Multi-Particle Effects in Particle Accelerators (I)

A. Hofmann and G. Rumolo,

Thanks to: D. Brandt, W. Herr

G. Arduini, G. Bellodi, H. Burkhardt, R. Calaga, W. Fischer, E. Métral, G. Papotti, T. Pieloni, D. Quatraro, S. Redaelli, B. Salvant, E. Shaposhnikova, R. Tomás

Overview on the course

3 hours to explain all the basics of multi-particle effects and reasons why they are object of study

- ½ hour (GR)

  → Intuitive introduction to different types of effects involving many particles, what multi-particle effects lead to

  → Glossary, classifications

- 1 ½ hours (AH)

  → Space charge

  → Impedance: longitudinal effects, transverse effects

- 1 hour (GR)

  → Numerical modeling of multi-particle effects

  → Examples of experimental observations

General definition of *multi-particle processes* in an accelerator or storage ring

Class of phenomena in which the evolution of the particle beam cannot be studied as if the beam was a single particle (as is done in beam optics), but depends on the combination of external fields and interaction between particles. Particles can interact between them through

- Self generated fields:

  $\rightarrow$ Direct space charge fields

  $\rightarrow$ Electromagnetic interaction of the beam with the surrounding environment through the beam's own images and the wake fields (impedances)

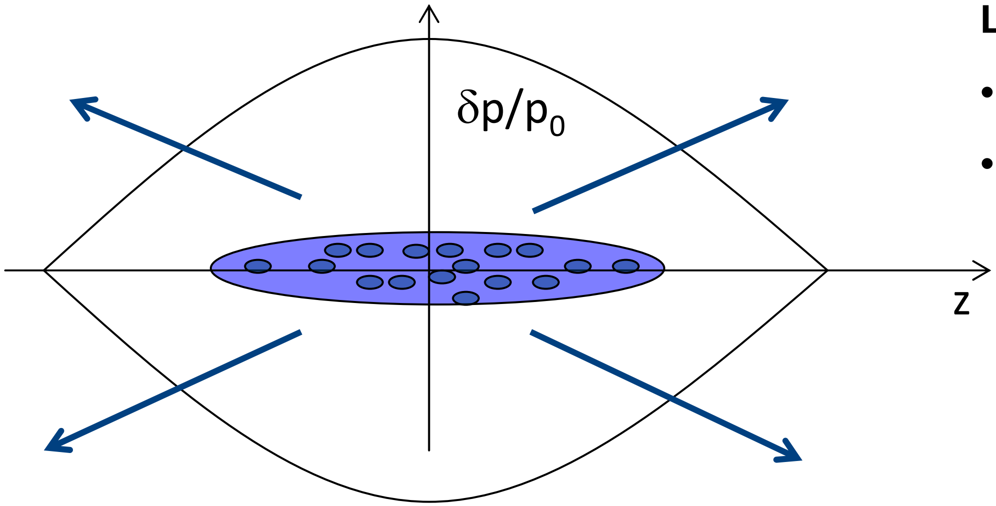
  $\rightarrow$ Interaction with the beam's own synchrotron radiation

- Long- and short-range Coulomb collisions, associated to intra-beam scattering and Touschek effect, respectively

- Interaction of electron beams with trapped ions, proton/positron/ion beams with electron clouds, beam-beam in a collider ring, electron cooling for ions

Multi-particle processes are detrimental for the beam (degradation and loss, see next slides)

Direct space charge forces



Longitudinal space charge

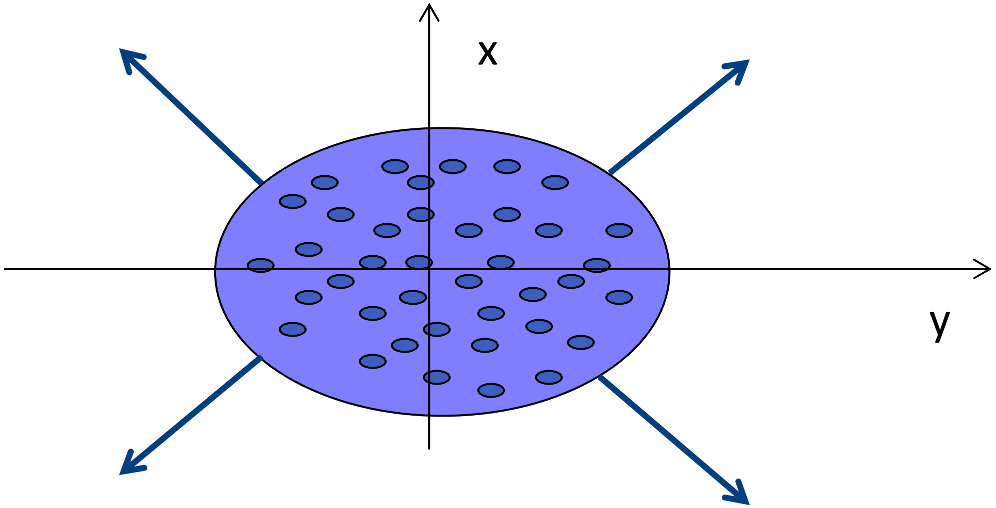
•Force decays like $1/\gamma^2$

•it can be attactive above transition

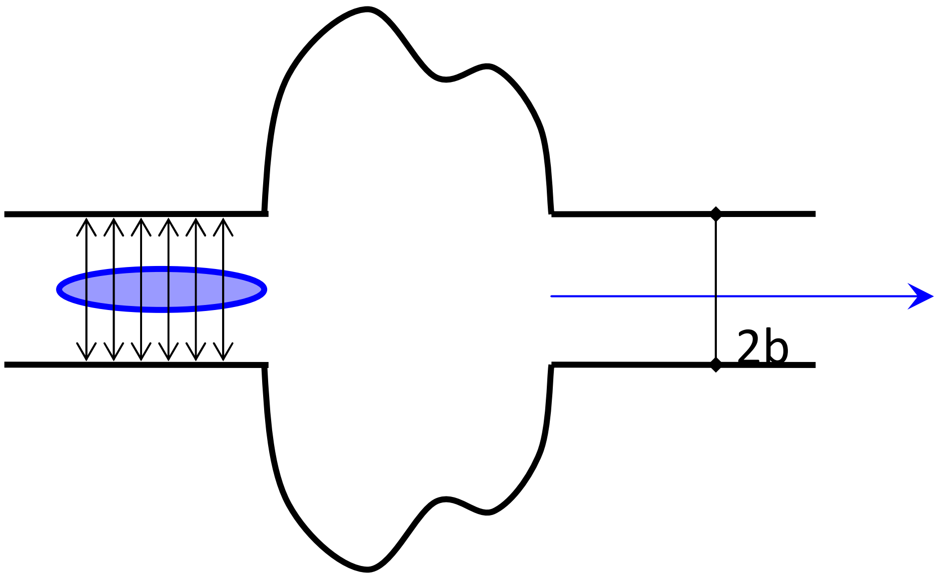
Transverse space charge

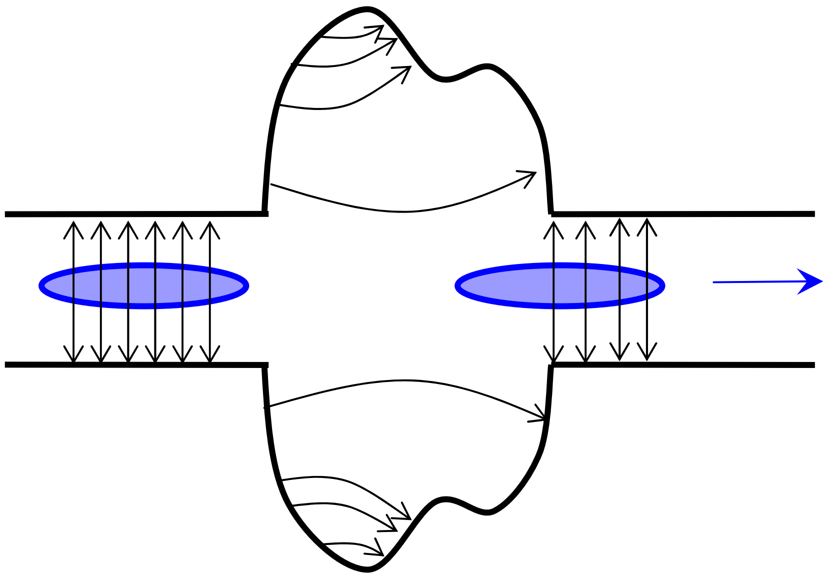
•Force decays like $1/\gamma^2\beta$

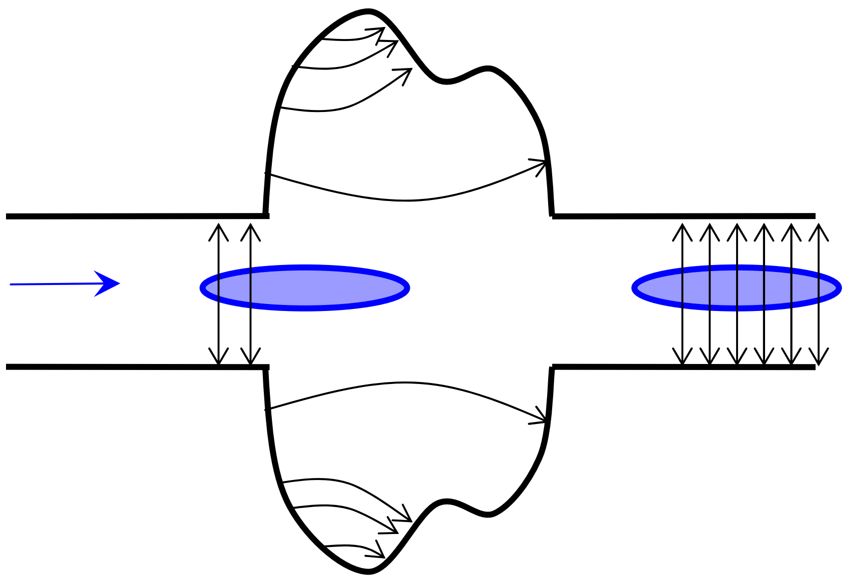
•It is always repulsive



Wake fields (impedances)





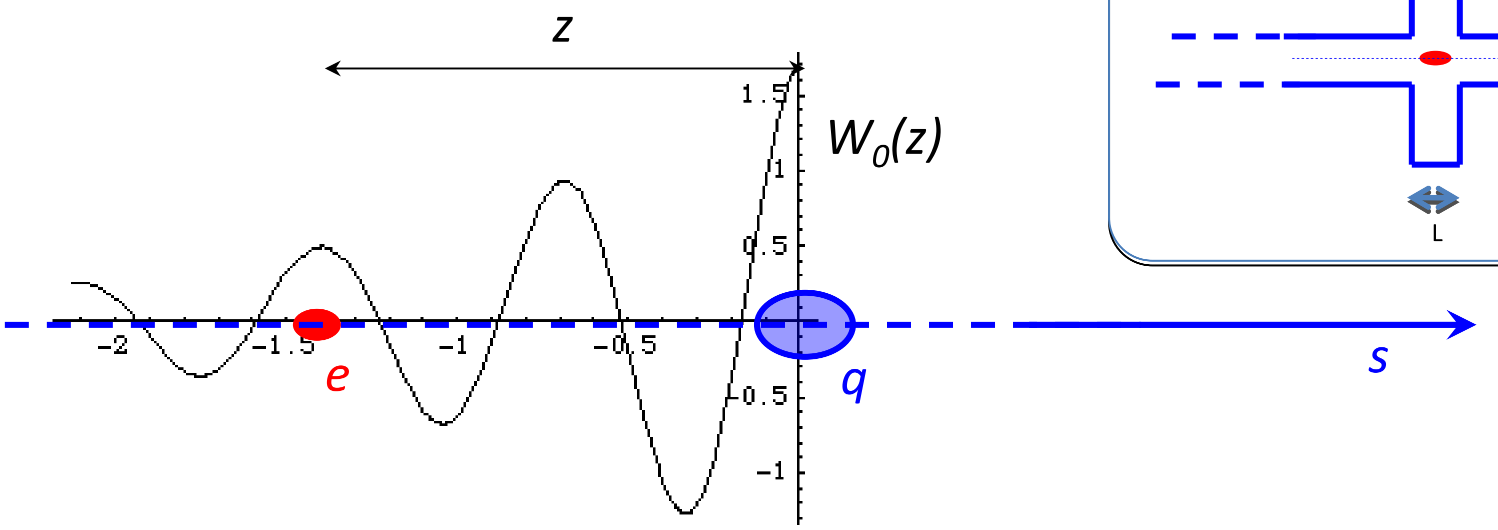


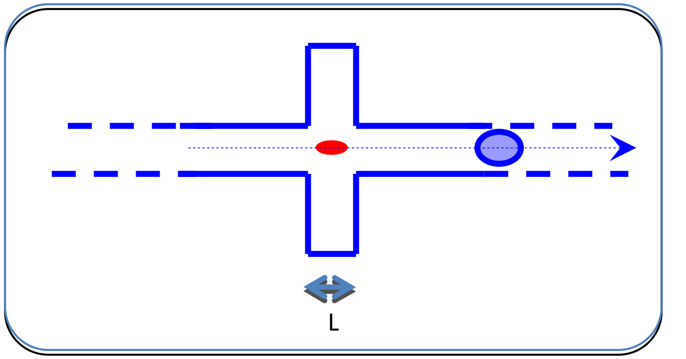
When the beam goes through a discontinuity, it induces an e-m field which keeps ringing after the beam has passed:

→ Energy loss

→ Effect intra-bunch and on following bunches

Wake fields (impedances)





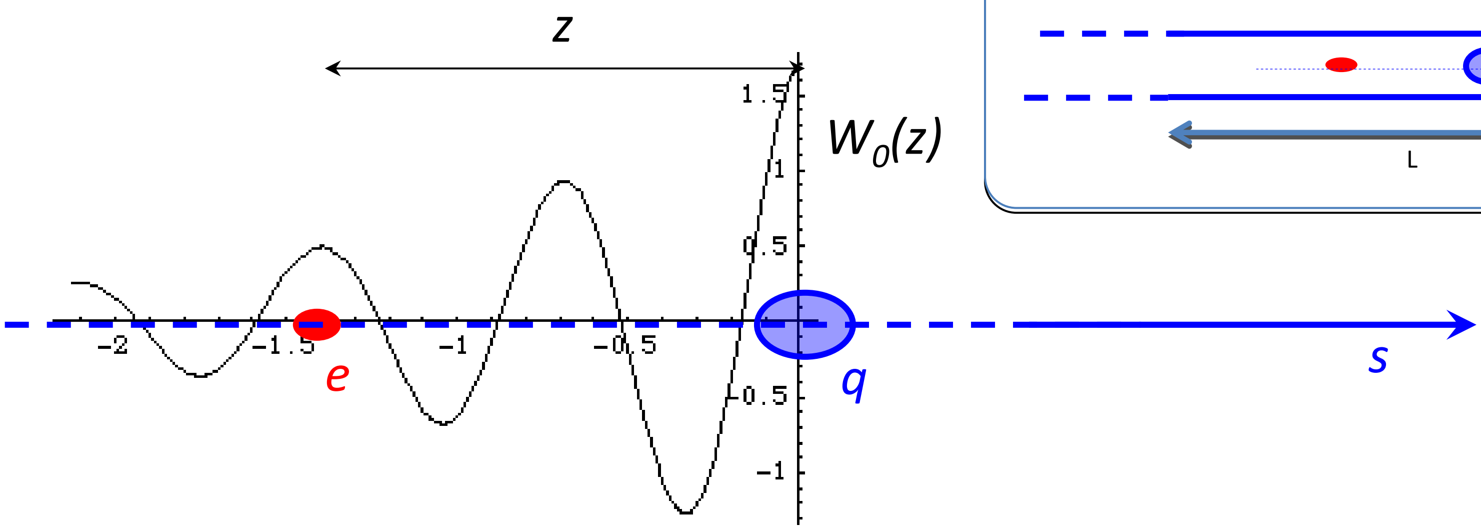
Model:

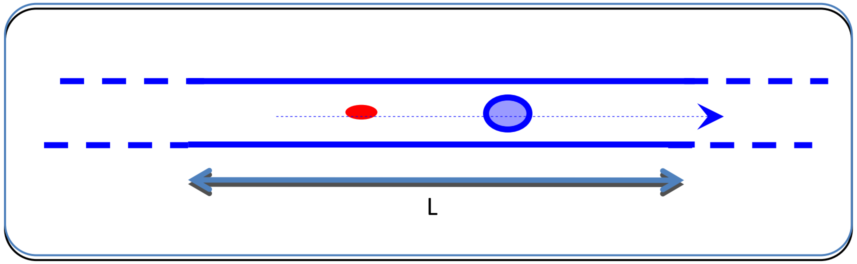
A particle ***q*** going through a device of length L, ***s (0,L),*** leaves behind an oscillating field and a probe charge ***e*** at distance ***z*** feels a net force as a result. The integral of this force over the device defines the ***wake field*** and its Fourier transform is called the ***impedance of the device of length L***.

$$\int_0^L F_{||}(s,z)ds = -eqW_{||}(z) \qquad \int_0^L F_{\perp}(s,z)ds = -eqxW_{\perp}(z)$$

$$Z_{||\,(\perp)}(\omega) \equiv \frac{1}{c}\int_{-\infty}^{\infty} dz\, e^{-i\omega z/c} W_{||\,(\perp)}$$

Wake fields (impedances)





Model (cont‘s):

The device of length L can also be a segment of accelerator (defined by the simple beam pipe) and the wake is generated by the finite conductivity of the pipe material. In this case the wake field and the impedance are said to be of **resistive wall** type and the integration can be done over L=C

$$\int_0^L F_{||}(s,z)ds = -eqW_{||}(z) \qquad \int_0^L F_{\perp}(s,z)ds = -eqxW_{\perp}(z)$$

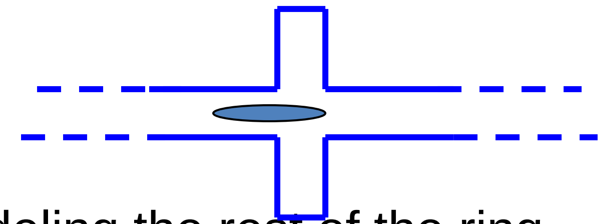
$$Z_{||\,(\perp)}(\omega) \equiv \frac{1}{c}\int_{-\infty}^{\infty} dz\, e^{-i\omega z/c} W_{||\,(\perp)}$$

Wake fields (impedances)

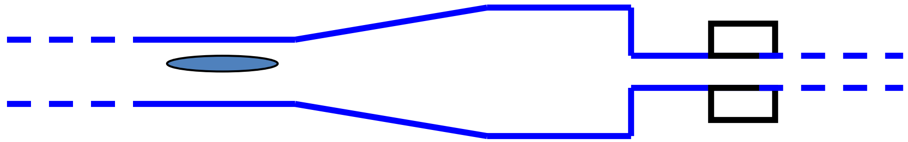
The full ring is usually modeled with a so called total impedance made of three main components:

- Resistive wall impedance

- Several narrow-band resonators at lower frequencies than the pipe cutoff frequency *c/b*



- One broad band resonator at $\omega_r \sim c/b$ modeling the rest of the ring (pipe discontinuities, tapers, other non-resonant structures like pick-ups, kickers bellows, etc.)



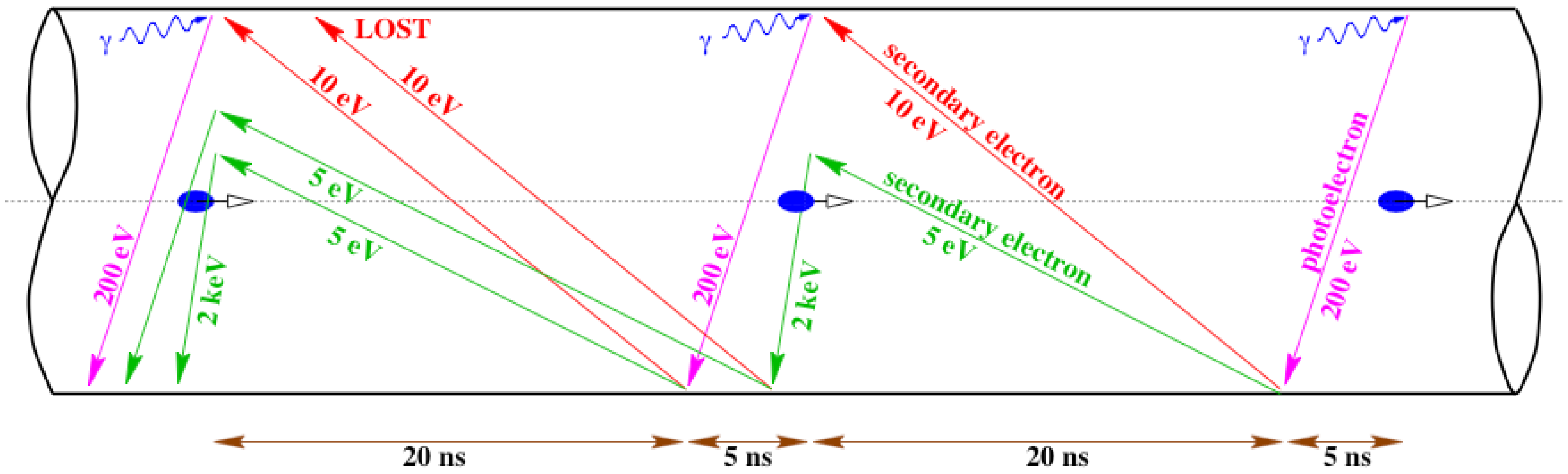
$\Rightarrow$ The total impedance is **allocated to the single ring elements** by means of off-line calculation prior to construction/installation

$\Rightarrow$ **Total impedance** designed such that the nominal intensity is stable

Electron cloud

Principle of electron multipacting:

Example of LHC

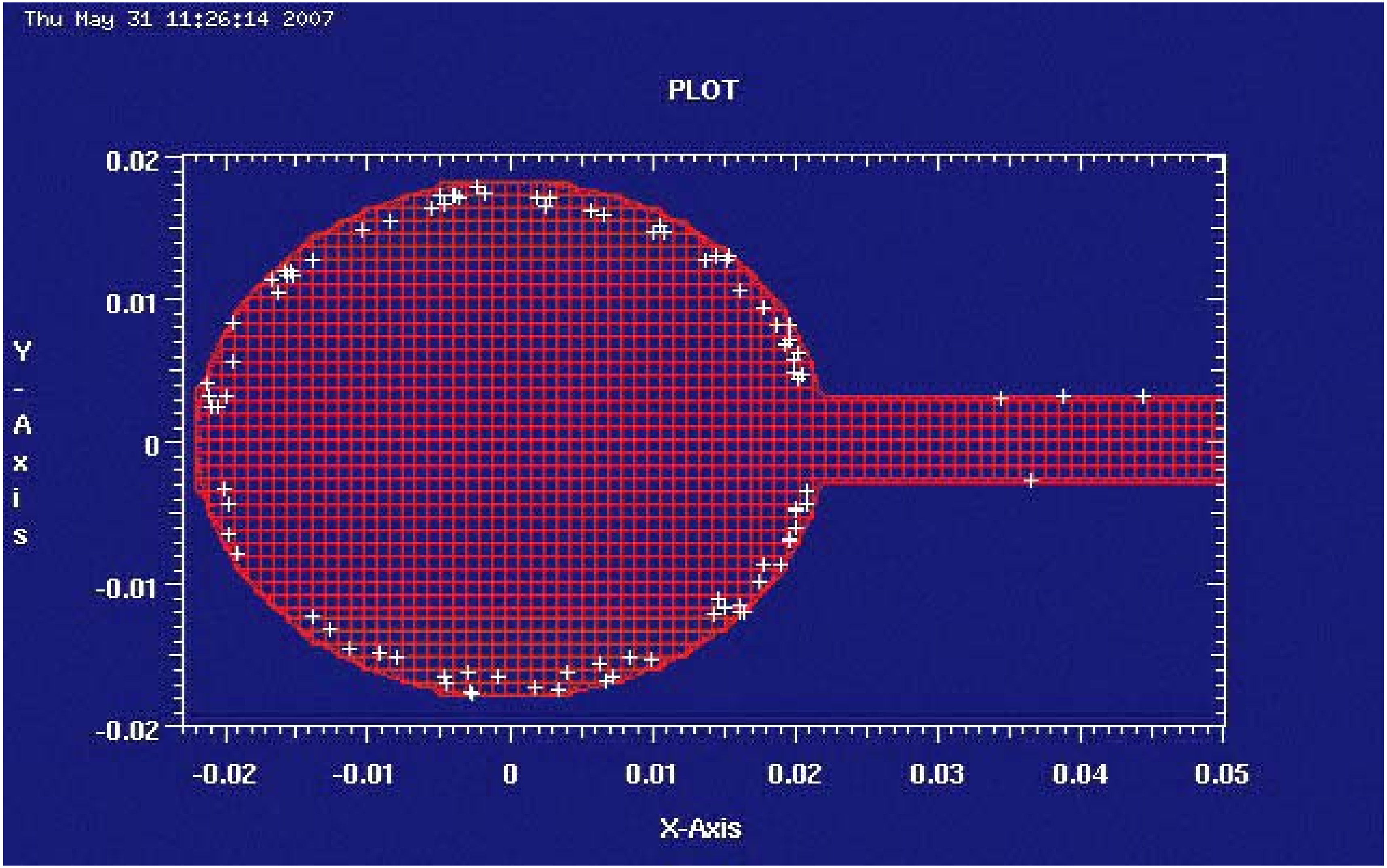


Electron multiplication is made possible by:

- ✓ Electron generation due to photoemission, but also residual gas ionization
- ✓ Electron acceleration in the field of the passing bunches
- ✓ Secondary emission with efficiency larger than one, when the electrons hit the inner pipe walls with high enough energy

Electron cloud

Photoelectrons produced by the synchrotron radiation are accelerated by the bunches and quickly accumulate in the vacuum chamber (example, wiggler of a CLIC damping ring)



Several names to describe these effects...

‚Multi-particle‘ is the most generic attribute. ***‚High-current‘, ‚high-intensity‘, ‚high brightness‘*** are also used because these effects are important when the beam has a high density in phase space (many particles in little volume)

Other labels are also used to refer to different subclasses

- **Collective** effects (**coherent)**:

  $\rightarrow$ The beam resonantly responds to a self-induced electromagnetic excitation

  $\rightarrow$ Are **fast** and visible in the beam **centroid motion** (tune shift, instability)

- **Collective** effects **(incoherent)**:

  $\rightarrow$ Excitation moves with the beam, spreads the frequencies of particle motion.

  $\rightarrow$ Lead to particle diffusion in phase space and slow emittance growth

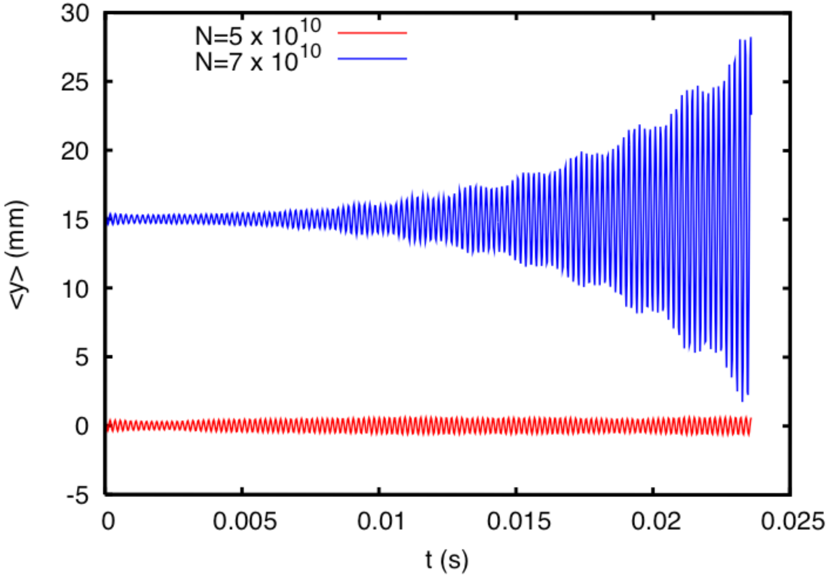
- **Collisional** effects **(incoherent)**:

  $\rightarrow$ Isolated two-particle encounters have a global effect on the beam dynamics (diffusion and emittance growth, lifetime)

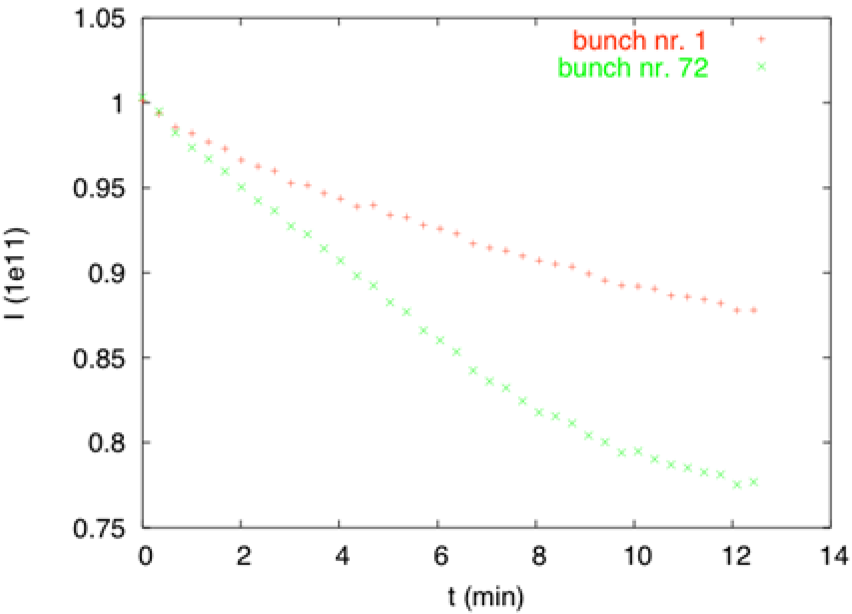
- **Two-stream** phenomena:

  $\rightarrow$ Two component plasmas needed (beam-beam, pbeam-ecloud, ebeam-ions) and the beam reacts to an excitation caused by another „beam“

Some examples….



Coherent effect:
When the bunch current exceeds a certain limit (current threshold), the centroid of the beam, e.g. as seen by a BPM, exhibits an exponential growth and the beam is lost within few milliseconds (simulation of an SPS bunch)

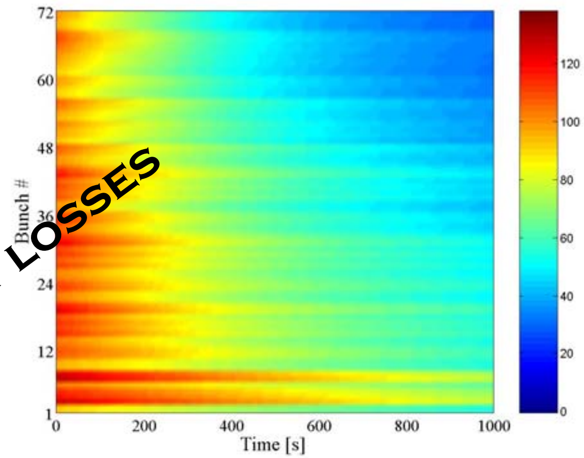


Incoherent effect:
Slow beam loss. First and last bunch of an SPS train of 72 bunches gradually lose their particles (10% and 25%, respectively) over several minutes (data from SPS, 2004)

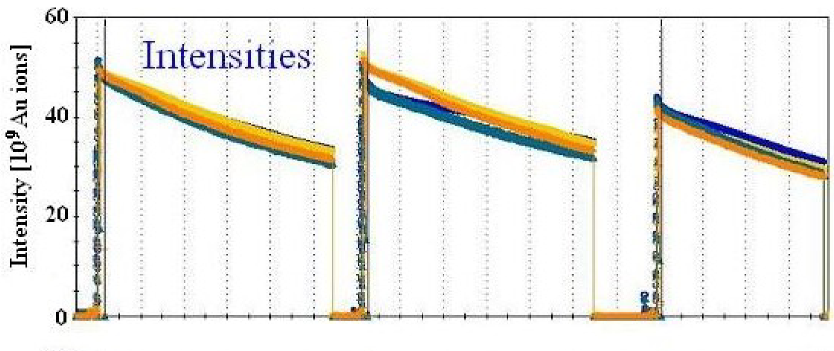
More examples… (plots show measured beam intensity vs time)

CERN-SPS (I) Bunch by bunch life times with LHC beam (2004)

SLOW LOSSES



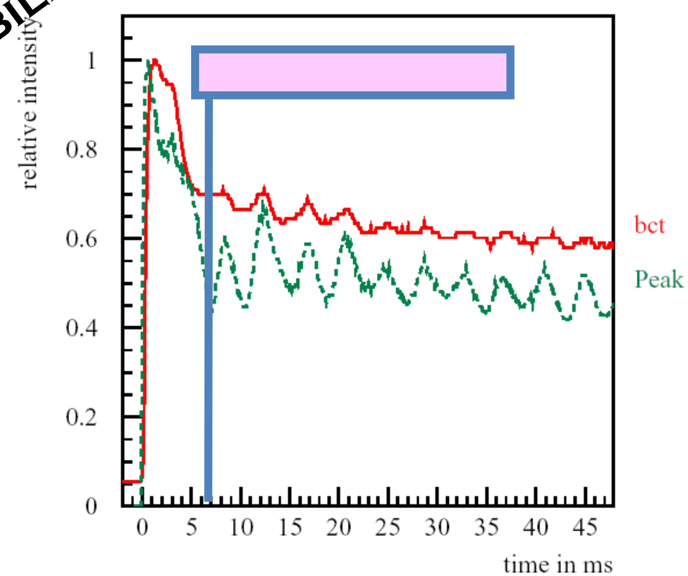
BNL-RHIC, Au-Au operation, Run-4 (2004)



16h

FAST LOSSES: INSTABILITY

CERN-SPS (II) TMC Instability (2003)



Note the different time scale on these plots!

Why are multi-particle effects important ? (I)

The performances of accelerators and storage rings are usually limited by multi-particle effects!!

➔ When the the beam current in a machine is pushed above a certain limit (intensity threshold), losses or beam quality degradation appear due to these phenomena and they can become intolerable!

➔ These effects can affect the beam in several possible ways

- Collective coherent effects can be:
  - $\rightarrow$ Longitudinal or transverse
  - $\rightarrow$ Single- or multi-bunch
- Incoherent phenomena can be:
  - $\rightarrow$ Longitudinal or transverse
  - $\rightarrow$ Single bunch
- Two-stream phenomena are:
  - $\rightarrow$ Mainly transverse
  - $\rightarrow$ Multi-bunch (or single-bunch mechanism, but relying on multi-bunch build up)

Why are multi-particle effects important ? (II)

Multi-particle effects usually determine, together with the machine aperture, the intensity limitations of an accelerator!!

➔ How each of these effects can affect the beam:

- Coherent effects:

  $\rightarrow$ Transverse (single- or multi-bunch) result in fast beam loss

  $\rightarrow$ Longitudinal (single- or multi-bunch) may cause beam loss, but are more usually responsible for energy loss and beam degradation in the longitudinal phase space

- Incoherent effects:

  $\rightarrow$ Transverse cause diffusion and slow emittance growth

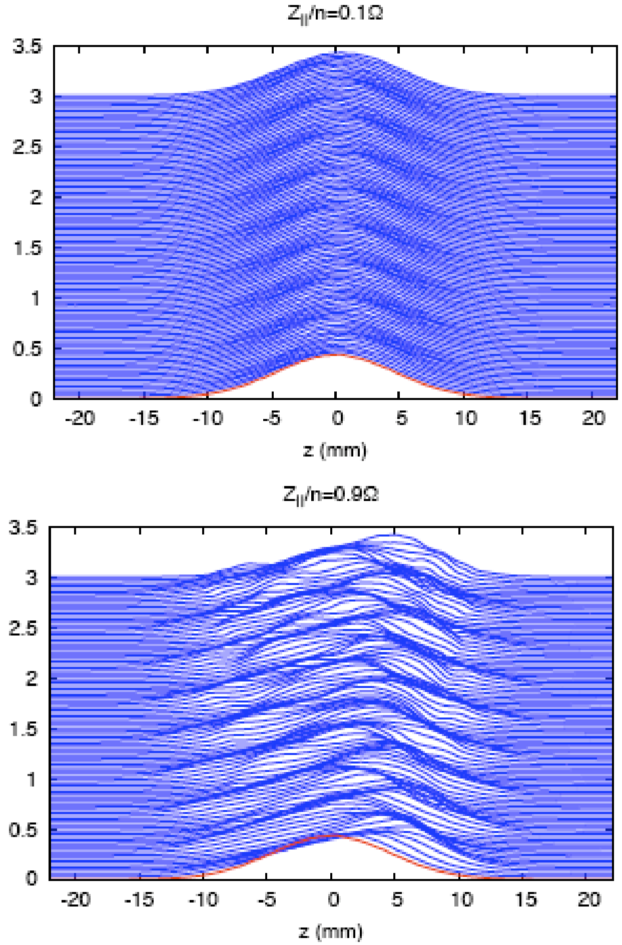
  $\rightarrow$ Longitudinal cause degradation of the beam lifetime

- Two-stream phenomena:

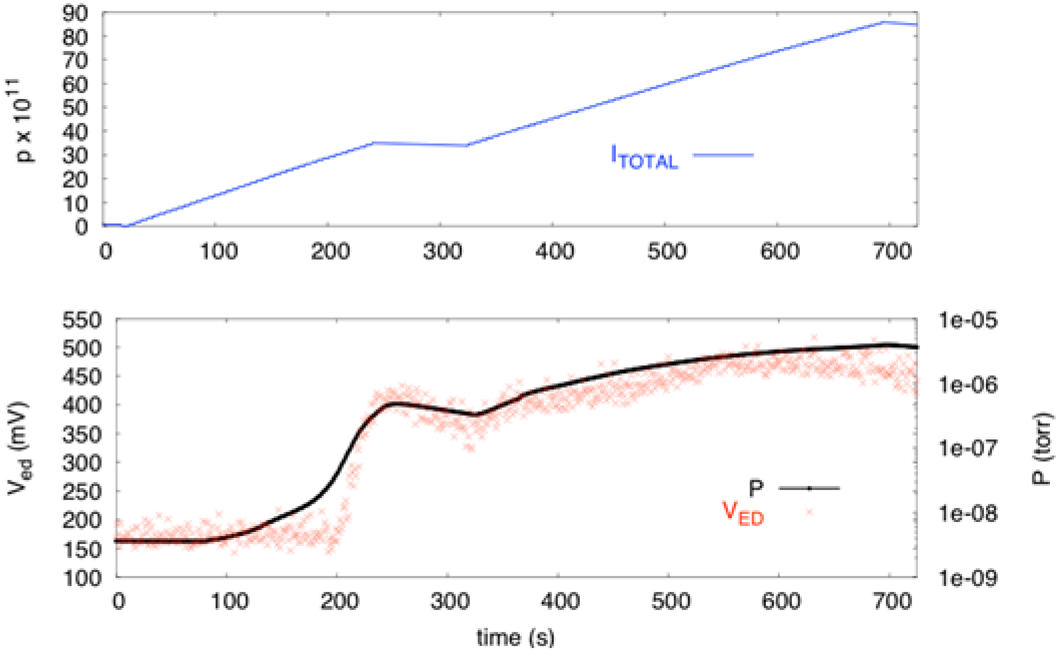
  $\rightarrow$ Can result into transverse coherent or incoherent effects (see above)

  $\rightarrow$ Electron clouds or trapped ions can also be not dense enough to significantly affect the beam evolution, but they are still undesired for other reasons (pressure rise, interference with instrumentation, additional heat load)

Some examples….



Longitudinal single-bunch effect:
When the impedance is above a certain value, the bunch shape can be strongly affected (simulation of a bunch from the ALBA storage ring, 2005)



Pressure rise:
When filling RHIC, after a certain value of current there is a pressure rise due to electron cloud (data from RHIC, 2004)

Why are multi-particle effects important ? (III)

The ultimate performance of an accelerator or storage ring is usually defined by a specific multi-particle effect!!

➔ There exist other signatures of a multi-particle effect acting on the beam (observables often used to quantify space charge, impedance, ecloud/ion densities)

- Collective coherent effects:

  → Transverse => Coherent tune shift (usually tune reduction)

  → Longitudinal => Bunch lengthening, synchronous phase shift, coherent synchrotron tune shift

- Incoherent phenomena:

  → Transverse => Incoherent tune spread

  → Longitudinal => Incoherent synchrotron tune spread

- Two-stream phenomena:

  → Coherent tune shift increasing along a train of bunches

  → Bunch shortening due to loss of high synchrotron amplitude particles.

Types of collective effects (I)

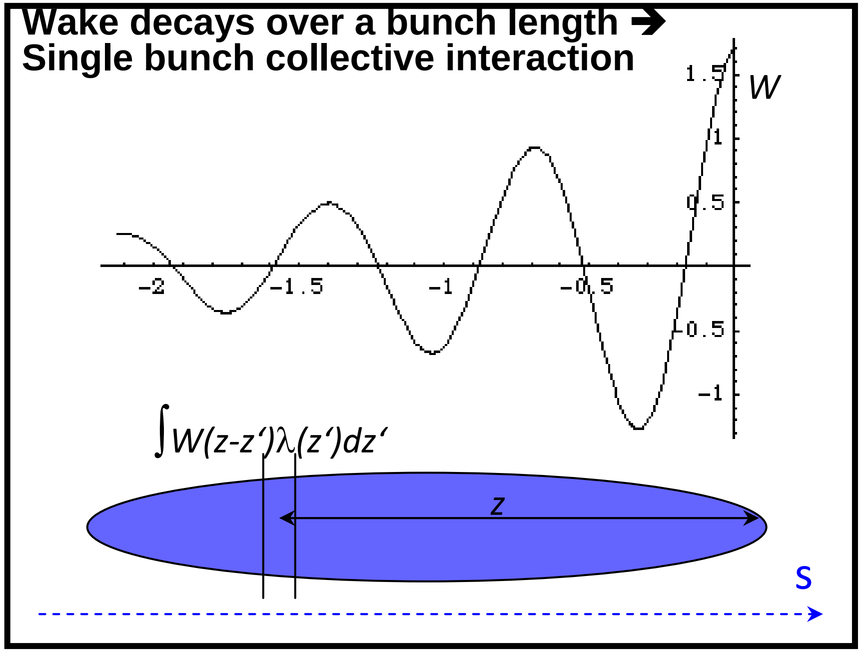
- Transverse:
  - → Single bunch
    - ✓ Head-tail instability
    - ✓ Transverse Mode Coupling Instability (TMCI), also referred to as beam break-up or strong (fast) head-tail instability
    - ✓ Electron Cloud Instability (ECI)
  - → Coupled bunch instability
    - ✓ Coupling between subsequent bunches through wake fields or electron cloud, Ion Instability, Fast-Ion Instability
- Longitudinal:
  - → Single bunch
    - ✓ Potential well distortion, energy loss
    - ✓ Microwave instability, also referred to as turbulent bunch lengthening
  - → Coupled bunch instability

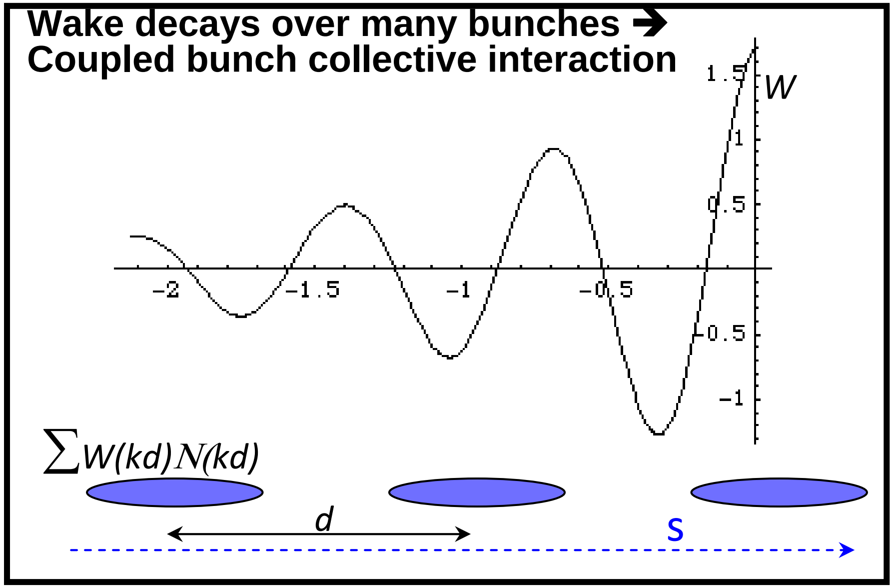
Types of collective effects (II)

- Single or multi-bunch behavior depends on the range of action of the wake fields

  → Single bunch effects are usually caused by short range wake fields (broad-band impedances)

  → Multi bunch effects are usually associated to long range wake fields (narrow-band impedances)





Difference between multi-particle effects and single-particle resonances

- Both **coherent instabilities** and **resonances** can make the amplitudes of some particles grow in phase space till they are lost hitting the machine aperture

- Both **incoherent effects** and **resonances** can cause some beam particles to migrate to the large amplitudes and produce emittance growth

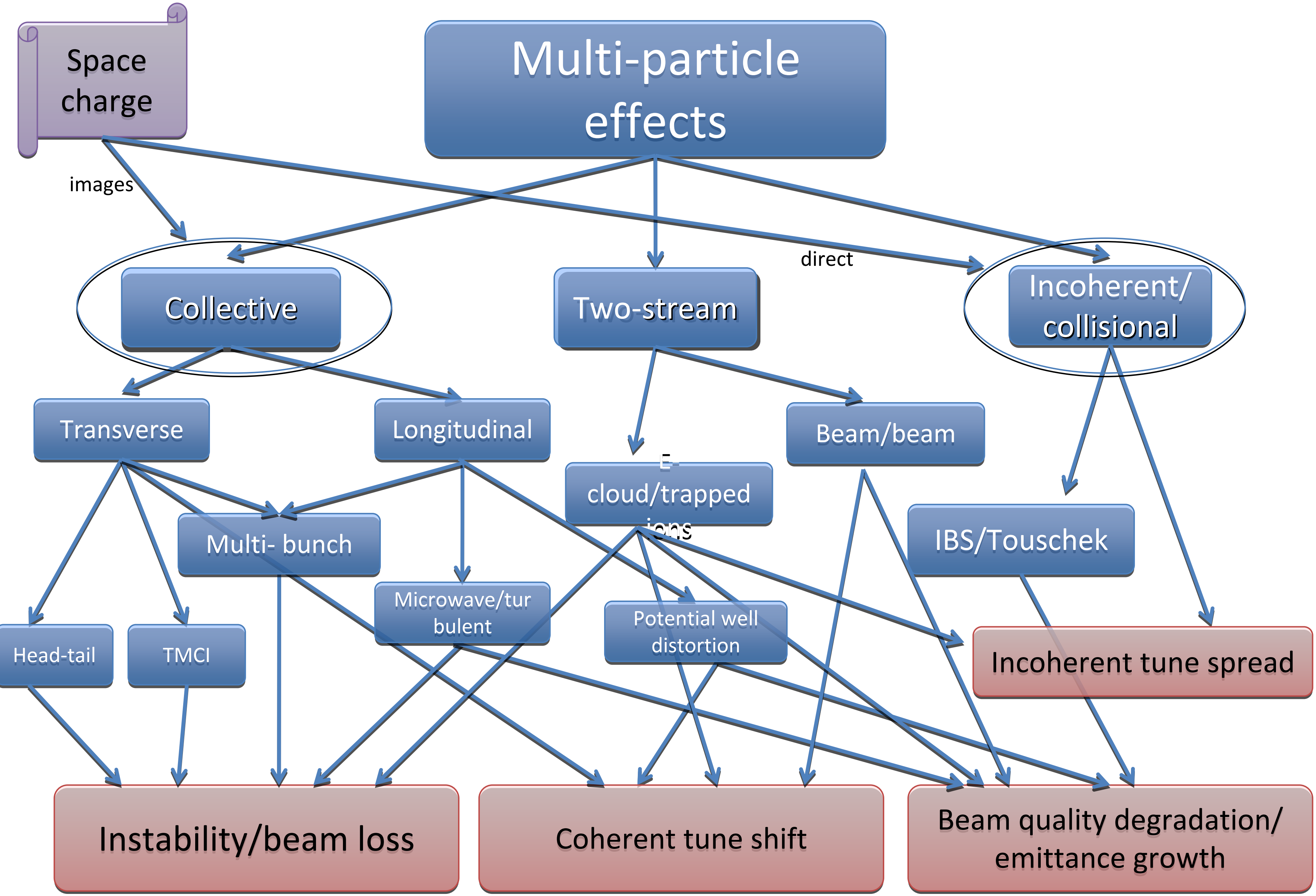
- BUT:

  $\rightarrow$ Resonances only depend on the distribution of linear/nonlinear errors in the lattice of a machine and are not the response of a beam to a self-excitation

  $\rightarrow$ The unstable motion due to a resonance is not intensity dependent and occurs whenever the working point of a machine is badly placed with respect to the resonance lines excited by the distribution of the lattice errors.

  $\rightarrow$ The theory of linear/nonlinear resonances is based on the single particle dynamics, whereas multi-particle effects can be only described with a kinetic/macroparticle approach (Vlasov equation, two- or more-particle models)

  $\rightarrow$ However, interplays between resonances and multi-particle effects are possible due to the intensity dependent tune shift/spread, which can lead to resonance crossing (static, dynamic or periodic)


Space charge
Multi-particle effects
images
direct
Collective
Two-stream
Incoherent/ collisional
Transverse
Longitudinal
Beam/beam
E-cloud/trapped ions
IBS/Touschek
Multi- bunch
Microwave/turbulent
Potential well distortion
Head-tail
TMCI
Incoherent tune spread
Instability/beam loss
Coherent tune shift
Beam quality degradation/ emittance growth

1) SPACE-CHARGE

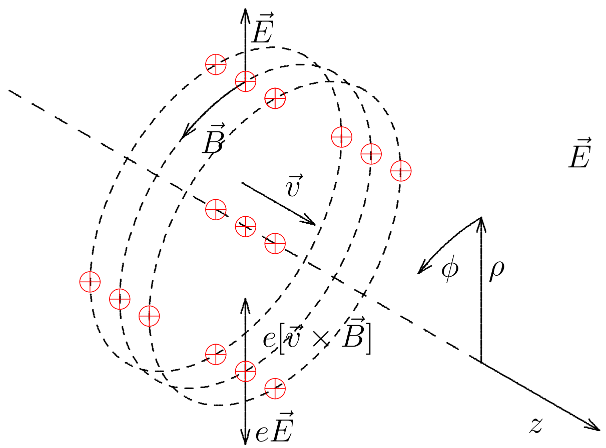
Introduction

The many charged particles in a high intensity beam represent a space-charge and produce electromagnetic self-fields which affect the beam dynamics being otherwise determined by the guide fields of the magnetic lattice and RF-system. Assuming weak self-fields we treat their effects as a perturbation and concentrate on the transverse case where this shifts the betatron frequencies (tunes).

For the **direct space charge effect** the conducting vacuum chamber is neglected, E and B-fields are obtained directly. The first is repelling and defocuses while the Lorentz force of the second focuses. The balance between them becomes more perfect as the particle velocity v approaches c.

Conducting boundaries modify the field giving an **indirect space-charge effect** which is calculated with image charges. Here the balance between E and B-effects is perturbed and is important also for $v \to c$

For a rigid, **coherent**, oscillation of the beam as a whole, the direct-space charge represents an internal force which does not influence this motion, however the indirect wall effect does.



Direct space-charge effect

Fields and forces

Continuous (unbunched) beam of circular cross section, radius a, uniform charge/current densities η, $\vec{J} = \eta\beta c$ with total charge per unit length $\lambda = \pi a^2 \eta$ and current $I = \beta c \lambda$, produces cylindrically symmetric fields $\vec{E} = [E_\rho,\, 0,\, 0]$ and $\vec{B} = [0,\, E_\phi,\, 0]$ at radial distance ρ:

$$\text{div}\,\vec{E} = \eta/\epsilon_0 \qquad \iiint \text{div}\,\vec{E}\,\mathrm{d}V = \iint \vec{E}\mathrm{d}\vec{S}_E$$

$$\text{curl}\,\vec{B} = \mu_0 \vec{J} \qquad \oint \vec{B}\cdot\mathrm{d}\vec{s} = \iint \text{curl}\,\vec{B}\cdot\mathrm{d}\vec{S}_J$$

$\mathrm{d}\vec{S}_E = 2\pi\rho[\mathrm{d}\rho,\, 0,\, 0]$, $\mathrm{d}\vec{S}_J = 2\pi\rho\mathrm{d}\rho[0,\, 0,\, 1]$,
$\mathrm{d}\vec{s} = \rho[0,\, \mathrm{d}\phi,\, 0]$, $\mathrm{d}V = 2\pi\rho\mathrm{d}s\mathrm{d}\rho$. Integrate $\int_0^\rho \eta(\rho')\mathrm{d}\rho'$

in $\rho \le a$:

$$2\pi\rho\ell E_\rho = \pi a^2 \ell\eta/\epsilon_0 \qquad E_\rho = \frac{\eta\rho}{2\epsilon_0} = \frac{\lambda}{2\pi\epsilon_0}\frac{\rho}{a^2}$$

$$2\pi\rho B_\phi = \pi a^2 \mu_0 J_s \qquad B_\phi = \frac{\beta\eta\rho}{2\epsilon_0 c} = \frac{\mu_0 I}{2\pi}\frac{\rho}{a^2}$$

out $\rho \ge a$:

$$2\pi\rho\ell E_\rho = \pi\rho^2 \ell\eta/\epsilon_0 \qquad E_\rho = \frac{\eta a^2}{2\epsilon_0\rho} = \frac{\lambda}{2\pi\epsilon_0}\frac{1}{\rho}$$

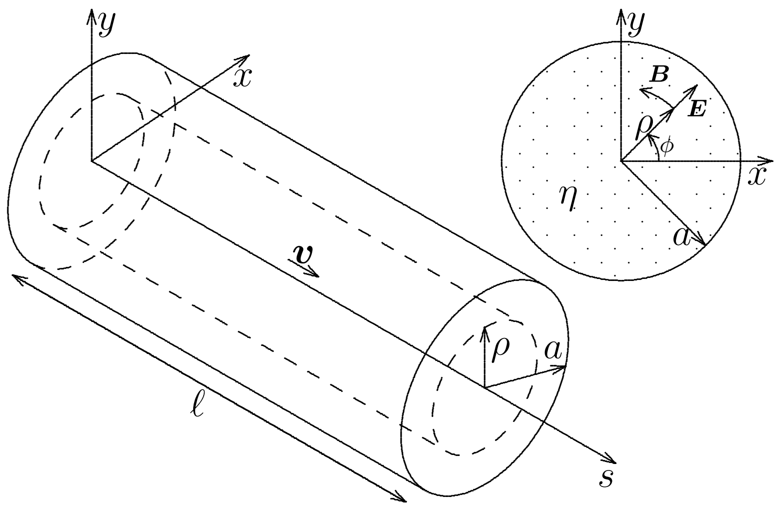
$$2\pi\rho B_\phi = \pi\rho^2 \mu_0 J_s \qquad B_\phi = \frac{\beta\eta a^2}{2\epsilon_0 c\rho} = \frac{\mu_0 I}{2\pi}\frac{1}{\rho}$$

.

Fields inside beam, $\rho \le a$, relevant for direct space-charge, only charges at $\rho' \le \rho$ contribute. Force on a particle

$$\vec{F} = F_E + F_B = e\left(\vec{E} + [\vec{v}\times\vec{B}]\right) = \frac{e\eta}{2\epsilon_0}(1-\beta^2)\vec{\rho} = \frac{eI}{2\pi\epsilon_0 c\beta\gamma^2}\frac{\vec{\rho}}{a^2}.$$

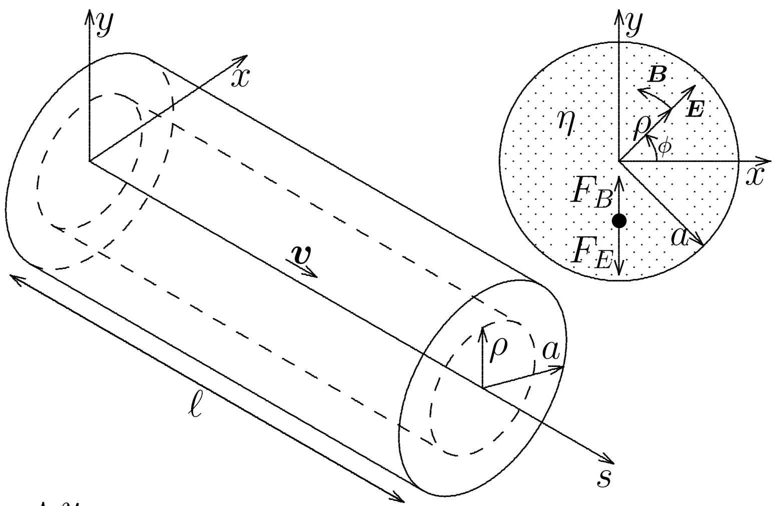
$\vec{F} \propto \vec{\rho}$ gives linear defocusing being $\propto 1/\gamma^2$ and vanishes as $\beta \to 1$.

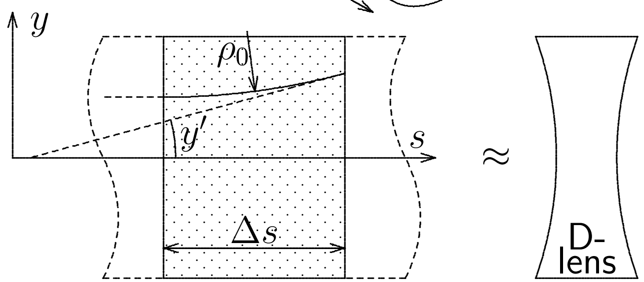


Space-charge defocusing

Uniform space-charge force on particle is linear, radial, repulsive and defocuses beam in x- and y-plane, changing tunes Q_x, $/Q_y$, (taking y):

$$\vec{F} = F_E + F_B = e\left(\vec{E} + [\vec{v} \times \vec{B}]\right)$$
$$= \frac{e\eta}{2\epsilon_0}(1-\beta^2)\vec{\rho} = \frac{eI}{2\pi\epsilon_0 c\beta\gamma^2}\frac{\vec{\rho}}{a^2}.$$





Force deflects by angle $\Delta y' = \propto y$

$$F_y \approx m_0\gamma \mathrm{d}^2 y/\mathrm{d}t^2 = m_0 c^2\beta^2 c^2 \mathrm{d}y'/\mathrm{d}s$$
$$\frac{\mathrm{d}y'}{y} = \mathrm{d}\left(\frac{1}{f}\right) = \frac{eI\mathrm{d}s}{2\pi\epsilon_0 a^2 m_0 c^3\beta^3\gamma^3} = \frac{2r_0 I ds}{ec\beta^3\gamma^3 a^2}$$
$$r_0 = \frac{e^2}{4\pi\epsilon_0 m_0 c^2} = \begin{matrix} 1.54\cdot 10^{-18}\,\mathrm{m\,protons} \\ 2.82\cdot 10^{-15}\,\mathrm{m\,electrons}\end{matrix}$$

$y' = \mathrm{d}y/\mathrm{d}s = \dot{y}/(\beta c) \ll 1$, focusing strength $1/f$, classical particle radius r_0. Tune change by element of length Δs, strength $1/f$.

$$\mathrm{d}Q_y = \frac{\beta_y(s)}{4\pi}\mathrm{d}\left(\frac{1}{f}\right) = \frac{-r_0 I}{2\pi c e\beta^3\gamma^3}\frac{\beta_y(s)\mathrm{d}s}{a^2(s)}$$
$$\Delta Q_y = \frac{-r_0 I}{2\pi c e\beta^3\gamma^3}\oint\frac{\beta_y(s)\mathrm{d}s}{a^2(s)} = \frac{-r_0 I R}{ce\beta^3\gamma^3\mathcal{E}_y}$$

using invariant emittance $\mathcal{E}_y = a^2/\beta_y$. Tune shift by local space-charge depends on $\mathcal{E}_y$, not on β_y and a separately. Small β_y gives small a and strong force but reduced effect.

Approx.: $\mathcal{E}_y = a^2/\beta_y$; no change of β_y.

Elliptic beam cross section

Uniform η and elliptic cross section with half-axes $a,\ b$ give fields and forces inside (L. Teng)

$$\vec{E} = [E_x,\ E_y]\ =\ \frac{I}{\pi\epsilon_0(a+b)\beta c}\left[\frac{x}{a}, \frac{y}{b}\right]$$

$$\vec{B} = [B_x,\ B_y]\ =\ \frac{\mu_0 I}{\pi(a+b)}\left[-\frac{y}{b}, \frac{x}{a}\right]$$

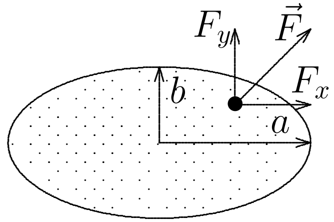
which satisfies div $\vec{E} = \eta/\epsilon_0$, curl $\vec{B} = \mu_0 \vec{J}$.

$$\vec{F} = e\left[\vec{E} + [\vec{v}\times\vec{B}]\right] = \frac{I\,[(x/a),\,(y/b)]\,.}{\pi\epsilon_0\beta c\gamma^2(a+b)}$$

This force is $F_x \propto x$, $F_y \propto y$ and gives linear defocusing in the two directions.

$$\Delta Q_x = \frac{-r_0 I}{\pi e c\beta^3\gamma^3\mathcal{E}_x}\oint\frac{a}{a+b}\mathrm{d}s$$

$$\Delta Q_y = \frac{-r_0 I}{\pi e c\beta^3\gamma^3\mathcal{E}_y}\oint\frac{b}{a+b}\mathrm{d}s$$



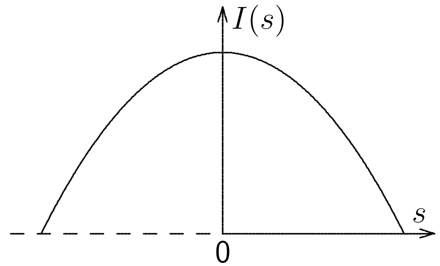
Since a/b depends on s the local tune shift contribution depends also weakly on s.

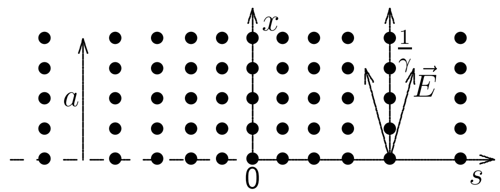
Bunched beams

Current $I(s)$ depends on longitudinal distance s from bunch center. Relativistic field has small opening angle $\approx 1/\gamma$ and depends on local $I(s)$ if this changes little over $\Delta s = a/\gamma$

$$\Delta Q_y = -\frac{r_0 R I(s)}{ec\mathcal{E}_y\beta^3\gamma^3}\ ,\quad \Delta Q_x = -\frac{r_0 R I(s)}{ec\mathcal{E}_x\beta^3\gamma^3},$$

Tune shift depends on particle position s in bunch giving to a tune spread, and, through synchrotron oscillations, to a tune modulation.





Non-uniform distribution

General charge distribution is not uniform, has radial dependence $\eta(\rho)$ giving non-linear force, making tune shift depend on betatron oscillation amplitude and leading to a tune spread.

Conducting vacuum chamber

Vacuum chamber has conducting plate boundaries at $\pm h$ from beam. To get there $E_\parallel = 0$ need not only image charges of beam but also secondary images of those with alternating polarities. Field of all λ_{in} calculated close to beam, first order in x and y (quadrupole field). Vertical field of nth image pair at $\pm 2nh$ and sum over n are

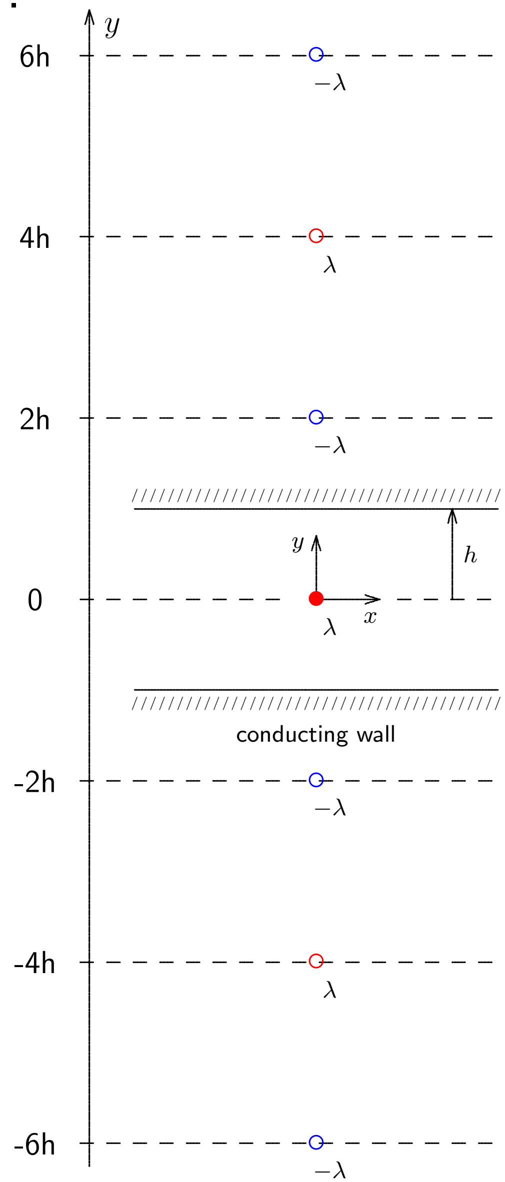
$$E_{iny} = \frac{(-1)^n\lambda}{2\pi\epsilon_0}\left(\frac{1}{2nh+y} - \frac{1}{2nh-y}\right) \approx -\frac{\lambda y}{4\pi\epsilon_0 h^2}\frac{(-1)^n}{n^2}$$

$$E_{iy} = \sum_1^\infty E_{iny} = \frac{\lambda y}{4\pi\epsilon_0 h^2}\frac{\pi^2}{12}, \quad \text{div}\vec{E}_i = 0 \to E_{i_x} = -\frac{\lambda x}{4\pi\epsilon_0 h^2}\frac{\pi^2}{12},$$

$$F_x = \frac{2e\lambda x}{2\pi\epsilon_0}\left(\frac{1}{2a^2\gamma^2} - \frac{\pi^2}{48h^2}\right), \quad F_y = \frac{2e\lambda y}{2\pi\epsilon_0}\left(\frac{1}{2a^2\gamma^2} + \frac{\pi^2}{48h^2}\right)$$

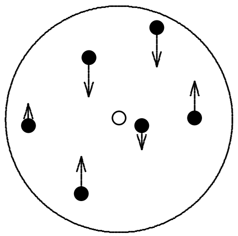
$$\Delta Q_{x/y} = -\frac{2r_0 I R\langle\beta_{x/y}\rangle}{ec\beta^3\gamma}\left(\frac{1}{2a^2\gamma^2} \mp \frac{\pi^2}{48h^2}\right), \quad \text{with } I = \lambda\beta c.$$

Conducting boundary does not affect B-field unless ferromagnetic. No relativistic E/B-force compensation for indirect space charge. ΔQ: First term, direct space-charge, $\propto 1/\gamma^2$, depends on size a; second term, indirect effect of wall, depends on distance h, both have additional factor $1/\gamma$ due to particle rigidity.

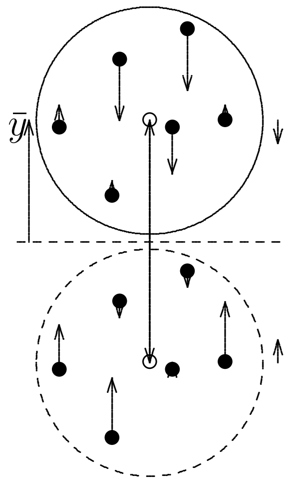


Incoherent and coherent motion

Direct space-charge effect



Individual particles move center-of-mass not: **incoherent** motion



Center of mass moves **coherent** motion

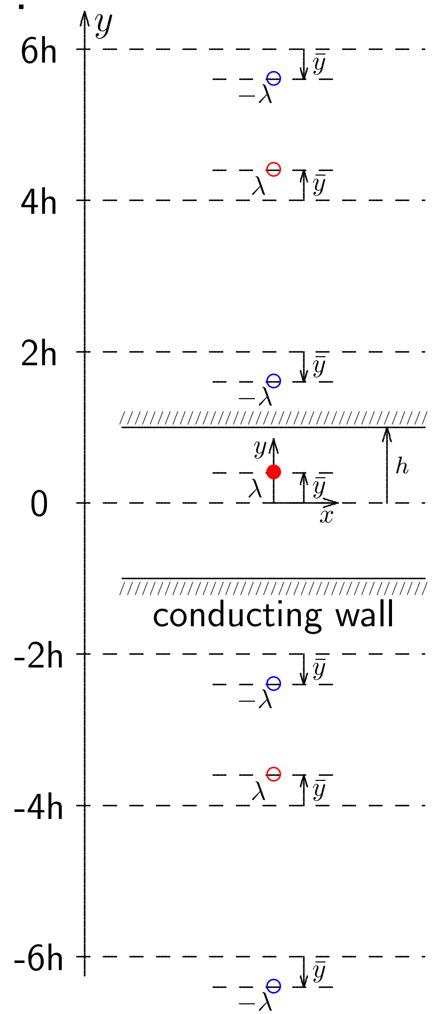
For an incoherent motion particles have a space-charge tune shift

$$\Delta Q_{\text{inc.}} = -\frac{r_0 I R \beta_y}{c e a^2 \beta^3 \gamma^3}$$

For a coherent motion space-charge force is intern, moves with beam and does not affect center-of-mass motion $\Delta Q_{\text{coh.}} = 0$

Indirect space-charge effect

Space-charge field with a conducting wall at distance h was obtained by image line charge at h behind wall. A coherent beam motion by $\bar{y}$ moves first images to $\pm 2h - \bar{y}$ with a field at the beam



$$E_{c1y} = \frac{-\lambda}{2\pi\epsilon_0}\left(\frac{1}{2h+2\bar{y}} - \frac{1}{2h-2\bar{y}}\right).$$

Equidistant 2nd images cancel, general

$$E_{cny} = -\frac{(-1)^n \lambda \bar{y}}{4\pi\epsilon_0 h^2}\left(\frac{1}{n^2} - \frac{(-1)^n}{n^2}\right)$$

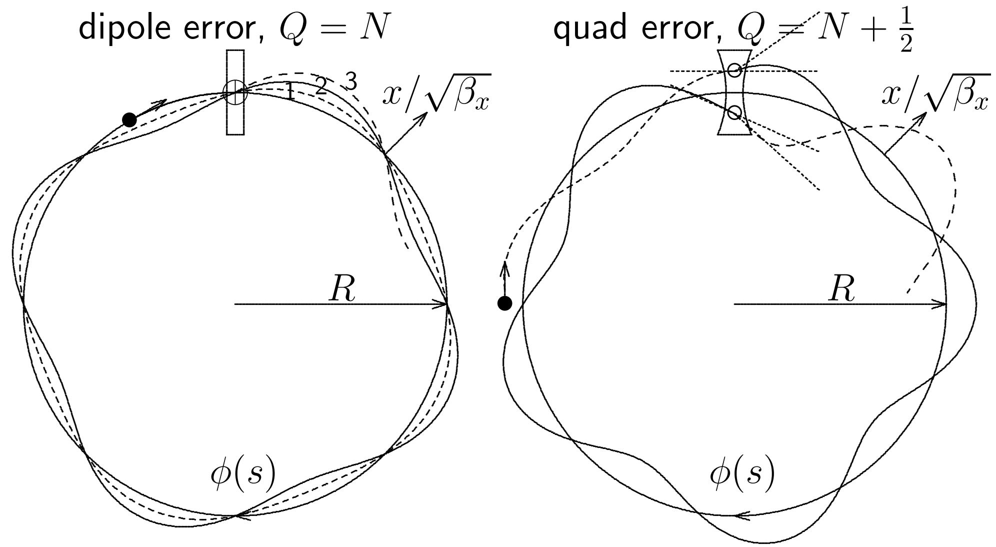
$$E_{cy} = \sum_1^\infty E_{cny} = \frac{\lambda \bar{y}}{4\pi\epsilon_0 h^2}\left(\frac{\pi^2}{12} + \frac{\pi^2}{6}\right)$$

$$Q_{ycoh.} = Q_0 - \frac{\pi^2}{16}\frac{2 r_0 I R \langle\beta_y\rangle}{e c \beta^3 \gamma h^2}$$

$$Q_{ycoh.} - Q_{yinc.} = \frac{2 r_0 I R \langle\beta_y\rangle}{e c \beta^3 \gamma}\left(\frac{1}{2a^2\gamma^2} - \frac{\pi^2}{24 h^2}\right).$$

Problems caused by space-charge in rings

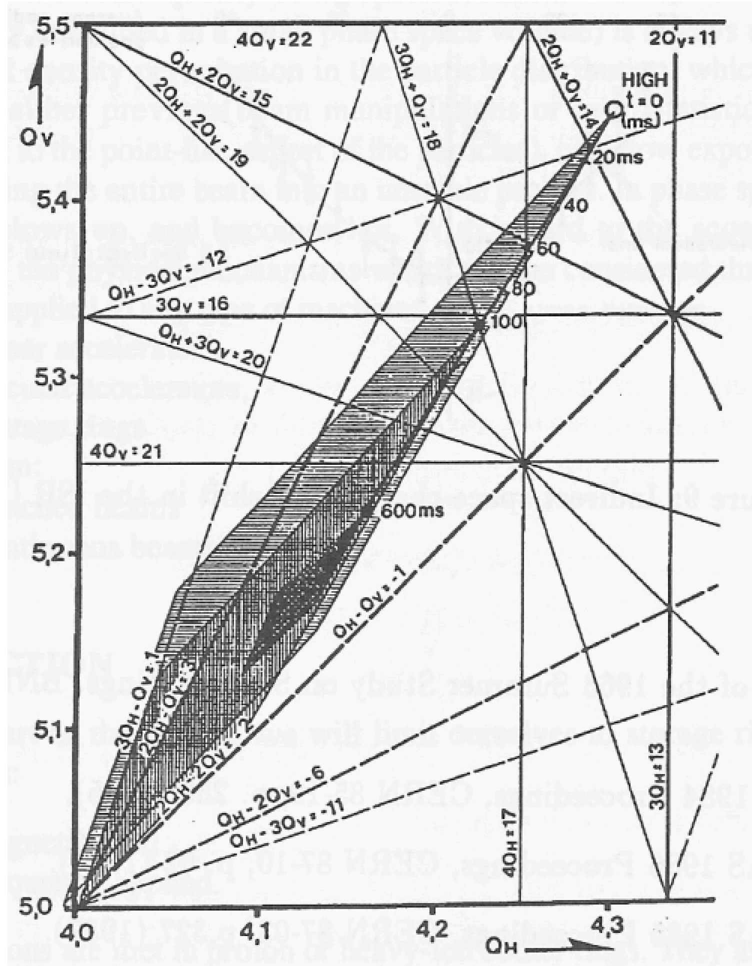
In rings space-charge can shift tunes into resonances where $Q = N/M$ is a simple rational fraction. Dipole imperfection deflects particle each turn in phase if Q =integer and for a quadrupole error this happens if Q =half integer. Since space-charge shifts coherent and incoherent tunes differently and produces spread it may be difficult to avoid all resonances.



Related effects

Beam-beam effect: electric and magnetic forces ad.

Ions and electron clouds: don't move no B-force.

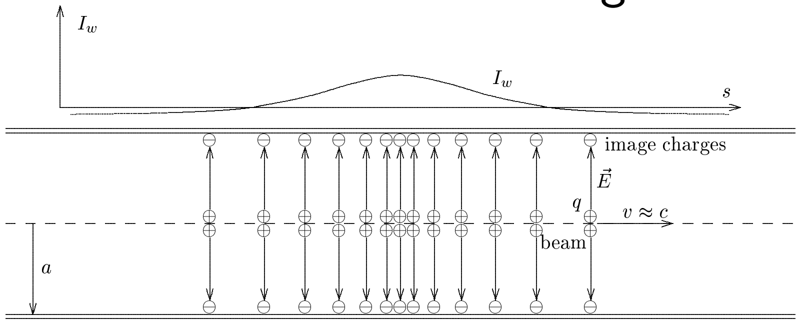


Direct space-charge tune shift and spread at different γ during acceleration (shaded areas), CERN booster, E. Brouzet, K.H. Schindl.

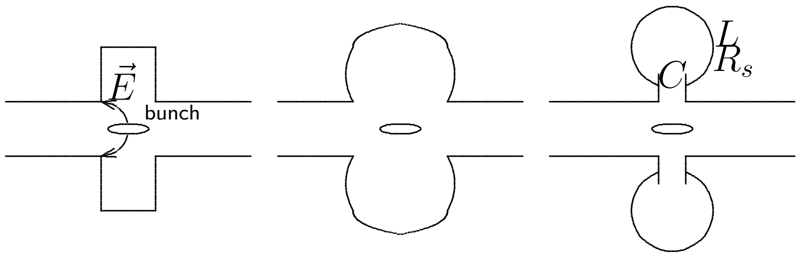
2) IMPEDANCES AND WAKE FUNCTIONS

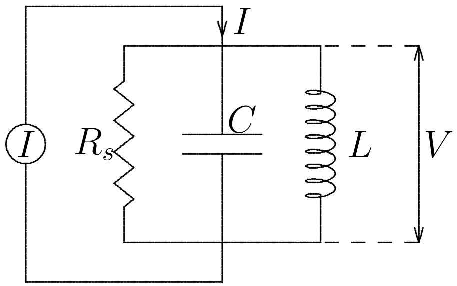
Resonator

For space charge a perfectly conducting wall of uniform cross section was assumed and treated with electrostatic methods. We generalize.



Beam induces wall current $I_w = -(I_b - \langle I_b \rangle)$



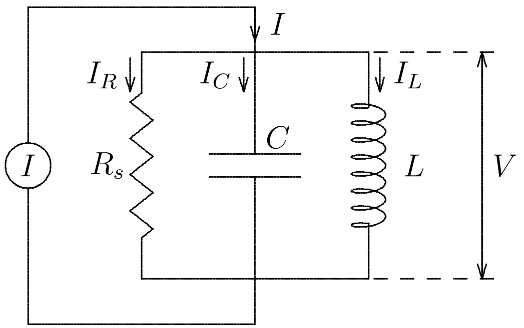


Cavities have narrow band oscillation modes which can drive coupled bunch instabilities. Each resembles an **RCL - circuit** and can, in good approximation, be treated as such. This circuit has a shunt impedance R_s, an inductance L and a capacity C. In a real cavity these parameters cannot easily be separated and we use others which can be measured directly: The **resonance frequency** ω_r, the **quality factor** Q and the **damping rate** α:

$$\omega_r = \frac{1}{\sqrt{LC}}, \quad Q = R_s\sqrt{\frac{C}{L}} = \frac{R_s}{L\omega_r} = R_s C \omega_r$$

$$\alpha = \frac{\omega_r}{2Q}, \quad L = \frac{R_s}{Q\omega_r}, \; C = \frac{Q}{\omega_r R_s}.$$

Driving this circuit with a current I gives the voltages and currents across the elements



$$V_R = I_R R_s$$
$$V_C = \frac{1}{C}\int I_C dt$$
$$V_L = L\frac{dI_L}{dt}$$

$$V_R = V_C = V_L = V,\ I_R + I_C + I_L = I$$

$$\dot{I} = \dot{I}_R + \dot{I}_C + \dot{I}_L = \dot{V}/R_s + C\ddot{V} + V/L.$$

Using $L = R_s/(\omega_r Q)$, $C = Q/(\omega_r R_s)$ gives

differential eqn. $\ddot{V} + \frac{\omega_r}{Q}\dot{V} + \omega_r^2 V = \frac{\omega_r R_s}{Q}\dot{I}$

Homogeneous solution is damped oscillation

$$V(t) = \mathrm{e}^{-\alpha \mathrm{t}}\left(\mathrm{A}\cos\left(\omega_\mathrm{r}\sqrt{1-\frac{1}{4\mathrm{Q}^2}}\mathrm{t}\right)\right.$$
$$\left.+B\sin\left(\omega_r\sqrt{1-\frac{1}{4Q^2}}t\right)\right),\ \alpha = \frac{\omega_r}{2Q}$$

Wake/Green – function, pulse response

$I(t) = q\delta t$, charge q gives capacity voltage

$$V(0^+) = \frac{q}{C} = \frac{\omega_r R_s}{Q}q \text{ using } C = \frac{Q}{\omega_r R_s}$$

Energy stored in C = energy lost by q

$$U = \frac{q^2}{2C} = \frac{\omega_r R_s}{2Q}q^2 = \frac{V(0^+)}{2}q = k_{pm}q^2$$

parasitic mode loss factor $k_{pm} = \omega_r R_s/2Q$

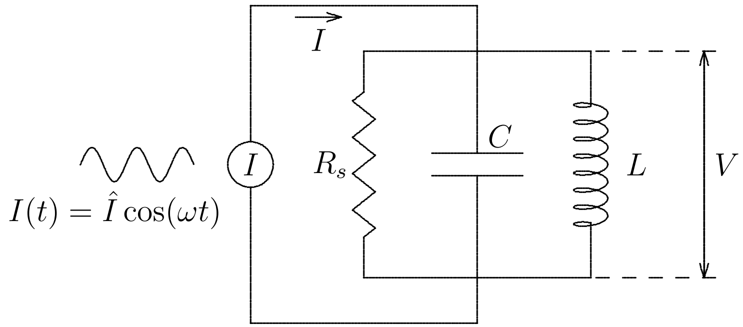
Capacitor discharges first through resistor

$$-\dot{V}(0^+) = \frac{\dot{q}}{C} = \frac{I_R}{C} = \frac{V(0^+)}{CR_s} = -\frac{2\omega_r k_{pm}}{Q}q.$$

$$V(0^+),\ \dot{V}(0^+) \rightarrow A = 2qk_{pm},\ B = \frac{-A}{\sqrt{4Q^2-1}}$$

$$V(t) = 2qk_{pm}\mathrm{e}^{-\alpha\mathrm{t}}\left(\cos\left(\omega_\mathrm{r}\sqrt{1-\frac{1}{4\mathrm{Q}^2}}\mathrm{t}\right)\right.$$
$$\left.-\frac{\sin\left(\omega_r\sqrt{1-\frac{1}{4Q^2}}t\right)}{\sqrt{4Q^2-1}}\right) \approx 2qk_{pm}\mathrm{e}^{-\alpha\mathrm{t}}\cos(\omega_\mathrm{r}\mathrm{t}).$$

Impedance



A **harmonic** excitation of circuit with current $I = \hat{I}\cos(\omega t)$ gives differential equation

$$\ddot{V} + \frac{\omega_r}{Q}\dot{V} + \omega_r^2 V = \frac{\omega_r R_s}{Q}\dot{I} = -\frac{\omega_r R_s}{Q}\hat{I}\omega\sin(\omega t).$$

Homogeneous solution damps leaving particular one $V(t) = A\cos(\omega t) + B\sin(\omega t)$. Put into differential equation, separating cosine and sine

$$\begin{aligned} -(\omega^2 - \omega_r^2)A + \frac{\omega_r\omega}{Q}B &= 0 \\ (\omega^2 - \omega_r^2)B + \frac{\omega_r\omega}{Q}A &= \frac{\omega_r\omega R_s}{Q}\hat{I}. \end{aligned}$$

Voltage induced by current $\hat{I}\cos(\omega t)$ is

$$V(t) = \hat{I}R_s \frac{\cos(\omega t) + Q\frac{\omega^2-\omega_r^2}{\omega_r\omega}\sin(\omega t)}{1 + Q^2\left(\frac{\omega^2-\omega_r^2}{\omega_r\omega}\right)^2}$$

Cosine term is **in phase** with exciting current, absorbs energy, **resistive**. Sine term is **out of phase**, does not absorb energy, **reactive**. Voltage/current ratio is **impedance** as **function of frequency** ω

$$\begin{aligned} Z_r(\omega) &= R_s \frac{1}{1 + Q^2\left(\frac{\omega_r^2-\omega^2}{\omega_r\omega}\right)^2} \\ Z_i(\omega) &= -R_s \frac{Q\frac{\omega^2-\omega_r^2}{\omega_r\omega}}{1 + Q^2\left(\frac{\omega^2-\omega_r^2}{\omega_r\omega}\right)^2}. \end{aligned}$$

Resistive part $Z_r(\omega) \geq 0$, reactive part $Z_i(\omega)$ positive below, negative above ω_r.

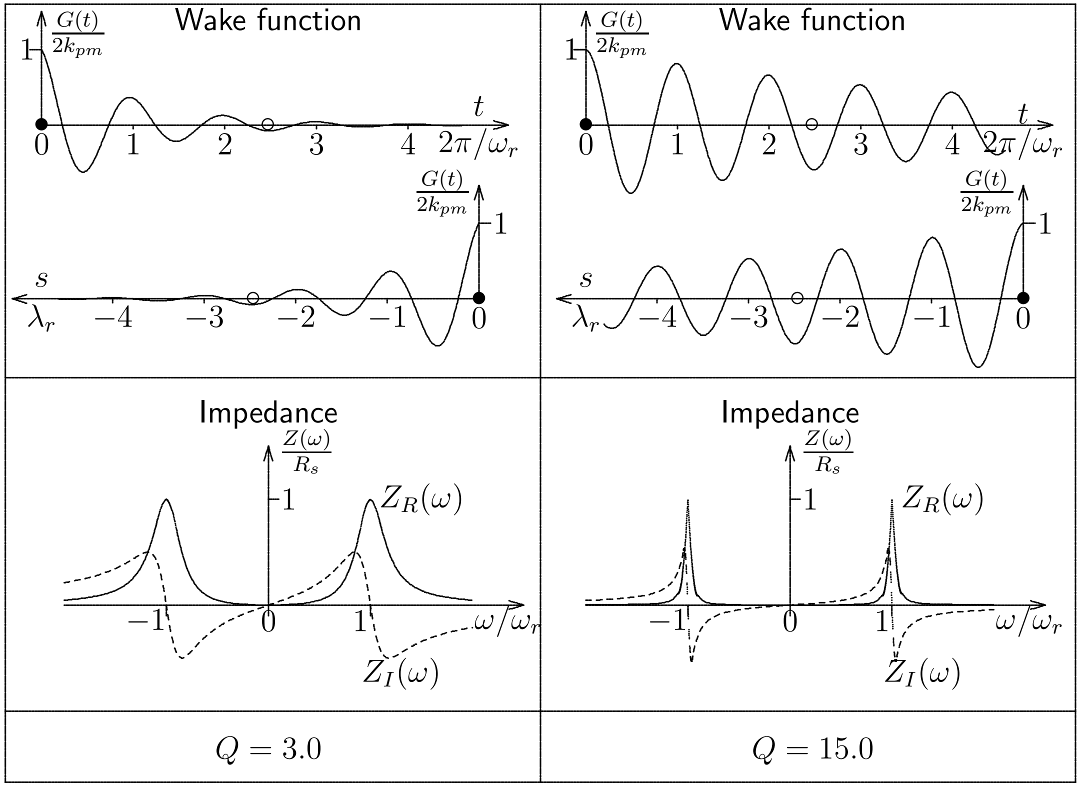
$$\hat{I}\cos(\omega t) \to V = \hat{I}[Z_r\cos(\omega t) - Z_i\sin(\omega t)]$$
$$\hat{I}\sin(\omega t) \to V = \hat{I}[Z_r\sin(\omega t) + Z_i\cos(\omega t)]$$

Complex notation

Excitation: $I(t) = \hat{I}\cos(\omega t) = \hat{I}\frac{e^{j\omega t} + e^{-j\omega t}}{2}$

with $0 \le \omega \le \infty$

$I(t) = \hat{I}e^{j\omega t}/2$ with $-\infty \le \omega \le \infty$



$$Z(\omega) = R_s \frac{1 - jQ\frac{\omega^2 - \omega_r^2}{\omega\omega_r}}{1 + Q^2\left(\frac{\omega^2 - \omega_r^2}{\omega\omega_r}\right)^2} = Z_r + jZ_i$$

$$\approx R_s \frac{1 - j2Q\Delta\omega/\omega_r}{1 + 4Q^2\left(\Delta\omega/\omega_r\right)^2} \quad \text{for } Q \gg 1$$

$\omega \approx \omega_r, |\omega - \omega_r|/\omega_r = |\Delta\omega|/\omega_r \ll 1.$

Resonator impedance properties:

at $\omega = \omega_r \ \rightarrow \ Z_r(\omega_r)$ max., $Z_i(\omega_r) = 0$

$0 < \omega < \omega_r \ \rightarrow \ Z_i(\omega) > 0$ (inductive)

$\omega > \omega_r \ \rightarrow \ Z_i(\omega) < 0$ (capacitive)

General impedance or wake properties

$Z_r(\omega) = Z_r(-\omega)$, $Z_i(\omega) = -Z_i(-\omega)$

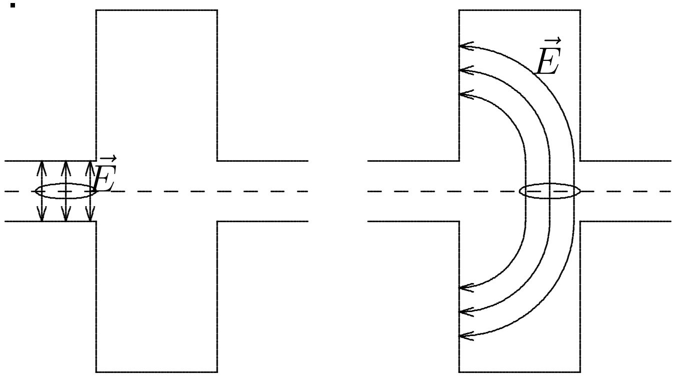
$Z(\omega) = \int_{-\infty}^{\infty} G(t)e^{-j\omega t}dt$

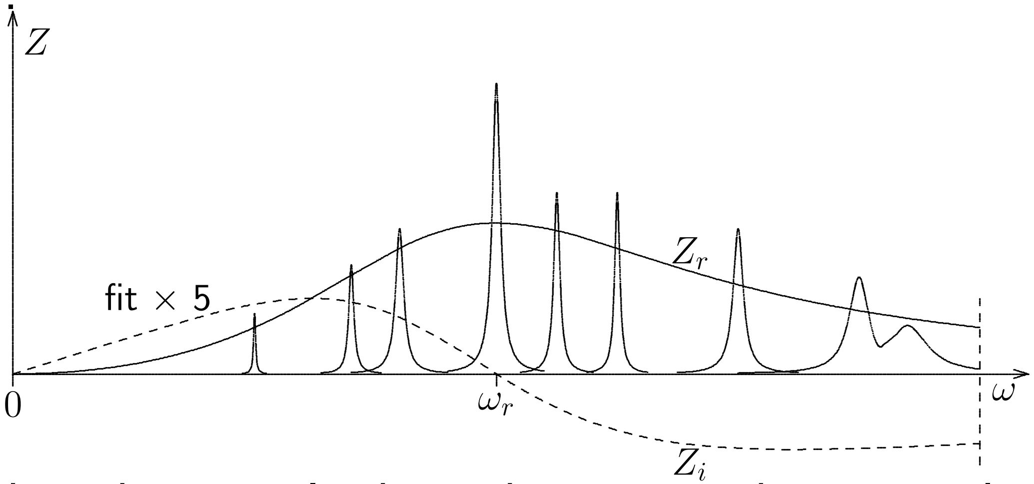
$Z(\omega) \propto$ Fourier transform of $G(t)$

for $t < 0 \ \rightarrow \ G(t) = 0,$

no fields before particle arrives, $\beta \approx 1.$

Typical impedance of a ring





Aperture changes form cavity-like objects with $\omega_r,\ R_s$ and Q and impedance $Z(\omega)$ developed for $\omega < \omega_r$, where it is inductive

$$Z(\omega) = R_s \frac{1 - jQ\frac{\omega^2-\omega_r^2}{\omega\omega_r}}{1+\left(Q\frac{\omega^2-\omega_r^2}{\omega\omega_r}\right)^2} \approx j\frac{R_s\omega}{Q\omega_r} + ...$$

Sum impedance at $\omega \ll \omega_{rk}$ divided by mode number $n = \omega/\omega_0$ is with inductance L

$$\left|\frac{Z}{n}\right|_0 = \sum_k \frac{R_{sk}\omega_0}{Q_k\omega_{rk}} = L\omega_0 = L\frac{\beta c}{R}.$$

It depends on impedance per length, $\approx$ 15 Ω in older, 1 Ω in newer rings. The shunt impedances R_{sk} increase with ω up to cut-off frequency where wave propagation starts and become wider and smaller. A broad band resonator fit helps to characterize impedance giving $Z_r,\ Z_i,\ G(t)$ useful for single traversal effects. However, for multi-traversal instabilities narrow resonances at ω_{rk} must be used.