Laser Wakefield Accelerators

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Wakefield generation

$$\frac{1}{{k_p}^2}\frac{\partial^2\phi}{\partial\zeta^2} = \frac{n_b}{n_0} + {\gamma_p}^2\left\{\beta_p\left[1 - \frac{(1+a^2)}{{\gamma_p}^2(1+\phi)^2}\right]^{-1/2} - 1\right\}$$

beam laser

Non-linear **quasistatic** 1D wakefield equation.

Can be simplified by noting $\gamma_p \to 1$ in most cases.

Taking $\gamma_p \gg 1$, $\beta_p = (1 - \frac{1}{{\gamma_p}^2})^{1/2} \approx (1 - \frac{1}{2}\frac{1}{{\gamma_p}^2})$

and expanding the square bracket gives:

$$\frac{1}{{k_p}^2}\frac{\partial^2\phi}{\partial\zeta^2} = \frac{1}{2}\left[\frac{(1+a^2)}{(1+\phi)^2} - 1\right]$$

Wakefield generation

$$\frac{1}{{k_p}^2}\frac{\partial^2\phi}{\partial\zeta^2} = \frac{1}{2}\left[\frac{(1+a^2)}{(1+\phi)^2} - 1\right]$$

Take $\phi \ll 1$,

$$\left(\frac{\partial^2}{\partial\zeta^2} + {k_p}^2\right)\phi = \frac{1}{2}{k_p}^2 a^2$$

A simple force oscillator. Take for example a laser intensity profile that goes as:

$$a^2 = {a_0}^2 \sin^2(\pi\zeta/L)$$

Wakefield generation

Solving (in 1D):

$$\frac{\partial E}{\partial \zeta} = -n_1 \qquad \text{(Gauss' Law)}$$

$$\frac{\partial n_1}{\partial \zeta} = \frac{\partial (n_e \beta)}{\partial \zeta} \qquad \text{(Continuity)}$$

$$(1-\beta)\frac{\partial \beta}{\partial \zeta} = eE - \frac{\partial (a^2)}{\partial \zeta} \qquad \text{(Motion)}$$

where $\beta = v/c$, $n_1 = \delta n / n_0$, and $E = E_{wf}/E_0$

(or alternatively m_e, c, ϵ_0, c all normalised to 1).

Wakefield generation

Solving (in 1D):

$$\frac{\partial E}{\partial \zeta} = -n_1 \qquad \text{(Gauss' Law)}$$

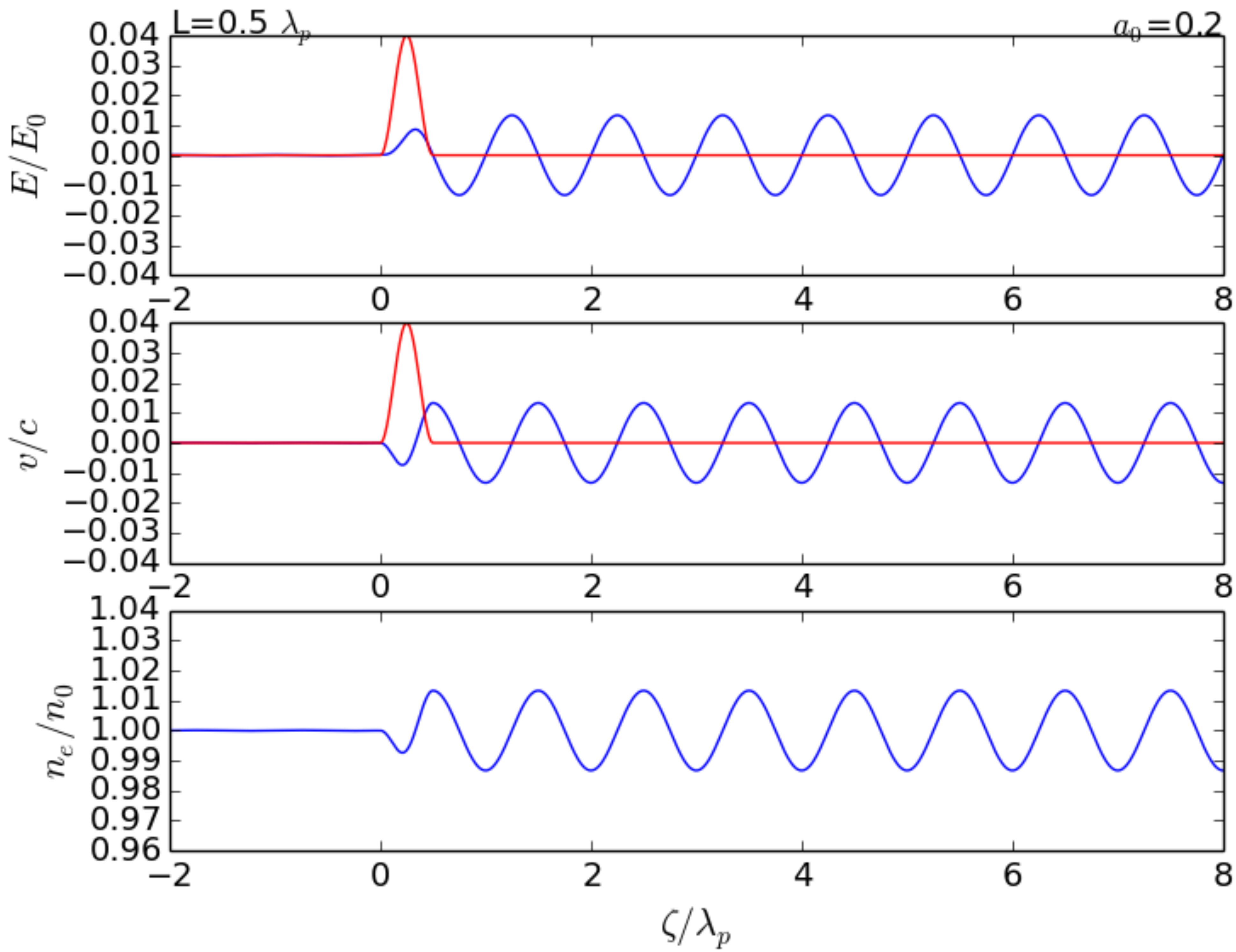
$$n_1 = n_0 \beta \qquad \text{(Continuity)}$$

$$\frac{\partial \beta}{\partial \zeta} = eE - \frac{\partial (a^2)}{\partial \zeta} \qquad \text{(Motion)}$$

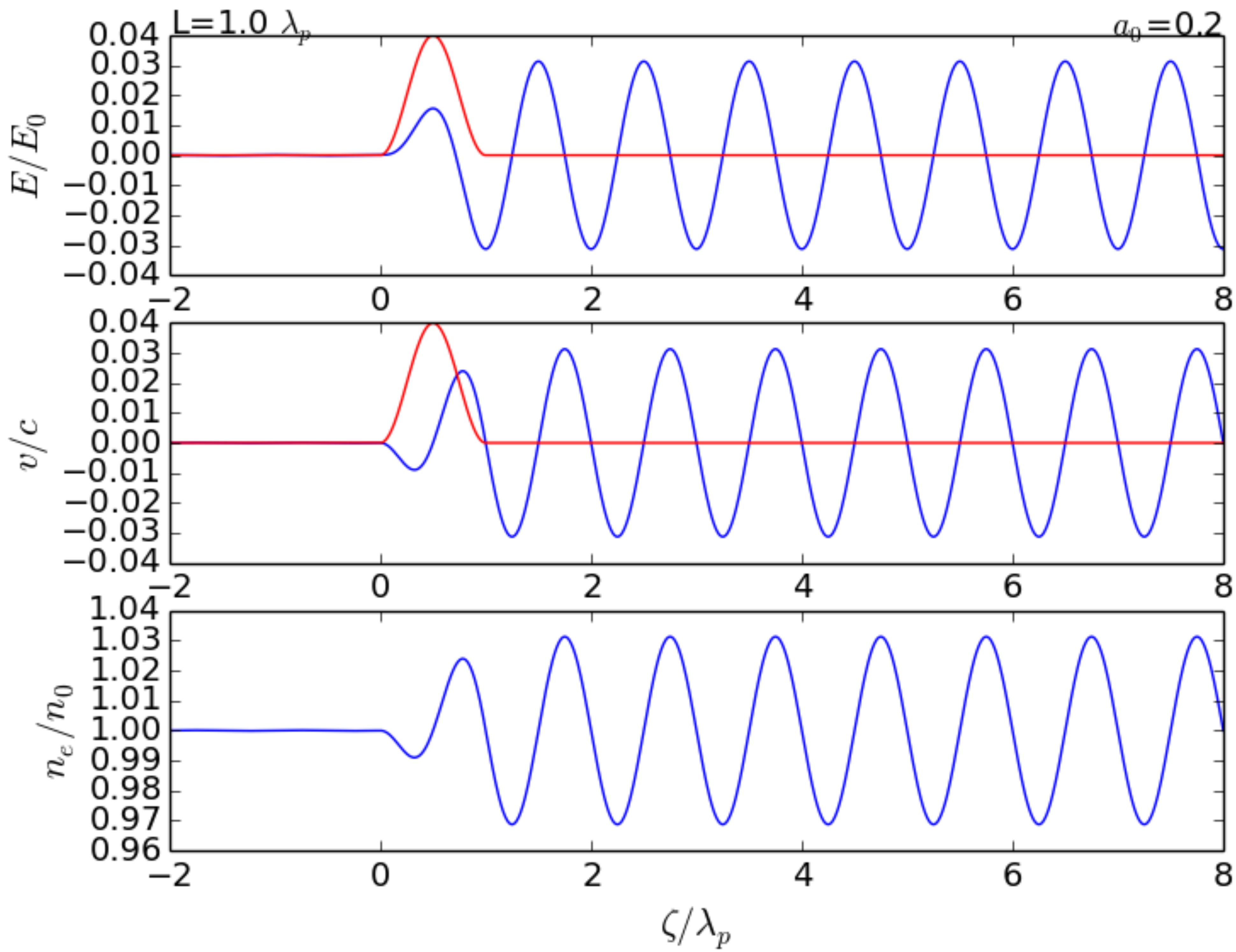
Assuming $\beta \ll 1$, $n_1 \ll n_0$ $\qquad n_e = n_0(1+\beta)$

Have coupled equations in E and β to solve

Wakefield generation



Wakefield generation



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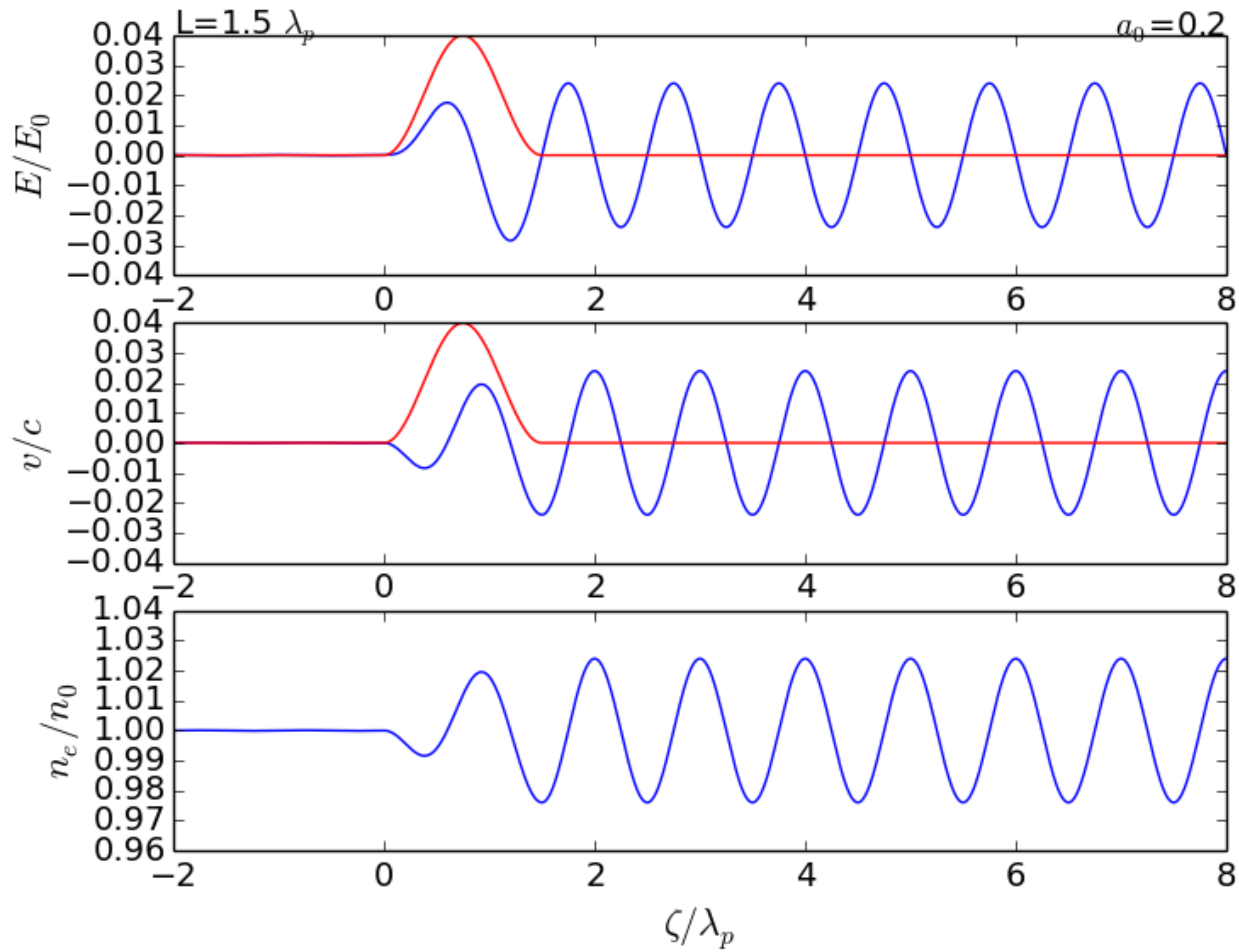

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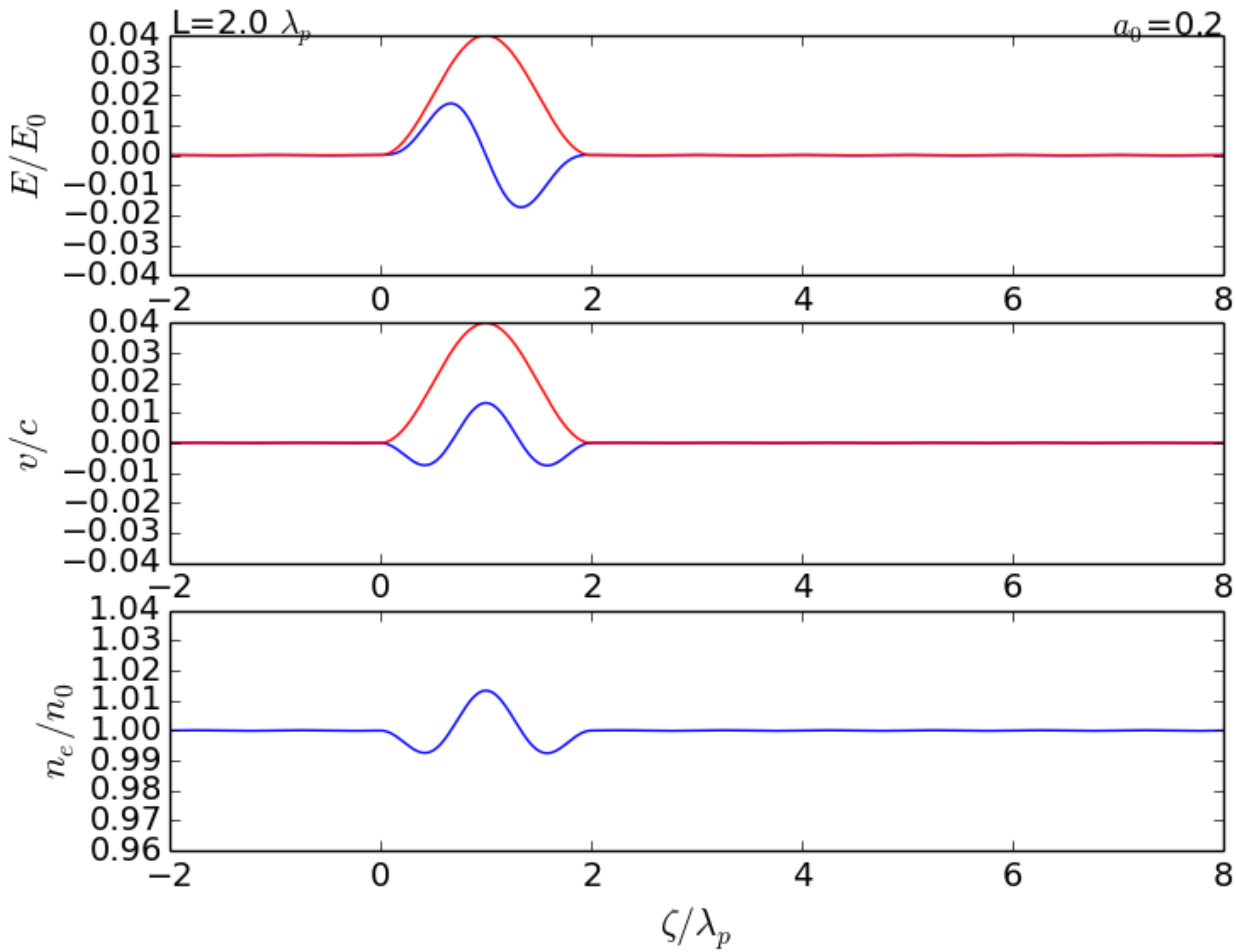

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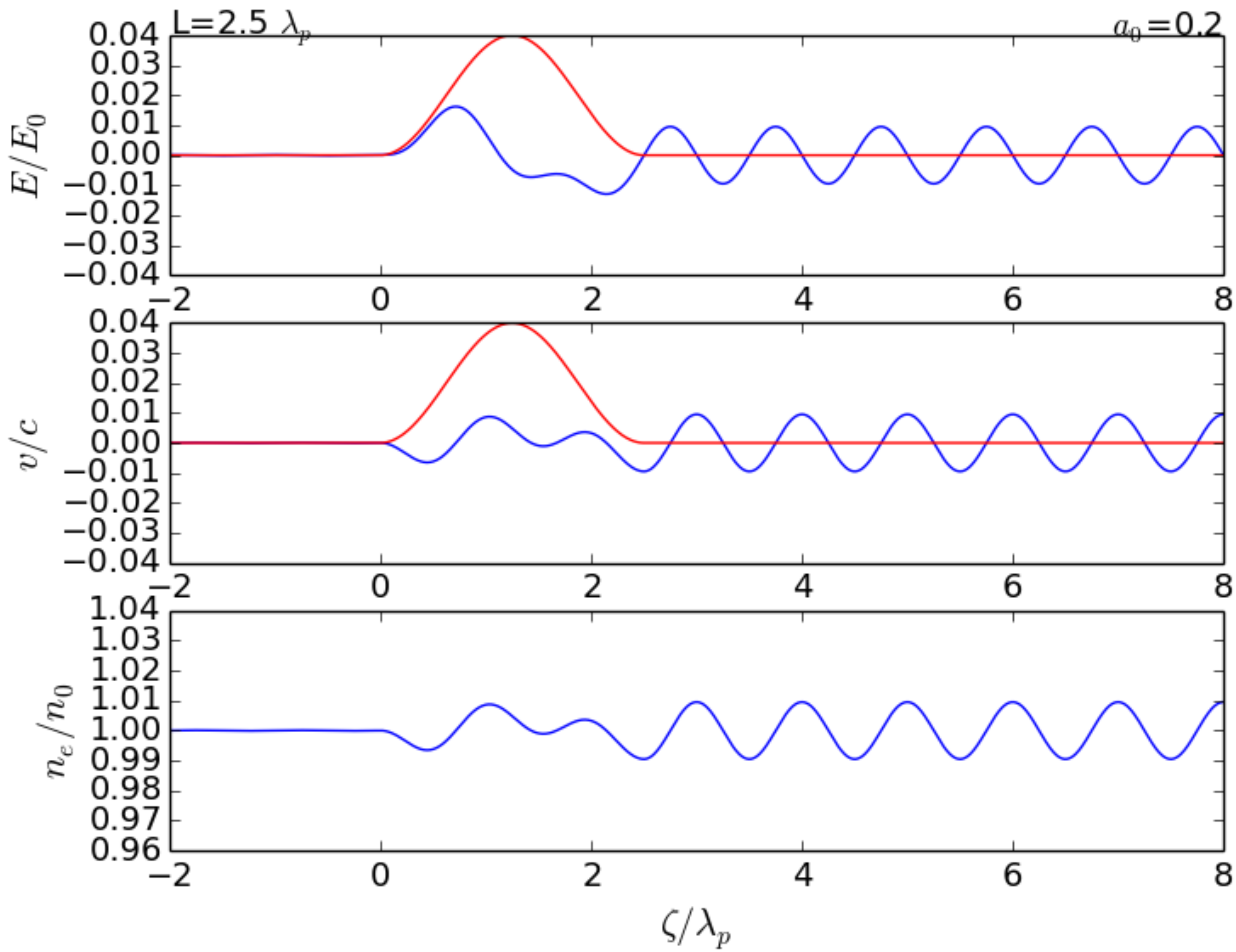

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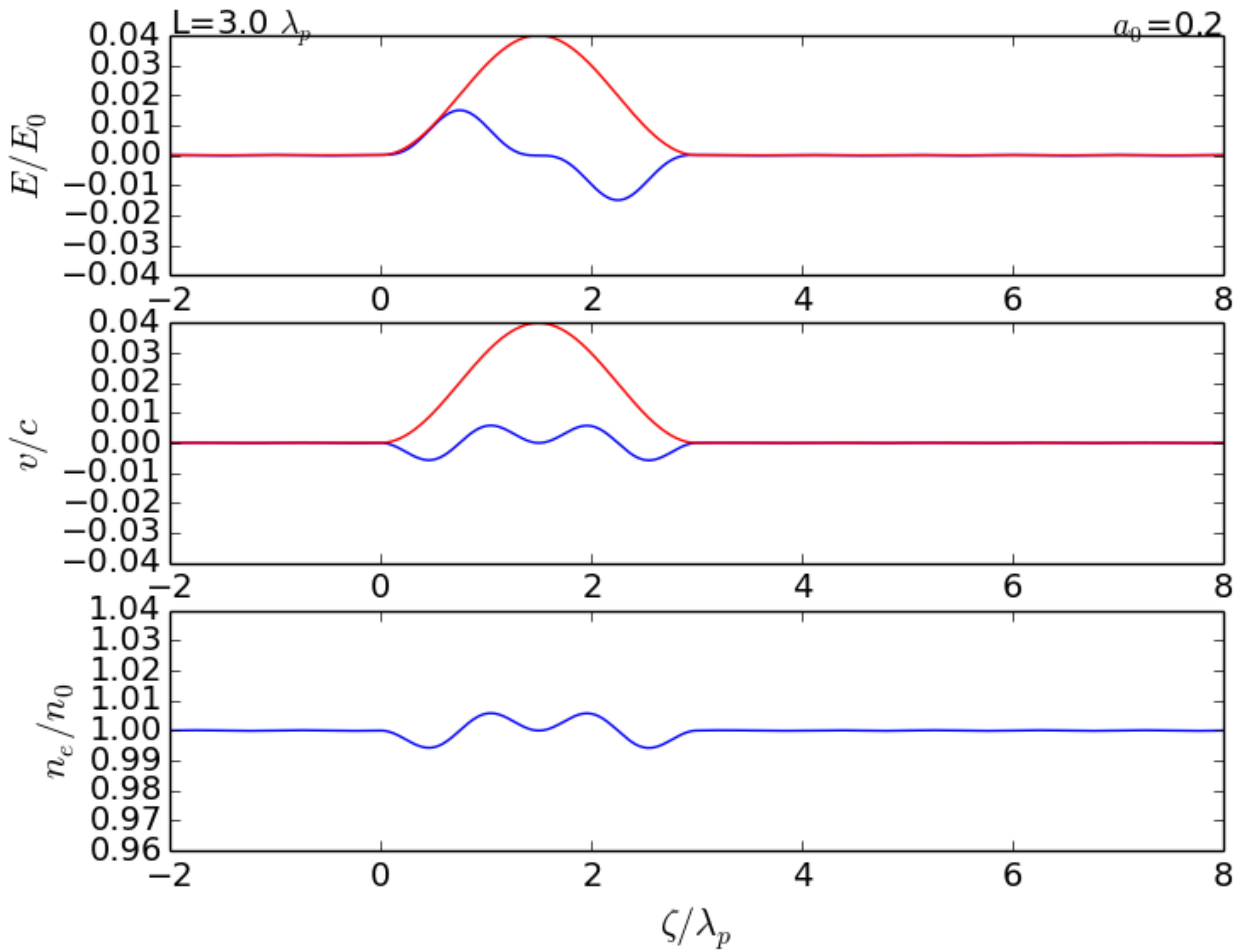

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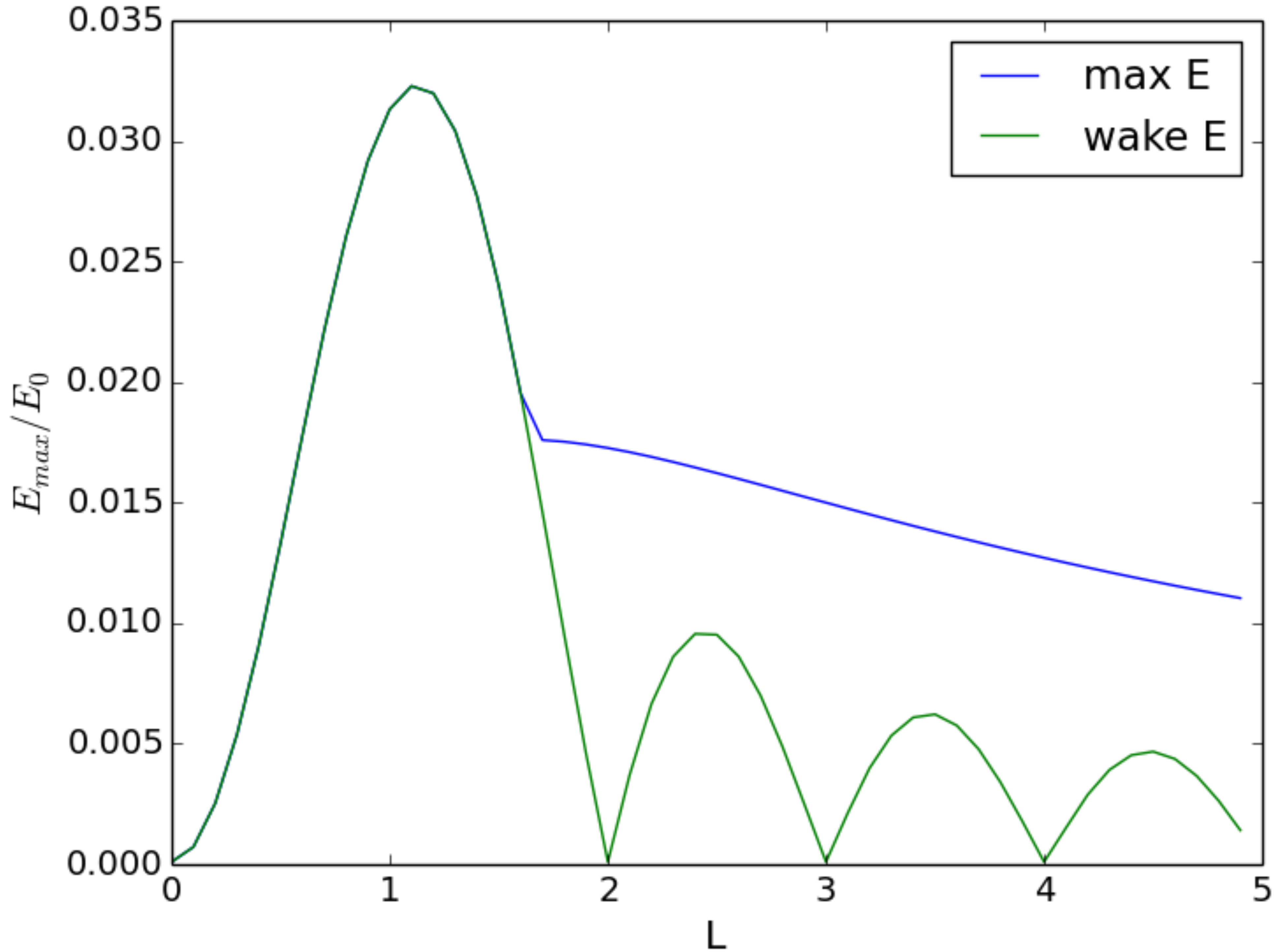

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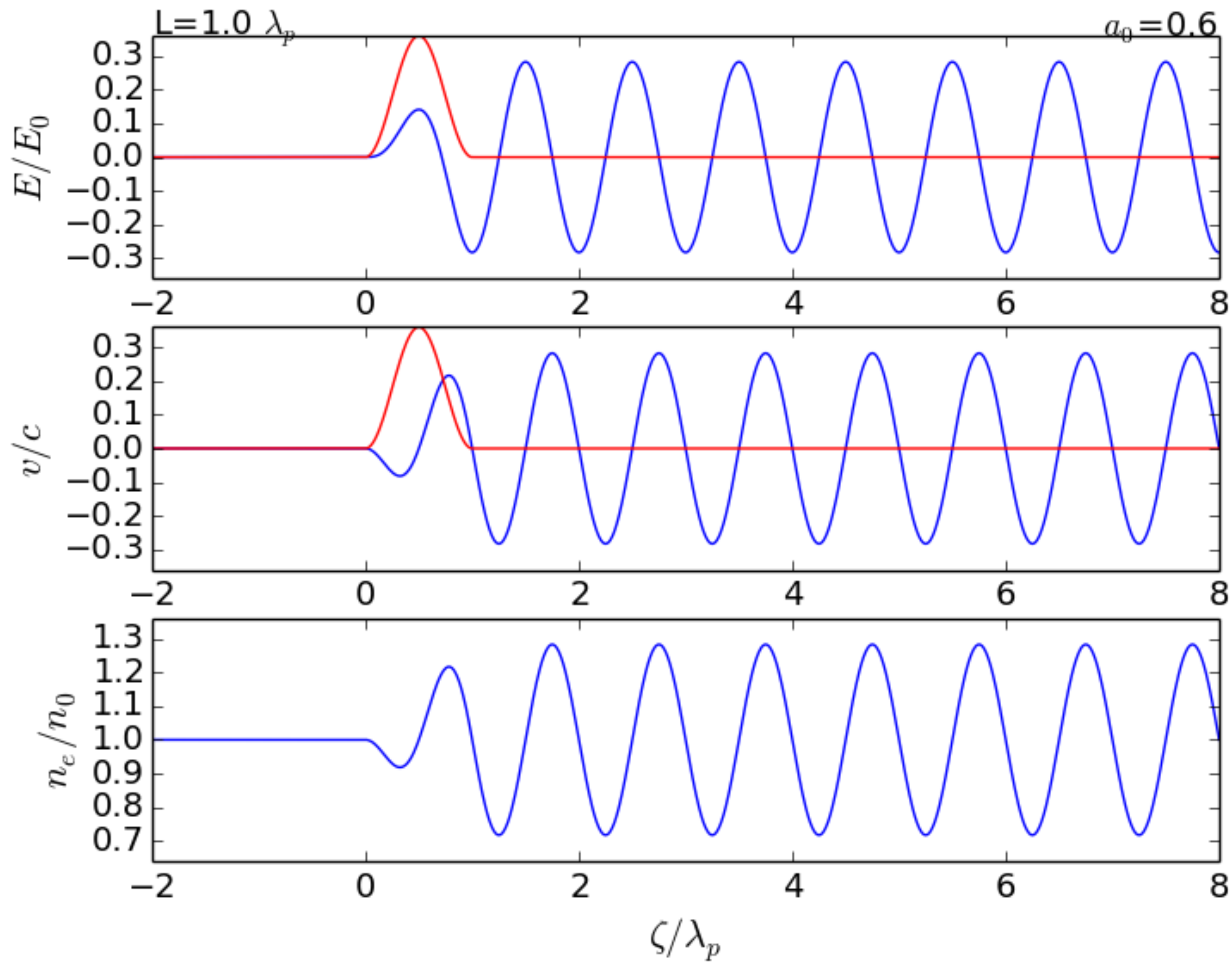

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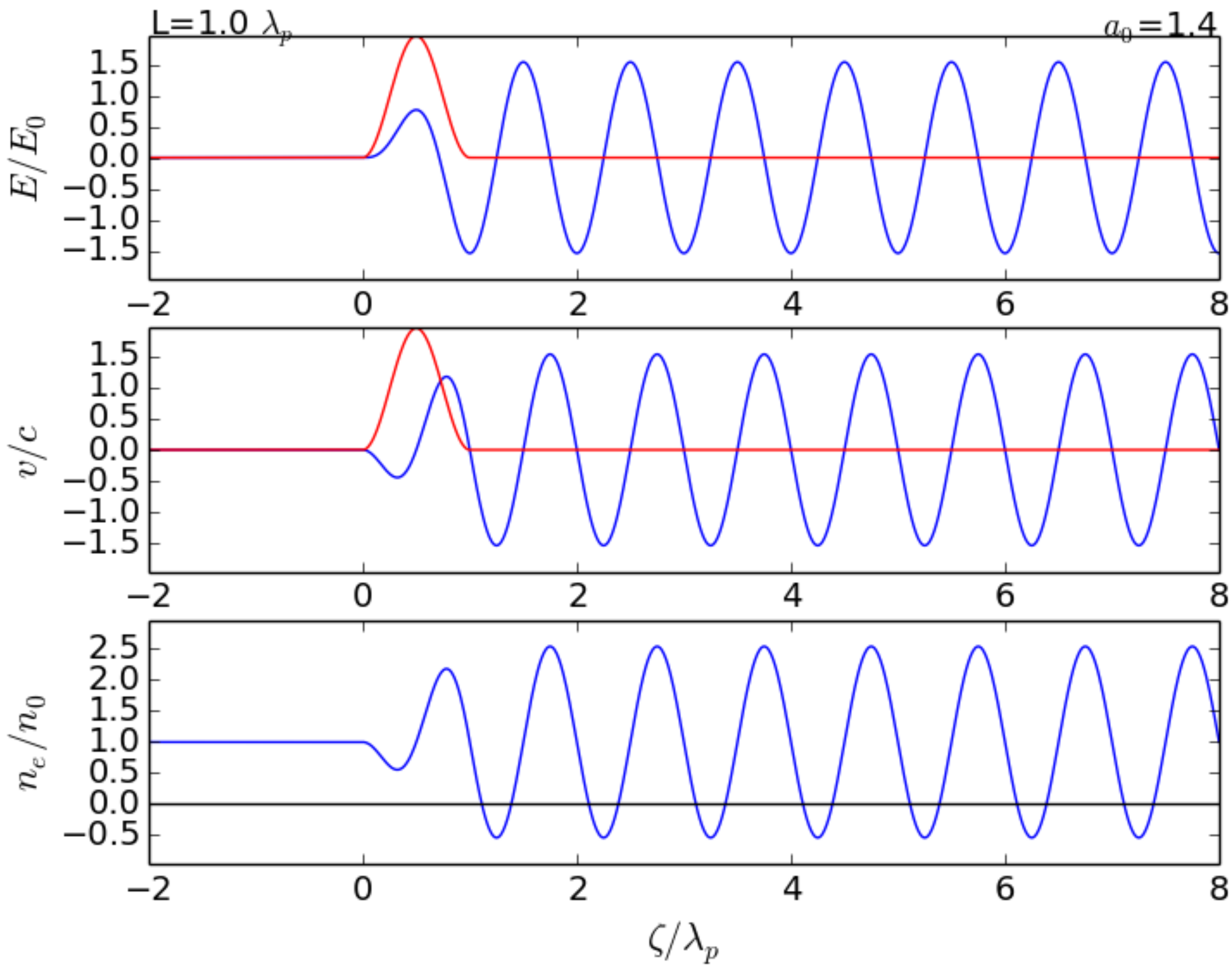


Wakefield generation

Now including plasma wave non-linearity

$$\frac{\partial E}{\partial \zeta} = -n_1 \qquad \text{(Gauss' Law)}$$

$$n_1 = (n_0 + n_1)\beta \qquad \text{(Continuity)}$$

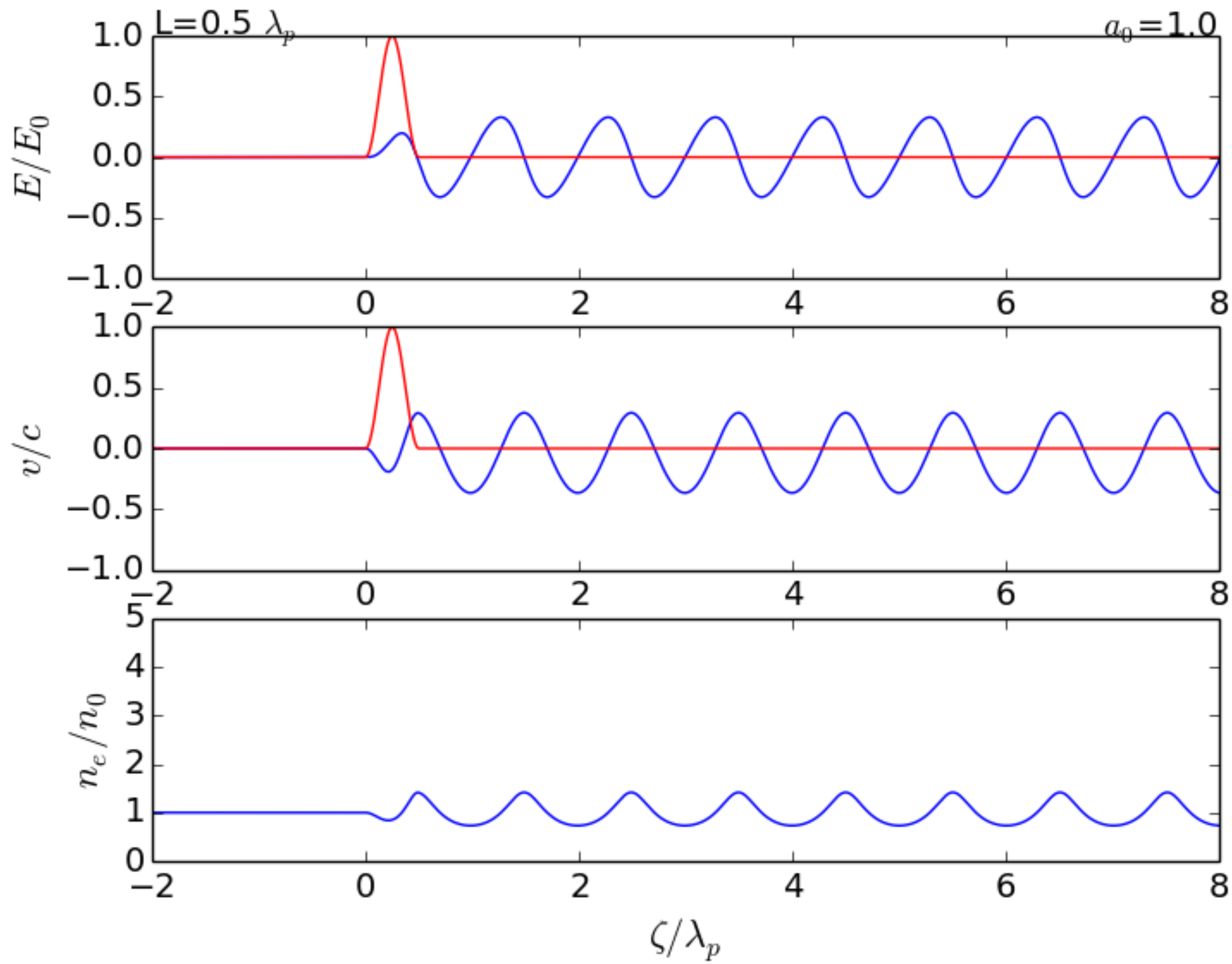
$$\frac{\partial \beta}{\partial \zeta} = eE - \frac{\partial (a^2)}{\partial \zeta} \qquad \text{(Motion)}$$

Rearranging Continuity equation:

$$n_e = n_0 + n_1 = \frac{n_0}{1-\beta}$$

This implies that $n_{min} = \frac{1}{2}$ in 1D, though complete cavitation is possible in 3D.

Wakefield generation



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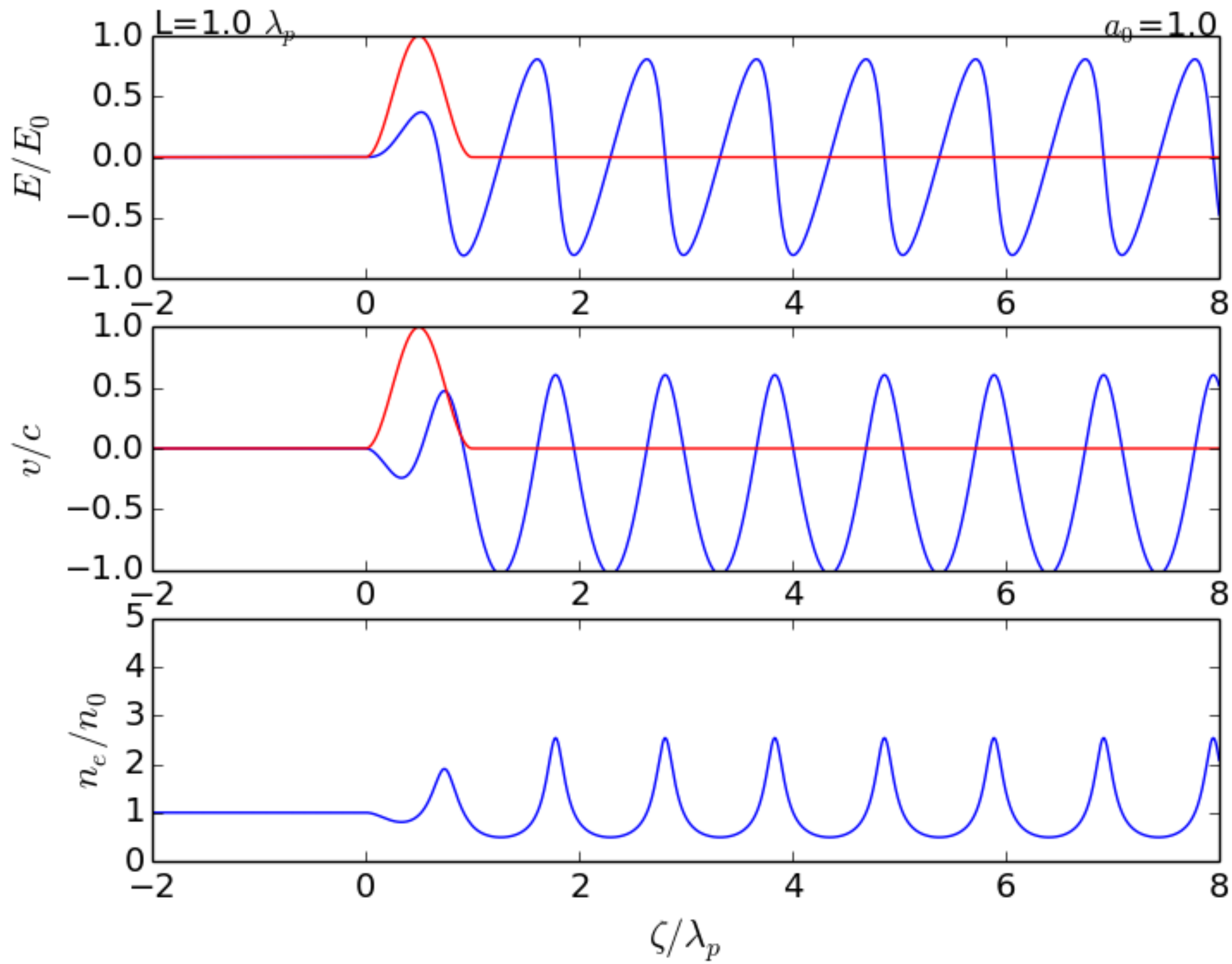

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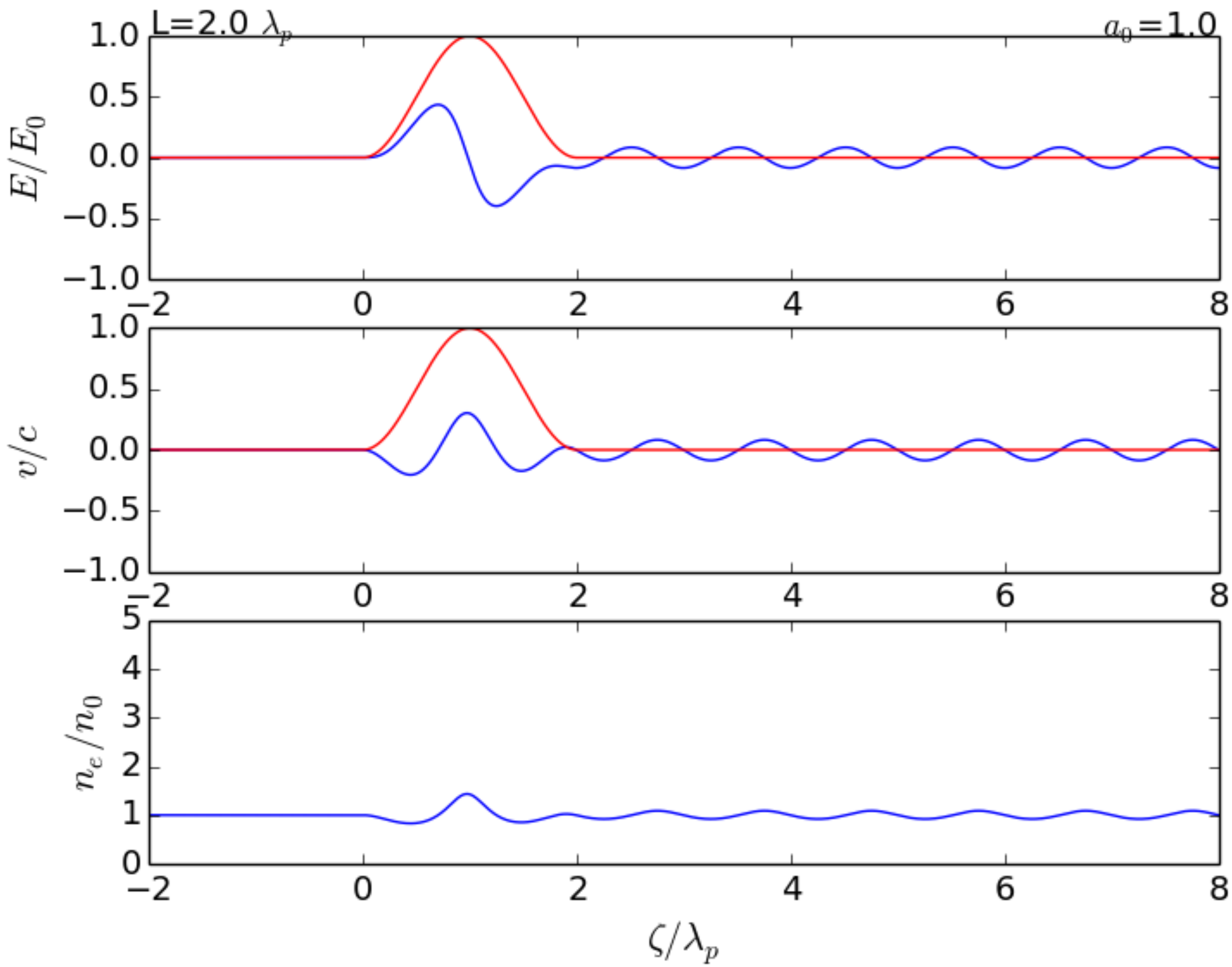

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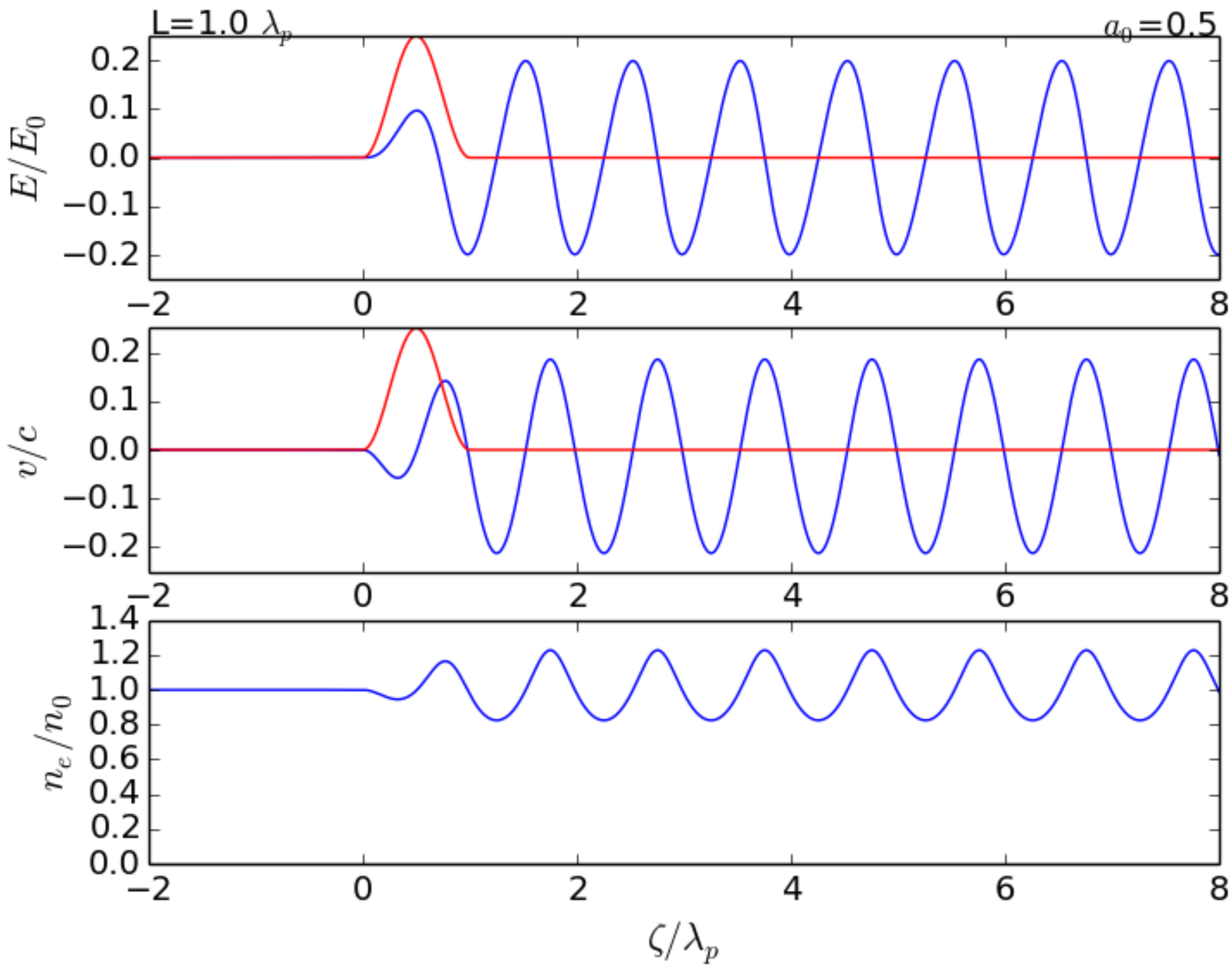

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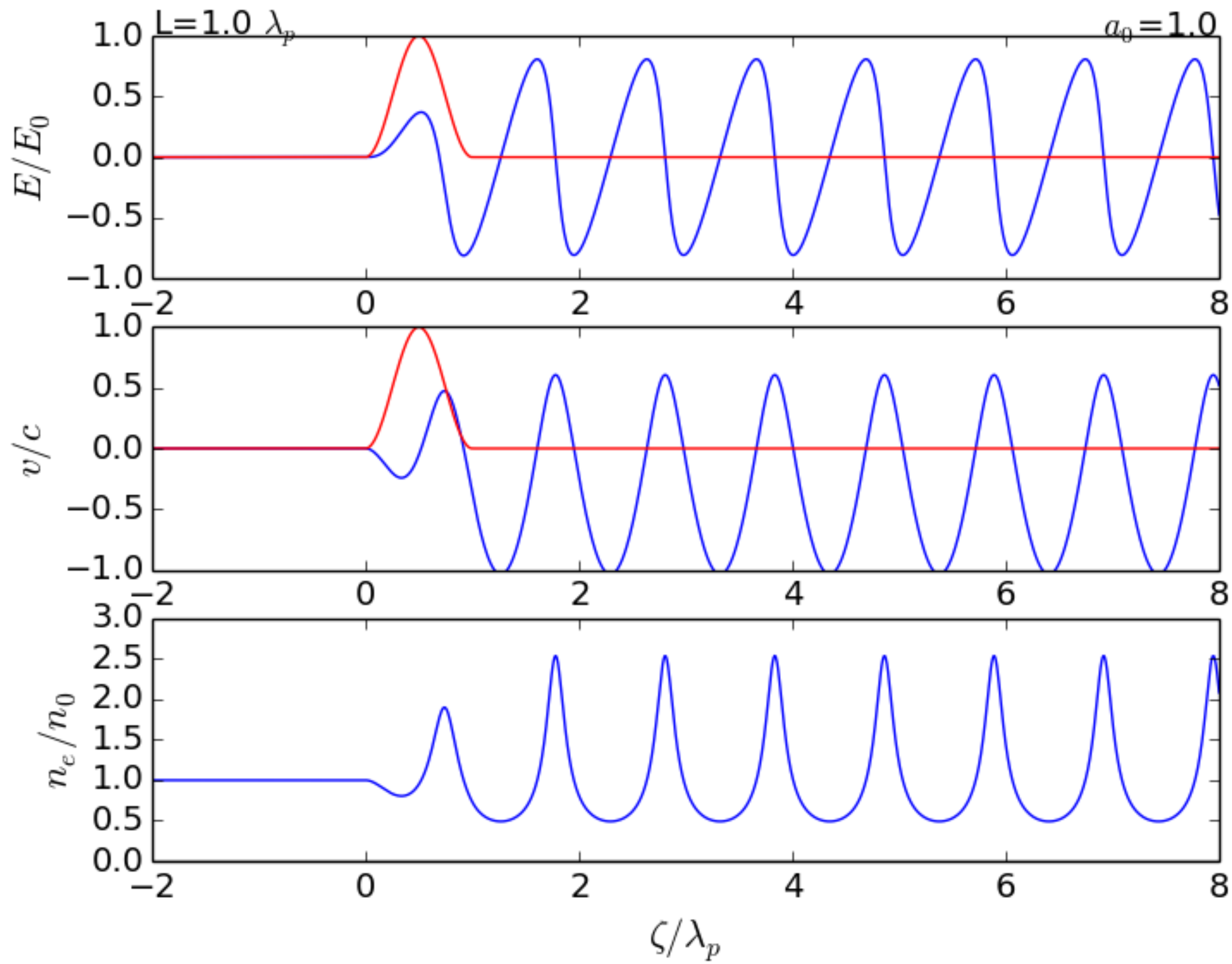

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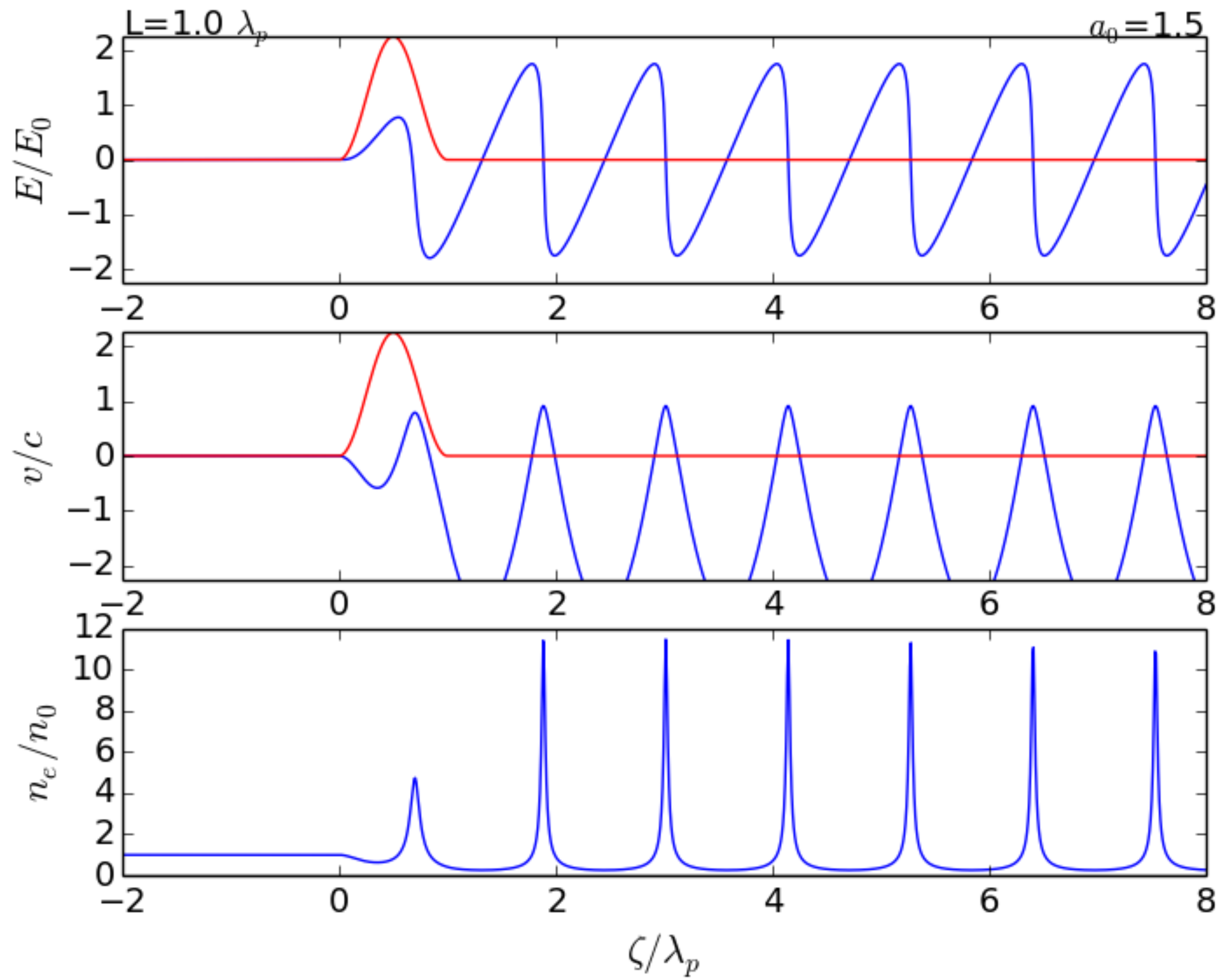


Wakefield generation

Include relativity (and convection):

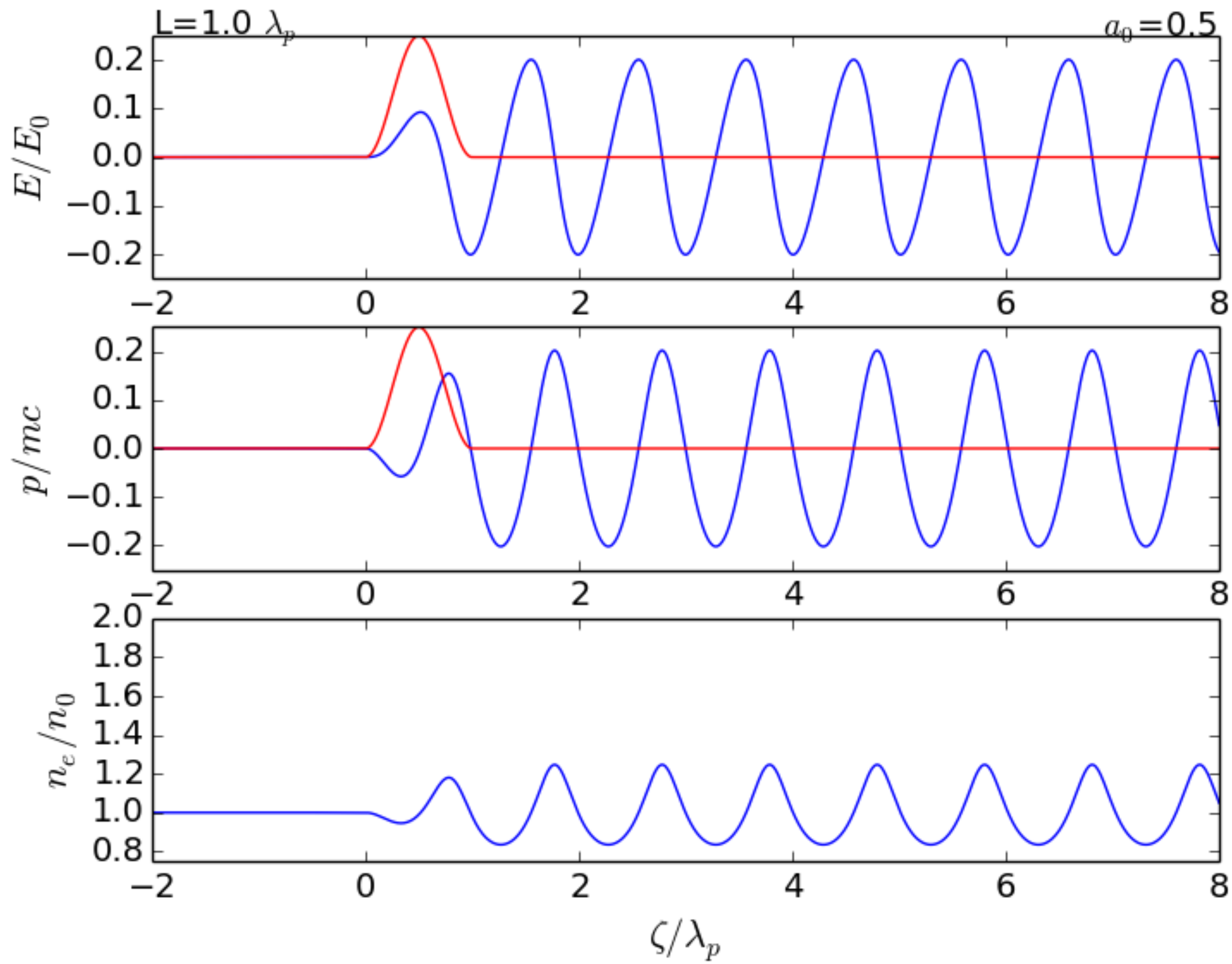
$$\frac{\partial E}{\partial \zeta} = -n_1 \qquad \text{(Gauss' Law)}$$

$$n_1 = (n_0 + n_1)\beta \qquad \text{(Continuity)}$$

$$(1-\beta)\frac{\partial p}{\partial \zeta} = eE - \frac{\partial (a^2)}{\partial \zeta} \qquad \text{(Motion)}$$

$$\text{and using } \beta = \frac{p}{(1+{p_T}^2)^{1/2}} = \frac{p}{(1+p^2+a^2)^{1/2}}$$

Wakefield generation



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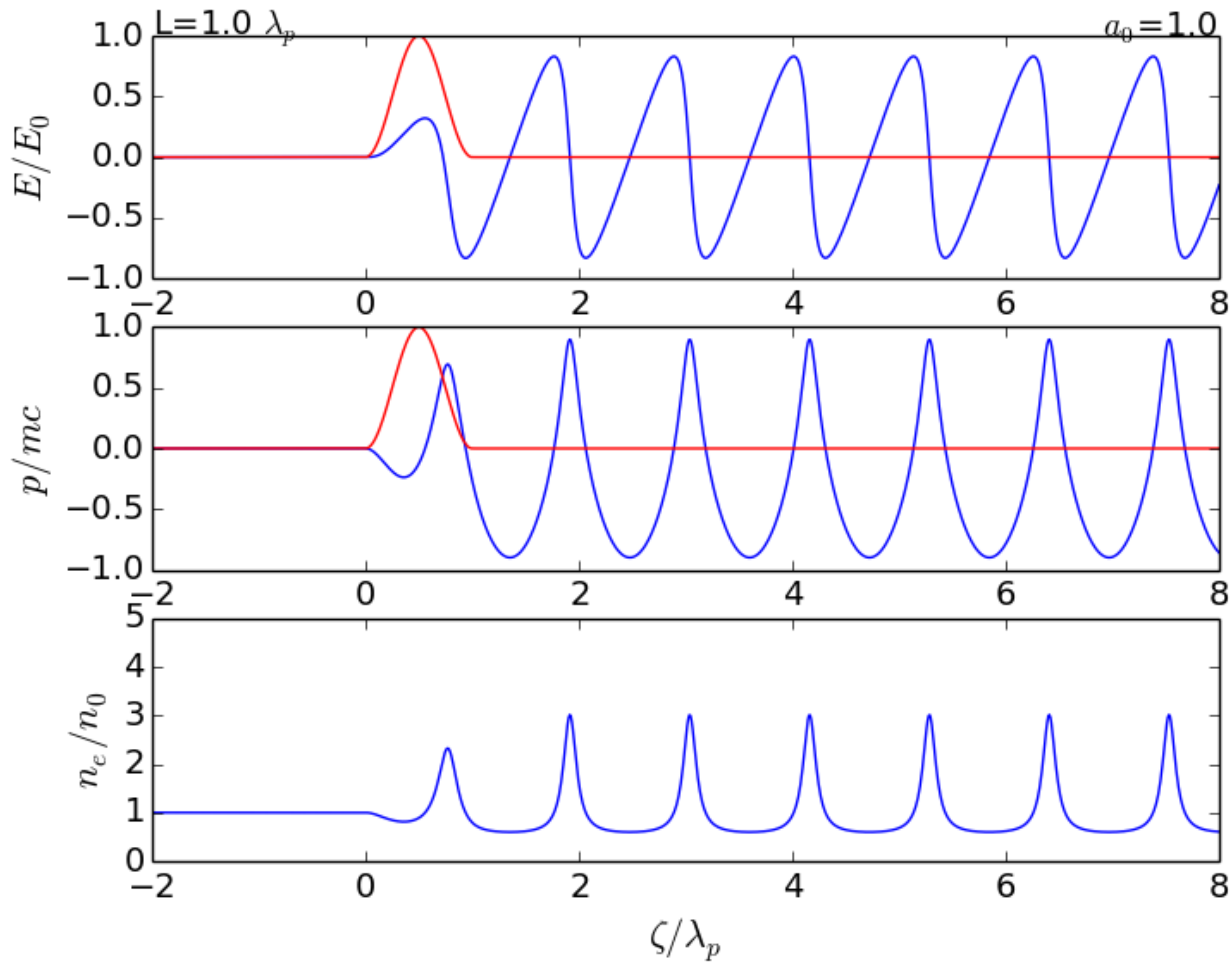

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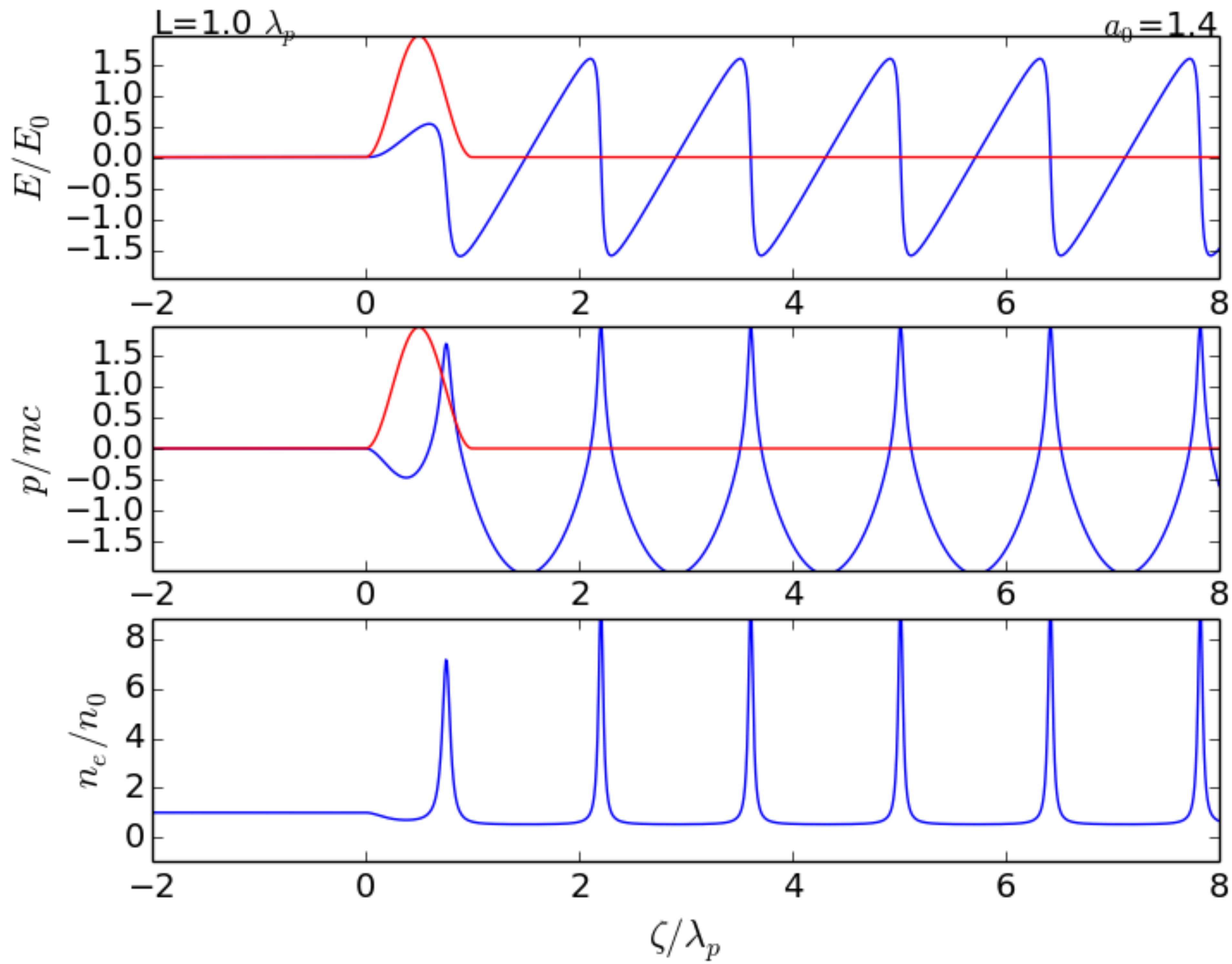

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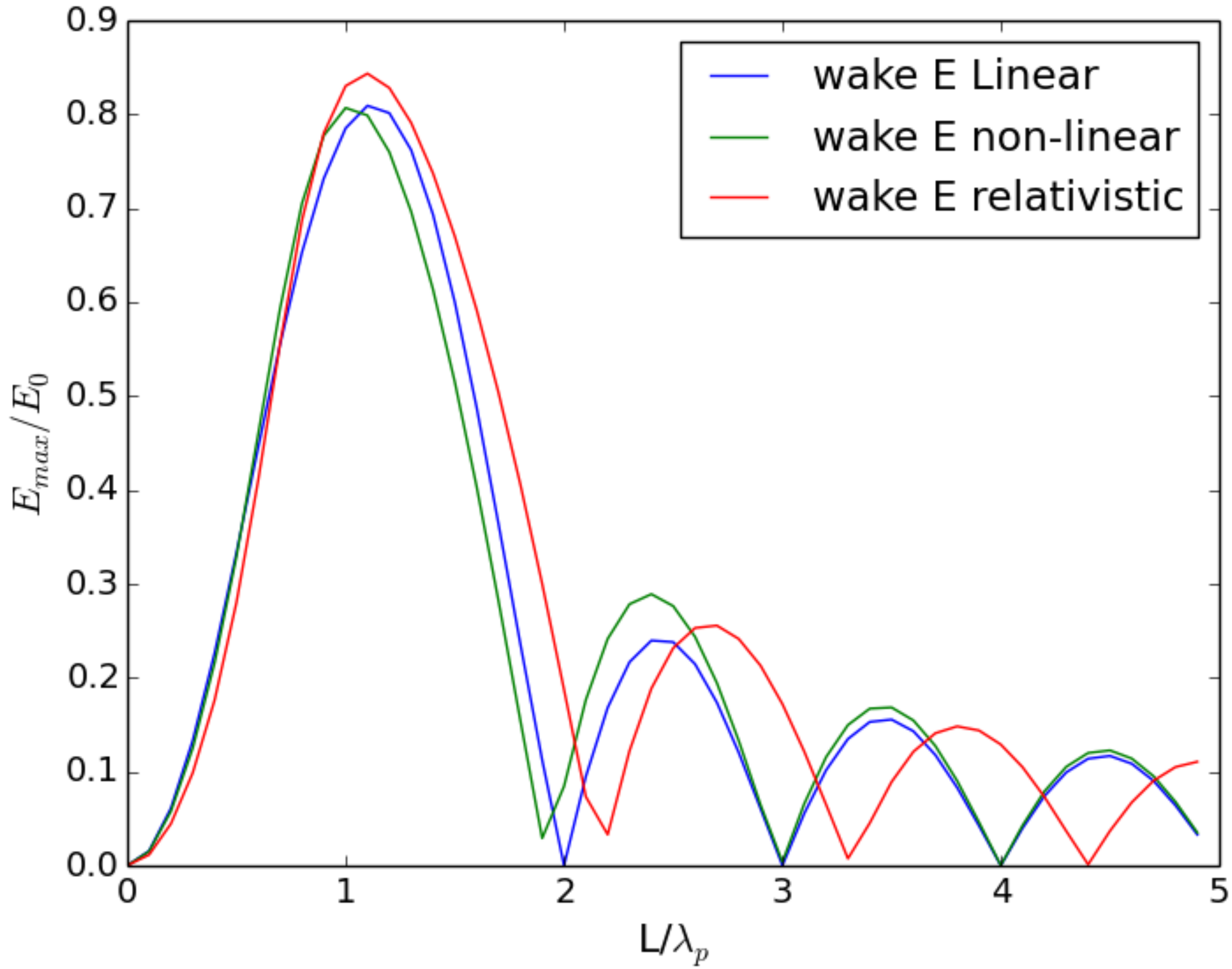

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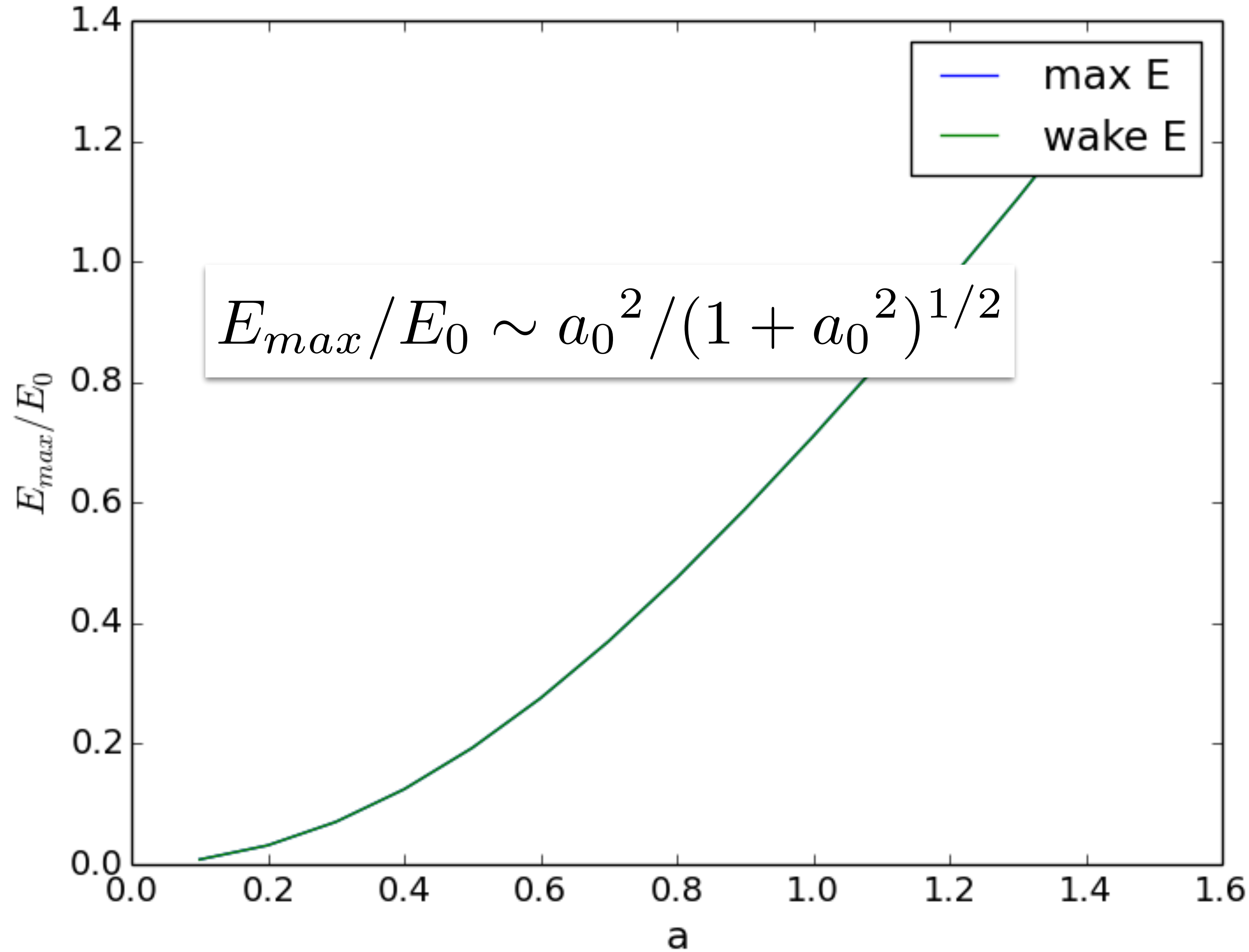


Wakefield Generation Summary

Wake amplitude maximised for $L_{fwhm} \sim \lambda_p/2$

Continuity forces steepening of plasma waves, and sawtoothing of E-field

Non-linearities cause lengthening of plasma wave amplitude (and shortening of growing phase)

Relativity cause flattening of wakes but broadening of peaks.

$$E_{max}/E_0 \sim a_0^2/(1+a_0^2)^{1/2}$$

Focussing conditions?

Plasma wave amplitude grows with increasing intensity, so is it best to aim for as high an intensity as possible?

Gaussian Focussing

Starting from the wave equation:

$$\nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

For a solution of the form (fast oscillations only in *z*):

$$\mathbf{E} = E(x,y,z)\exp(i(kz-\omega t))\hat{\mathbf{x}}$$

Leads to paraxial wave equation:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)E - 2ik\frac{\partial E}{\partial z} = 0$$

Gaussian Focussing

$$\frac{\partial^2 E}{\partial r^2} + \frac{1}{r}\frac{\partial E}{\partial r} - 2ik\frac{\partial E}{\partial z} = 0$$

Gaussian envelope solution

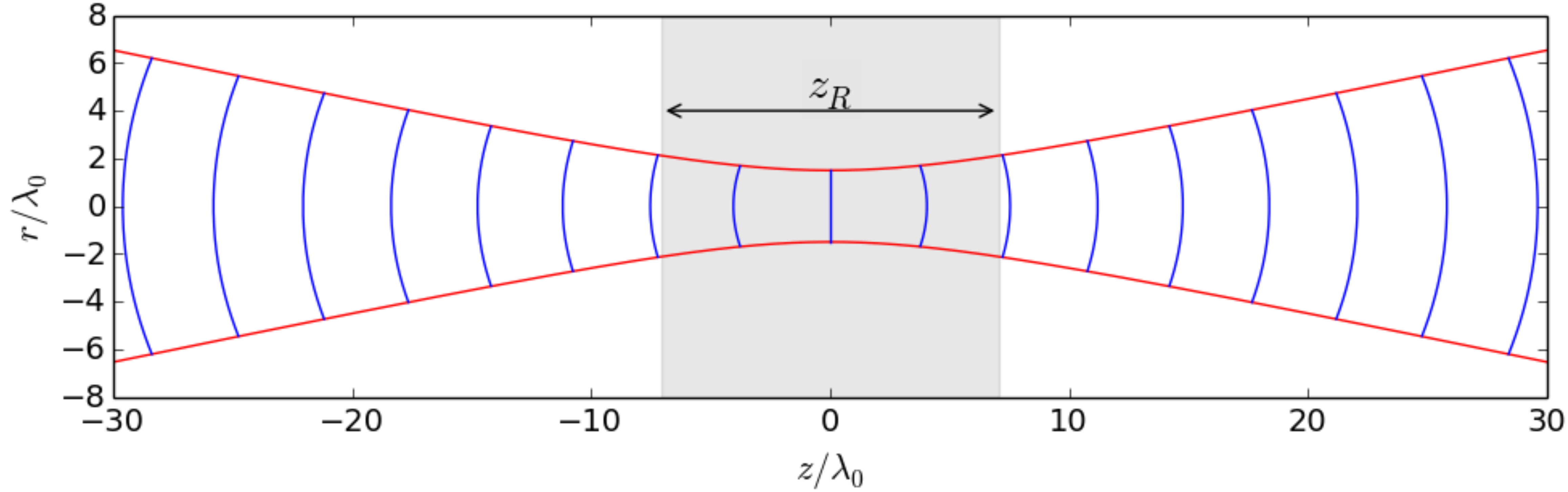
$$E(r,z)/E_0 = \frac{w_0}{w}\exp\left[\frac{-r^2}{w^2} - \frac{i\pi r^2}{\lambda R} + i\phi_0\right]$$

$$w = w_0\sqrt{1+\left(\frac{z}{z_R}\right)^2} \qquad \text{(beam waist)}$$

$$R = \frac{1}{z}\left(z^2 + {z_R}^2\right) \qquad \text{(radius of curvature)}$$

$$\tan\phi_0 = \frac{\lambda z}{\pi {w_0}^2} \qquad \text{(Gouy phase)}$$

Gaussian Focussing



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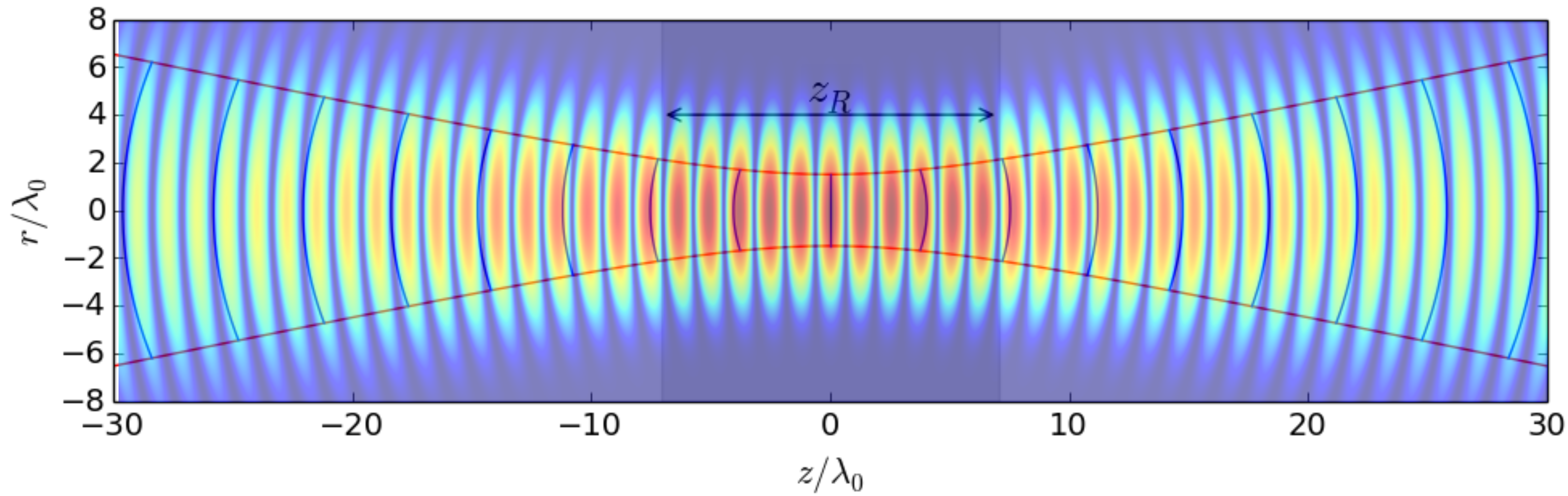
Gaussian Focussing



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Gaussian Focussing

$$I(r,z)/I_0 = \frac{w_0^2}{w^2}\exp\left[\frac{-2r^2}{w^2}\right]$$

At $z = z_R$, $I = \frac{1}{2} I_0$, so z_R is effective interaction length

For $w_0 \sim 30\ \mu m$, $\lambda_0 \sim 1\ \mu m$, $z_R \sim 3$ mm

But for $n_e \sim 10^{18}\ cm^{-3}$, ($\lambda_p \sim 30\ \mu m$), $L_{deph} \sim 3$ cm

$$w = w_0\sqrt{1+\left(\frac{z}{z_R}\right)^2} \qquad \text{(beam waist)}$$

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$$z_R = \frac{\pi {w_0}^2}{\lambda} \qquad \text{(Rayleigh Range)}$$

Plasma Propagation

$$\nabla^2 \mathbf{E} - \boxed{\frac{\eta^2}{c^2}} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

density

relativity

$$\eta_R \simeq 1 - \frac{{\omega_p}^2}{2\omega^2} \frac{n(r)}{n_0 \gamma_r} \simeq 1 - \frac{{\omega_p}^2}{2\omega^2} \left(1 + \frac{\delta n}{n_0} - \frac{a^2}{2} \right)$$

where we used $\gamma = \sqrt{1 + {a_0}^2}$

For a gaussian pulse of beam width *R*

defocusing

focusing

$$\frac{d^2 R}{dz^2} = \frac{1}{{z_R}^2 R^3} \left(1 - \frac{P}{P_c} \right).$$

$$P_{cr} \simeq 17 \, (n_e / n_{cr}) \, \mathrm{GW}$$

P. Sprangle et al," PRL, **59**, 202 (1987)

Plasma Propagation

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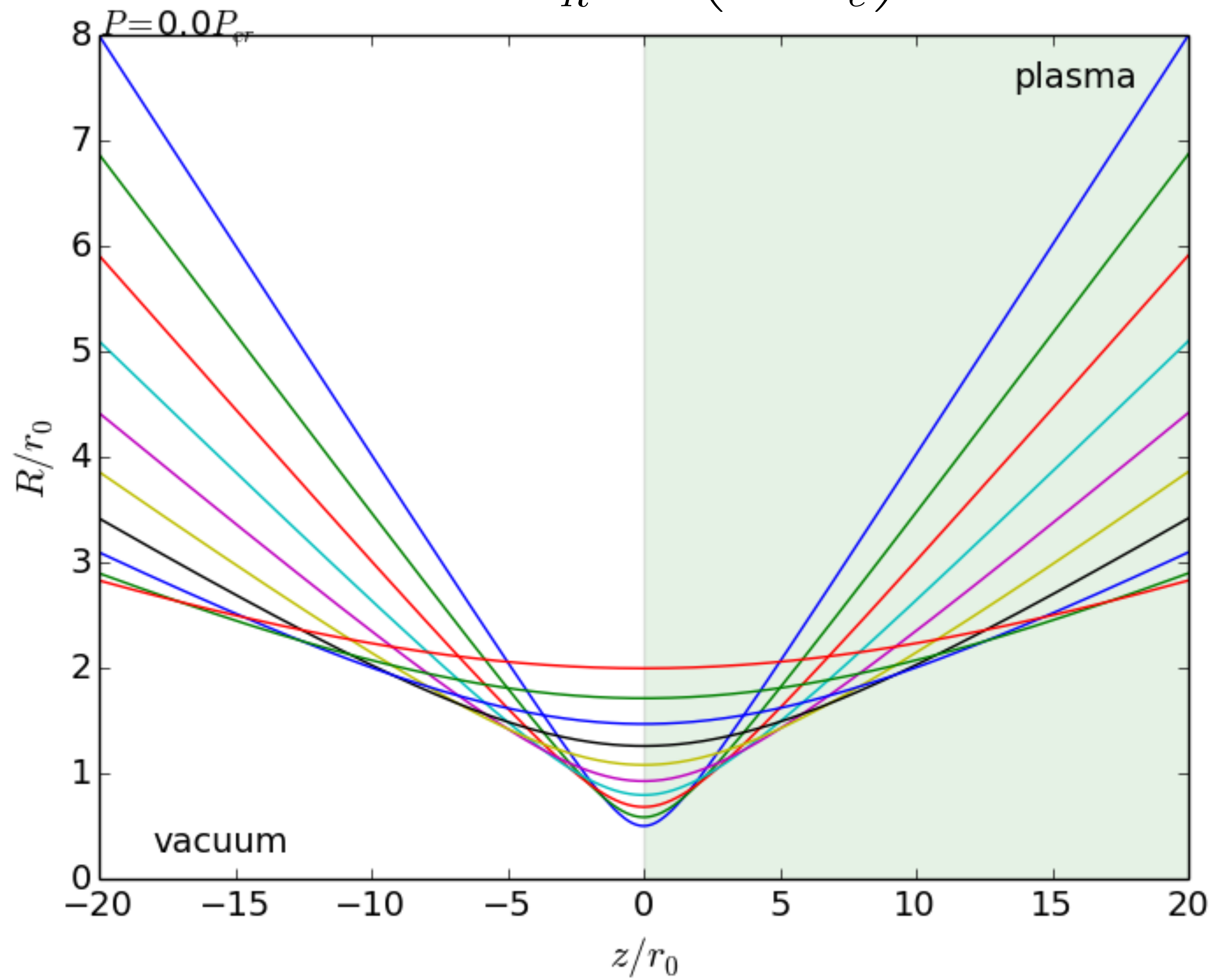

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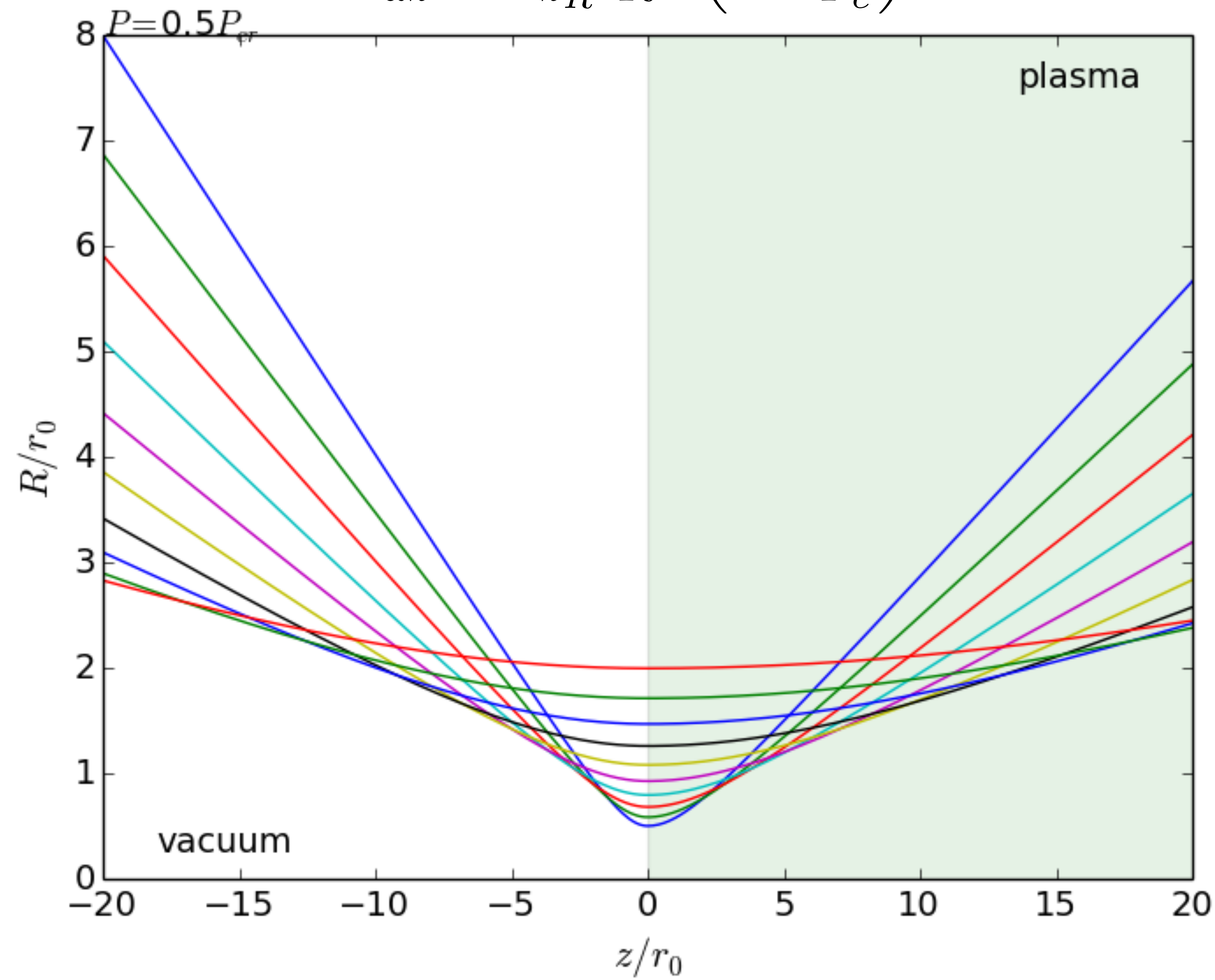

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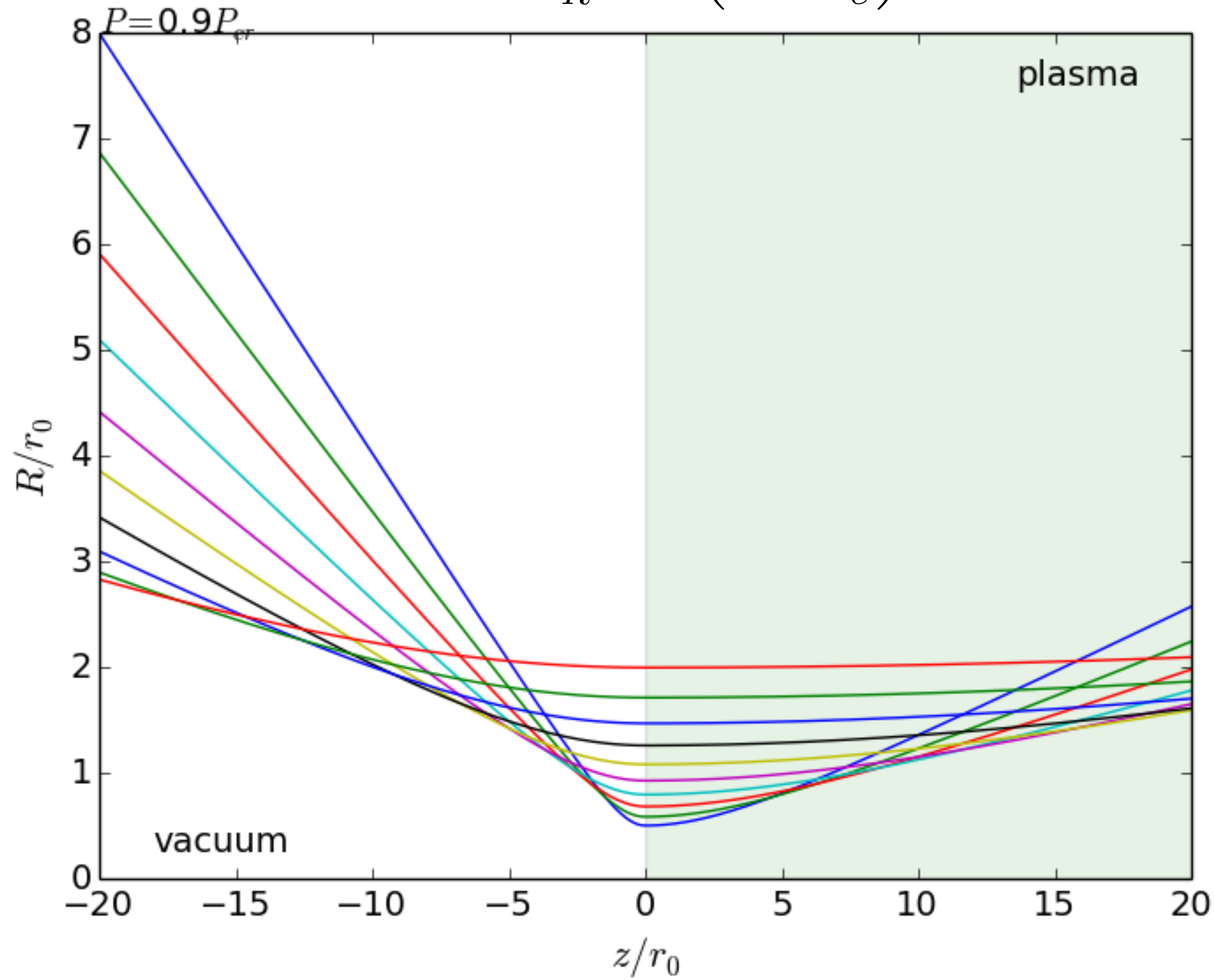

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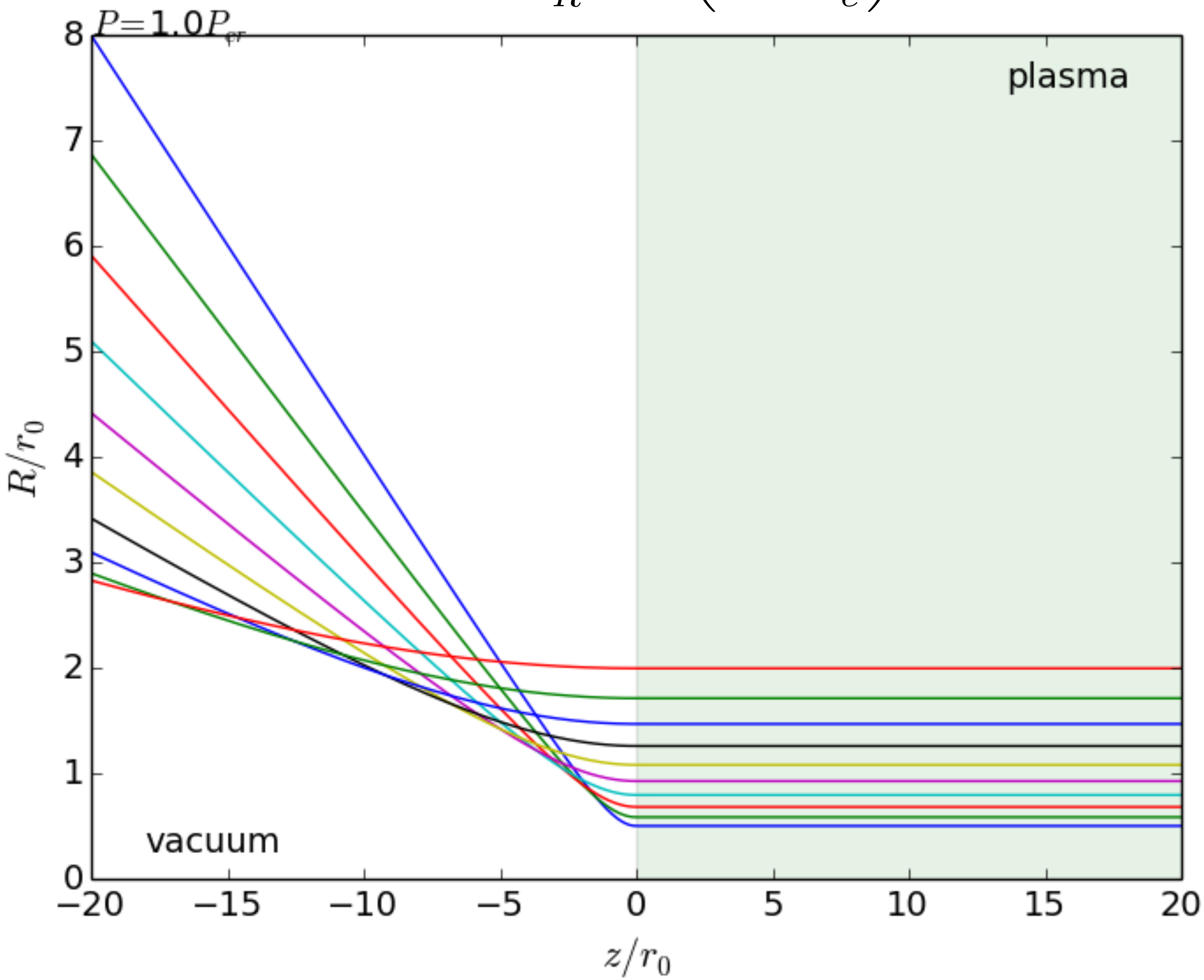

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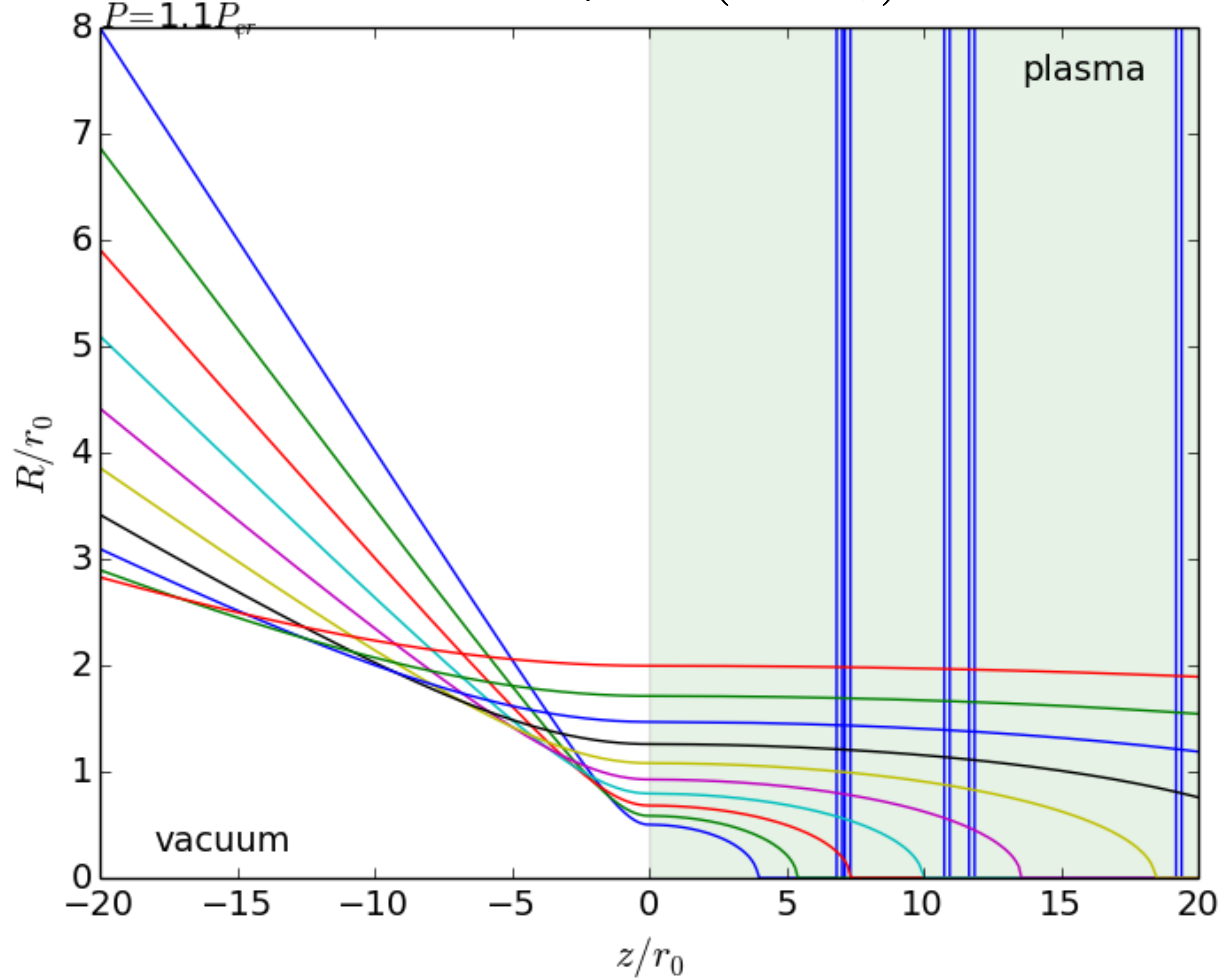
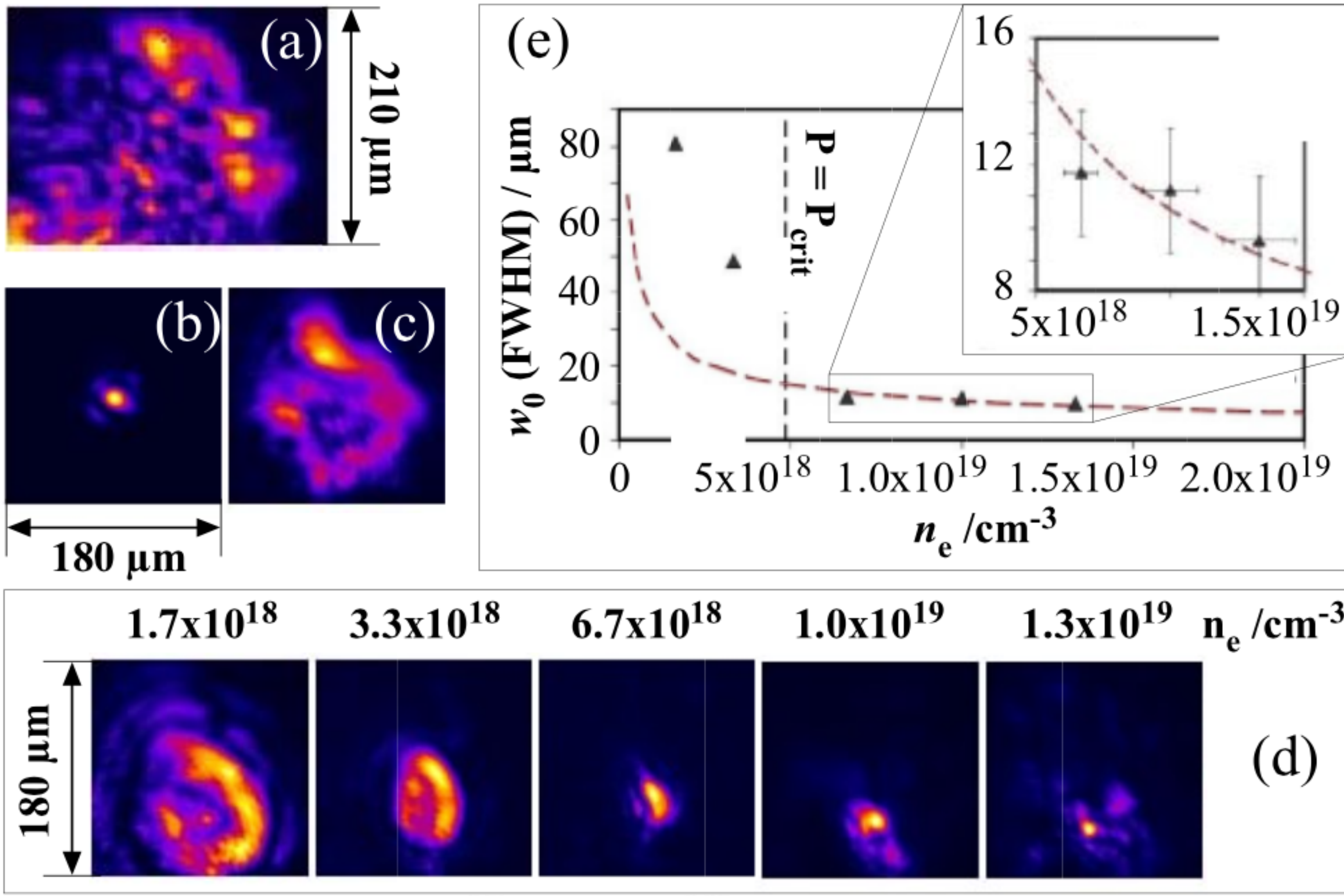


Plasma Propagation

f/3

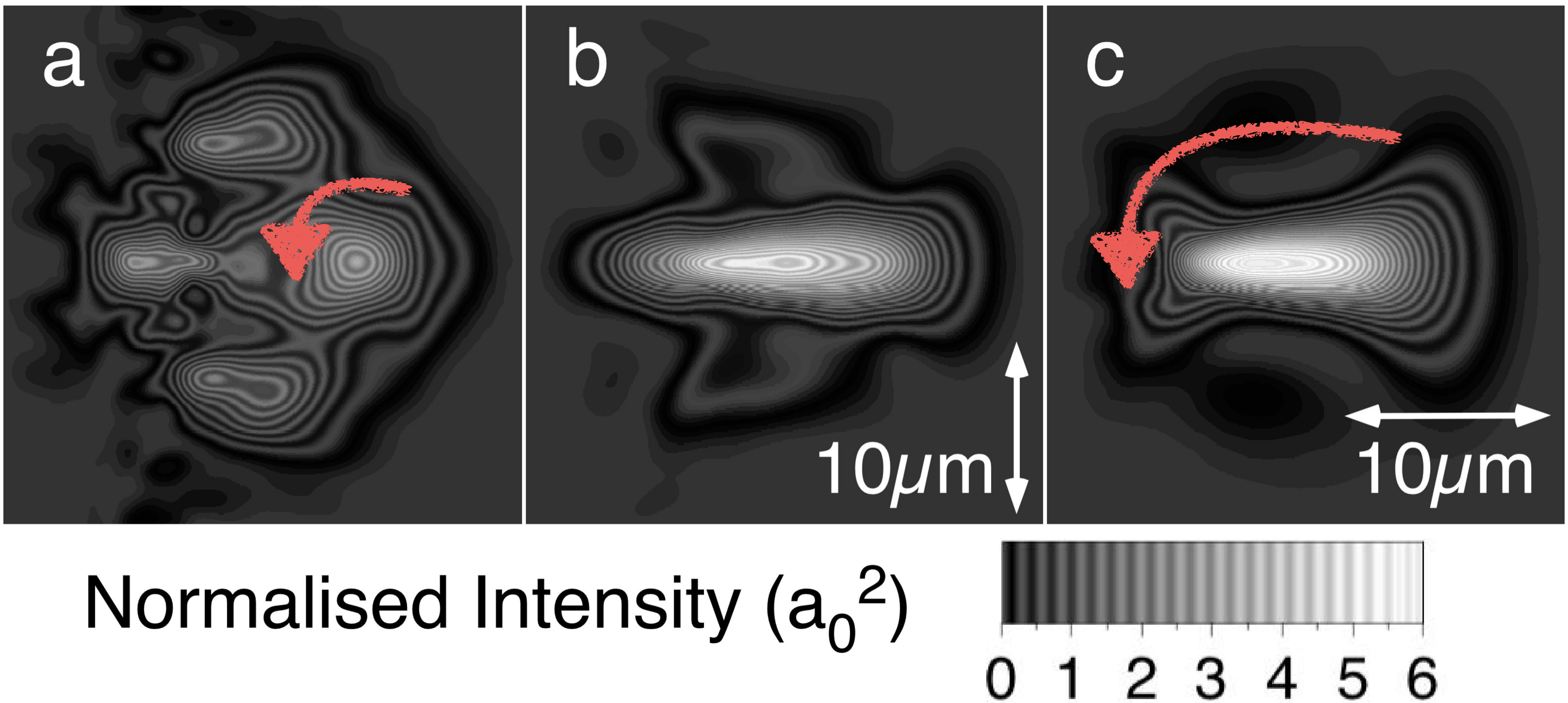
f/16



Thomas et al., PRL 98, 095004 (2007)

Plasma Propagation

Focal spots after 3 vacuum Rayleigh lengths



Normalised Intensity (a_0^2)

Thomas et al., PRL 98, 095004 (2007)

Plasma Propagation

Focussing in a guiding channel can be modelled with:

$$\frac{d^2R}{dz^2} = \frac{1}{{z_R}^2 R^3}\left(\underbrace{1}_{\text{defocusing}} - \underbrace{\frac{\delta n}{\delta n_c} R^4}_{\text{focusing}}\right)$$

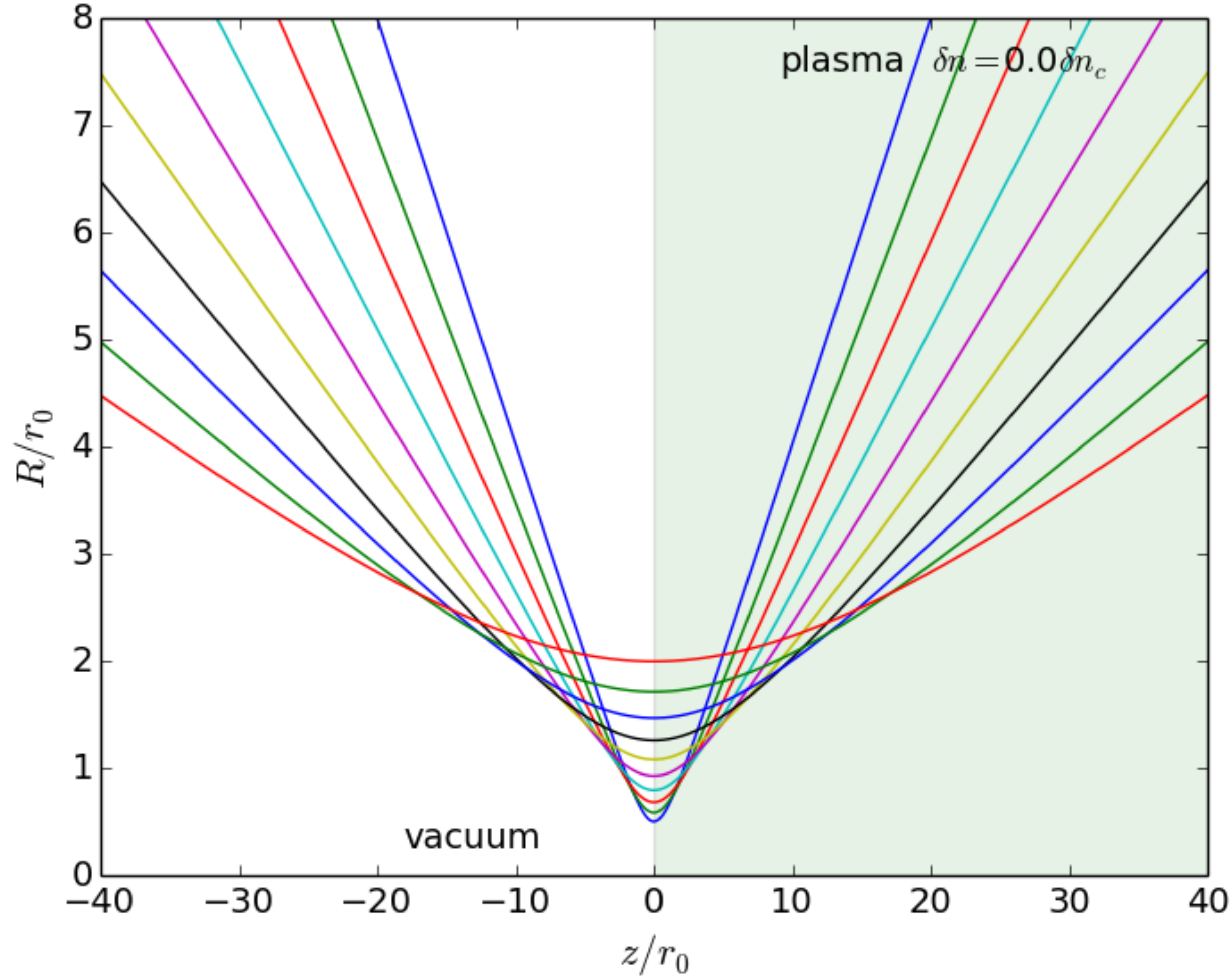
where the critical channel depth is defined by:

$$\delta n_c = \frac{1}{\pi r_e {r_0}^2}$$

here r_e is the classical radius of an electron $r_e = e^2/m_e^2c^2$

Plasma Propagation

$$\frac{d^2R}{dz^2} = \frac{1}{{z_R}^2R^3}\left(1 - \frac{\delta n}{\delta n_c}R^4\right)$$



Plasma Propagation

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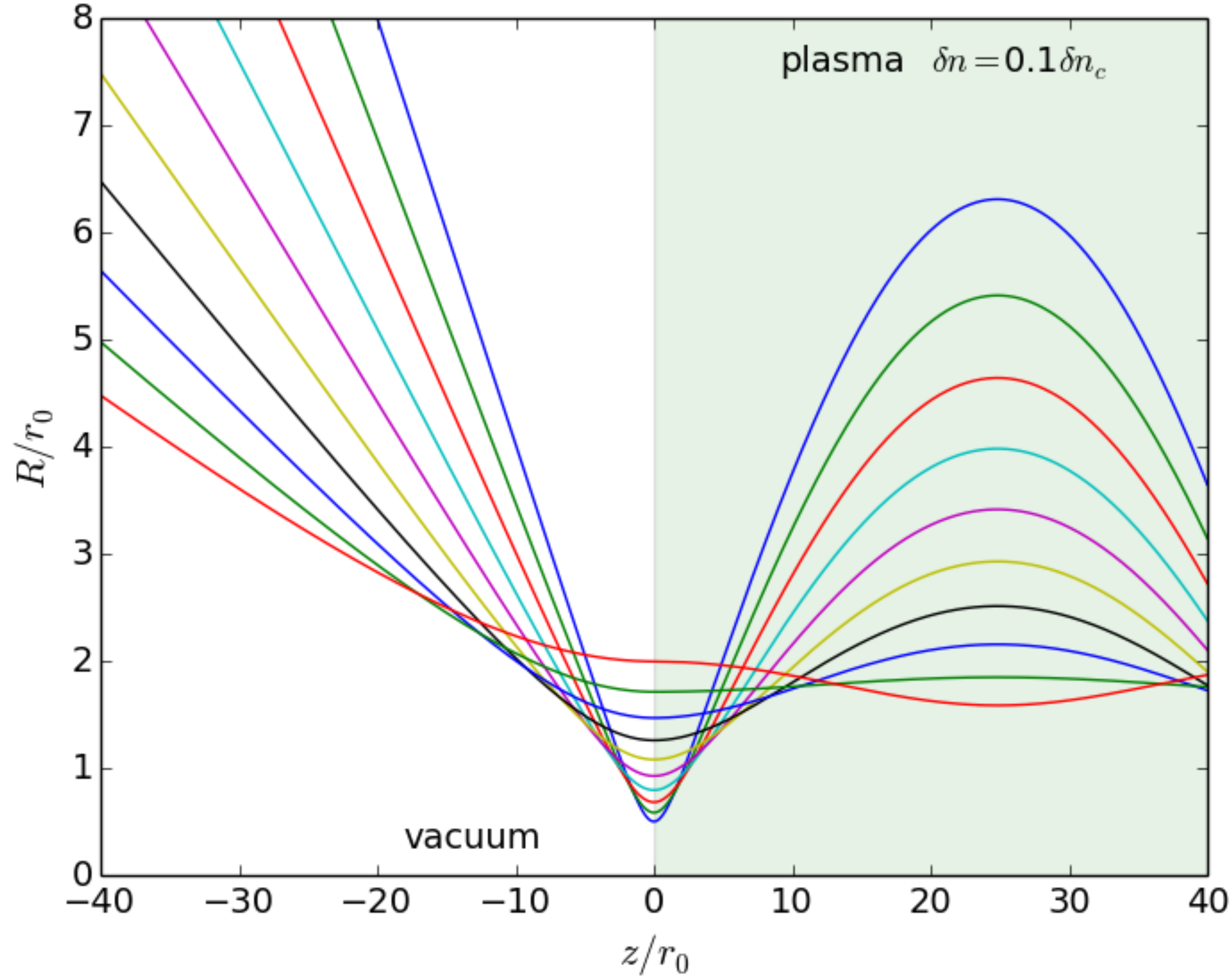

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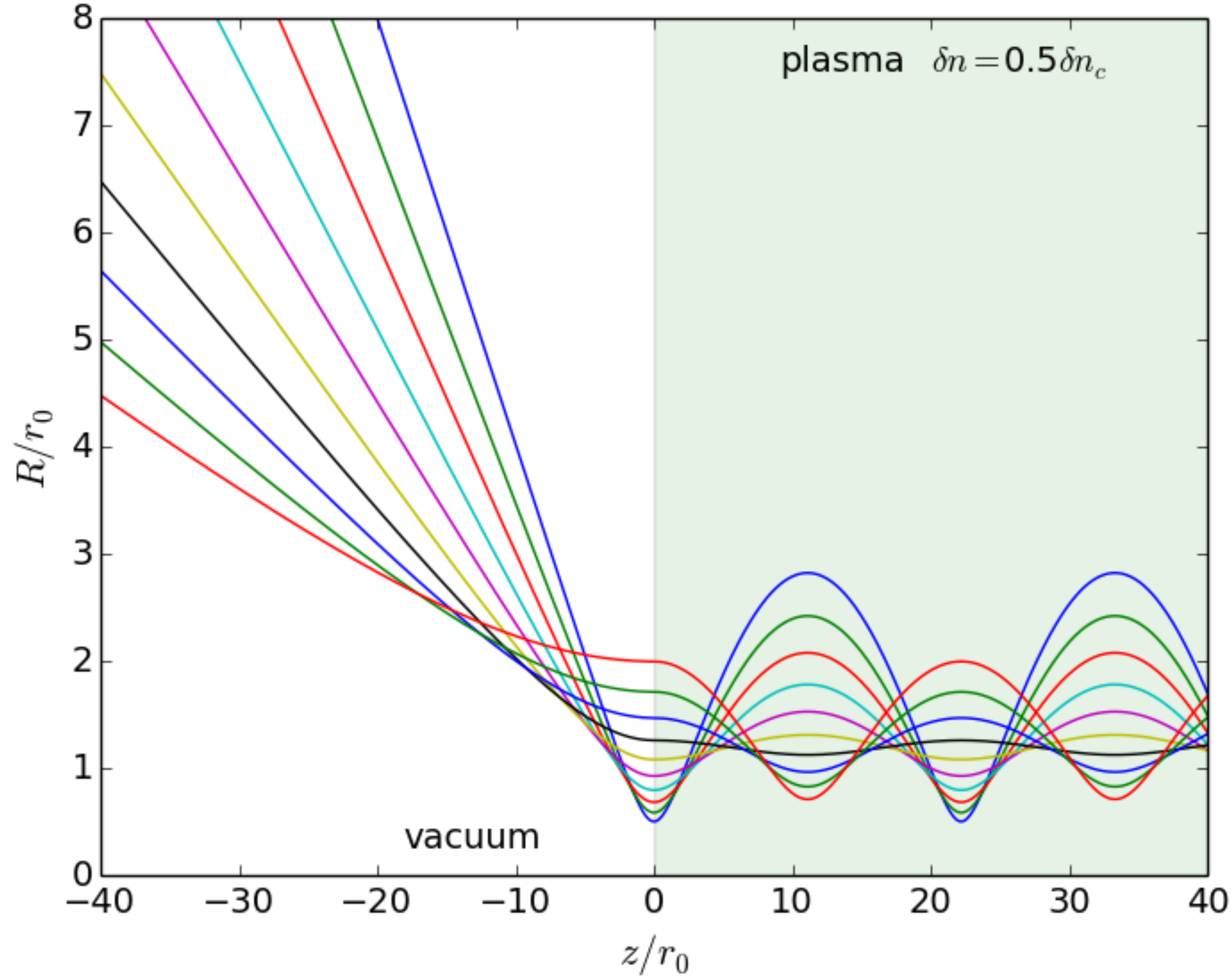

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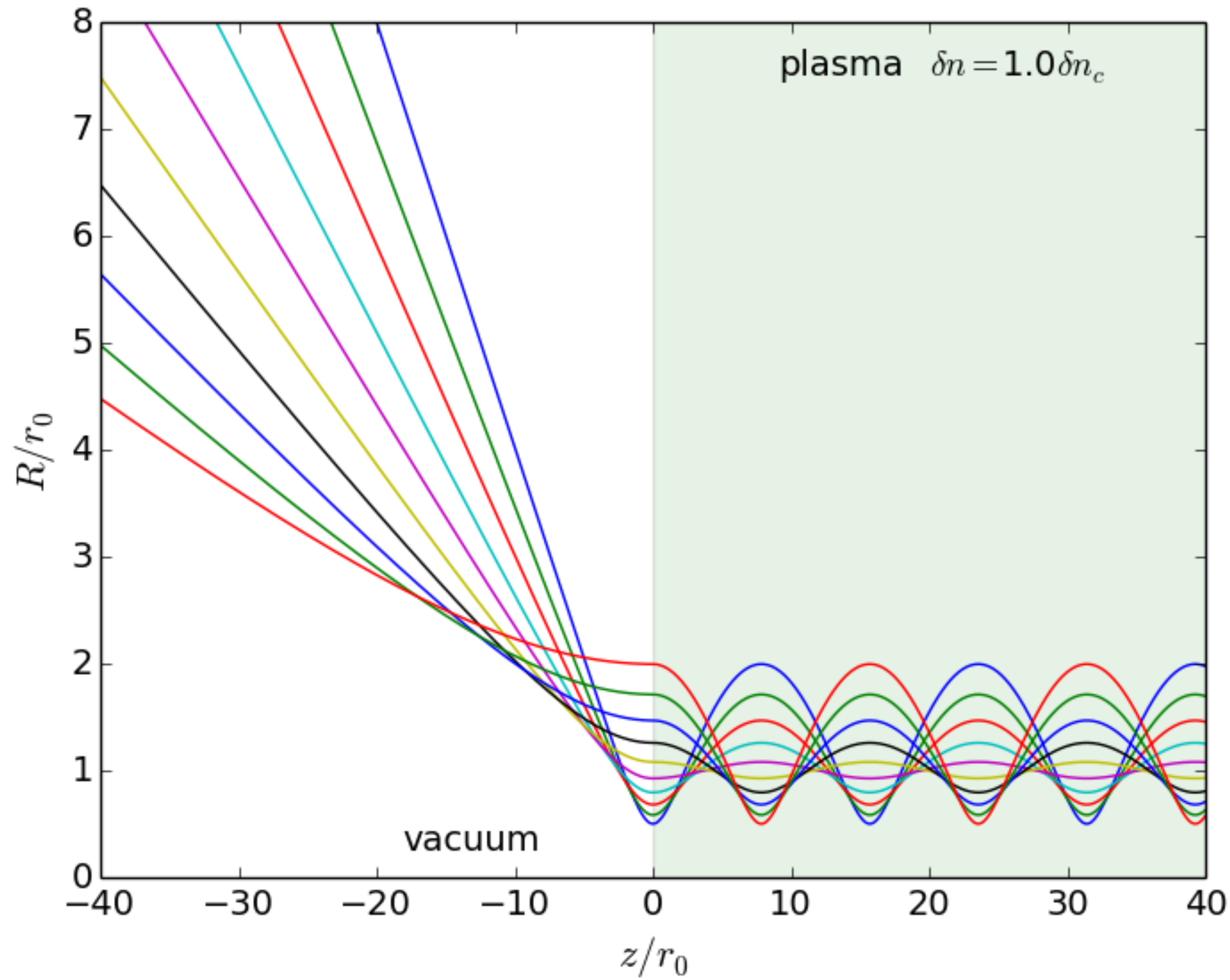

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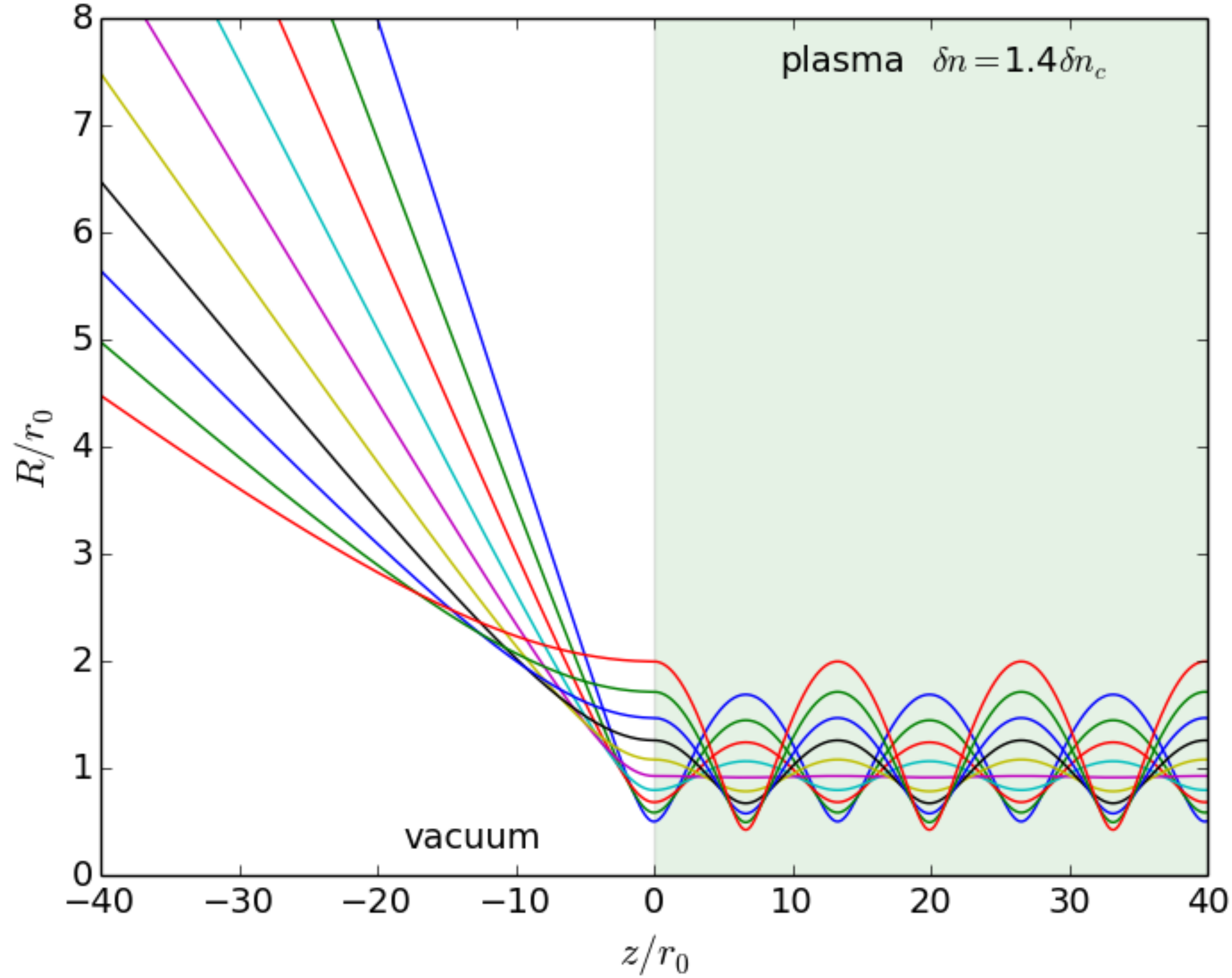
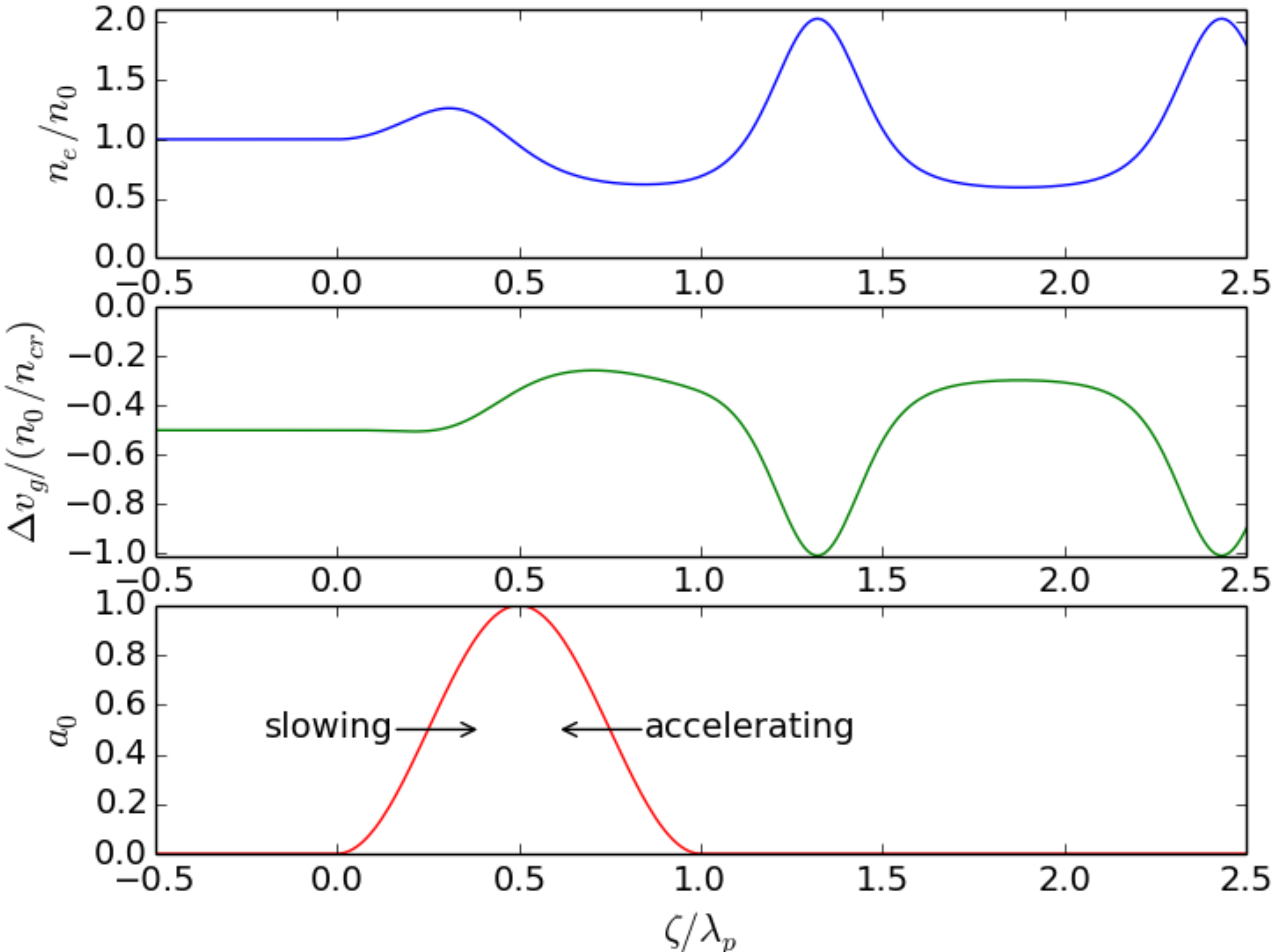


Pulse compression

$$\tau = \tau_0 - \frac{n_{e0} l}{2 n_{cr} c}$$



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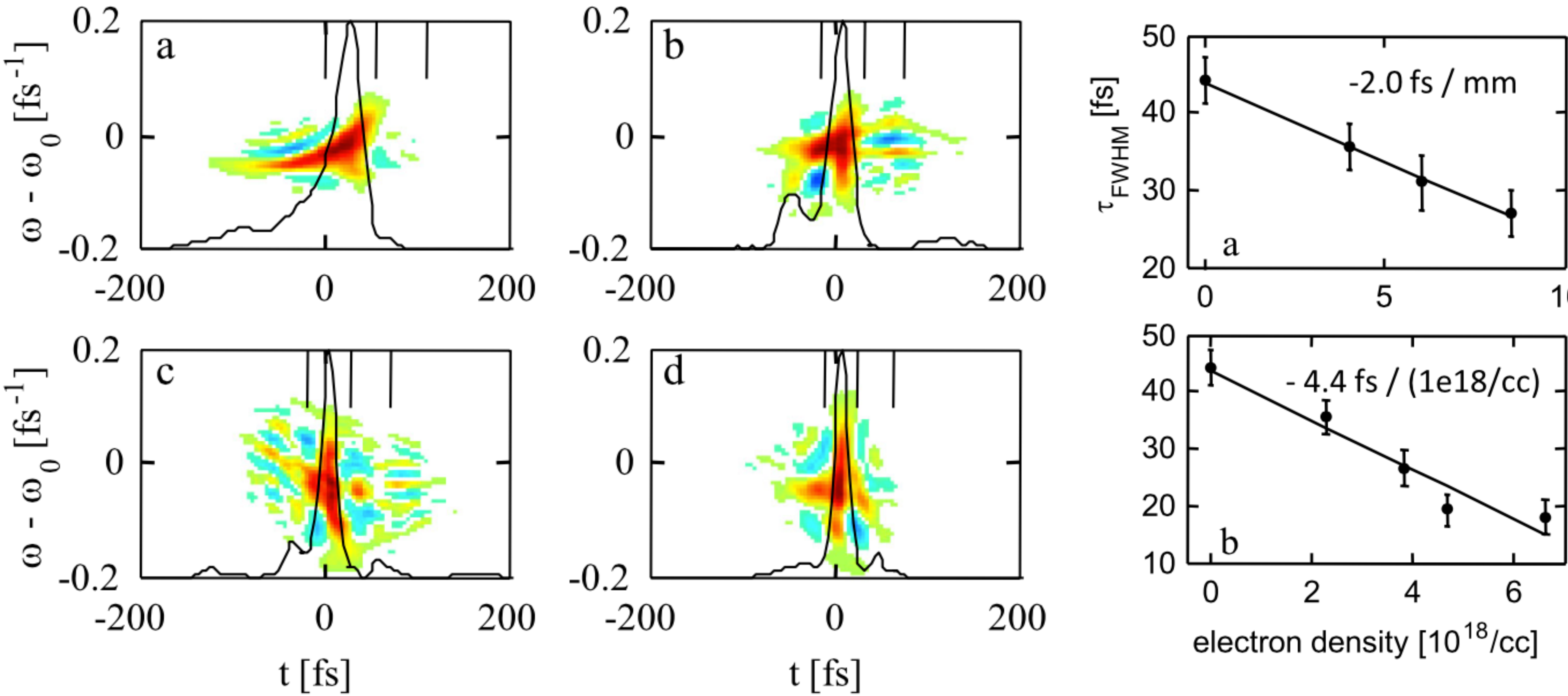
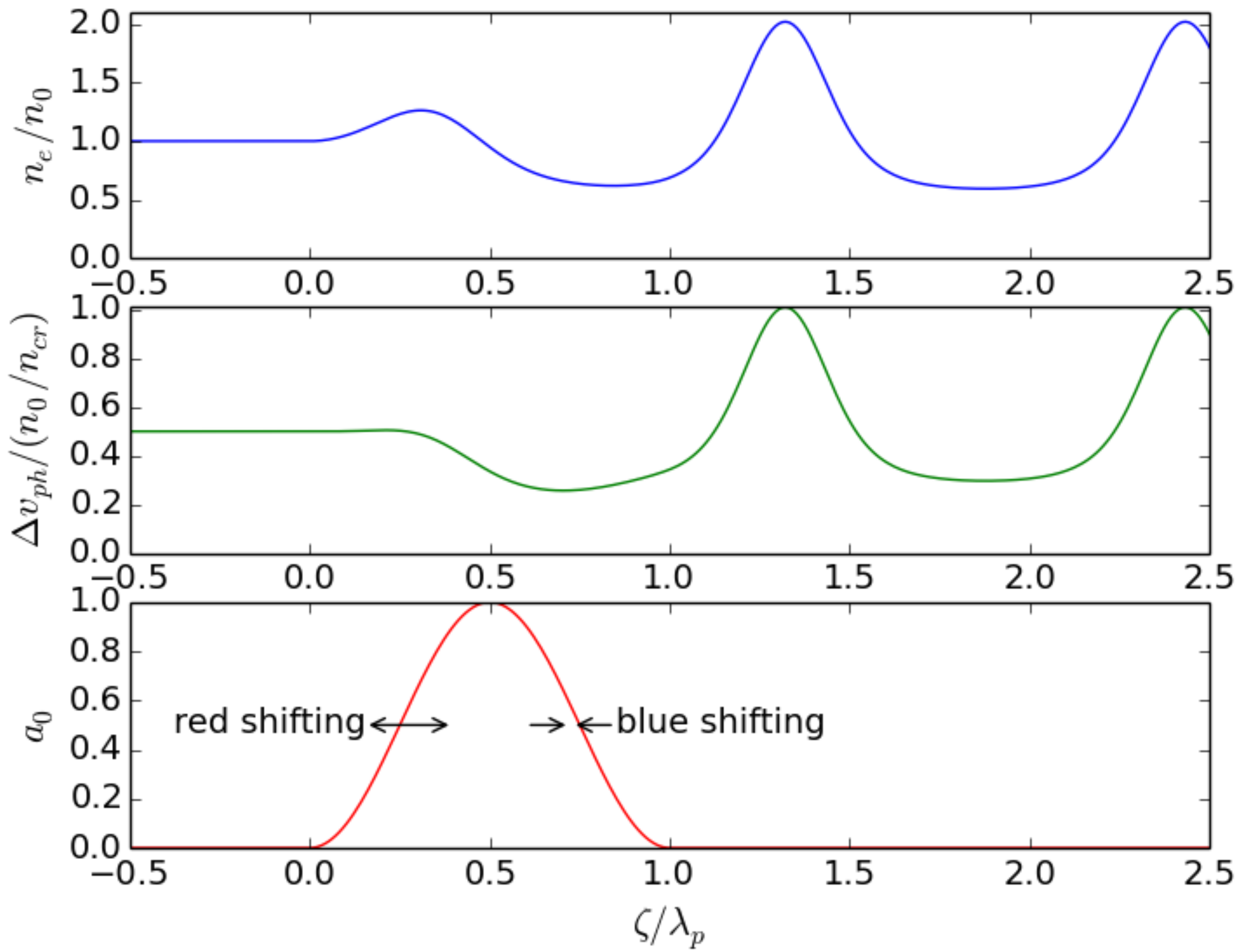


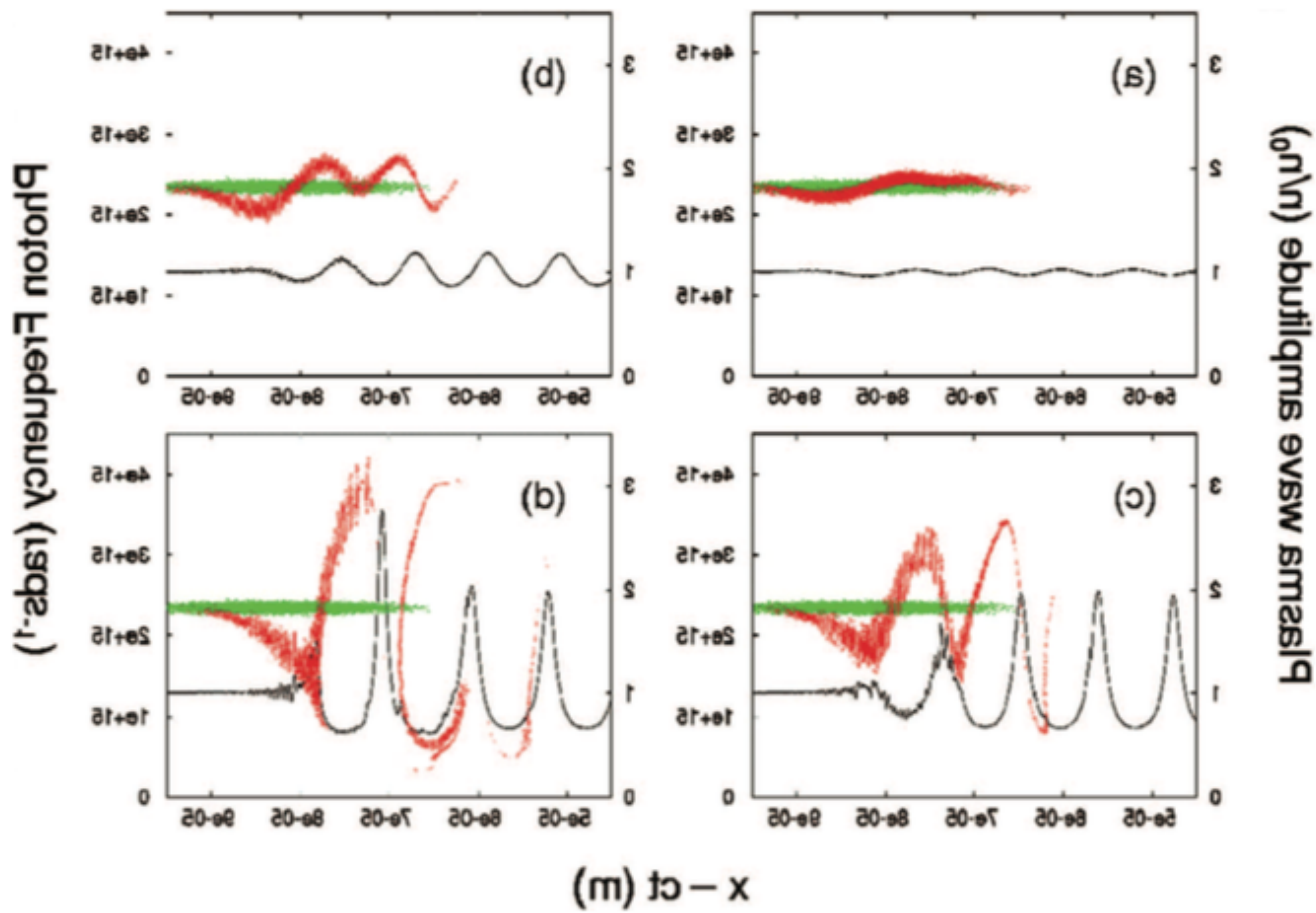
Schreiber et al., PRL 105, 235003 (2010)

Photon Acceleration

$$\delta\omega = \omega_0 \left(1 - z\frac{d\beta_p}{d\zeta}\right) \simeq \omega_0 \left(1 - z\frac{d}{d\zeta}\left(\frac{\delta n}{n_0}\right)\right)$$

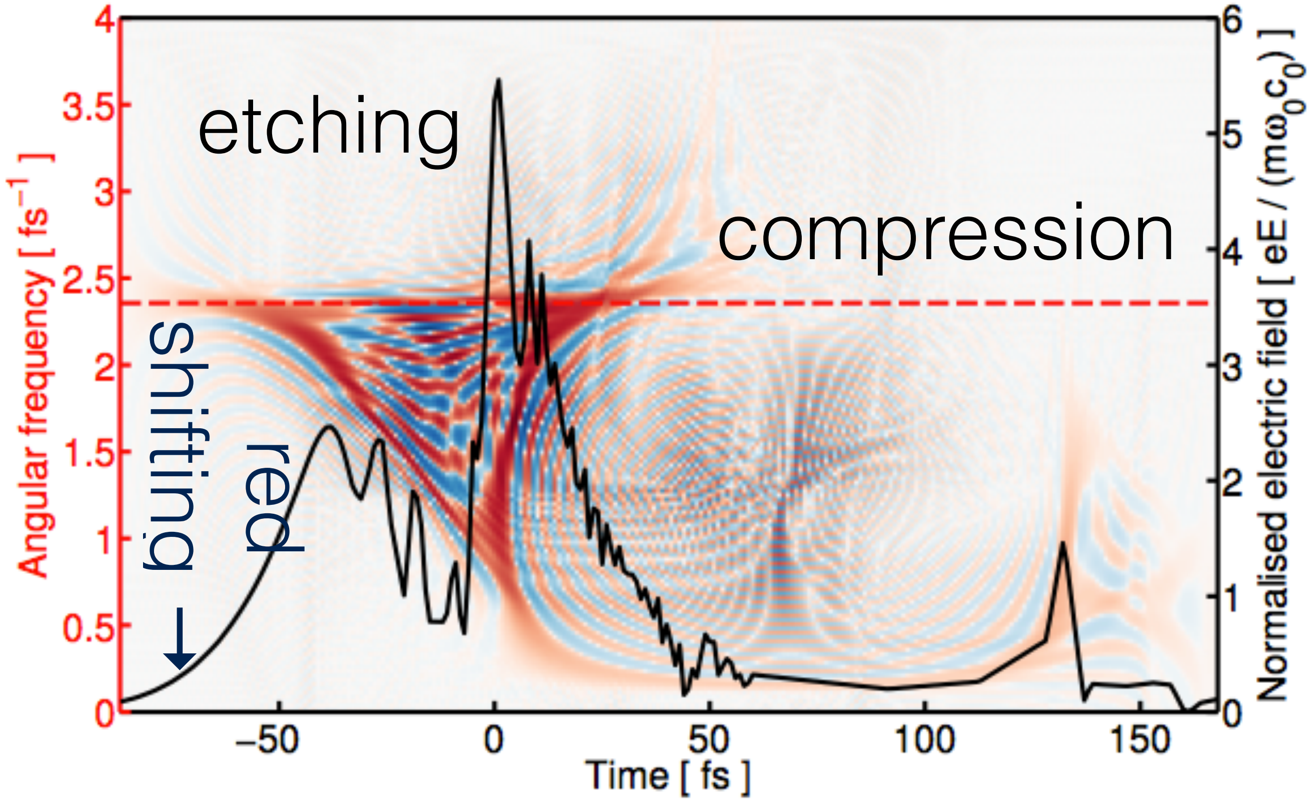


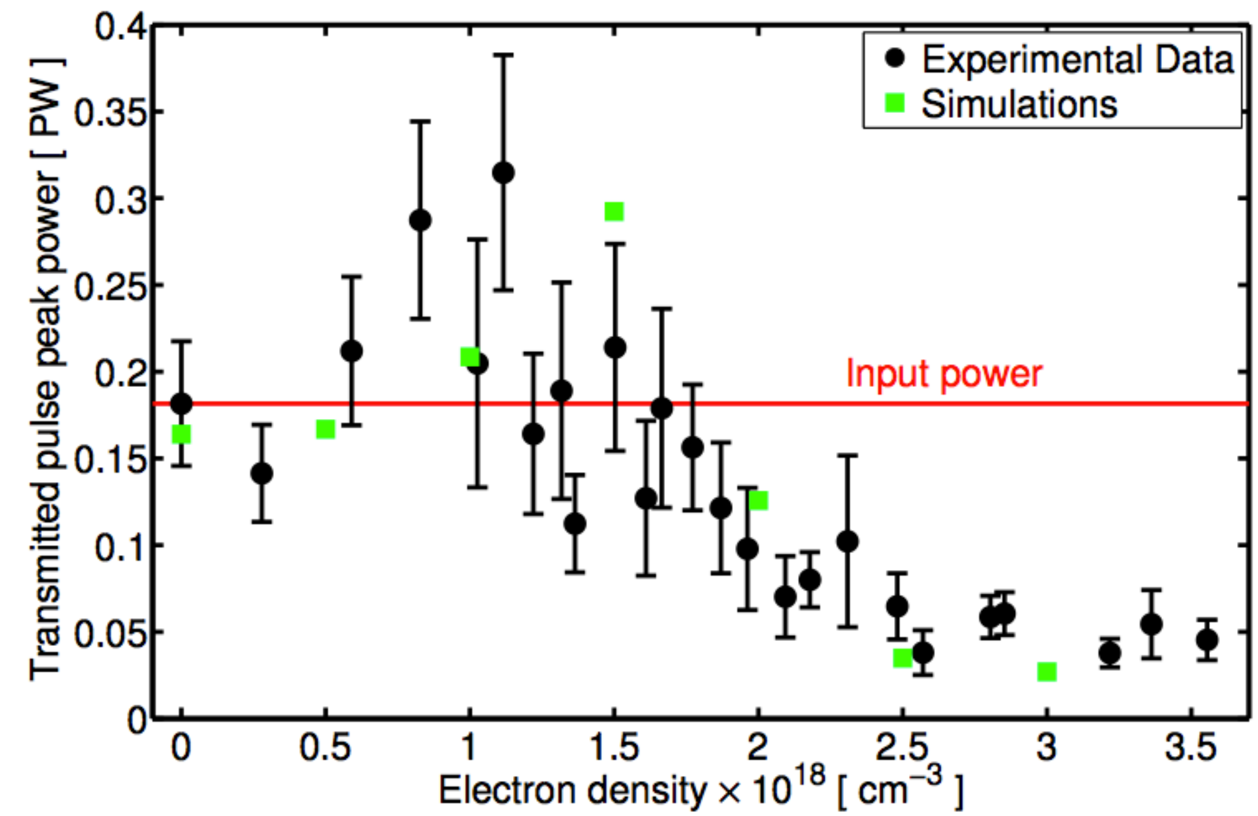
Photon Acceleration

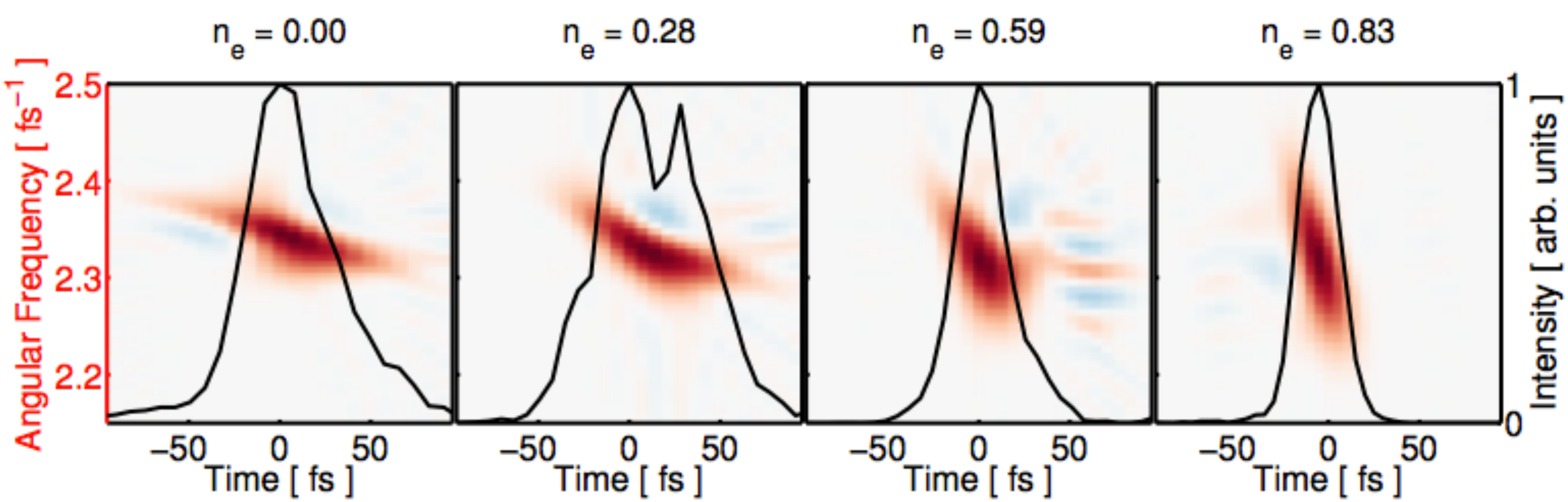


Murphy et al., POP 13, 033108 (2006)

Etching and Power







Streeter et al., In preparation (2014)

Focussing Summary

Laser pulses in vacuum only have high intensity over a Rayleigh range

Interaction can be extended for laser power $P > P_{cr}$ or by using a guiding profile $\delta n > \delta n_c$

Laser pulses lose energy to wakefield, in extreme case being etched from the front.

Compression can help maintain laser power even as laser pulse depletes.

Formula Summary

Regime	a_0	$k_p w_0$	$\delta n/n_0$	$k_p L_{deph}$	$k_p L_{depl}$	λ_W	γ_ϕ	$\Delta W/mc^2$
Linear:	<1	2π	${a_0}^2$	$\frac{{\omega_0}^2}{{\omega_p}^2}$	$\left(\frac{{\omega_0}^2}{{\omega_p}^2}\right)\left(\frac{\omega_p\tau}{{a_0}^2}\right)$	$\frac{2\pi}{k_p}$	$\frac{\omega_0}{\omega_p}$	${a_0}^2\left(\frac{{\omega_0}^2}{{\omega_p}^2}\right)$
1D NL:	>1	2π	a_0	$4{a_0}^2\left(\frac{{\omega_0}^2}{{\omega_p}^2}\right)$	$\frac{1}{3}\left(\frac{{\omega_0}^2}{{\omega_p}^2}\right)\omega_p\tau$	$\frac{4a_0}{k_p}$	$\sqrt{a_0}\left(\frac{\omega_0}{\omega_p}\right)$	$4{a_0}^2\left(\frac{{\omega_0}^2}{{\omega_p}^2}\right)$
3D NL:	>2	$2\sqrt{a_0}$	$\frac{1}{2}\sqrt{a_0}$	$\frac{4}{3}\sqrt{a_0}\left(\frac{{\omega_0}^2}{{\omega_p}^2}\right)$	$\left(\frac{{\omega_0}^2}{{\omega_p}^2}\right)\omega_p\tau$	$\frac{2\pi\sqrt{a_0}}{k_p}$	$\frac{1}{\sqrt{3}}\left(\frac{\omega_0}{\omega_p}\right)$	$\frac{2}{3}{a_0}^2\left(\frac{{\omega_0}^2}{{\omega_p}^2}\right)$
Bubble:	>20	$\sqrt{a_0}$	$\sqrt{a_0}$		$a_0\left(\frac{{\omega_0}^2}{{\omega_p}^2}\right)\omega_p\tau$			$4{a_0}^2\left(\frac{{\omega_0}^2}{{\omega_p}^2}\right)$

W. Lu et al, PR STAB 10, 061301 (2007)