Injection, Extraction & Matching

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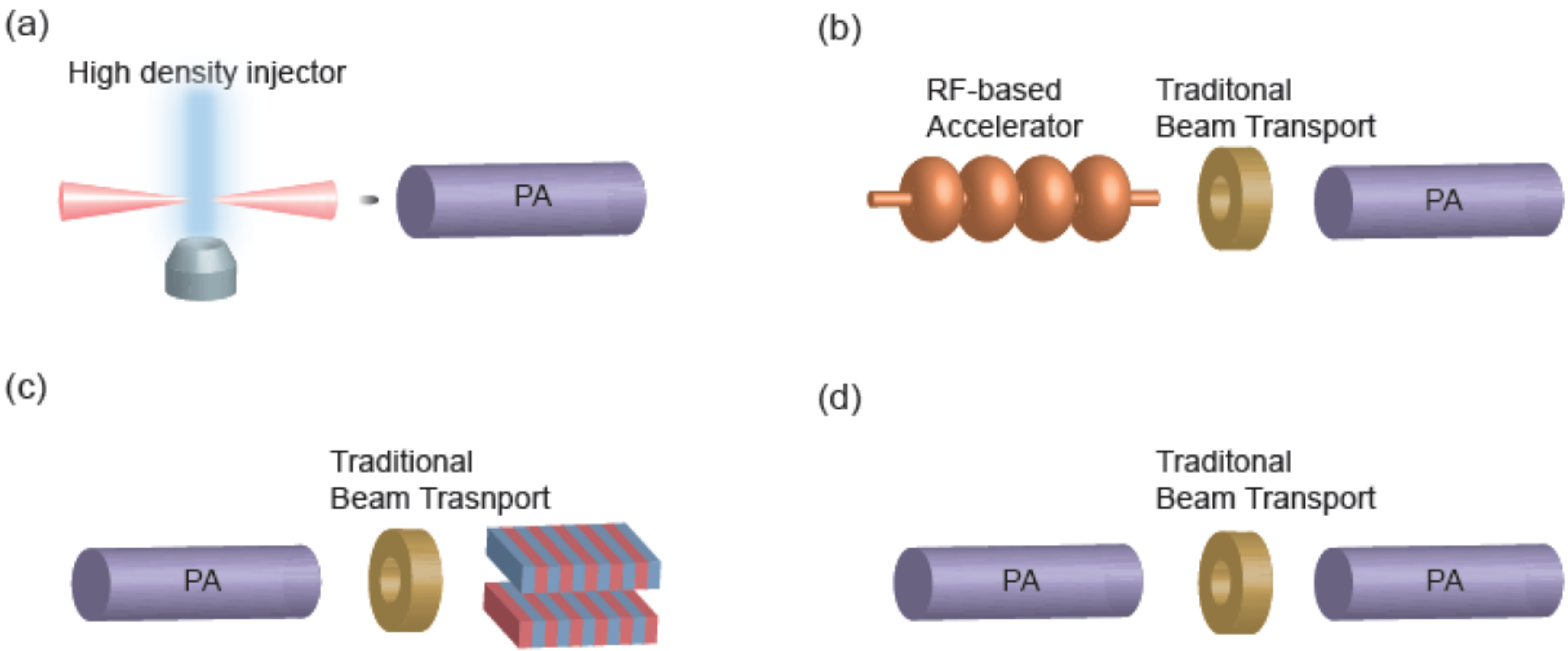


CERN Accelerator School on Plasma Wake Acceleration, CERN 25-11-2014

Modern accelerators require high quality beams:
==> High Luminosity & High Brightness

$$L = \frac{N_{e+} N_{e-} f_r}{4\pi\sigma_x\sigma_y}$$

- –N of particles per pulse => 10^9
- –High rep. rate f_r => bunch trains

- –Small spot size => low emittance
- –Low energy spread

$$B_n \approx \frac{2I}{\varepsilon_n^2}$$

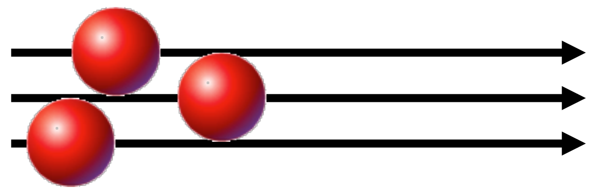
- –Short pulse (ps to fs)
- –High peak current

- –Little spread in transverse momentum and angle => low emittance
- –Low energy spread

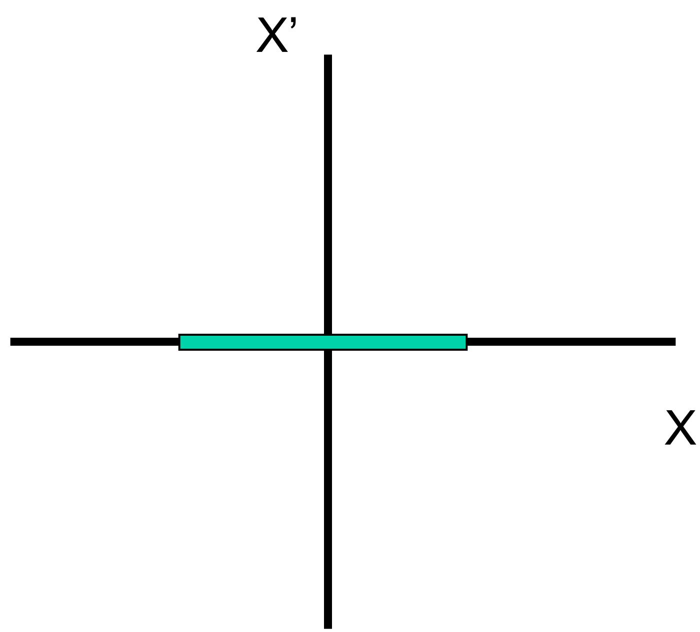
OUTLINE

- The rms emittance concept
- Energy spread contribution
- rms enevelope equation
- Beam emittance escillations and decoherence
- Beam/plasma matching conditions
- Adiabatic matching
- Living with energy spread

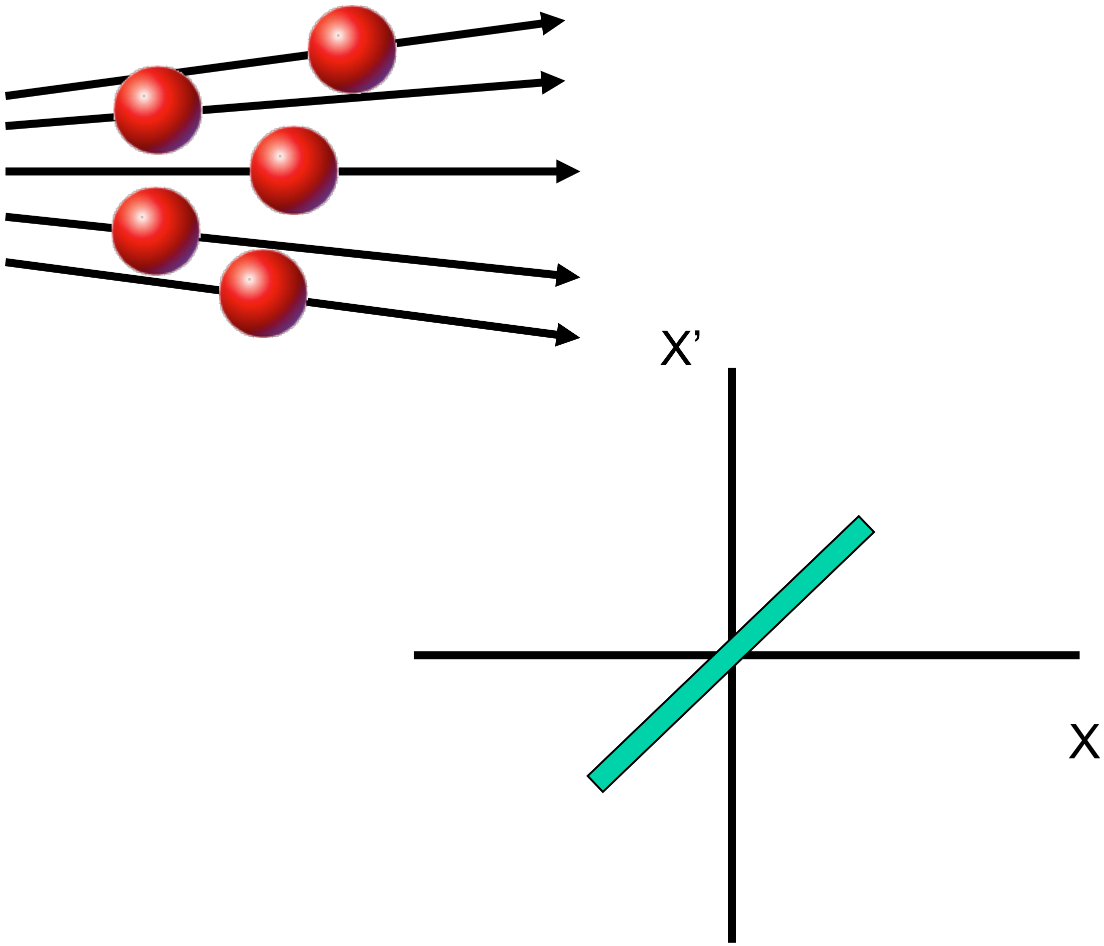
Trace space of an ideal laminar beam



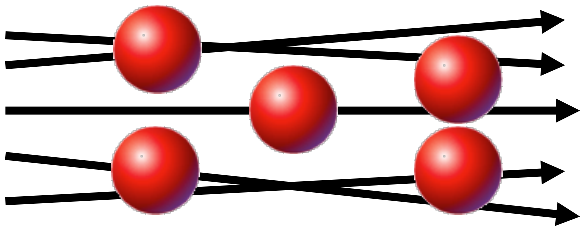
$$\begin{cases} x \\ x' = \dfrac{dx}{dz} = \dfrac{p_x}{p_z} \end{cases} \qquad p_x << p_z$$

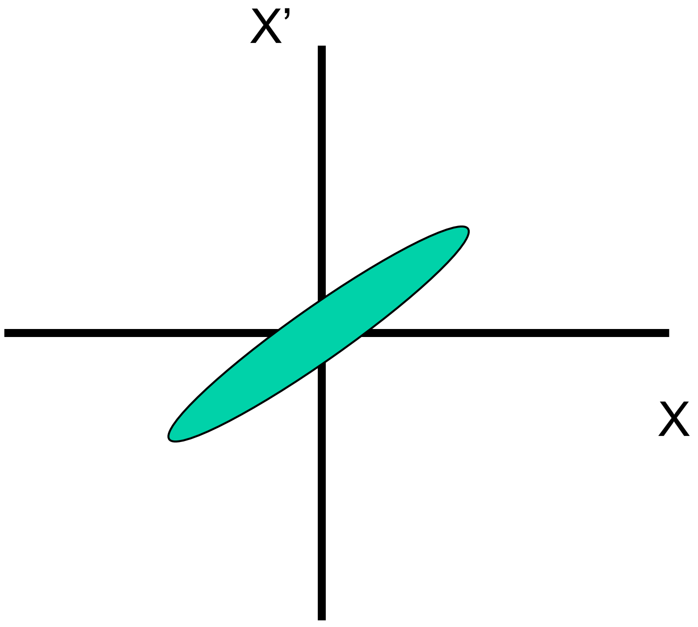


Trace space laminar beam



Trace space of non laminar beam



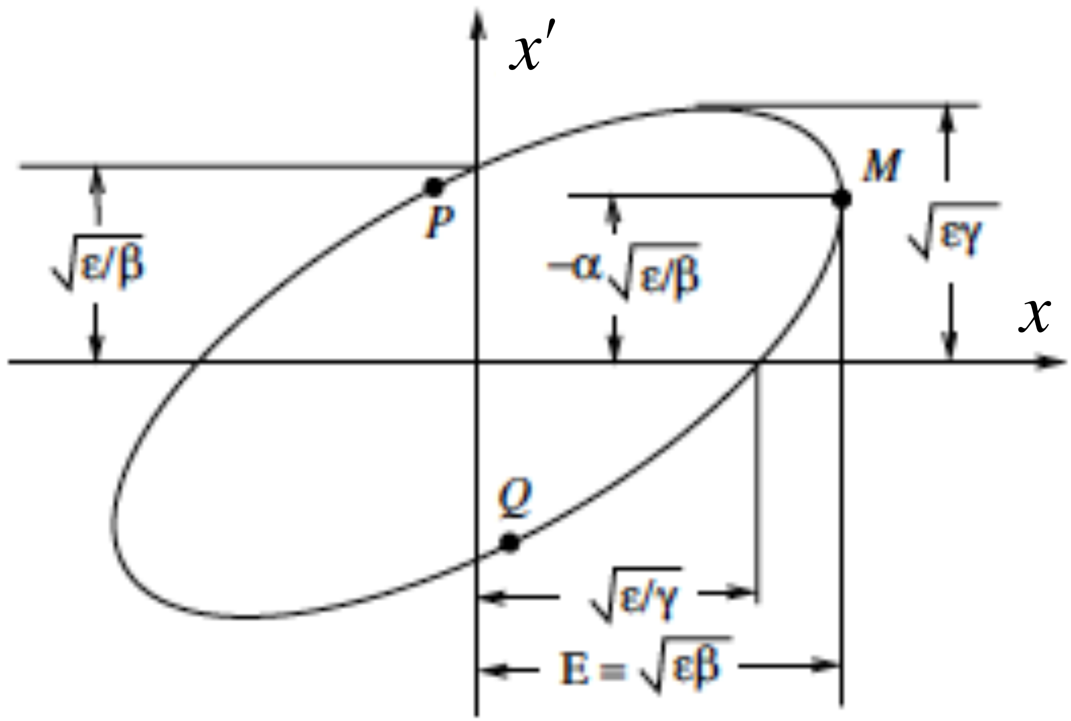


Geometric emittance (Liouville): ε_g

Ellipse equation: $$\gamma x^2 + 2\alpha x x' + \beta x'^2 = \varepsilon_g$$

Twiss parameters: $\beta\gamma - \alpha^2 = 1$ $\beta' = -2\alpha$

Ellipse area: $$A = \pi\varepsilon_g$$



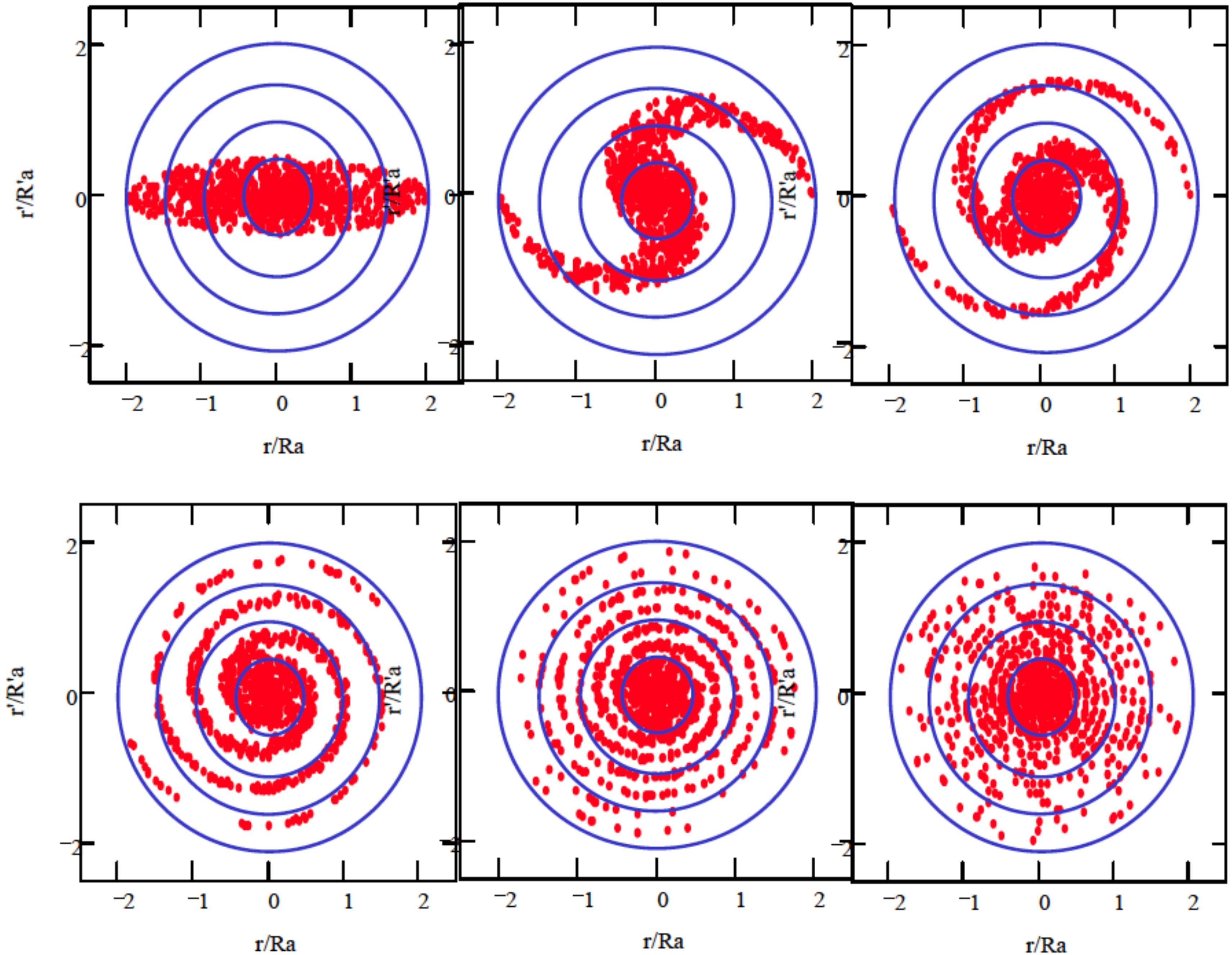
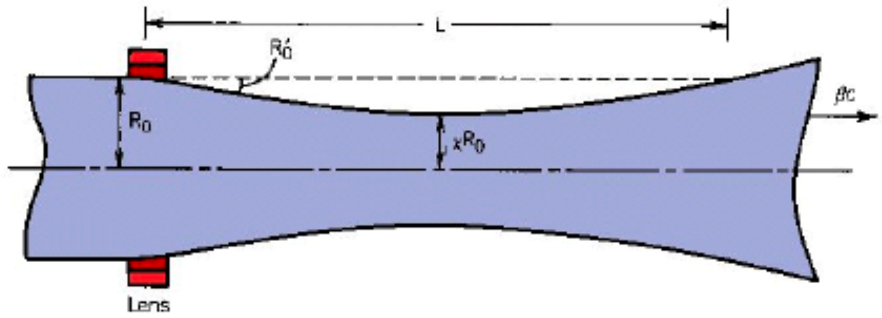
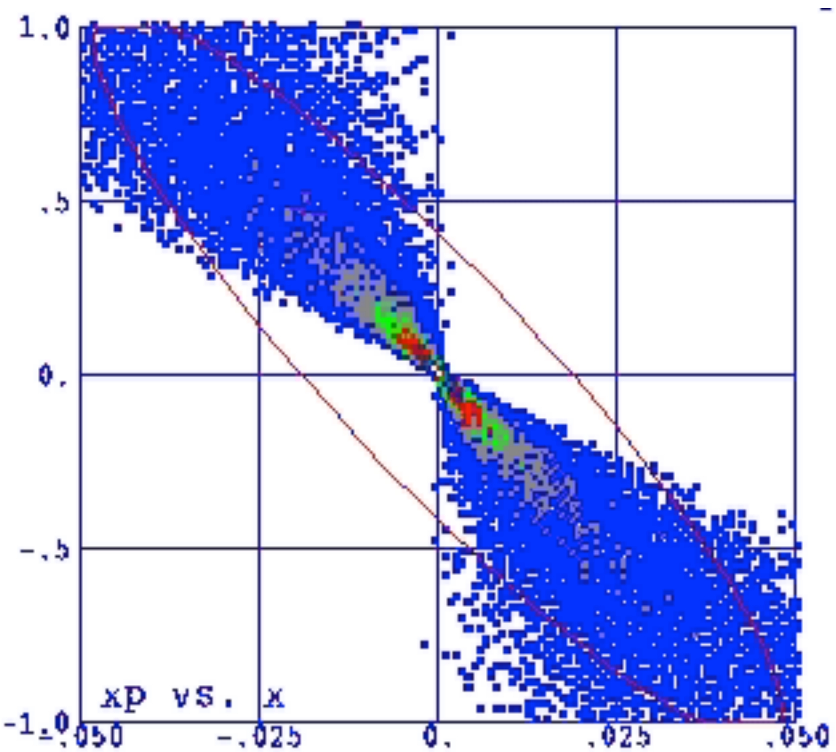


Fig. 17: Filamentation of mismatched beam in non-linear force

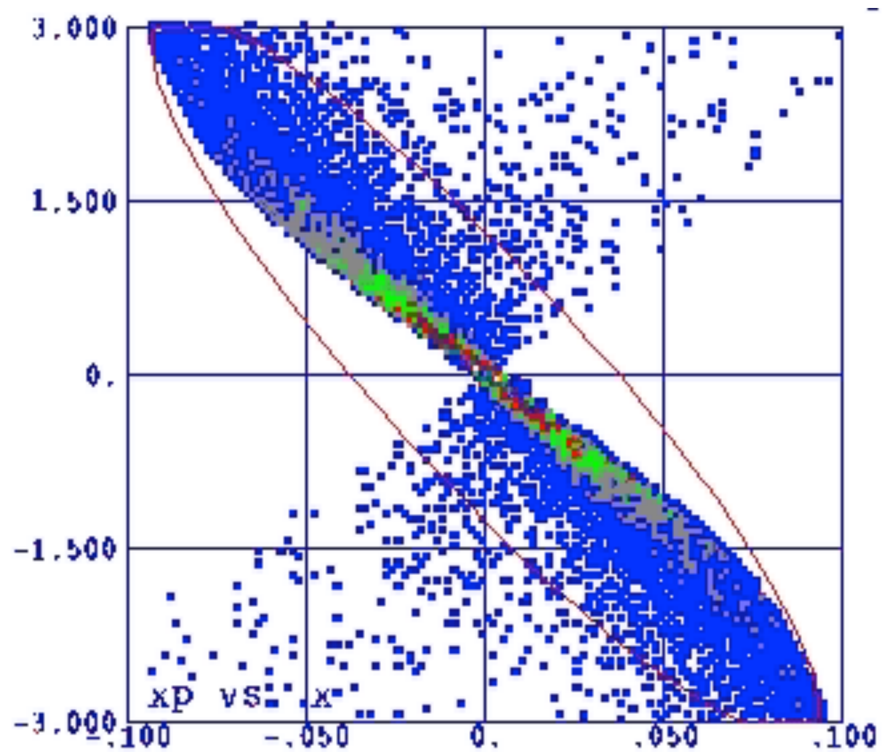
Phase space evolution



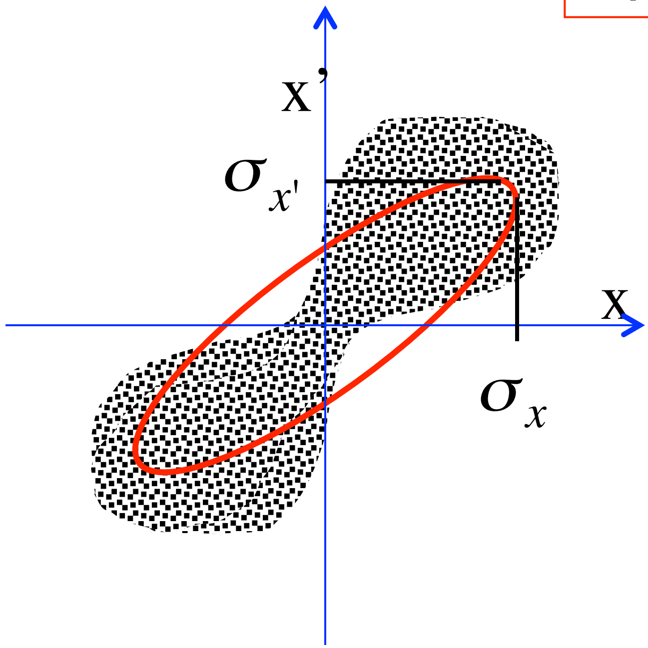
No space charge => cross over



With space charge => no cross over



rms emittance ε_{rms}



$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,x')\,dx\,dx' = 1$$

$$f'(x,x') = 0$$

rms beam envelope:

$$\sigma_x^2 = \langle x^2 \rangle = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x^2 f(x,x')dxdx'$$

Define rms emittance:

$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = \varepsilon_{rms}$$

such that:

$$\sigma_x = \sqrt{\langle x^2 \rangle} = \sqrt{\beta \varepsilon_{rms}}$$

$$\sigma_{x'} = \sqrt{\langle x'^2 \rangle} = \sqrt{\gamma \varepsilon_{rms}}$$

Since: $\alpha = -\dfrac{\beta'}{2}$

it follows: $\alpha = -\dfrac{1}{2\varepsilon_{rms}}\dfrac{d}{dz}\langle x^2 \rangle = -\dfrac{\langle x x' \rangle}{\varepsilon_{rms}} = -\dfrac{\sigma_{xx'}}{\varepsilon_{rms}}$

$$\sigma_x = \sqrt{\langle x^2 \rangle} = \sqrt{\beta \varepsilon_{rms}}$$

$$\sigma'_x = \sqrt{\langle x'^2 \rangle} = \sqrt{\gamma \varepsilon_{rms}}$$

$$\sigma_{xx'} = \langle xx' \rangle = -\alpha \varepsilon_{rms}$$

It holds also the relation: $\gamma\beta - \alpha^2 = 1$

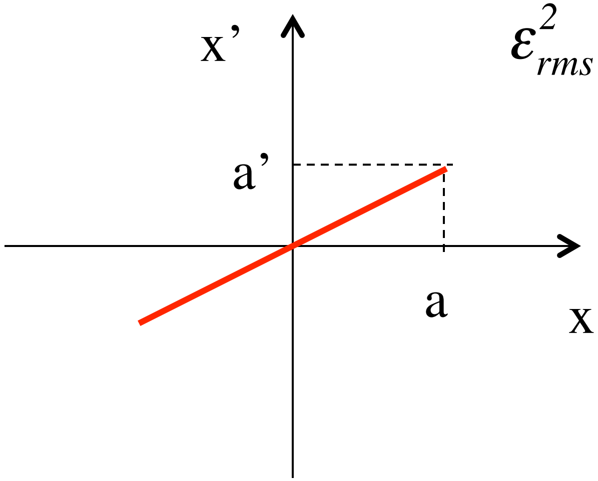
Substituting α, β, γ we get $\frac{\sigma_{x'}^2}{\varepsilon_{rms}} \frac{\sigma_x^2}{\varepsilon_{rms}} - \left(\frac{\sigma_{xx'}}{\varepsilon_{rms}}\right)^2 = 1$

We end up with the definition of rms emittance in terms of the second moments of the distribution:

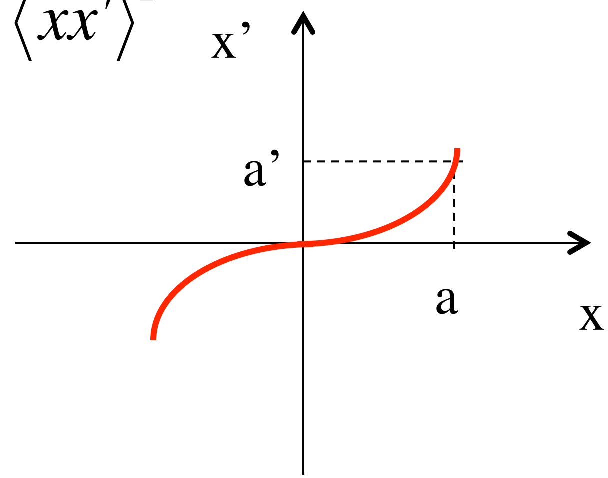
$$\varepsilon_{rms} = \sqrt{\sigma_x^2 \sigma_{x'}^2 - \sigma_{xx'}^2} = \sqrt{\left(\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2\right)}$$

$$x' = \frac{p_x}{p_z}$$

What does rms emittance tell us about phase space distributions under linear or non-linear forces acting on the beam?



$$\varepsilon_{rms}^2 = \langle x^2 \rangle \langle x'^2 \rangle - \langle x x' \rangle^2$$



Assuming a generic x, x' correlation of the type: $x' = C x^n$

$$\varepsilon_{rms}^2 = C^2 \left(\langle x^2 \rangle \langle x^{2n} \rangle - \langle x^{n+1} \rangle^2 \right)$$

When n = 1 ==> $\varepsilon_{rms} = 0$

When n ≠ 1 ==> $\varepsilon_{rms} \neq 0$

Constant under linear transformation only

$$\frac{d}{dz}\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2 = 2\langle xx'\rangle\langle x'^2\rangle + 2\langle x^2\rangle\langle x'\rangle\langle x''\rangle - 2\langle xx''\rangle\langle xx'\rangle = 0$$

For linear transformations, $x'' = -k_x^2 x$, and the right-hand side of the equation is

$$2k_x^2\langle x^2\rangle\langle xx'\rangle - 2\langle x^2\rangle\langle xx'\rangle k_x^2 = 0,$$

so

$$\frac{d}{dz}\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2 = 0$$

And without acceleration:

$$x' = \frac{p_x}{p_z}$$

Normalized rms emittance: $\varepsilon_{n,rms}$

Canonical transverse momentum: $p_x = p_z x' = m_o c \beta\gamma x'$ $\qquad p_z \approx p$

$$\varepsilon_{n,rms} = \sqrt{\sigma_x^2 \sigma_{p_x}^2 - \sigma_{xp_x}^2} = \frac{1}{m_o c}\sqrt{\left(\langle x^2\rangle\langle p_x^2\rangle - \langle xp_x\rangle^2\right)} \approx \langle\beta\gamma\rangle\varepsilon_{rms}$$

Liouville theorem: the density of particles n, or the volume V occupied by a given number of particles in phase space (x,p_x,y,p_y,z,p_z) remains invariant.

$$\frac{dn}{dt} = 0$$

It hold also in the projected phase spaces (x,p_x),(y,p_y)(,z,p_z) provided that there are no couplings

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WARNING
Energy spread contribution to rms emittance

$$\varepsilon_n^2 =< x^2 >< \beta^2\gamma^2 x'^2 > - < x\beta\gamma x' >^2$$

If the correlation between the energy and transverse position is negligible:

$$\varepsilon_n^2 =< \beta^2\gamma^2 >< x^2 >< x'^2 > - < \beta\gamma >^2 < xx' >^2$$

Using the definition of relative energy spread: $$\sigma_E^2 = \frac{< \beta^2\gamma^2 > - < \beta\gamma >^2}{< \gamma >^2}$$

Substituting in the previous equations and assuming relativistic electrons, yields to:

$$\varepsilon_n^2 = \langle\gamma\rangle^2(\sigma_E^2\sigma_x^2\sigma_{x'}^2 + \varepsilon^2).$$

Migliorati et al., Phys. Rev. STAB 16, 011302 (2013)

$$\varepsilon_n^2 = \langle\gamma\rangle^2(\sigma_E^2\sigma_x^2\sigma_{x'}^2 + \varepsilon^2).$$

Geometric emittance

At the plasma-vacuum interface is of the same order of magnitude as for conventional accelerators at low energies; however, due to the rapid increase of the bunch size, it becomes predominant compared to the second term.

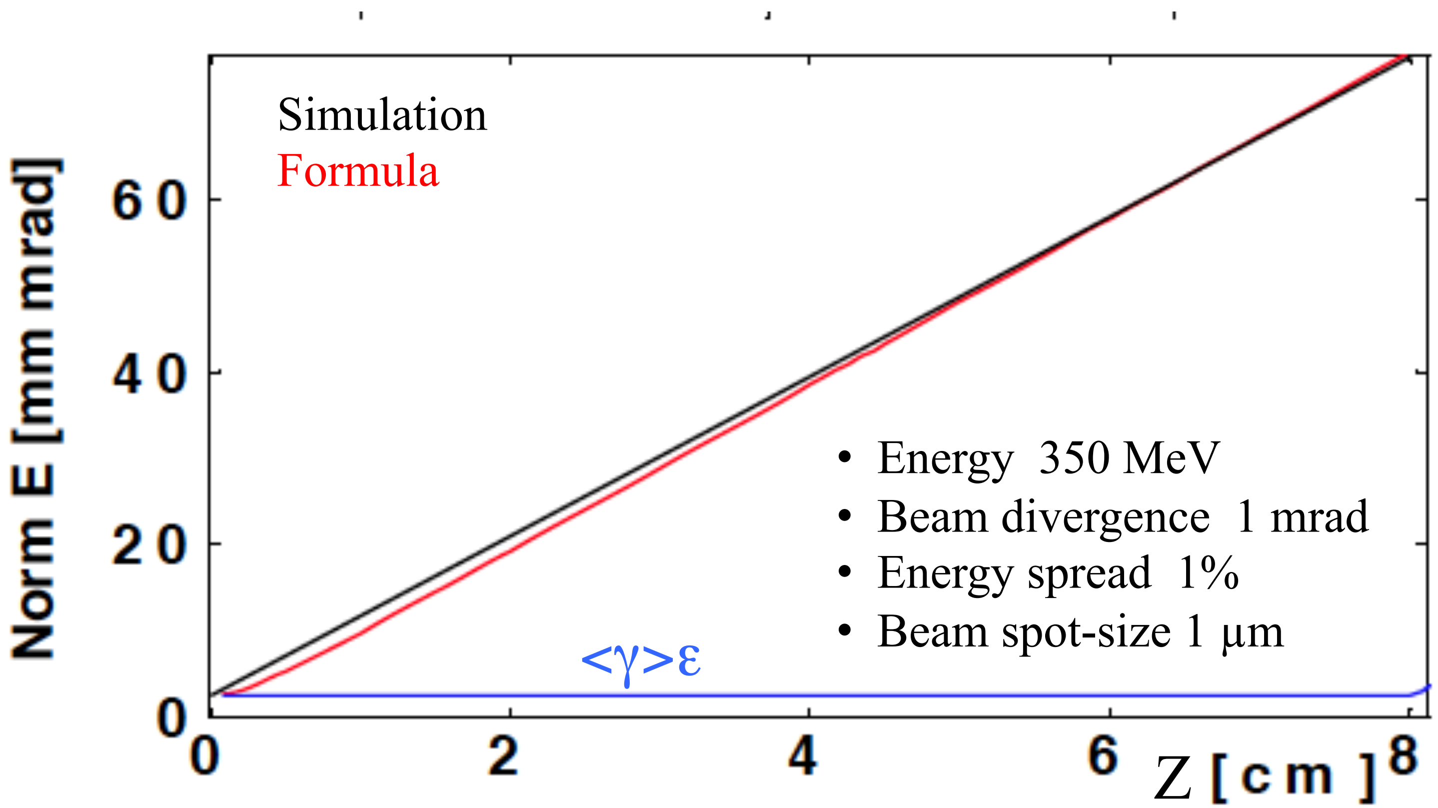
Considering the transverse beam size increase due to free diffraction, assuming as a starting condition a beam waist and a sufficiently long drift, we find that the bunch size becomes:

$$\sigma_x(s) \approx \sigma_{x'} s$$

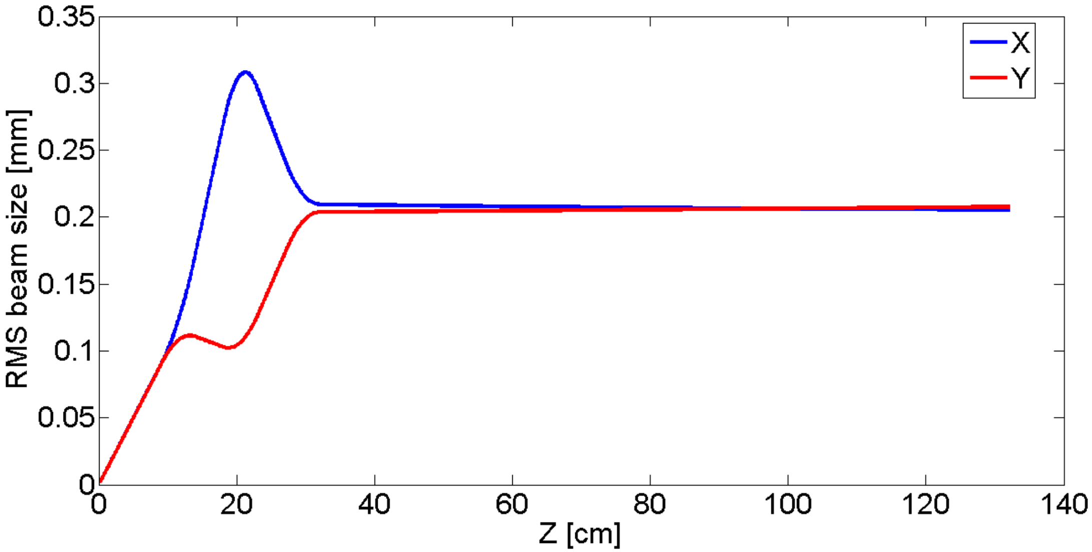
and the emittance:

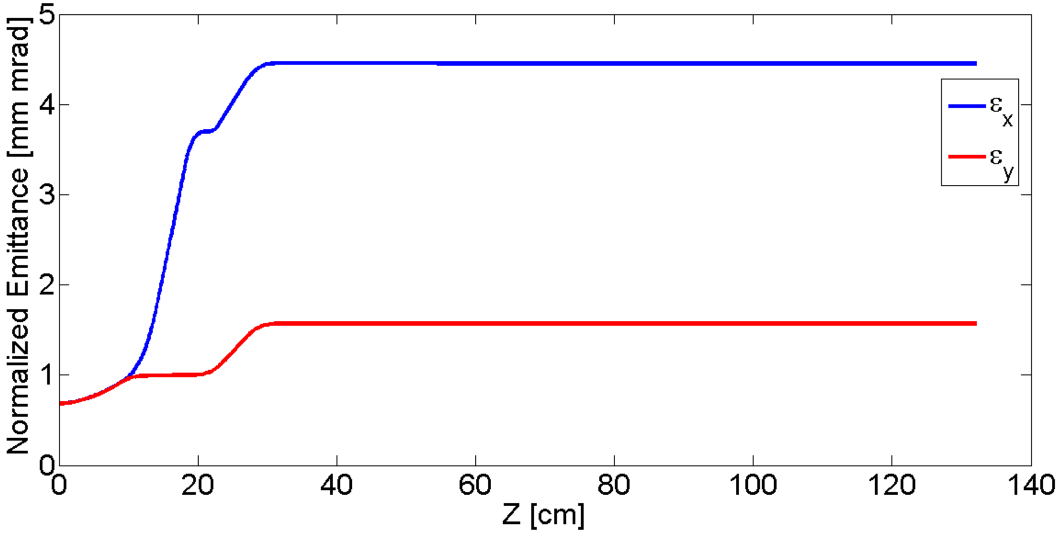
$$\varepsilon_n^2 = \langle\gamma\rangle^2(s^2\sigma_E^2\sigma_{x'}^4 + \varepsilon^2)$$

$$\varepsilon_n^2 = \langle\gamma\rangle^2(s^2\sigma_E^2\sigma_{x'}^4 + \varepsilon^2)$$



Beam transport line simulated with TSTEP

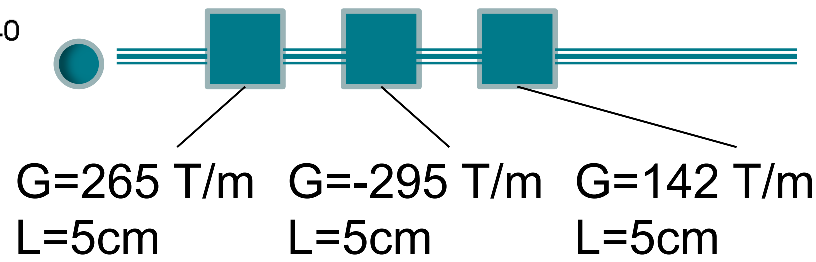




Beam transport line based on a triplet-lattice.

Beam parameters are:

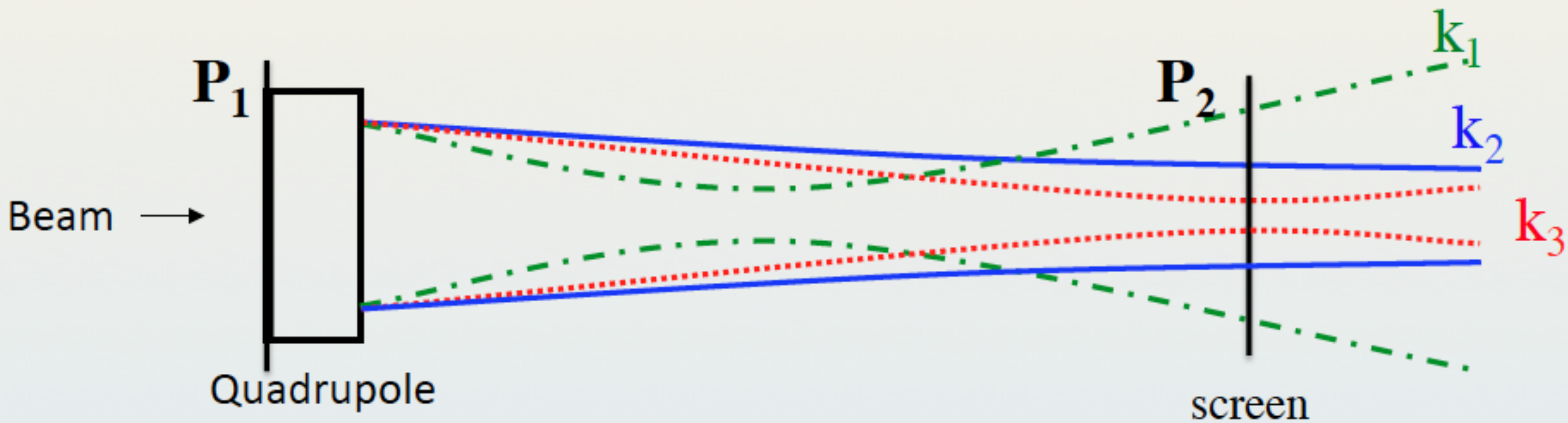
- Energy 350 MeV
- Beam divergence 1 mrad
- Energy spread 1%
- Beam spot-size 1 μm



Keeping the **beam size under control** is possible, **but normalized emittance grows** throughout the beamline.

$$\Delta\varepsilon_{n,rms} = \langle\gamma\rangle \left| \left(\sigma_\gamma k_q l_q + \sigma'_o\right)\sigma_o^2 + \sigma_o\sigma'_o \right|$$

Geometric emittance measurement ➔ Quad Scan



$$\sigma_{11} = C^2(k)\sigma_{11} + 2C(k)S(k)\sigma_{12} + S^2(k)\sigma_{22}$$

- Changing the strength of a magnetic lens is possible to measure the beam size
- With a least 3 different measurements is possible to retrieve the elements of the sigma matrix that are related with the emittance

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Envelope Equation without Acceleration

Now take the derivatives:

$$\frac{d\sigma_x}{dz} = \frac{d}{dz}\sqrt{\left\langle x^2 \right\rangle} = \frac{1}{2\sigma_x}\frac{d}{dz}\left\langle x^2 \right\rangle = \frac{1}{2\sigma_x} 2\left\langle xx' \right\rangle = \frac{\sigma_{xx'}}{\sigma_x}$$

$$\frac{d^2\sigma_x}{dz^2} = \frac{d}{dz}\frac{\sigma_{xx'}}{\sigma_x} = \frac{1}{\sigma_x}\frac{d\sigma_{xx'}}{dz} - \frac{\sigma_{xx'}^2}{\sigma_x^3} = \frac{1}{\sigma_x}\left(\left\langle x'^2 \right\rangle - \left\langle xx'' \right\rangle\right) - \frac{\sigma_{xx'}^2}{\sigma_x^3} = \frac{\sigma_{x'}^2 + \left\langle xx'' \right\rangle}{\sigma_x} - \frac{\sigma_{xx'}^2}{\sigma_x^3}$$

And simplify:

$$\sigma_x'' = \frac{\sigma_x^2\sigma_{x'}^2 - \sigma_{xx'}^2}{\sigma_x^3} - \frac{\left\langle xx'' \right\rangle}{\sigma_x} = \frac{\varepsilon_{rms}^2}{\sigma_x^3} + \frac{\left\langle xx'' \right\rangle}{\sigma_x}$$

We obtain the rms envelope equation in which the rms emittance enters as defocusing pressure like term.

$$\sigma_x'' - \frac{\left\langle xx'' \right\rangle}{\sigma_x} = \frac{\varepsilon_{rms}^2}{\sigma_x^3}$$

Lets now consider for example the simple case with $\langle xx'' \rangle = 0$ describing a beam drifting in the free space.

The envelope equation reduces to:

$$\sigma_x^3 \sigma_x'' = \varepsilon_{rms}^2$$

With initial conditions σ_o, σ_o' at z_o, depending on the upstream transport channel, the equation has a hyperbolic solution:

$$\sigma(z) = \sqrt{\left(\sigma_o + \sigma_o'(z - z_o)\right)^2 + \frac{\varepsilon_{rms}^2}{\sigma_o^2}(z - z_o)^2}$$

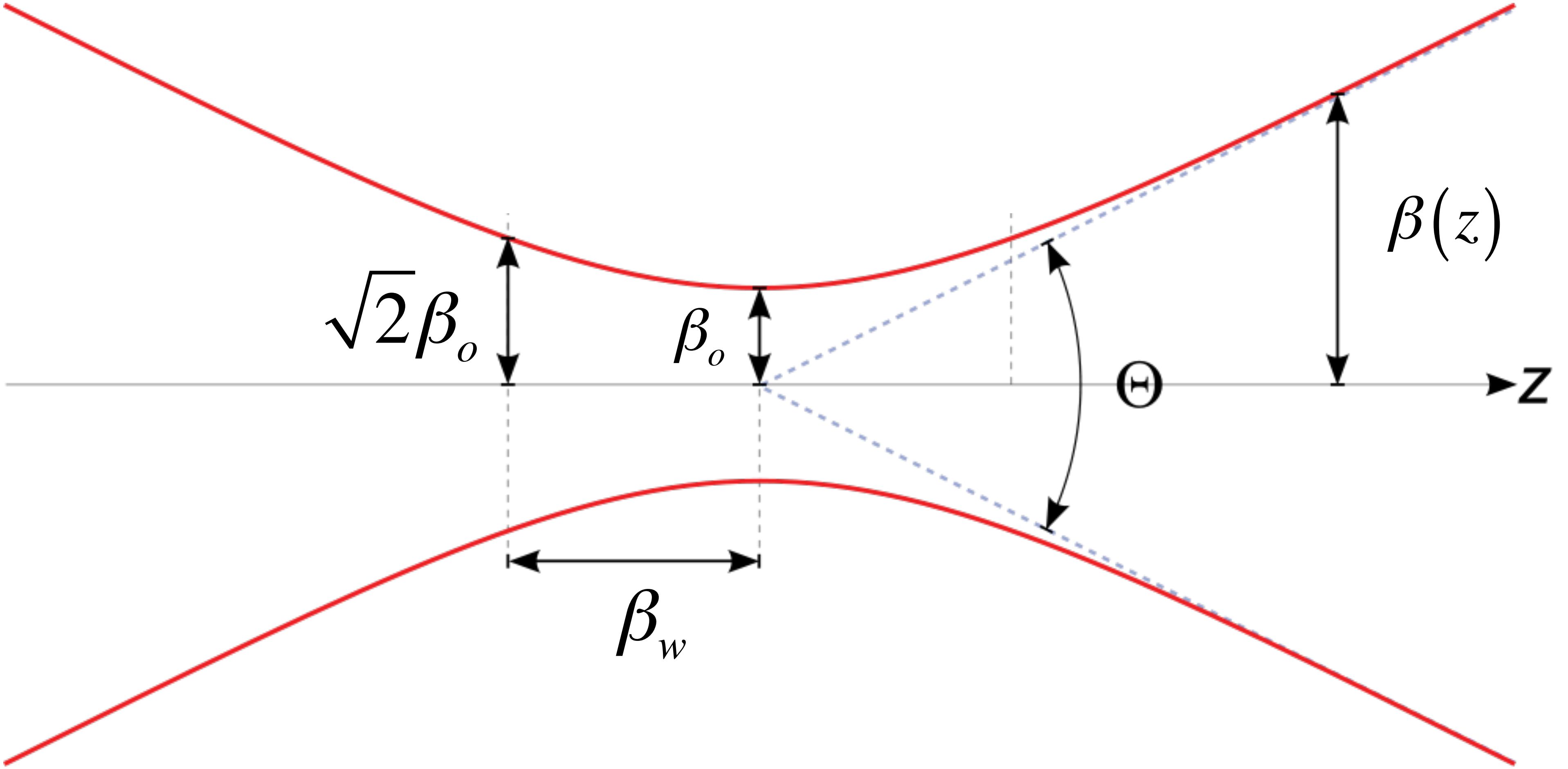
Considering the case $\sigma'_o = 0$ (beam at waist)

and using the definition $\sigma_x = \sqrt{\beta \varepsilon_{rms}}$

the solution is often written in terms of the β function as:

$$\sigma(z) = \sigma_o \sqrt{1 + \left(\frac{z - z_o}{\beta_w} \right)^2}$$

This relation indicates that without any external focusing element the beam envelope increases from the beam waist by a factor $\sqrt{2}$ with a characteristic length $\beta_w = \frac{\sigma_o^2}{\varepsilon_{rms}}$



For an effective transport of a beam with finite emittance is mandatory to make use of some external force providing beam confinement in the transport or accelerating line.

$$\sigma_x'' - \frac{\langle x x'' \rangle}{\sigma_x} = \frac{\varepsilon_{rms}^2}{\sigma_x^3}$$

Assuming that each particle is subject only to a linear focusing force, without acceleration: $x'' + k_x^2 x = 0$

take the average over the entire particle ensemble $\langle x x'' \rangle = -k_x^2 \langle x^2 \rangle$

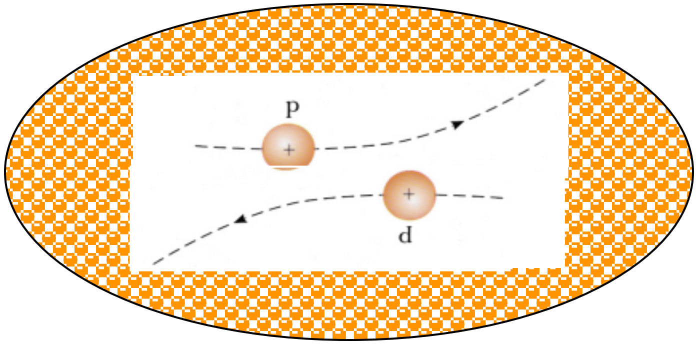
$$\sigma_x'' + k_x^2 \sigma_x = \frac{\varepsilon_{rms}^2}{\sigma_x^3}$$

We obtain the rms envelope equation with a linear focusing force in which the rms emittance enters as defocusing pressure like term.

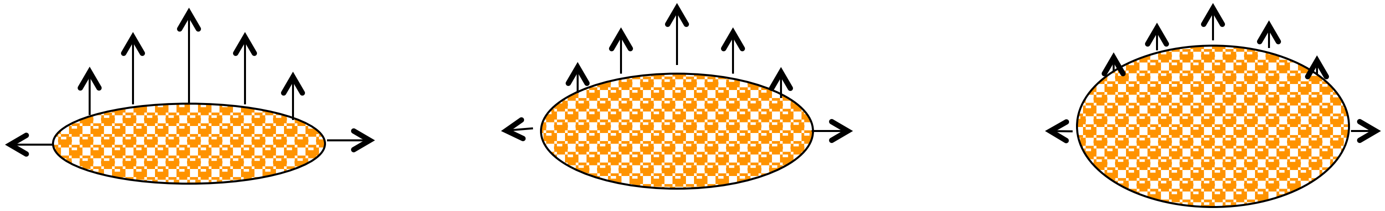
Space Charge: What does it mean?

The net effect of the **Coulomb** interactions in a multi-particle system can be classified into two regimes:

1) **Collisional Regime** ==> dominated by **binary collisions** caused by close particle encounters ==> **Single Particle Effects**

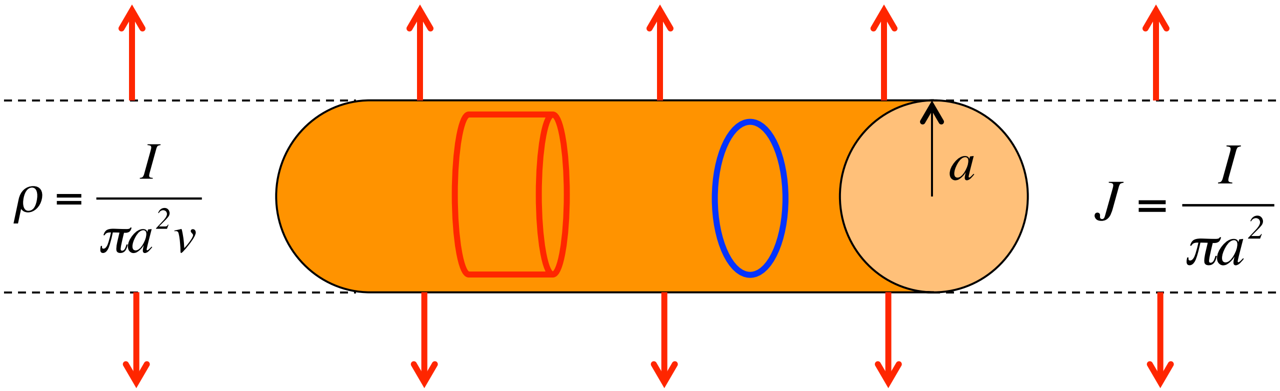


2) **Space Charge Regime** ==> dominated by the **self field** produced by the particle distribution, which varies appreciably only over large distances compare to the average separation of the particles ==> **Collective Effects**



Continuous Uniform Cylindrical Beam Model

$$\rho = \frac{I}{\pi a^2 v}$$



$$J = \frac{I}{\pi a^2}$$

Gauss's law

$$\int \varepsilon_o E \cdot dS = \int \rho dV$$

$$E_r = \frac{I}{2\pi\varepsilon_o a^2 v} r \quad \textbf{for} \quad \boldsymbol{r \le a}$$

$$E_r = \frac{I}{2\pi\varepsilon_o v}\frac{1}{r} \quad \textbf{for} \quad \boldsymbol{r > a}$$

$$B_\vartheta = \frac{\beta}{c} E_r$$

Ampere's law

$$\int B \cdot dl = \mu_o \int J \cdot dS$$

$$B_\vartheta = \mu_o \frac{Ir}{2\pi a^2} \quad \textbf{for} \quad \boldsymbol{r \le a}$$

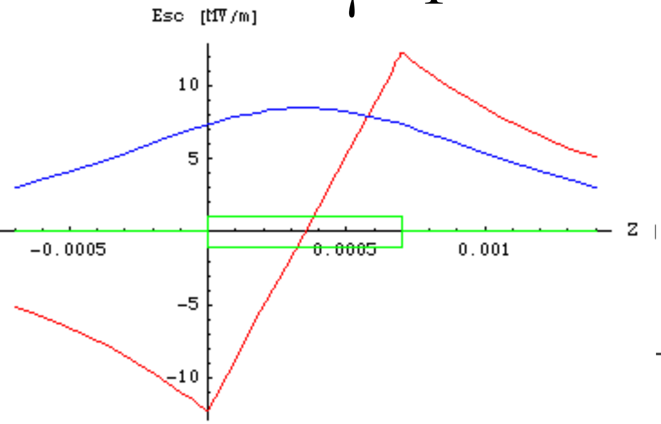
$$B_\vartheta = \mu_o \frac{I}{2\pi r} \quad \textbf{for} \quad \boldsymbol{r > a}$$

Bunched Uniform Cylindrical Beam Model

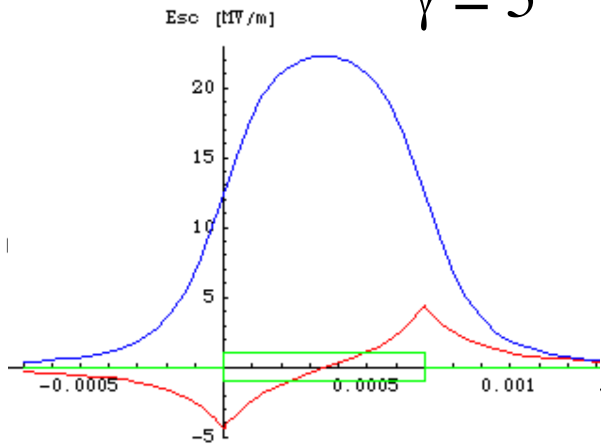
$$E_z(0,s,\gamma) = \frac{I}{2\pi\gamma\varepsilon_0 R^2 \beta c} h(s,\gamma)$$

$$E_r(r,s,\gamma) = \frac{Ir}{2\pi\varepsilon_0 R^2 \beta c} g(s,\gamma)$$

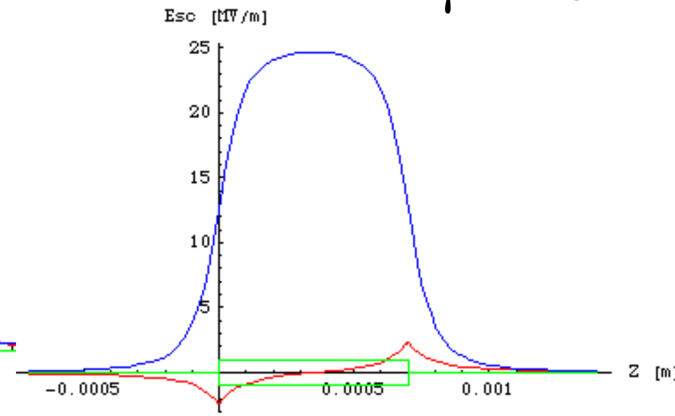
γ= 1



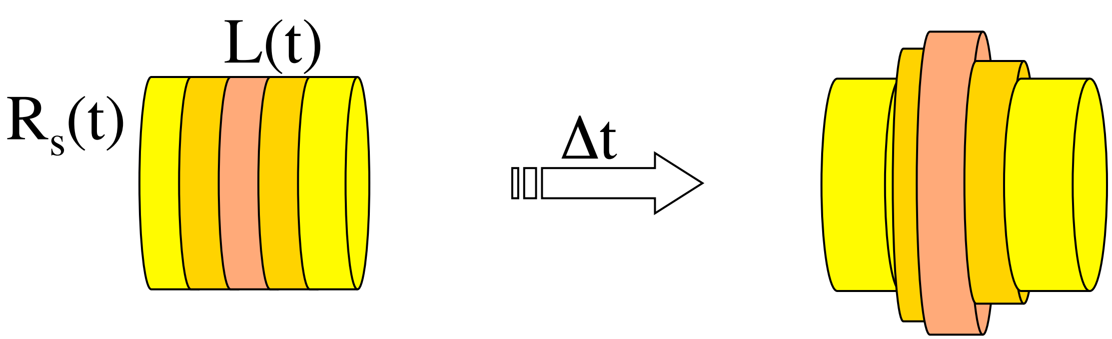
γ = 5



γ = 10



$$F_r = \frac{eE_r}{\gamma^2} = \frac{eIr}{2\pi\gamma^2\varepsilon_0 R^2 \beta c} g(s,\gamma)$$



$$B_\vartheta = \frac{\beta}{c} E_r$$

Lorentz Force

$$E_r(r,s,\gamma) = \frac{Ir}{2\pi\varepsilon_0 R^2 \beta c} g(s,\gamma)$$

$$F_r = e\left(E_r - \beta c B_\vartheta\right) = e\left(1-\beta^2\right) E_r = \frac{eE_r}{\gamma^2}$$

is a **linear** function of the transverse coordinate

$$\frac{dp_r}{dt} = F_r = \frac{eE_r}{\gamma^2} = \frac{eIr}{2\pi\gamma^2\varepsilon_0 R^2 \beta c} g(s,\gamma)$$

The attractive magnetic force , which becomes significant at high velocities, tends to compensate for the repulsive electric force. Therefore space charge defocusing is primarily a non-relativistic effect.

$$F_x = \frac{eIx}{2\pi\gamma^2\varepsilon_0\sigma_x^2\beta c} g(s,\gamma)$$

Envelope Equation with Space Charge

Single particle transverse motion:

$$\frac{dp_x}{dt} = F_x \qquad p_x = p\,x' = \beta\gamma m_o c x'$$

$$\frac{d}{dt}(px') = \beta c \frac{d}{dz}(p\,x') = F_x$$

$$x'' = \frac{F_x}{\beta c p}$$

$$x'' = \frac{k_{sc}(s,\gamma)}{\sigma_x^2} x$$

Space Charge de-focusing force

Generalized perveance

$$k_{sc}(s,\gamma) = \frac{2I}{I_A(\beta\gamma)^3} g(s,\gamma)$$

$$I_A = \frac{4\pi\varepsilon_o m_o c^3}{e} = 17kA$$

Now we can calculate the term $\langle xx'' \rangle$ that enters in the envelope equation

$$\sigma_x'' = \frac{\varepsilon_{rms}^2}{\sigma_x^3} - \frac{\langle xx'' \rangle}{\sigma_x}$$

$$\langle xx'' \rangle = \frac{k_{sc}}{\sigma_x^2} \langle x^2 \rangle = k_{sc}$$

Including all the other terms the envelope equation reads:

Space Charge De-focusing Force

$$\sigma_x'' + k^2 \sigma_x = \frac{\varepsilon_n^2}{(\beta\gamma)^2 \sigma_x^3} + \frac{k_{sc}}{\sigma_x}$$

Emittance Pressure

External Focusing Forces

Laminarity Parameter: $\rho = \frac{(\beta\gamma)^2 k_{sc} \sigma_x^2}{\varepsilon_n^2}$

The beam undergoes two regimes along the accelerator

$$\sigma_x'' + k^2\sigma_x = \frac{\varepsilon_n^2}{(\beta\gamma)^2\sigma_x^3} + \frac{k_{sc}}{\sigma_x}$$

ρ>>1 Laminar Beam

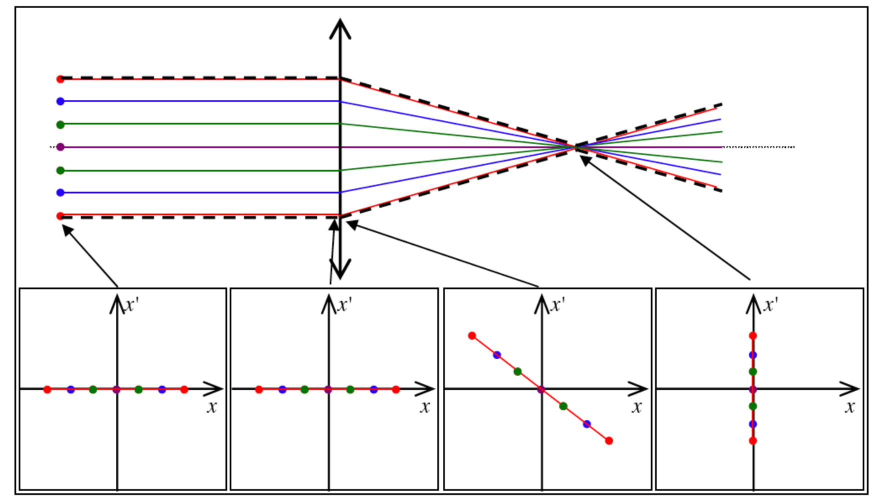


Fig. 10: Particle trajectories in laminar beam

$$\sigma_x'' + k^2\sigma_x = \frac{\varepsilon_n^2}{(\beta\gamma)^2\sigma_x^3} + \frac{k_{sc}}{\sigma_x}$$

ρ<<1 Thermal Beam

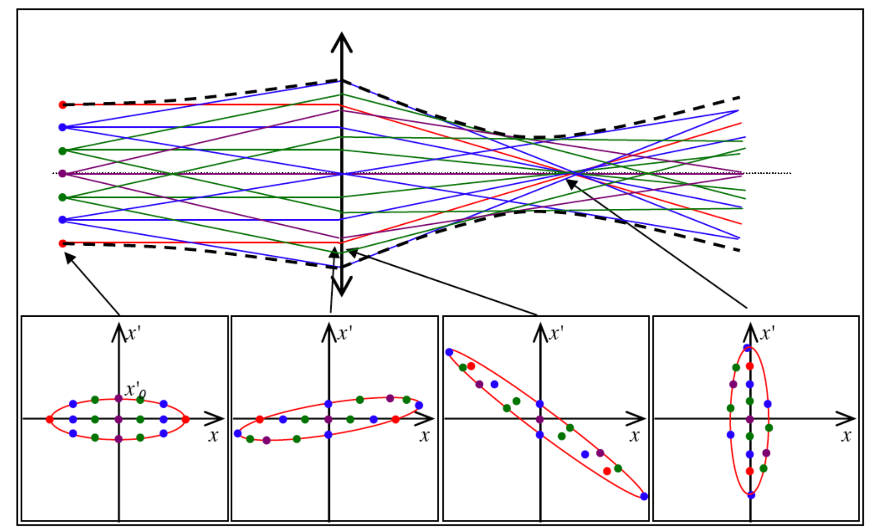


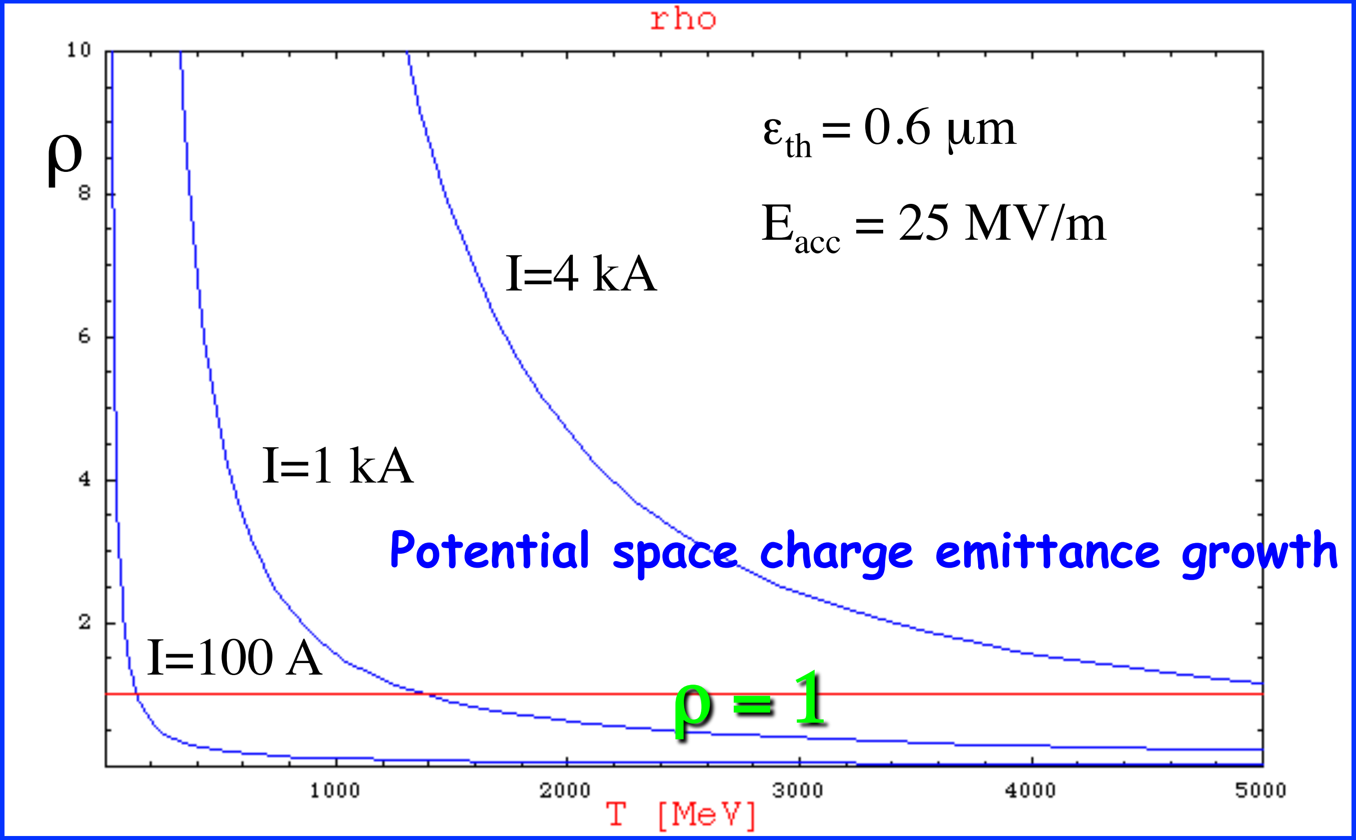
Fig. 11: Particle trajectories in non-zero emittance beam

Laminarity parameter

$$\rho = \frac{2I\sigma^2}{\gamma I_A \varepsilon_n^2} \equiv \frac{2I\sigma_q^2}{\gamma I_A \varepsilon_n^2} = \frac{4I^2}{\gamma'^2 I_A^2 \varepsilon_n^2 \gamma^2}$$

Transition Energy (ρ=1)

$$\gamma_{tr} = \frac{2I}{\gamma' I_A \varepsilon_n}$$



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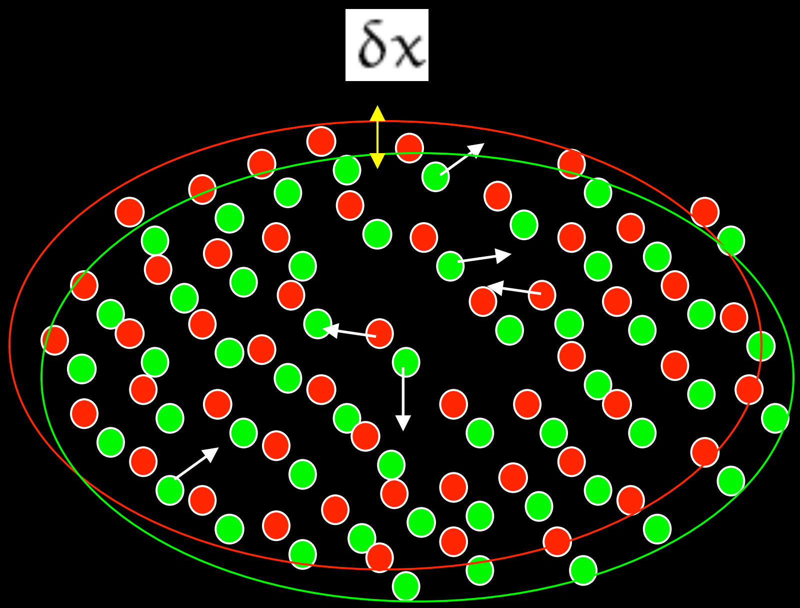
Surface charge density

$$\sigma = e\,n\,\delta x$$



Surface electric field

$$E_x = -\sigma/\epsilon_0 = -e\,n\,\delta x/\epsilon_0$$

Restoring force

$$m\,\frac{d^2\delta x}{dt^2} = e\,E_x = -m\,\omega_p^{\,2}\,\delta x$$

Plasma frequency

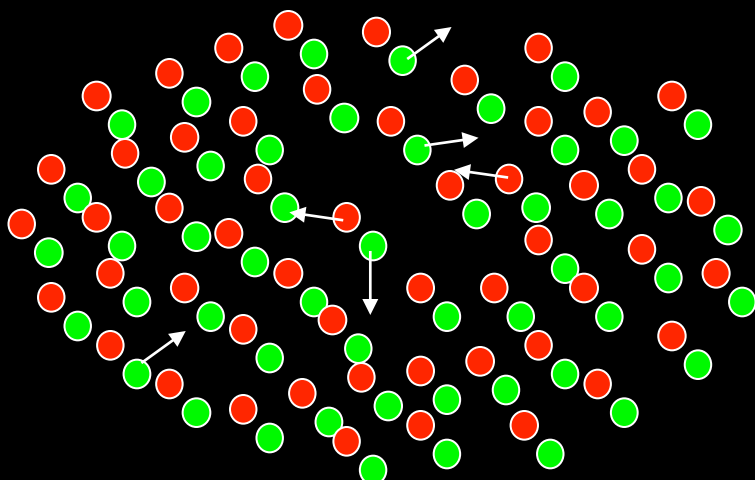
$$\omega_p^{\,2} = \frac{n\,e^2}{\epsilon_0\,m}$$

Plasma oscillations

$$\delta x = (\delta x)_0\,\cos\left(\omega_p\,t\right)$$

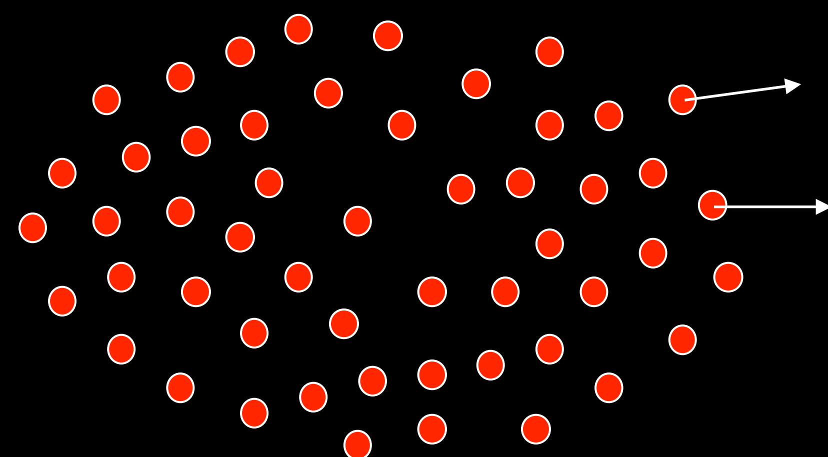
Neutral Plasma

- Oscillations
- Instabilities
- EM Wave propagation



Single Component Cold Relativistic Plasma

Magnetic focusing



Magnetic focusing

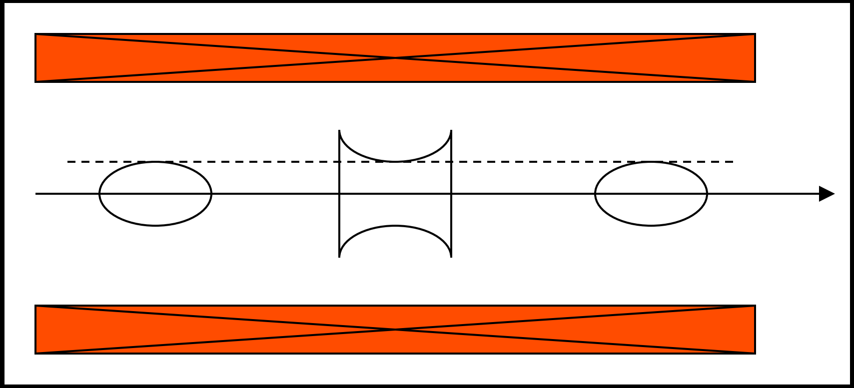
Single Component Relativistic Plasma

$$\sigma'' + k_s^2 \sigma = \frac{k_{sc}(s,\gamma)}{\sigma}$$

$$k_s = \frac{qB}{2mc\beta\gamma}$$

Equilibrium solution:

$$\sigma_{eq}(s,\gamma) = \frac{\sqrt{k_{sc}(s,\gamma)}}{k_s}$$



Small perturbation:

$$\sigma(\zeta) = \sigma_{eq}(s) + \delta\sigma(s)$$

$$\delta\sigma''(s) + 2k_s^2 \delta\sigma(s) = 0$$

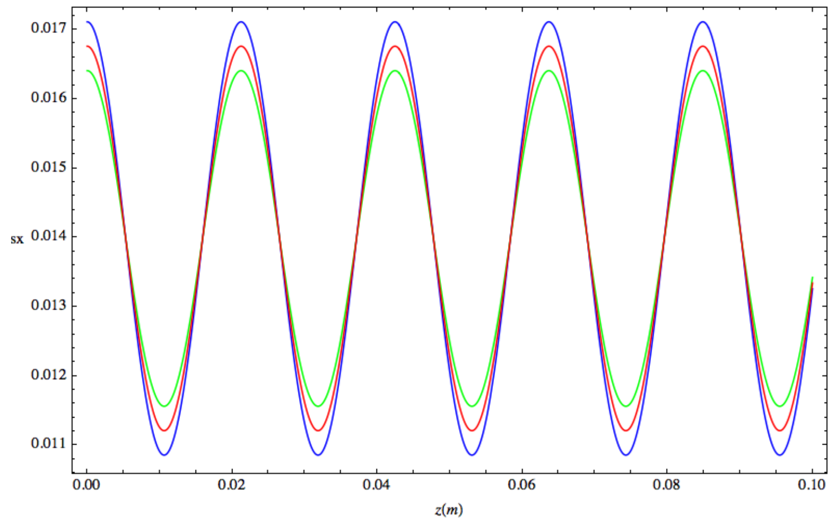
$$\delta\sigma(s) = \delta\sigma_o(s)\cos\left(\sqrt{2}k_s z\right)$$

Perturbed trajectories oscillate around the equilibrium with the same frequency but with different amplitudes:

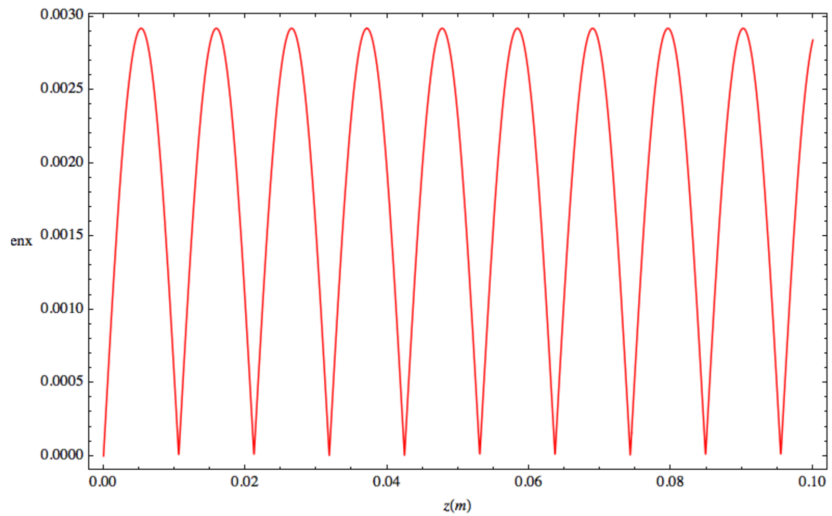
$$\sigma(s) = \sigma_{eq}(s) + \delta\sigma_o(s)\cos\left(\sqrt{2}k_s z\right)$$

Envelope oscillations drive Emittance oscillations

$\sigma(z)$



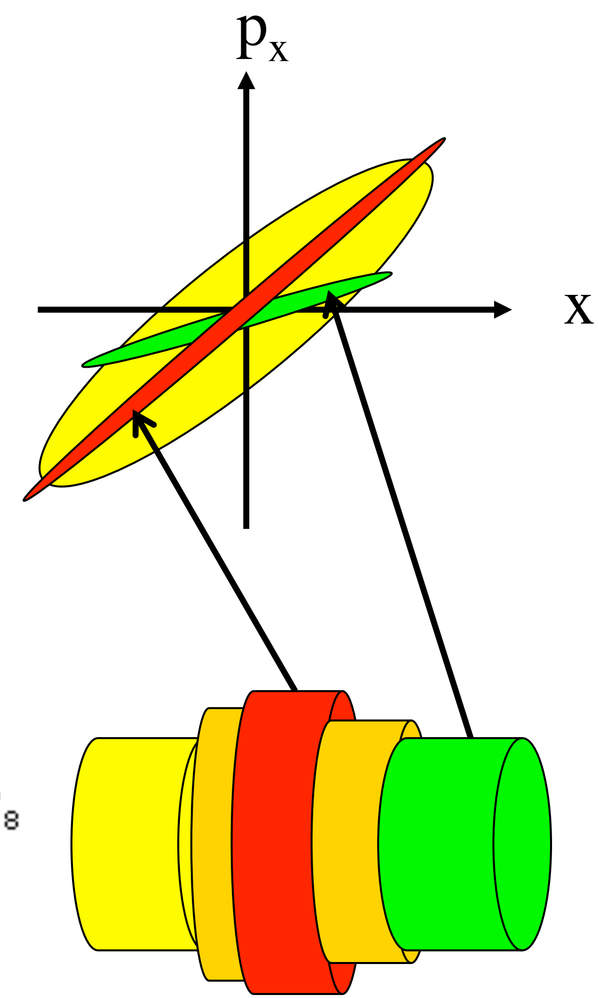
$\varepsilon(z)$



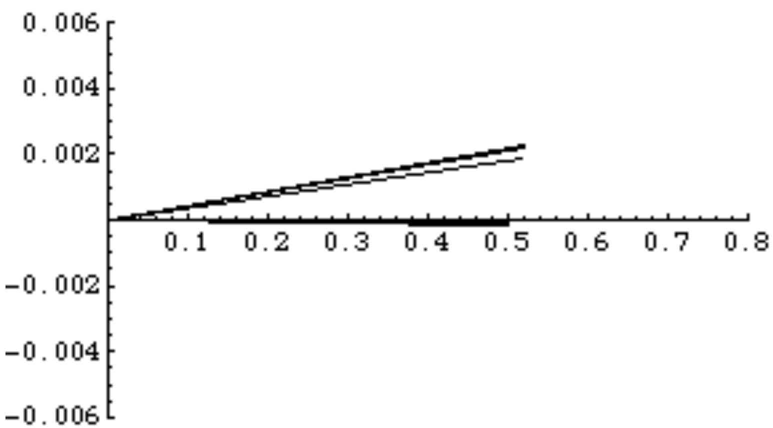
$$\varepsilon_{rms} = \sqrt{\sigma_x^2 \sigma_{x'}^2 - {\sigma_{xx'}}^2} = \sqrt{\left(\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2 \right)} \approx \left| sin\left(\sqrt{2} k_s z \right) \right|$$

Emittance Oscillations are driven by space charge differential defocusing in core and tails of the beam

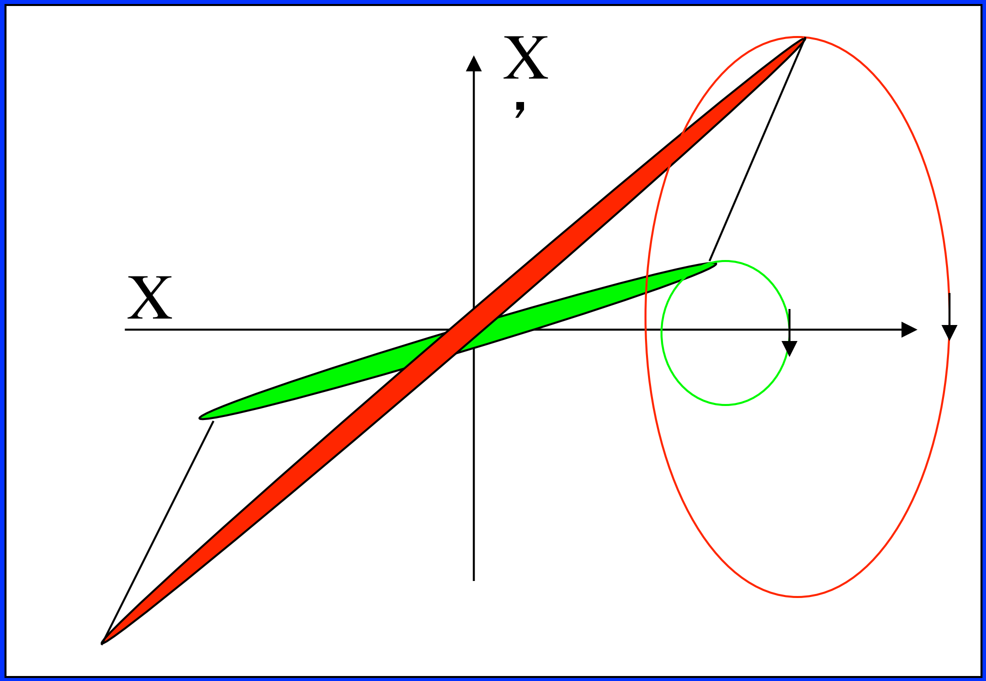
Projected Phase Space

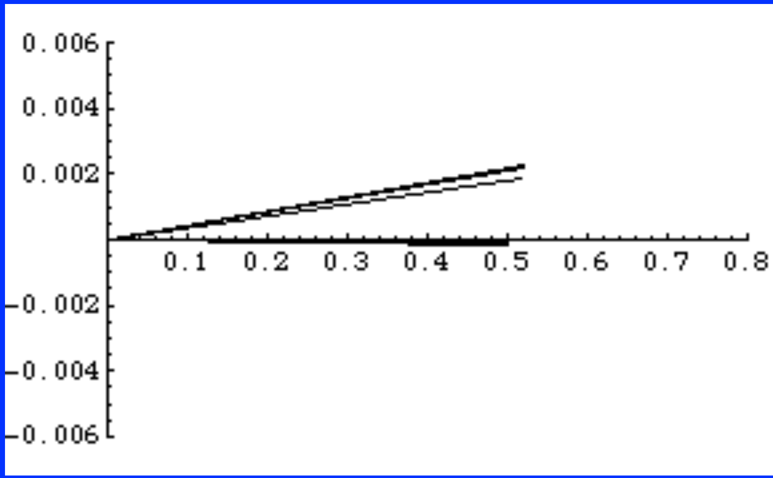


Slice Phase Spaces



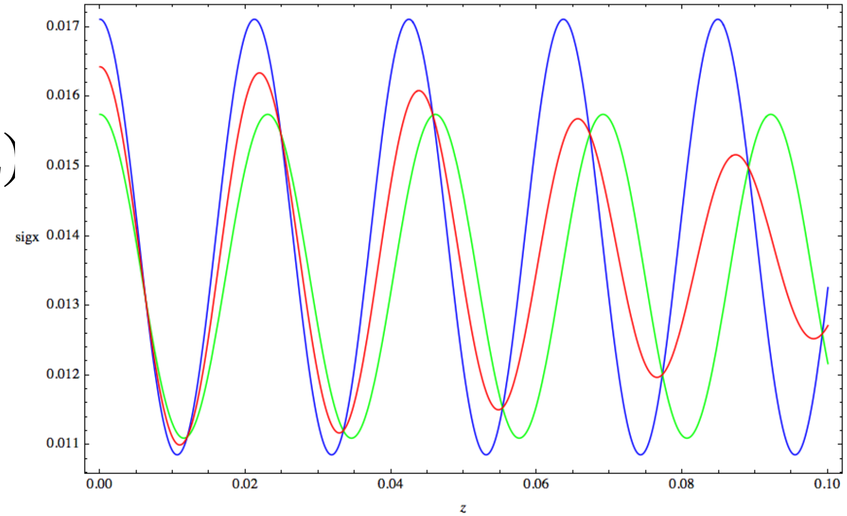
Perturbed trajectories oscillate around the equilibrium with the **same frequency** but with **different amplitudes**



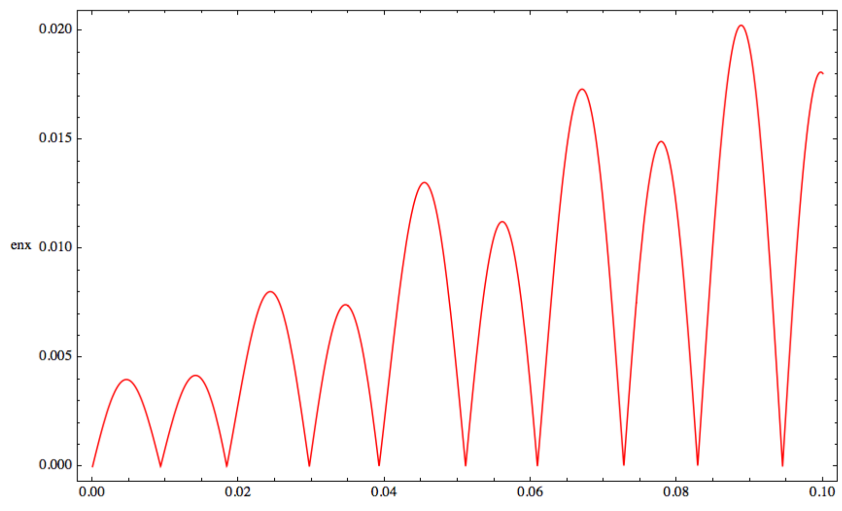


energy spread induces decoherence

σ(z)



ε(z)



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Envelope Equation with Longitudinal Acceleration

$$p_o = \gamma_o m_o \beta_o c$$

$$p_x << p_o$$

$$p = p_o + p' z$$

$$p' = (\beta\gamma)' m_o c$$

$$\frac{dp_x}{dt} = \frac{d}{dt}(px') = \beta c \frac{d}{dz}(px') = 0$$

$$x'' + \frac{p'}{p} x' = 0$$

$$x'' = -\frac{(\beta\gamma)'}{\beta\gamma} x'$$

$$\langle xx'' \rangle = -\frac{(\beta\gamma)'}{\beta\gamma} \langle xx' \rangle = -\frac{(\beta\gamma)'}{\beta\gamma} \sigma_{xx'}$$

$$\sigma_x'' = \frac{\varepsilon_{rms}^2}{\sigma_x^3} - \frac{\langle xx'' \rangle}{\sigma_x}$$

$$\frac{d\sigma_x}{dz} = \sigma_x' = \frac{\sigma_{xx'}}{\sigma_x}$$

Space Charge De-focusing Force

$$\sigma_x'' + \frac{(\beta\gamma)'}{\beta\gamma} \sigma_x' + k^2 \sigma_x = \frac{\varepsilon_n^2}{(\beta\gamma)^2 \sigma_x^3} + \frac{k_{sc}}{\sigma_x}$$

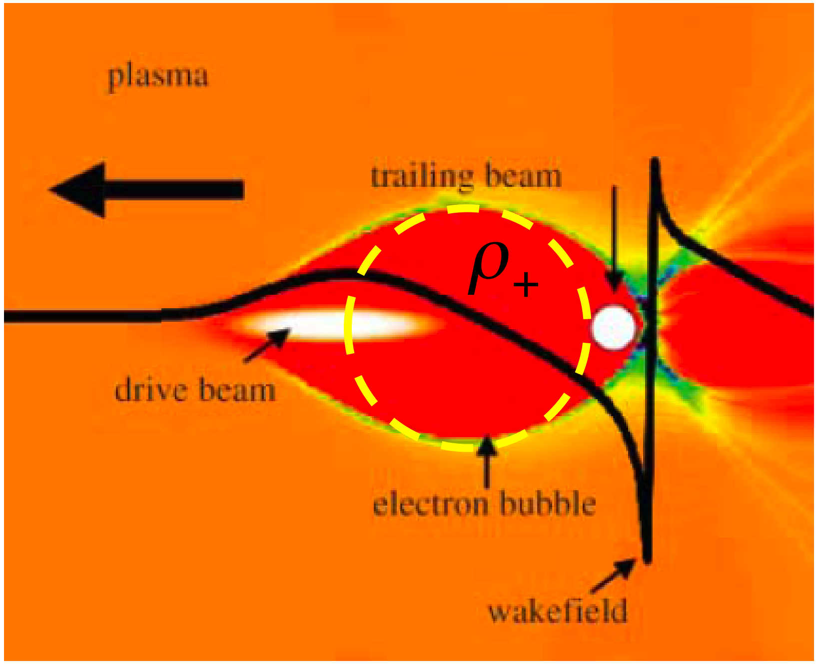
Adiabatic Damping

Emittance Pressure

Other External Focusing Forces

$$\varepsilon_n = \beta\gamma\varepsilon_{rms}$$

Envelope equation in a plasma accelerator



$$R_{sphere} \approx \frac{\lambda_p}{2}$$ Bubble radius

$$n_1 \approx n_{drive}$$ Bubble density

$$E_r = \frac{en_1}{3\varepsilon_o} r$$ Radial field

$$F_r = e\left(E_r - \beta c B_\vartheta\right) = eE_r = \frac{e^2 n_1}{3\varepsilon_0} r$$

$$x'' = \frac{F_x}{\beta c p} = \frac{e^2 n_1 x}{3\varepsilon_o \gamma m c^2} = \frac{k_p^2}{3\gamma} x \qquad k_p^2 = \frac{e^2 n_1}{\varepsilon_o m c^2} \qquad \langle x x'' \rangle = \frac{k_p^2}{\gamma}\langle x^2 \rangle = \frac{k_p^2}{\gamma}\sigma_x^2$$

$$\sigma_x'' + \frac{\gamma'}{\gamma}\sigma_x' + \frac{k_p^2}{3\gamma}\sigma_x = \frac{\varepsilon_n^2}{\gamma^2 \sigma_x^3} + \frac{k_{sc}^o}{\gamma^3 \sigma_x}$$

$$\sigma_x'' + \frac{\gamma'}{\gamma}\sigma_x' + \frac{k_p^2}{3\gamma}\sigma_x = \frac{\varepsilon_n^2}{\gamma^2\sigma_x^3} + \frac{k_{sc}^o}{\gamma^3\sigma_x}$$

An equilibrium solution of the previous equation is difficult to find, nevertheless some simplification is possible and an approximated matching condition exists.

As one can see there are two focusing terms, the adiabatic damping and the ion focusing, and two defocusing terms, the emittance pressure and the space charge effects.

To compare the relative importance of the first two terms is more convenient to rewrite the previous equation with the new variable: $\tilde{\sigma}_x = \sqrt{\gamma}\sigma_x$
leading to the equation:

$$\tilde{\sigma}_x'' + \left(\left(\frac{\gamma'}{2\gamma}\right)^2 + \frac{k_p^2}{3\gamma}\right)\tilde{\sigma}_x = \frac{\varepsilon_n^2}{\tilde{\sigma}_x^3} + \frac{k_o^{sc}}{\gamma^2\tilde{\sigma}_x}$$

$$\tilde{\sigma}_x'' + \left(\left(\frac{\gamma'}{2\gamma} \right)^2 + \frac{k_p^2}{3\gamma} \right) \tilde{\sigma}_x = \frac{\varepsilon_n^2}{\tilde{\sigma}_x^3} + \frac{k_o^{sc}}{\gamma^2 \tilde{\sigma}_x}$$

The beam is space charge dominated, as already discussed, when:

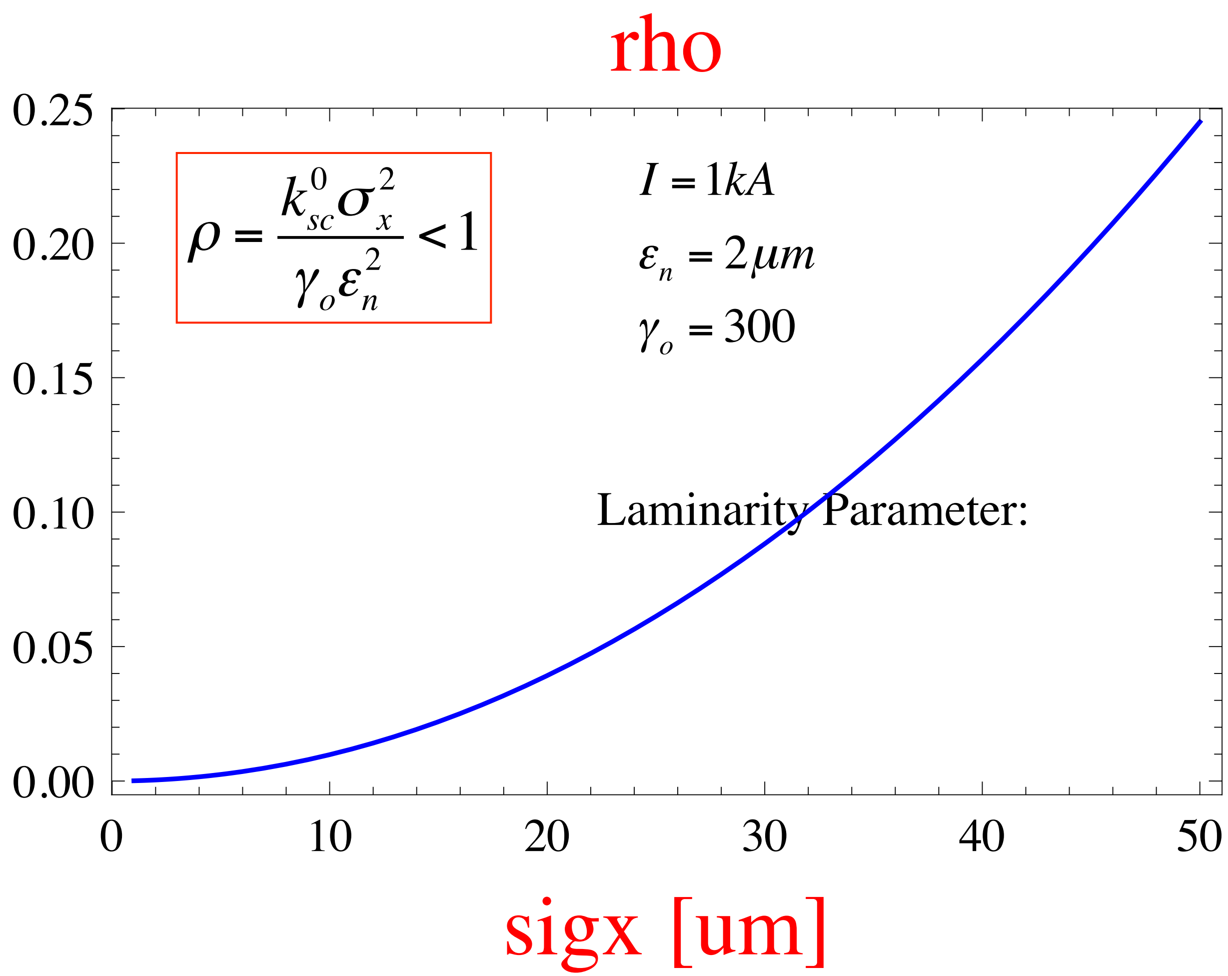
$$\rho = \frac{k_o^{sc} \tilde{\sigma}_x^2}{\varepsilon_n^2 \gamma^2} = \frac{k_o^{sc} \sigma_x^2}{\varepsilon_n^2 \gamma} >> 1$$

Space charge dominated

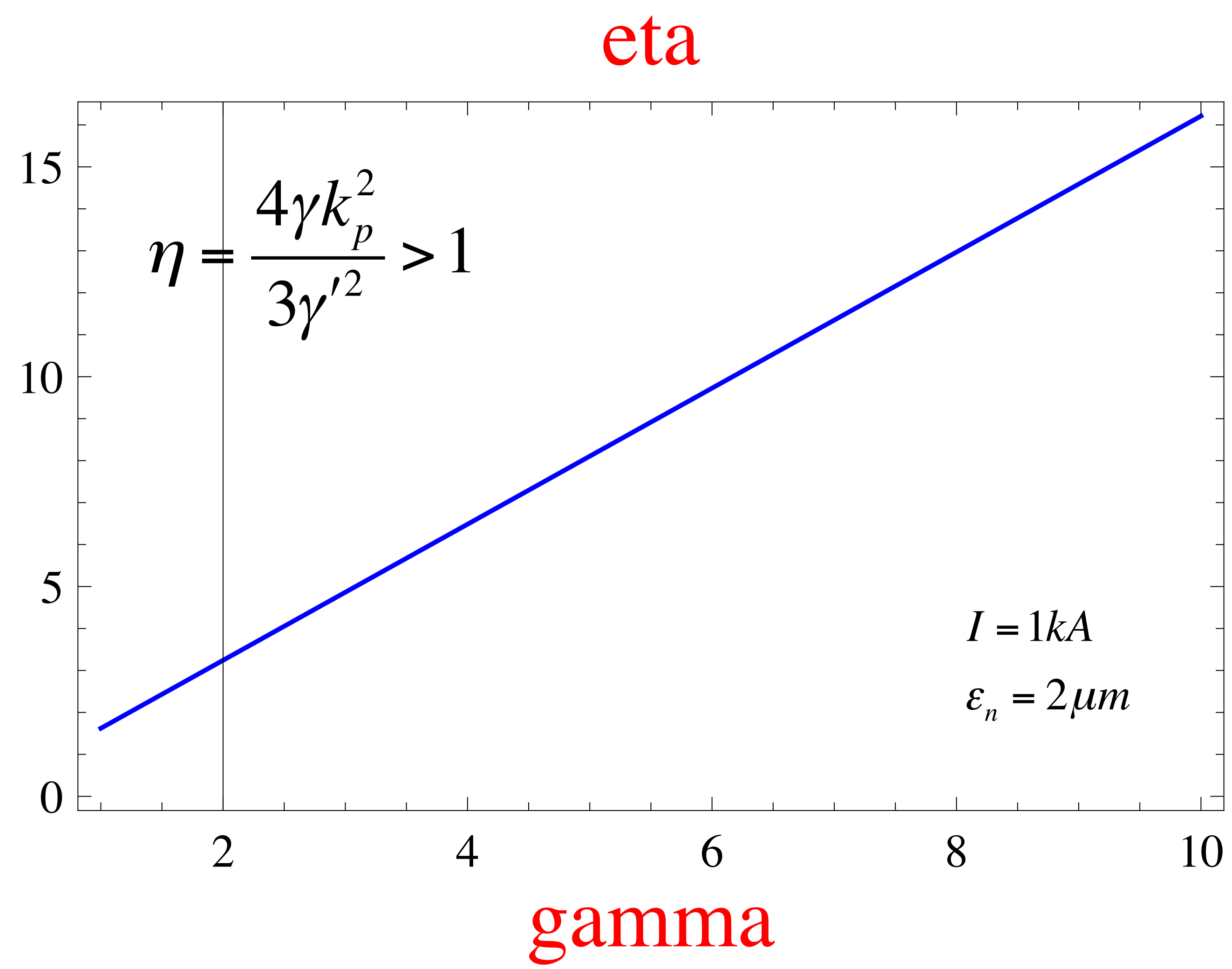
and ion focusing dominated when:

$$\eta = \frac{4\gamma k_p^2}{3\gamma'^2} >> 1$$

Plasma focussing dominated



rho
$$\rho = \frac{k_{sc}^0 \sigma_x^2}{\gamma_o \varepsilon_n^2} < 1$$
$I = 1kA$
$\varepsilon_n = 2\mu m$
$\gamma_o = 300$
Laminarity Parameter:
0.25
0.20
0.15
0.10
0.05
0.00
0
10
20
30
40
50
sigx [um]


eta
$\eta = \frac{4\gamma k_p^2}{3\gamma'^2} > 1$
$I = 1kA$
$\varepsilon_n = 2\mu m$
15
10
5
0
2
4
6
8
10
gamma

With the typical beam parameters: 1 kA peak current, 2 μm normalized emittance, injection energy γ_o=300 and spot size about 3 μm, we have ρ <1 and η >1.

It follows that the envelope equation can be well approximated by the reduced expression:

$$\sigma_x'' + \frac{k_p^2}{3\gamma}\sigma_x = \frac{\varepsilon_n^2}{\gamma^2\sigma_x^3}$$

$$\gamma'' = 0 \qquad \gamma' \neq 0$$

When $\eta = \frac{4\gamma k_p^2}{3\gamma'^2} >> 1$ $\qquad \rho = \frac{k_{sc}^0 \sigma_x^2}{\gamma_o \varepsilon_n^2} << 1$

$$\sigma_x'' + \frac{k_p^2}{3\gamma}\sigma_x = \frac{\varepsilon_n^2}{\gamma^2\sigma_x^3}$$

Looking for an "equilibrium" solution: $\sigma_x = \sigma_\varepsilon = \sigma_o \gamma^n$

==> all terms must have the same dependence on γ

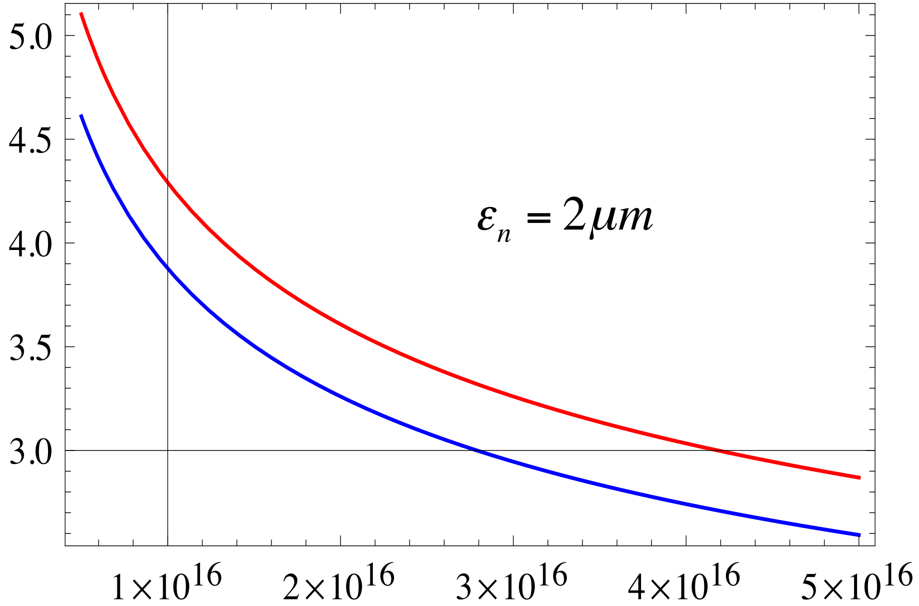
$$n = -\frac{1}{4} \quad \Rightarrow \quad \sigma_0^4 = \frac{3\varepsilon_n^2}{k_p^2}$$

$$\sigma_\varepsilon = \sqrt[4]{\frac{3}{\gamma}}\sqrt{\frac{\varepsilon_n}{k_p}}$$

We get the matching condition with acceleration:

$$\sigma_\varepsilon = \sqrt[4]{\frac{3}{\gamma}}\sqrt{\frac{\varepsilon_n}{k_p}}$$

sigma_r [um]



n [cm−3]

Perturbation around the equilibrium solution:

$$\sigma_\varepsilon = \sqrt[4]{\frac{3}{\gamma}}\sqrt{\frac{\varepsilon_n}{k_p}}$$

$$\sigma = \sigma_\varepsilon + \delta\sigma$$

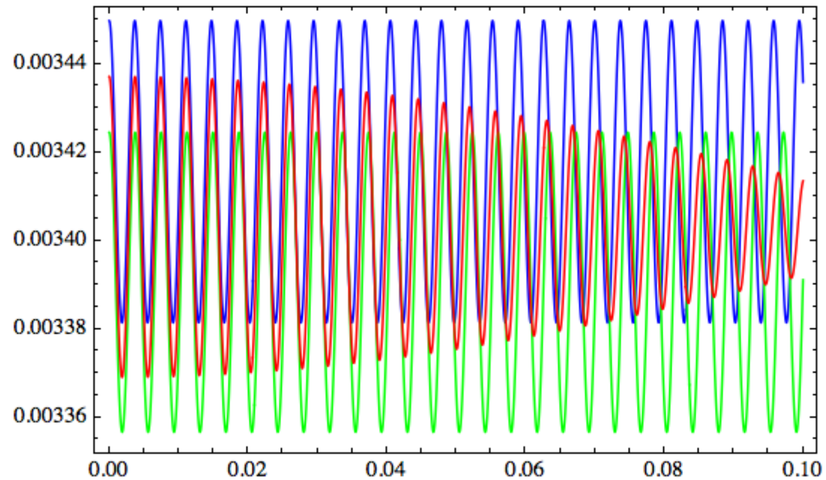
$$\sigma''_x + \frac{k_p^2}{3\gamma}\sigma_x = \frac{\varepsilon_n^2}{\gamma^2\sigma_x^3} \quad ==> \quad \delta\sigma''_x + \frac{4}{3}\frac{k_p^2}{\gamma}\delta\sigma_x = 0$$

$$\delta\sigma(z) = \delta\sigma_o \cos\left(\sqrt{\frac{4}{3\gamma}}k_p z\right)$$

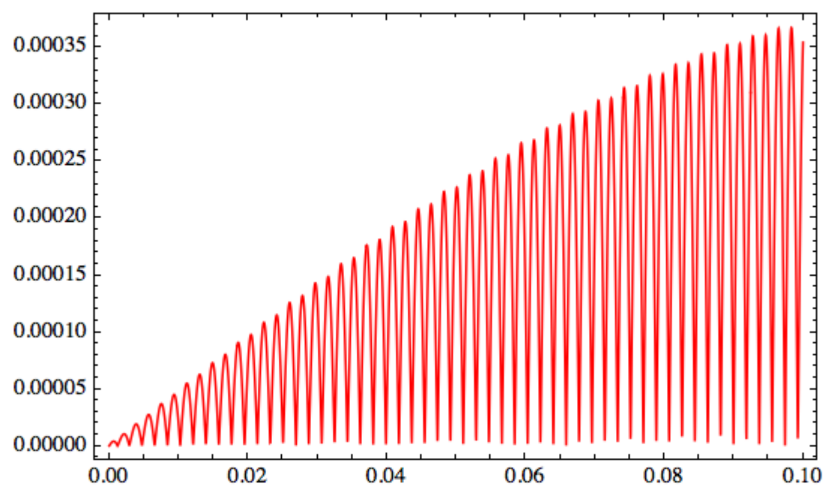
$$\sigma = \sqrt[4]{\frac{3}{\gamma}}\sqrt{\frac{\varepsilon_n}{k_p}} + \delta\sigma_o \cos\left(\sqrt{\frac{4}{3\gamma}}k_p z\right)$$

Energy spread (~ 1%) drives Emittance degradation

$\sigma(z)$



$\varepsilon(z)$



$$\varepsilon_{rms} = \sqrt{\sigma_x^2 \sigma_{x'}^2 - \sigma_{xx'}^2} = \sqrt{\left(\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2\right)} \approx \left| \sin\left(\sqrt{\frac{4}{3\gamma}} k_p z\right) \right|$$

PHYSICAL REVIEW SPECIAL TOPICS - ACCELERATORS AND BEAMS **15**, 111303 (2012)

Transverse emittance growth in staged laser-wakefield acceleration

T. Mehrling,[1] J. Grebenyuk,[2] F. S. Tsung,[3] K. Floettmann,[2] and J. Osterhoff[1,*]

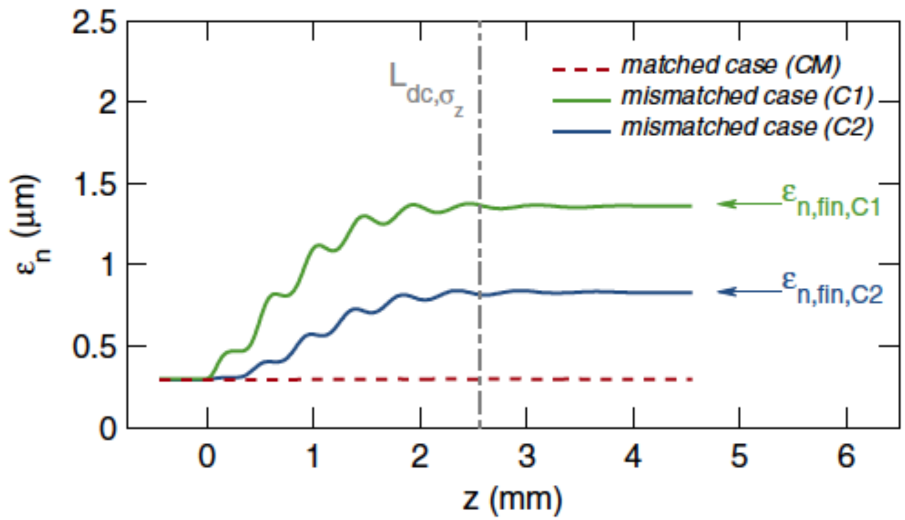


FIG. 3. Evolution of the normalized emittance ϵ_n in PIC simulations for the three considered cases. Arrows show the analytic predictions of the emittance growth. The betatron-decoherence length for the injection phase in the simulations $k_p\xi_0 = 1.00$ relative to position z_0 is indicated by the dash-dotted line.

$$\epsilon_{n,\mathrm{fin}} = \frac{\epsilon_{n,\mathrm{init}}}{2}\left(\frac{1+\alpha^2}{\beta^*} + \beta^*\right).$$

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- rms enevelope equation
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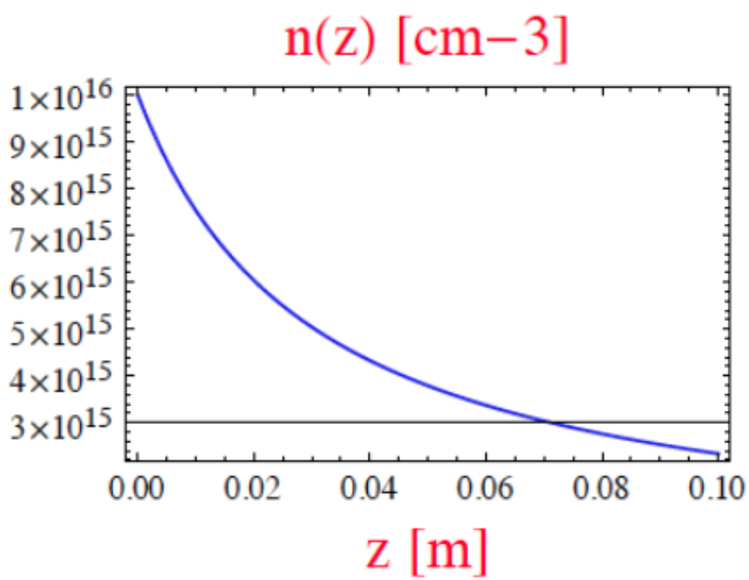
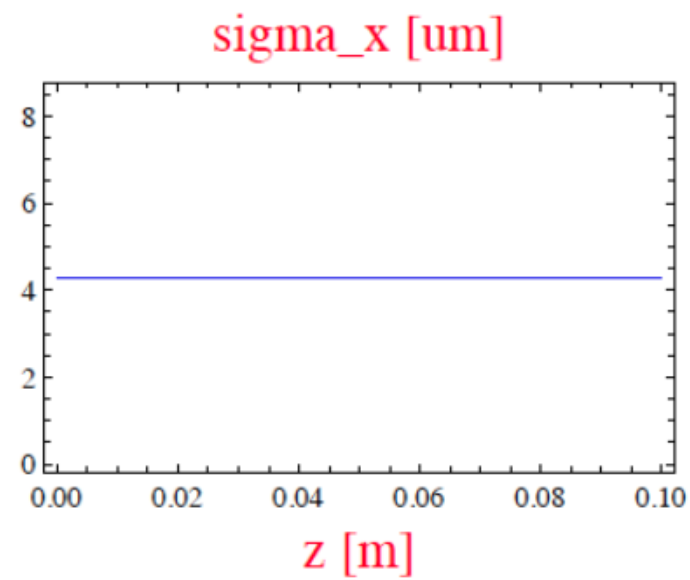
It is an interesting exercise to see the effect of a plasma density vanishing as $n(z) = \frac{\gamma_o}{\gamma(z)} n_o$, giving $k_p^2 = \frac{e^2 n_o}{\varepsilon_o m c^2}\frac{\gamma_o}{\gamma} = \frac{\gamma_o}{\gamma} k_{o,p}^2$. In this case the envelope equation

$$\sigma_x'' + \frac{\gamma'}{\gamma}\sigma_x' + \frac{\gamma_o k_{o,p}^2}{3\gamma^2}\sigma_x = \frac{\varepsilon_n^2}{\gamma^2 \sigma_x^3}$$

admits a constant equilibrium solution:

$$\sigma_x = \sqrt[4]{\frac{3}{\gamma_o}}\sqrt{\frac{\varepsilon_{n,}}{k_{o,p}}}$$





A laser-plasma lens for laser-wakefield accelerators

R. Lehe,* C. Thaury,† E. Guillaume, A. Lifschitz, and V. Malka

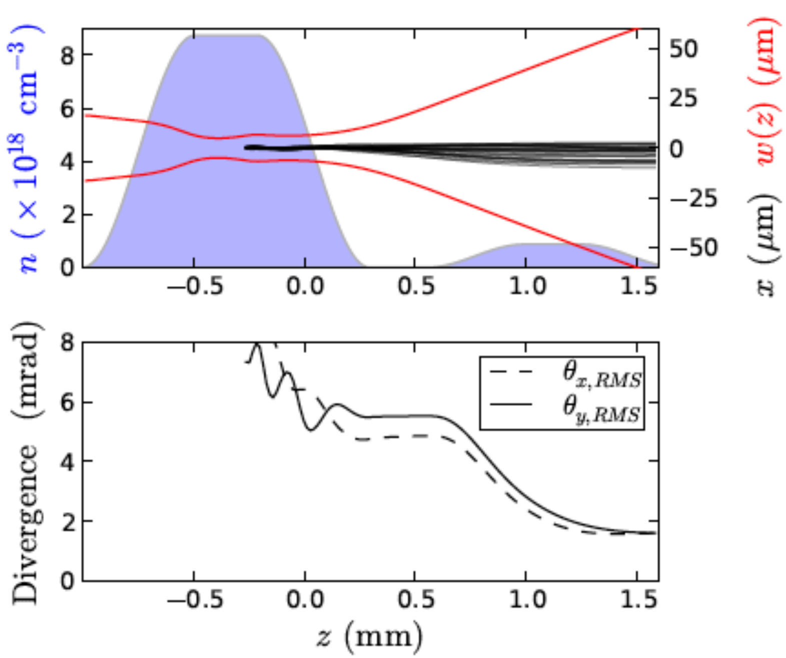


FIG. 2. Upper panel: density profile (blue), laser waist (red), and trajectories of a few injected electrons (black) in the PIC simulation. Lower panel: RMS divergence of the bunch in the x and y directions. (The laser is polarized along x.)

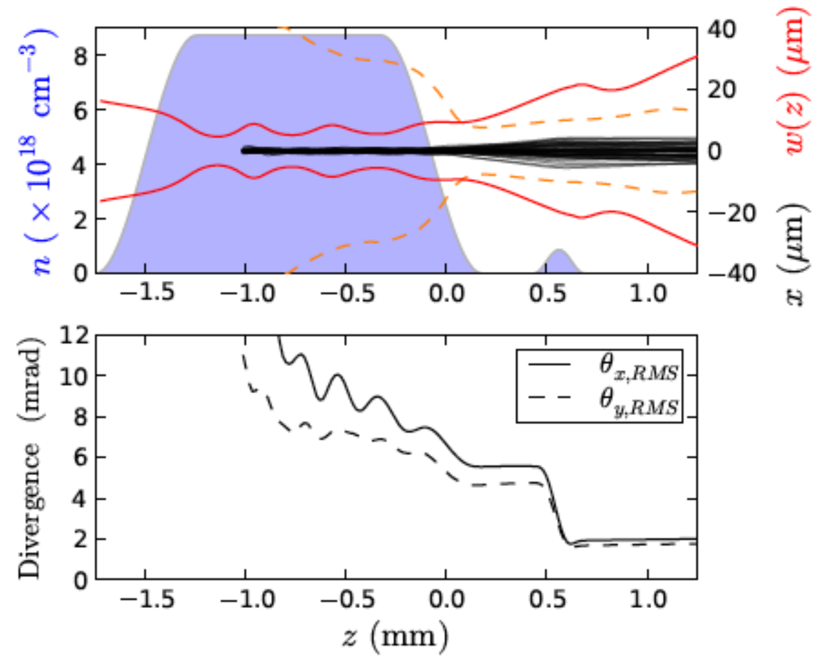
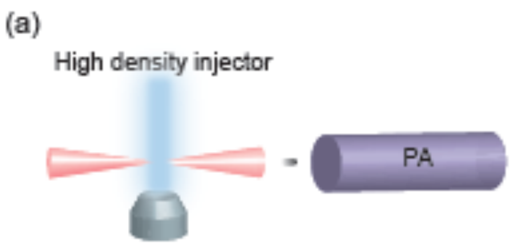


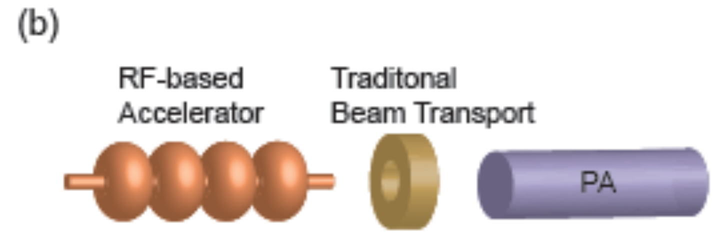
FIG. 4. Upper panel: density profile (blue), waists of the first (red, solid) and second (orange, dashed) laser pulse, and trajectories of a few injected electrons (black) in the PIC simulation. Lower panel: divergence of the bunch.

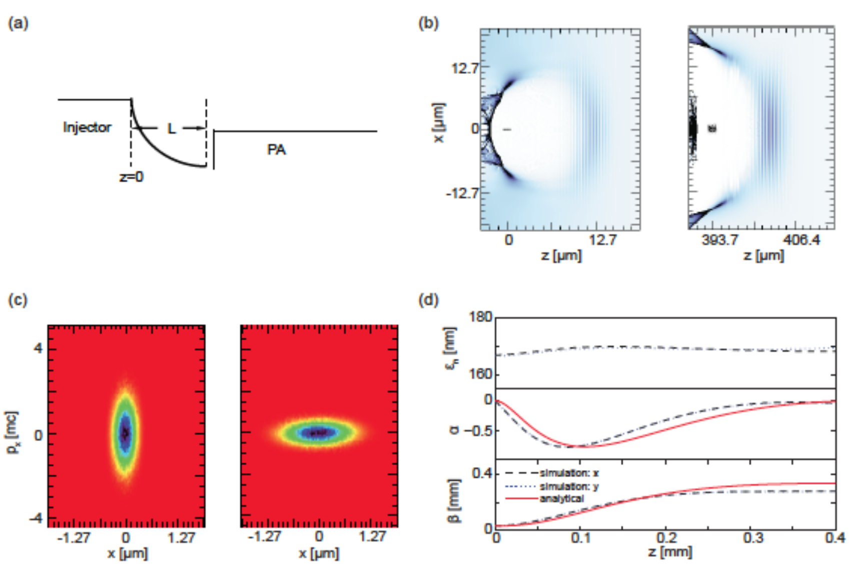
$$\frac{\langle k_{\rm foc}\rangle Z_R^2}{L_{\rm d}+L_2}\tan\left(\frac{\langle k_{\rm foc}\rangle Z_R^2}{L_{\rm d}}-\frac{\langle k_{\rm foc}\rangle Z_R^2}{L_{\rm d}+L_2}\right)=1,$$

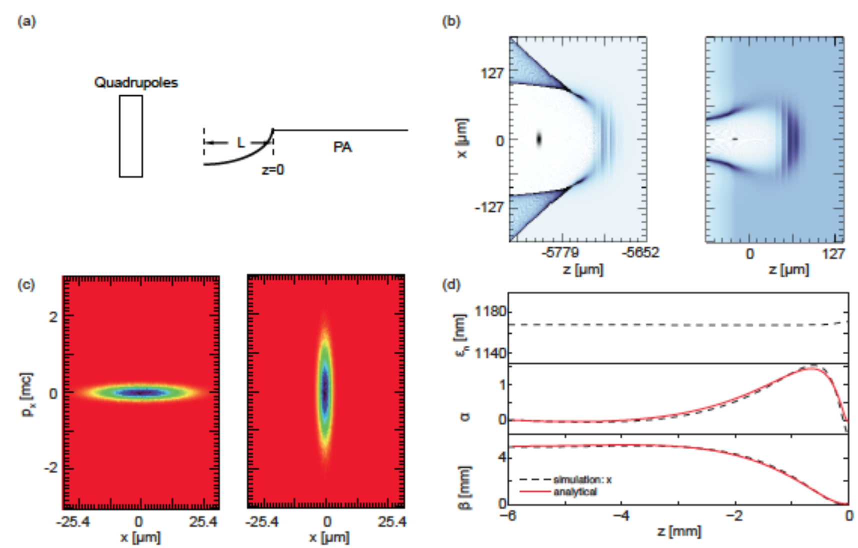
Coherent phase space matching for staging plasma and traditional accelerator using longitudinally tailored plasma structure

X. L. Xu,[1] Y. P. Wu,[1] C. J. Zhang,[1] F. Li,[1] Y. Wan,[1] J. F. Hua,[1] C.-H. Pai,[1] W. Lu,[1,*] P. Yu,[2] W. An,[2] W. B. Mori,[2] C. Joshi,[2] and M. J. Hogan[3]









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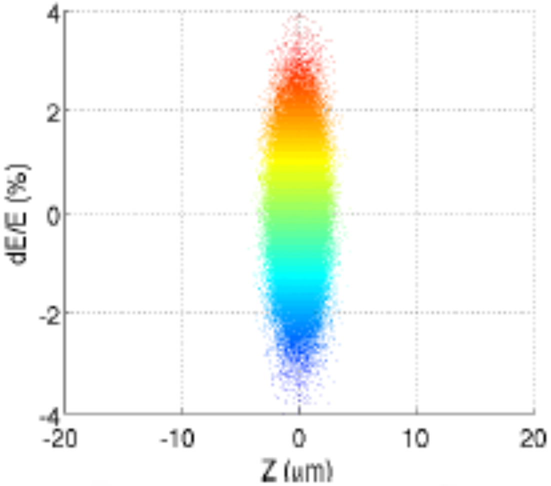
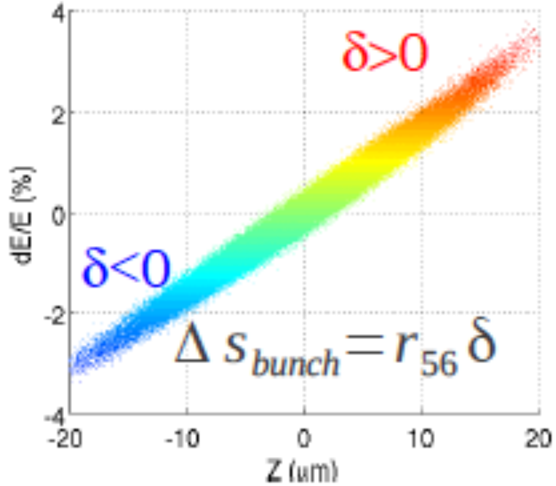
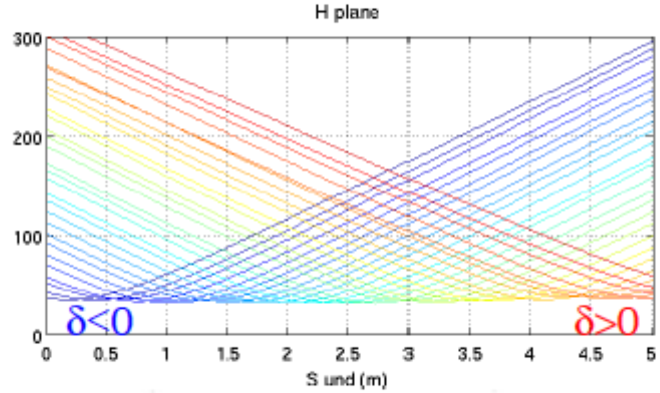


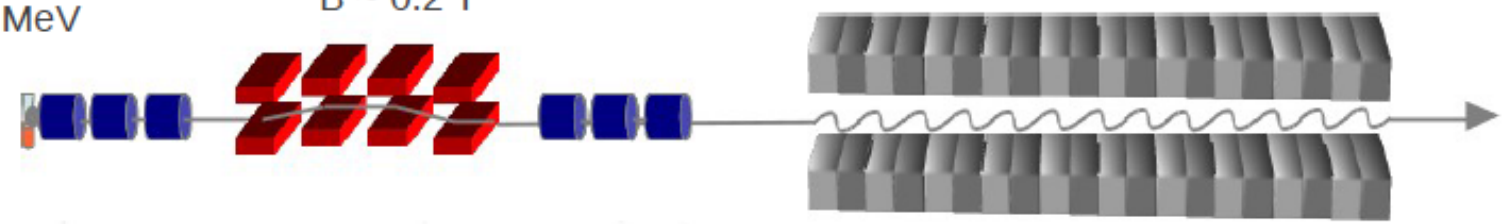




yo ready Undulator (PrFeB)
15 mm period
B ~ 1.5 T @ gap=3.6 mm

r_{56}~ 1 mm
B ~ 0.2 T

Source
400 MeV



First triplet
Re-focusing
G < 200 T/m
Bore = 100 mm length
10 mm radius

Second triplet
Chromatic matching
G < 40 T/m
(+ ~1.5 m)

~ 10 m