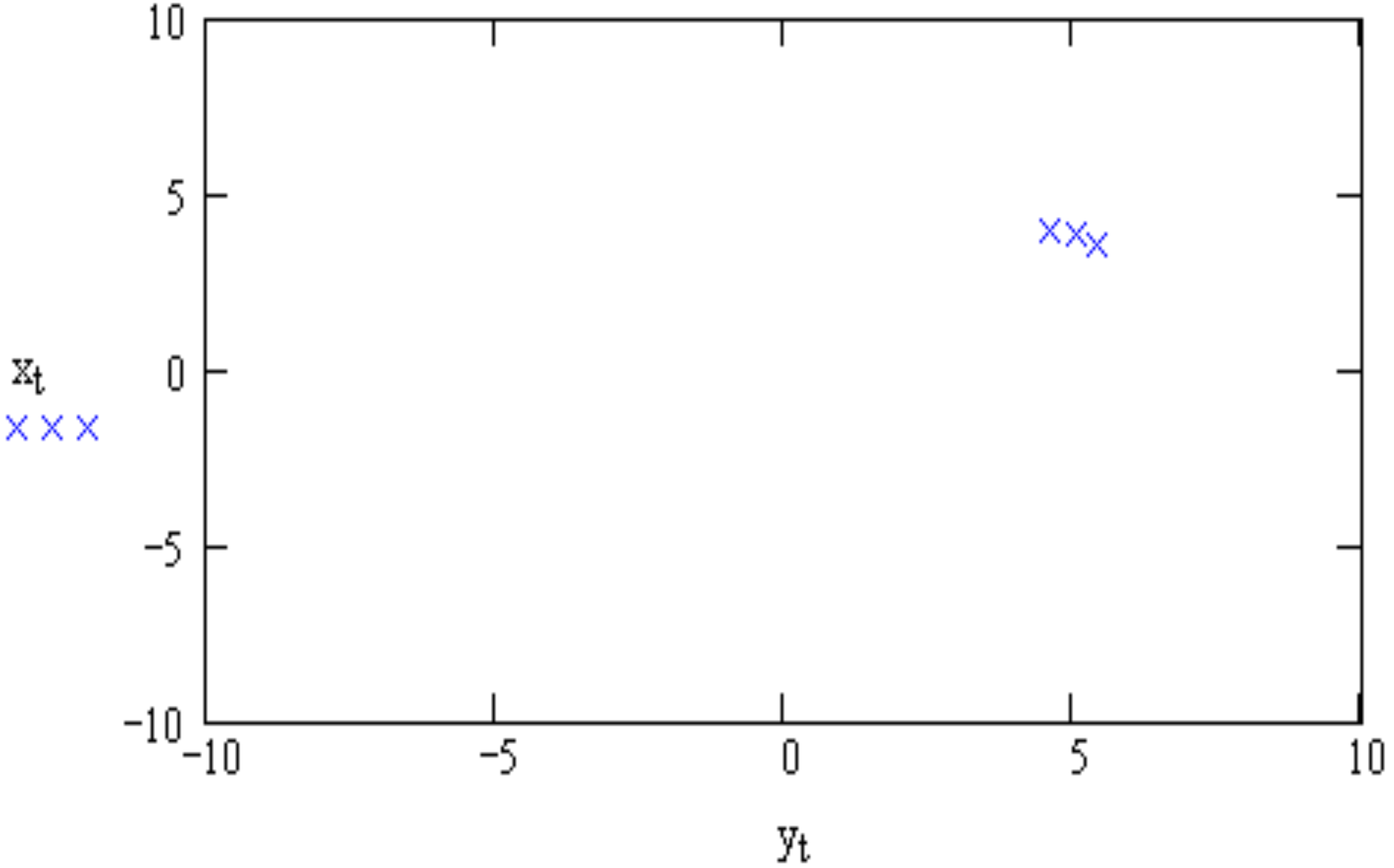






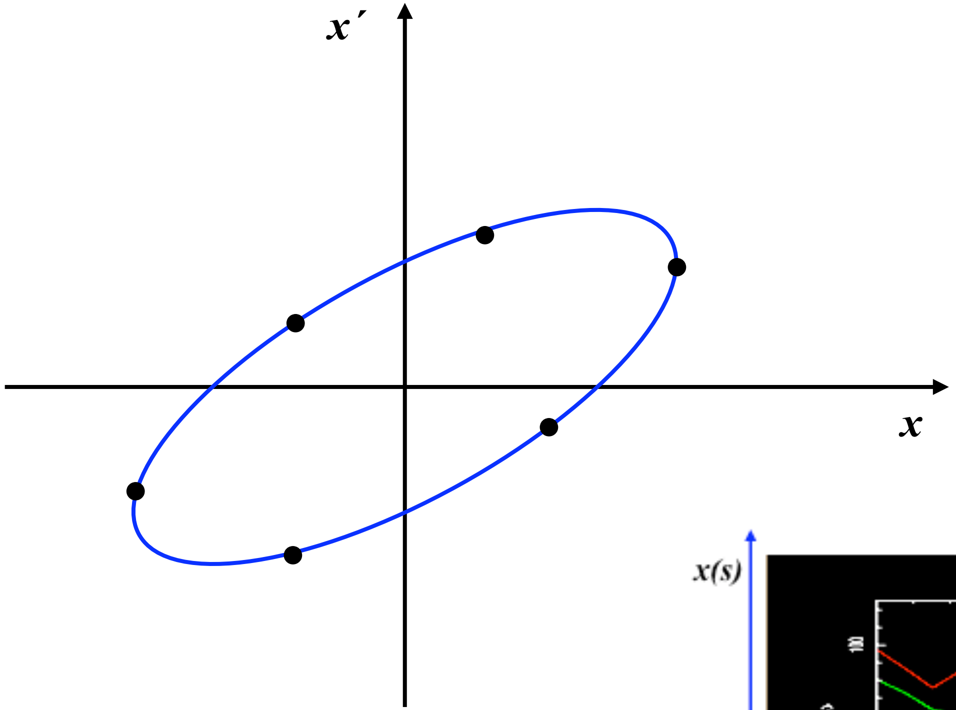
… and now the ellipse:

note for each turn x, x´at a given position „s_1" and plot in the phase space diagram



8.) Beam Emittance and Phase Space Ellipse

$$\varepsilon = \gamma(s)\, x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)\, x'^2(s)$$



***Liouville:* in reasonable storage rings**
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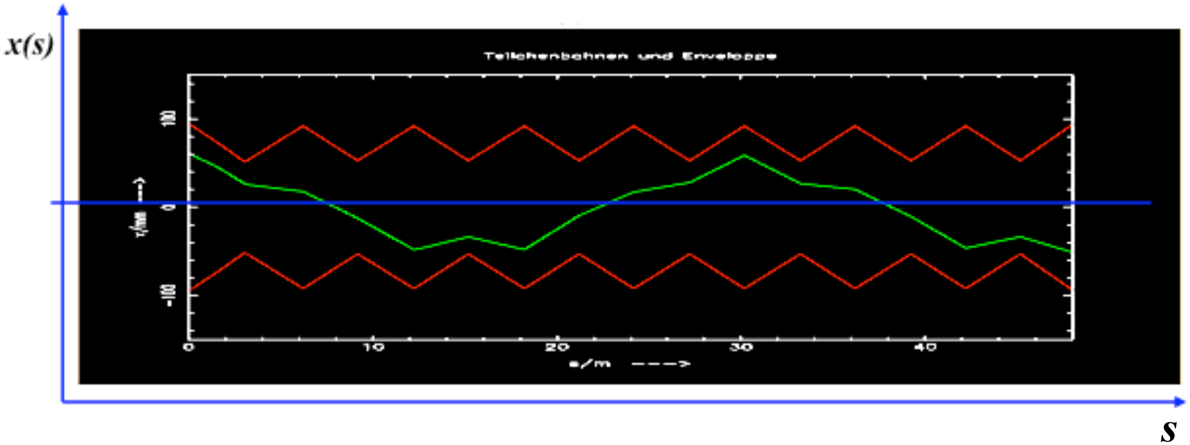


***ε beam emittance* = *woozilycity* of the particle ensemble, *intrinsic beam parameter*,**
cannot be changed by the foc. properties.
***Scientifiquely spoken:* area covered *in transverse x, x´ phase space* … and it is constant !!!**

Phase Space Ellipse

particel trajectory: $x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}\cos\{\psi(s) + \phi\}$

max. Amplitude: $\hat{x}(s) = \sqrt{\varepsilon\beta}$ → *x´ at that position …?*

… put $\hat{x}(s)$ *into* $\varepsilon = \gamma(s)\, x^2(s) + 2\alpha(s) x(s) x'(s) + \beta(s)\, x'^2(s)$ *and solve for x´*

$$\varepsilon = \gamma \cdot \varepsilon\beta + 2\alpha\sqrt{\varepsilon\beta}\cdot x' + \beta x'^2$$

$$\rightarrow \quad x' = -\alpha \cdot \sqrt{\varepsilon / \beta}$$

- *A high β-function means a large beam size and a small beam divergence.* **!**
  … et vice versa !!!

- *In the middle of a quadrupole β = maximum,*
  α = zero
  } $x' = 0$

… and the ellipse is flat

Phase Space Ellipse

$$\varepsilon = \gamma(s)\, x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)\, x'^2(s)$$

$$\alpha(s) = \frac{-1}{2}\beta'(s)$$

$$\gamma(s) = \frac{1+\alpha(s)^2}{\beta(s)}$$

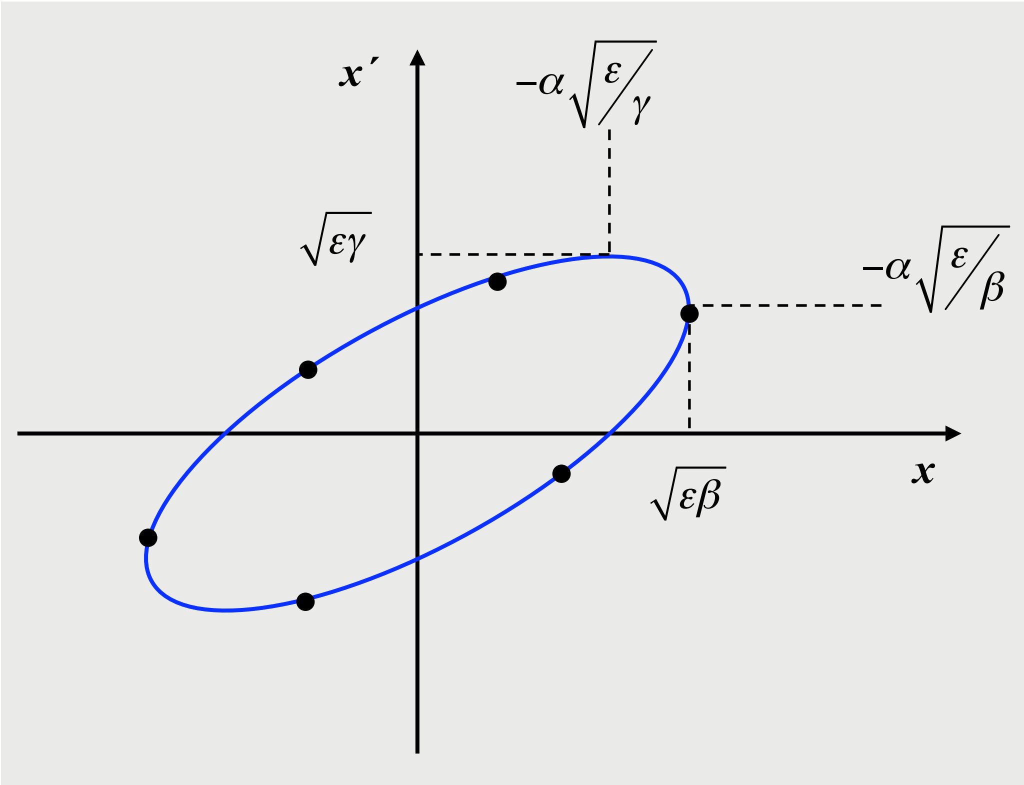
$$\rightarrow \quad \varepsilon = \frac{x^2}{\beta} + \frac{\alpha^2 x^2}{\beta} + 2\alpha \cdot xx' + \beta \cdot x'^2$$

... solve for x´ $\quad x'_{1,2} = \dfrac{-\alpha \cdot x \pm \sqrt{\varepsilon\beta - x^2}}{\beta}$

... and determine $\hat{x}'$ ***via:*** $\quad \dfrac{dx'}{dx} = 0$

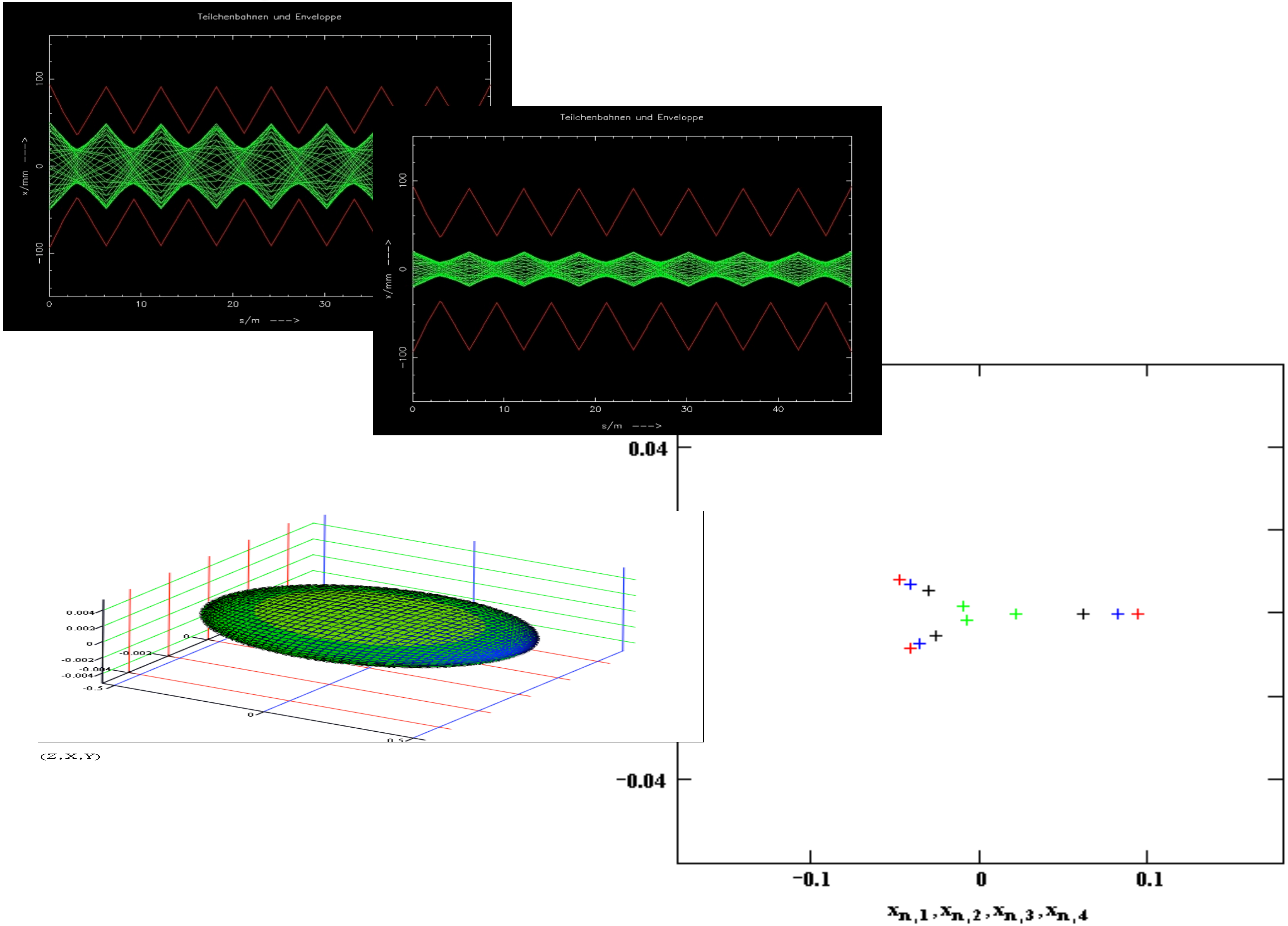
$$\rightarrow \quad \hat{x}' = \sqrt{\varepsilon\gamma}$$

$$\rightarrow \quad \hat{x} = \pm\alpha\sqrt{\varepsilon/\gamma}$$



shape and orientation of the phase space ellipse depend on the Twiss parameters β α γ

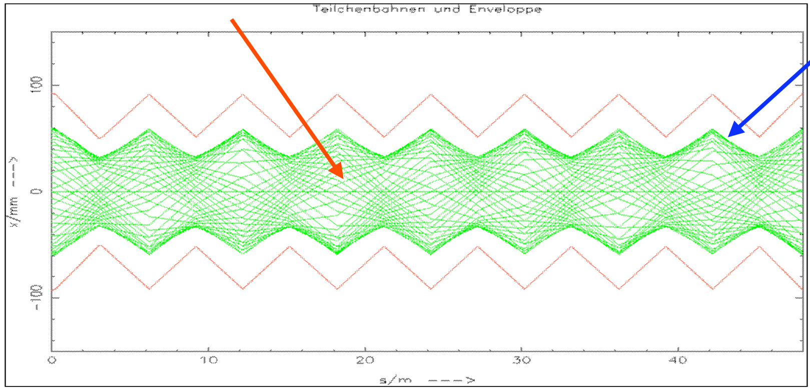
Emittance of the Particle Ensemble:



Emittance of the Particle Ensemble:

$$x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)} \cdot \cos(\Psi(s) + \phi)$$

$$\hat{x}(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}$$



single particle trajectories, $N \approx 10^{11}$ per bunch

Gauß Particle Distribution:

$$\rho(x) = \frac{N \cdot e}{\sqrt{2\pi}\sigma_x} \cdot e^{-\frac{1}{2}\frac{x^2}{\sigma_x^2}}$$

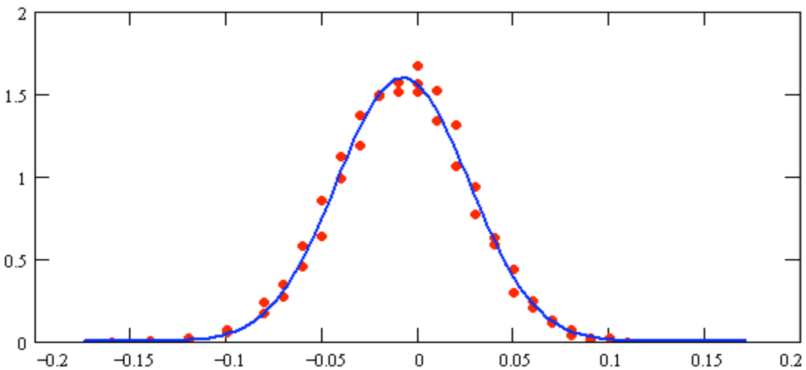
particle at distance 1 σ from centre
↔ 68.3 % of all beam particles

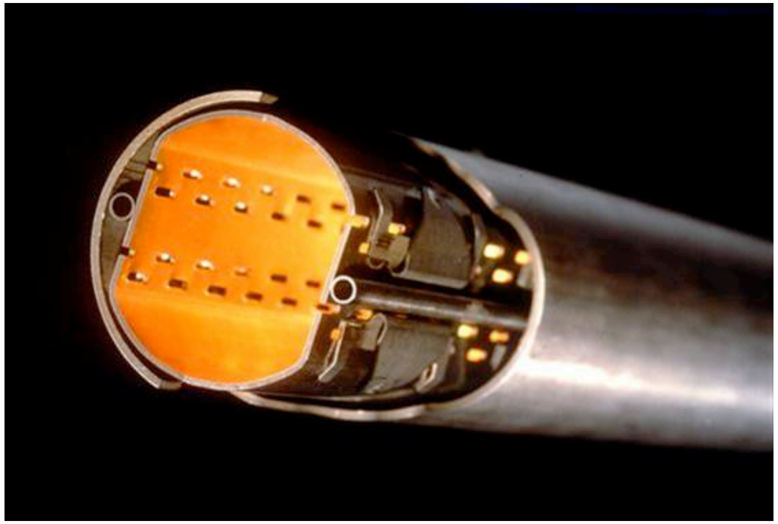
LHC:

$$\beta = 180\, m$$

$$\varepsilon = 5 * 10^{-10}\, m\, rad$$

$$\sigma = \sqrt{\varepsilon * \beta} = \sqrt{5 * 10^{-10}\, m * 180\, m} = 0.3\, mm$$





aperture requirements: r_0 = 12 * σ

9.) Transfer Matrix M   *… yes we had the topic already*

general solution of Hill´s equation

$$\begin{cases} x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}\cos\{\psi(s)+\phi\} \\ x'(s) = \dfrac{-\sqrt{\varepsilon}}{\sqrt{\beta(s)}}\left[\alpha(s)\cos\{\psi(s)+\phi\}+\sin\{\psi(s)+\phi\}\right] \end{cases}$$

remember the trigonometrical gymnastics: sin(a + b) = … etc

$$x(s) = \sqrt{\varepsilon}\sqrt{\beta_s}\left(\cos\psi_s\cos\phi - \sin\psi_s\sin\phi\right)$$

$$x'(s) = \frac{-\sqrt{\varepsilon}}{\sqrt{\beta_s}}\left[\alpha_s\cos\psi_s\cos\phi - \alpha_s\sin\psi_s\sin\phi + \sin\psi_s\cos\phi + \cos\psi_s\sin\phi\right]$$

starting at point s(0) = s_0, where we put $\Psi(0) = 0$

$$\left.\begin{array}{l} \cos\phi = \dfrac{x_0}{\sqrt{\varepsilon\beta_0}} \quad , \\ \sin\phi = -\dfrac{1}{\sqrt{\varepsilon}}\left(x_0'\sqrt{\beta_0} + \dfrac{\alpha_0 x_0}{\sqrt{\beta_0}}\right) \end{array}\right\}$$

inserting above …

$$x(s)=\sqrt{\frac{\beta_s}{\beta_0}}\left\{\cos\psi_s+\alpha_0\sin\psi_s\right\}x_0+\left\{\sqrt{\beta_s\beta_0}\sin\psi_s\right\}x_0'$$

$$x'(s)=\frac{1}{\sqrt{\beta_s\beta_0}}\left\{(\alpha_0-\alpha_s)\cos\psi_s-(1+\alpha_0\alpha_s)\sin\psi_s\right\}x_0+\sqrt{\frac{\beta_0}{\beta_s}}\left\{\cos\psi_s-\alpha_s\sin\psi_s\right\}x_0'$$

which can be expressed ... for convenience ... in matrix form

$$\begin{pmatrix}x\\x'\end{pmatrix}_s=M\begin{pmatrix}x\\x'\end{pmatrix}_0$$

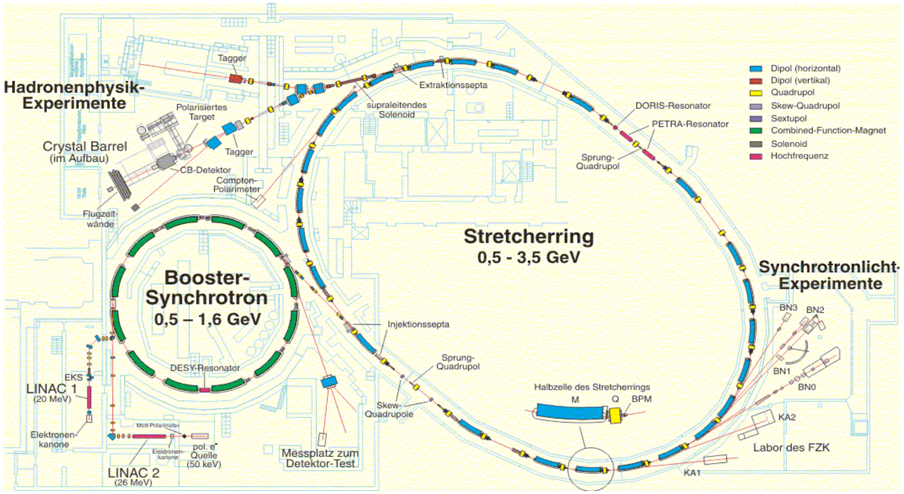
$$M=\begin{pmatrix}\sqrt{\frac{\beta_s}{\beta_0}}\left(\cos\psi_s+\alpha_0\sin\psi_s\right) & \sqrt{\beta_s\beta_0}\sin\psi_s\\ \frac{(\alpha_0-\alpha_s)\cos\psi_s-(1+\alpha_0\alpha_s)\sin\psi_s}{\sqrt{\beta_s\beta_0}} & \sqrt{\frac{\beta_0}{\beta s}}\left(\cos\psi_s-\alpha_s\sin\psi_s\right)\end{pmatrix}$$

* ***we can calculate the single particle trajectories between two locations in the ring, if we know the α β γ at these positions.***
* ***and nothing but the α β γ at these positions.***
* ***... !***

*** Äquivalenz der Matrizen***

10.) Periodic Lattices

$$M=\begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}\left(\cos\psi_s+\alpha_0\sin\psi_s\right) & \sqrt{\beta_s\beta_0}\,\sin\psi_s \\ \frac{(\alpha_0-\alpha_s)\cos\psi_s-(1+\alpha_0\alpha_s)\sin\psi_s}{\sqrt{\beta_s\beta_0}} & \sqrt{\frac{\beta_0}{\beta s}}\left(\cos\psi_s-\alpha_s\sin\psi_s\right) \end{pmatrix}$$



ELSA Electron Storage Ring

„This** rather formidable looking matrix **simplifies considerably if we** consider one complete revolution **…“

$$\boldsymbol{M(s)}=\begin{pmatrix} \cos\psi_{turn}+\alpha_s\sin\psi_{turn} & \beta_s\sin\psi_{turn} \\ -\gamma_s\sin\psi_{turn} & \cos\psi_{turn}-\alpha_s\sin\psi_{turn} \end{pmatrix}$$

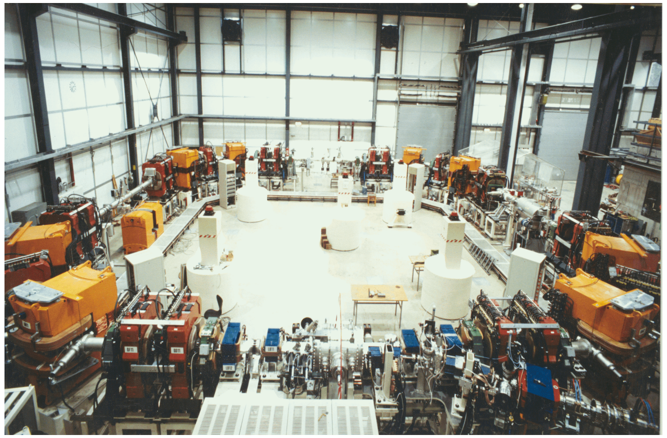
$$\psi_{turn}=\int_s^{s+L}\frac{ds}{\beta(s)}$$

ψ_{turn} = phase advance per period

***Tune:* Phase advance per turn in units of 2π**

$$\boldsymbol{Q}=\frac{1}{2\pi}\oint\frac{\boldsymbol{ds}}{\beta(s)}$$

Stability Criterion:



Question: ***what will happen, if we do not make too many mistakes and your particle performs one complete turn ?***

Matrix for 1 turn:

$$M = \begin{pmatrix} \cos\psi_{turn} + \alpha_s \sin\psi_{turn} & \beta_s \sin\psi_{turn} \\ -\gamma_s \sin\psi_{turn} & \cos\psi_{turn} - \alpha_s \sin\psi_{turn} \end{pmatrix} = \cos\psi \cdot \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\mathbf{1}} + \sin\psi \underbrace{\begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}}_{\mathbf{J}}$$

Matrix for N turns:

$$M^N = \left(1 \cdot \cos\psi + J \cdot \sin\psi\right)^N = 1 \cdot \cos N\psi + J \cdot \sin N\psi$$

The motion for N turns remains bounded, if the elements of M^N remain bounded

$$\psi = real \quad \leftrightarrow \quad \left|\cos\psi\right| \le 1 \quad \leftrightarrow \quad Tr(M) \le 2$$

stability criterion …. proof for the disbelieving collegues !!

Matrix for 1 turn:

$$M = \begin{pmatrix} \cos\psi_{turn} + \alpha_s \sin\psi_{turn} & \beta_s \sin\psi_{turn} \\ -\gamma_s \sin\psi_{turn} & \cos\psi_{turn} - \alpha_s \sin\psi_{turn} \end{pmatrix} = \cos\psi \cdot \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\boldsymbol{I}} + \sin\psi \underbrace{\begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}}_{\boldsymbol{J}}$$

Matrix for 2 turns:

$$\boldsymbol{M}^2 = (\boldsymbol{I}\cos\psi_1 + \boldsymbol{J}\sin\psi_1)(\boldsymbol{I}\cos\psi_2 + \boldsymbol{J}\sin\psi_2)$$

$$= \boldsymbol{I}^2 \cos\psi_1 \cos\psi_2 + \boldsymbol{IJ}\cos\psi_1 \sin\psi_2 + \boldsymbol{JI}\sin\psi_1 \cos\psi_2 + \boldsymbol{J}^2 \sin\psi_1 \sin\psi_2$$

now …

$$\boldsymbol{I}^2 = \boldsymbol{I}$$

$$\boldsymbol{I}\,\boldsymbol{J} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}$$

$$\boldsymbol{J}\,\boldsymbol{I} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} * \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}$$

$$\boldsymbol{I}\,\boldsymbol{J} = \boldsymbol{J}\,\boldsymbol{I}$$

$$\boldsymbol{J}^2 = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} * \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} = \begin{pmatrix} \alpha^2 - \gamma\beta & \alpha\beta - \beta\alpha \\ -\gamma\alpha + \alpha\gamma & \alpha^2 - \gamma\beta \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\boldsymbol{I}$$

$$\boldsymbol{M}^2 = \boldsymbol{I}\cos(\psi_1 + \psi_2) + \boldsymbol{J}\sin(\psi_1 + \psi_2)$$

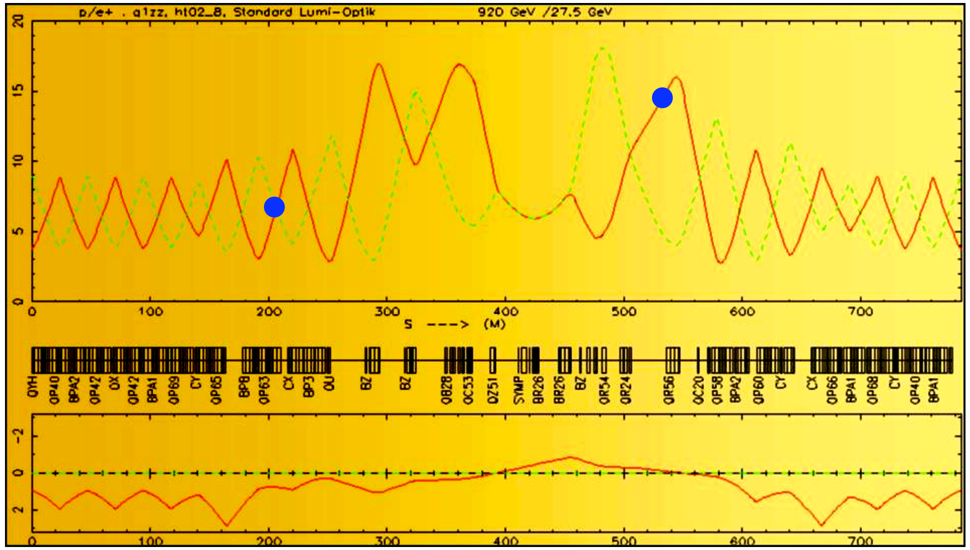
$$\boldsymbol{M}^2 = \boldsymbol{I}\cos(2\psi) + \boldsymbol{J}\sin(2\psi)$$

11.) Transformation of α, β, γ

consider two positions in the storage ring: s_0 , s

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M * \begin{pmatrix} x \\ x' \end{pmatrix}_{s0}$$

$$M = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix}$$



Betafunction in a storage ring

since ε = const (Liouville):

$$\varepsilon = \beta_s x'^2 + 2\alpha_s x x' + \gamma_s x^2$$

$$\varepsilon = \beta_0 x_0'^2 + 2\alpha_0 x_0 x_0' + \gamma_0 x_0^2$$

... remember W = CS´-SC´ = 1

$$\begin{pmatrix} x \\ x' \end{pmatrix}_0 = M^{-1} * \begin{pmatrix} x \\ x' \end{pmatrix}_s$$

$$M^{-1} = \begin{pmatrix} S' & -S \\ -C' & C \end{pmatrix}$$

$\rightarrow$

$$x_0 = S'x - Sx'$$

$$x_0' = -C'x + Cx'$$

... inserting into ε

$$\varepsilon = \beta_0 (Cx' - C'x)^2 + 2\alpha_0 (S'x - Sx')(Cx' - C'x) + \gamma_0 (S'x - Sx')^2$$

sort via x, x´and compare the coefficients to get ....

$$\beta(s) = C^2\beta_0 - 2SC\alpha_0 + S^2\gamma_0$$

$$\alpha(s) = -CC'\beta_0 + (SC' + S'C)\alpha_0 - SS'\gamma_0$$

$$\gamma(s) = C'^2\beta_0 - 2S'C'\alpha_0 + S'^2\gamma_0$$

in matrix notation:

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_s = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC' + CS' & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} \cdot \begin{pmatrix} \beta_0 \\ \alpha_0 \\ \gamma_0 \end{pmatrix}$$

!

1.) this expression is important

2.) given the twiss parameters α, β, γ at any point in the lattice we can transform them and calculate their values at any other point in the ring.

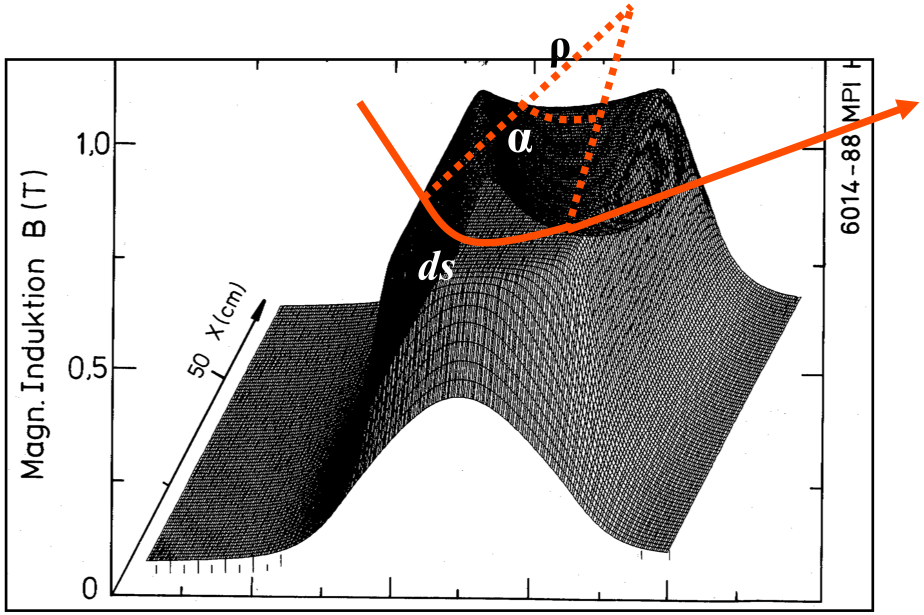
3.) the transfer matrix is given by the focusing properties of the lattice elements, the elements of M are just those that we used to calculate single particle trajectories.

4.) go back to point 1.)

12.) Lattice Design:

„… how to build a storage ring"

$$B\,\rho = p\,/\,q$$



field map of a storage ring dipole magnet

Circular Orbit: dipole magnets to define the geometry

$$\alpha = \frac{ds}{\rho} \approx \frac{dl}{\rho} = \frac{Bdl}{B\rho}$$

The angle run out in one revolution must be 2π, so

… for a full circle

$$\alpha = \frac{\int Bdl}{B\,\rho} = 2\pi \quad \rightarrow \quad \int Bdl = 2\pi\,\frac{p}{q}$$

… defines the integrated dipole field around the machine.

Nota bene: ♪ $\frac{\Delta B}{B} \approx 10^{-4}$ ***is usually required !!***

Example LHC:

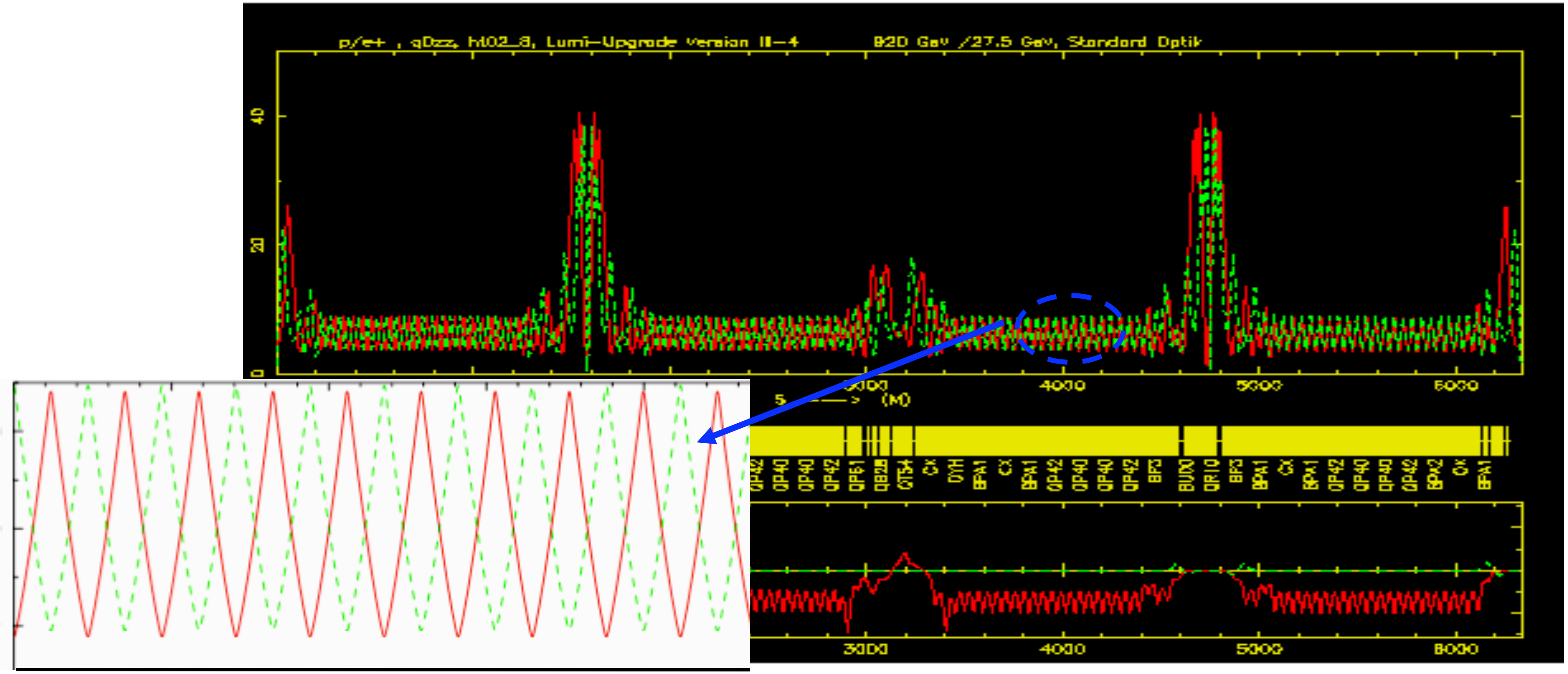
7000 GeV Proton storage ring
dipole magnets N = 1232
l = 15 m
q = +1 e

$$\int B\,dl \approx N\, l\, B = 2\pi\; p/e$$

$$B \approx \frac{2\pi \;\; 7000 \;\; 10^{9}\, eV}{1232 \;\; 15\, m \;\; 3 \;\; 10^{8}\, \frac{m}{s} \;\; e} = 8.3\; Tesla$$

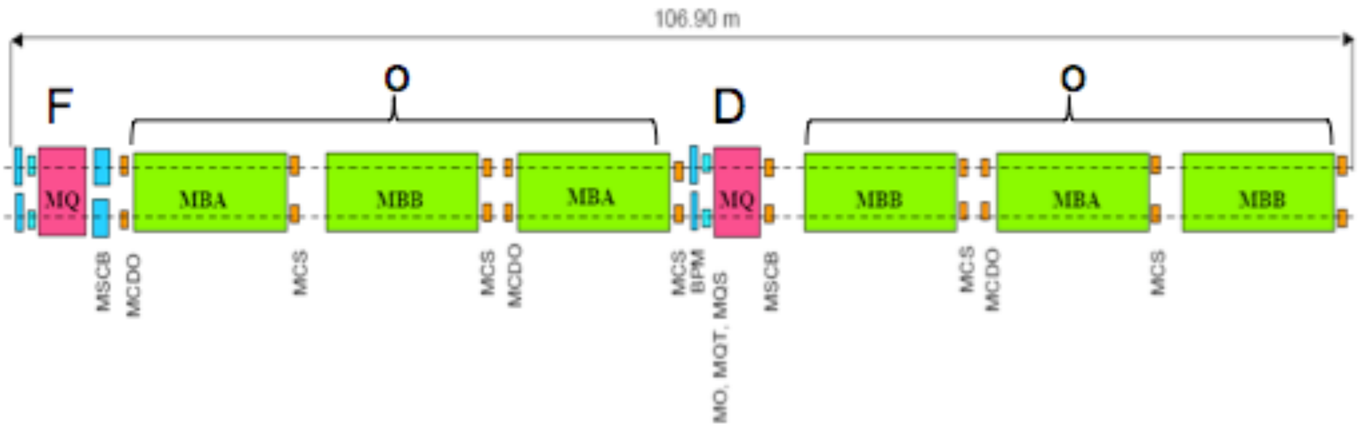
The FoDo-Lattice

A magnet structure consisting of focusing and defocusing quadrupole lenses in alternating order with nothing in between.
(Nothing = elements that can be neglected on first sight: drift, bending magnets, RF structures ... and especially experiments...)



Starting point for the calculation: in the middle of a focusing quadrupole
Phase advance per cell μ = 45°,
→ calculate the twiss parameters for a periodic solution

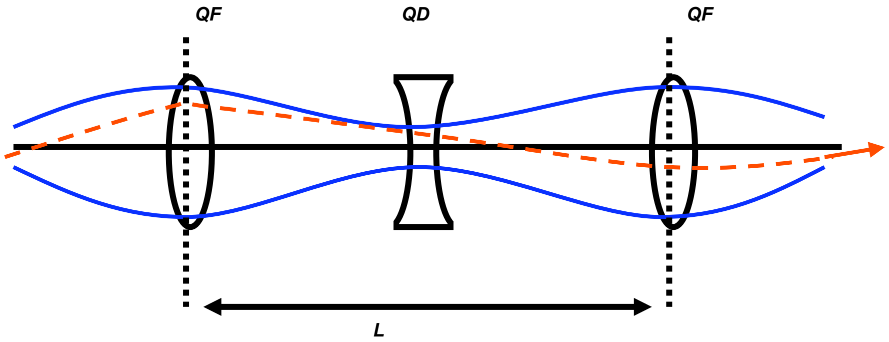
LHC: Lattice Design
the ARC 90° FoDo in both planes

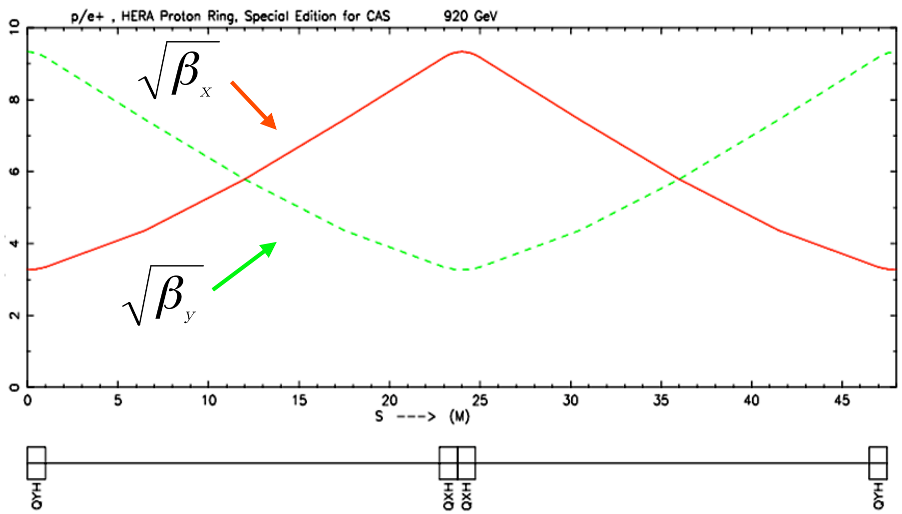


equipped with additional corrector coils

MB: main dipole
MQ: main quadrupole
MQT: Trim quadrupole
MQS: Skew trim quadrupole
MO: Lattice octupole (Landau damping)
MSCB: Skew sextupole
Orbit corrector dipoles
MCS: Spool piece sextupole
MCDO: Spool piece 8 / 10 pole
BPM: Beam position monitor + diagnostics

Periodic solution of a FoDo Cell





Output of the optics program:

Nr	*Type*	*Length*	*Strength*	β_x	α_x	ψ_x	β_y	α_y	ψ_y
		m	*1/m2*	*m*		*1/2π*	*m*		*1/2π*
0	*IP*	*0,000*	*0,000*	*11,611*	*0,000*	*0,000*	*5,295*	*0,000*	*0,000*
1	*QFH*	*0,250*	*-0,541*	*11,228*	*1,514*	*0,004*	*5,488*	*-0,781*	*0,007*
2	*QD*	*3,251*	*0,541*	*5,488*	*-0,781*	*0,070*	*11,228*	*1,514*	*0,066*
3	*QFH*	*6,002*	*-0,541*	*11,611*	*0,000*	*0,125*	*5,295*	*0,000*	*0,125*
4	*IP*	*6,002*	*0,000*	*11,611*	*0,000*	*0,125*	*5,295*	*0,000*	*0,125*

Q_X= 0,125 Q_Y= 0,125

$0.125 * 2\pi = 45°$

Can we understand, what the optics code is doing?

matrices

$$M_{foc} = \begin{pmatrix} \cos(\sqrt{|K|}l_q) & \frac{1}{\sqrt{|K|}}\sin(\sqrt{|K|}l_q) \\ -\sqrt{|K|}\sin(\sqrt{|K|}l_q) & \cos(\sqrt{|K|}l_q) \end{pmatrix}$$

$$M_{drift} = \begin{pmatrix} 1 & l_d \\ 0 & 1 \end{pmatrix}$$

strength and length of the FoDo elements

K = +/- 0.54102 m^{-2}
lq = 0.5 m
ld = 2.5 m

The matrix for the complete cell is obtained by multiplication of the element matrices

$$M_{FoDo} = M_{qfh} * M_{ld} * M_{qd} * M_{ld} * M_{qf}$$

Putting the numbers in and multiplying out ...

$$M_{FoDo} = \begin{pmatrix} 0.707 & 8.206 \\ -0.061 & 0.707 \end{pmatrix}$$

The transfer matrix for one period gives us all the information that we need !

Phase advance per cell

$$M(s) = \begin{pmatrix} \cos\psi + \alpha \sin\psi & \beta \sin\psi \\ -\gamma \sin\psi & \cos\psi - \alpha \sin\psi \end{pmatrix} \rightarrow$$

$$\cos(\psi) = \frac{1}{2} \boldsymbol{Trace}(\boldsymbol{M}) = 0.707$$

$$\psi = \boldsymbol{arc}\cos\left(\frac{1}{2} \boldsymbol{Trace}(\boldsymbol{M})\right) = 45°$$

hor β-function

$$\beta = \frac{\boldsymbol{M}_{1,2}}{\sin\psi} = 11.611\ \boldsymbol{m}$$

hor α-function ♪

$$\alpha = \frac{\boldsymbol{M}_{1,1} - \cos\psi}{\sin\psi} = 0$$

Resume´:

transfer matrix in Twiss form

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M \begin{pmatrix} x \\ x' \end{pmatrix}_0$$

$$M = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}\left(\cos\psi_s + \alpha_0 \sin\psi_s\right) & \sqrt{\beta_s \beta_0}\, \sin\psi_s \\ \frac{(\alpha_0 - \alpha_s)\cos\psi_s - (1 + \alpha_0 \alpha_s)\sin\psi_s}{\sqrt{\beta_s \beta_0}} & \sqrt{\frac{\beta_0}{\beta s}}\left(\cos\psi_s - \alpha_s \sin\psi_s\right) \end{pmatrix}$$

... and for the periodic case

$$\boldsymbol{M}(\boldsymbol{s}) = \begin{pmatrix} \cos\psi_{turn} + \alpha_s \sin\psi_{turn} & \beta_s \sin\psi_{turn} \\ -\gamma_s \sin\psi_{turn} & \cos\psi_{turn} - \alpha_s \sin\psi_{turn} \end{pmatrix}$$

beam emittance during acceleration

$$\varepsilon \propto \frac{1}{\beta\gamma}$$

dispersion

$$D(s) = \frac{x_i(s)}{\Delta p / p}$$