Linac Driven Free Electron Lasers (II)

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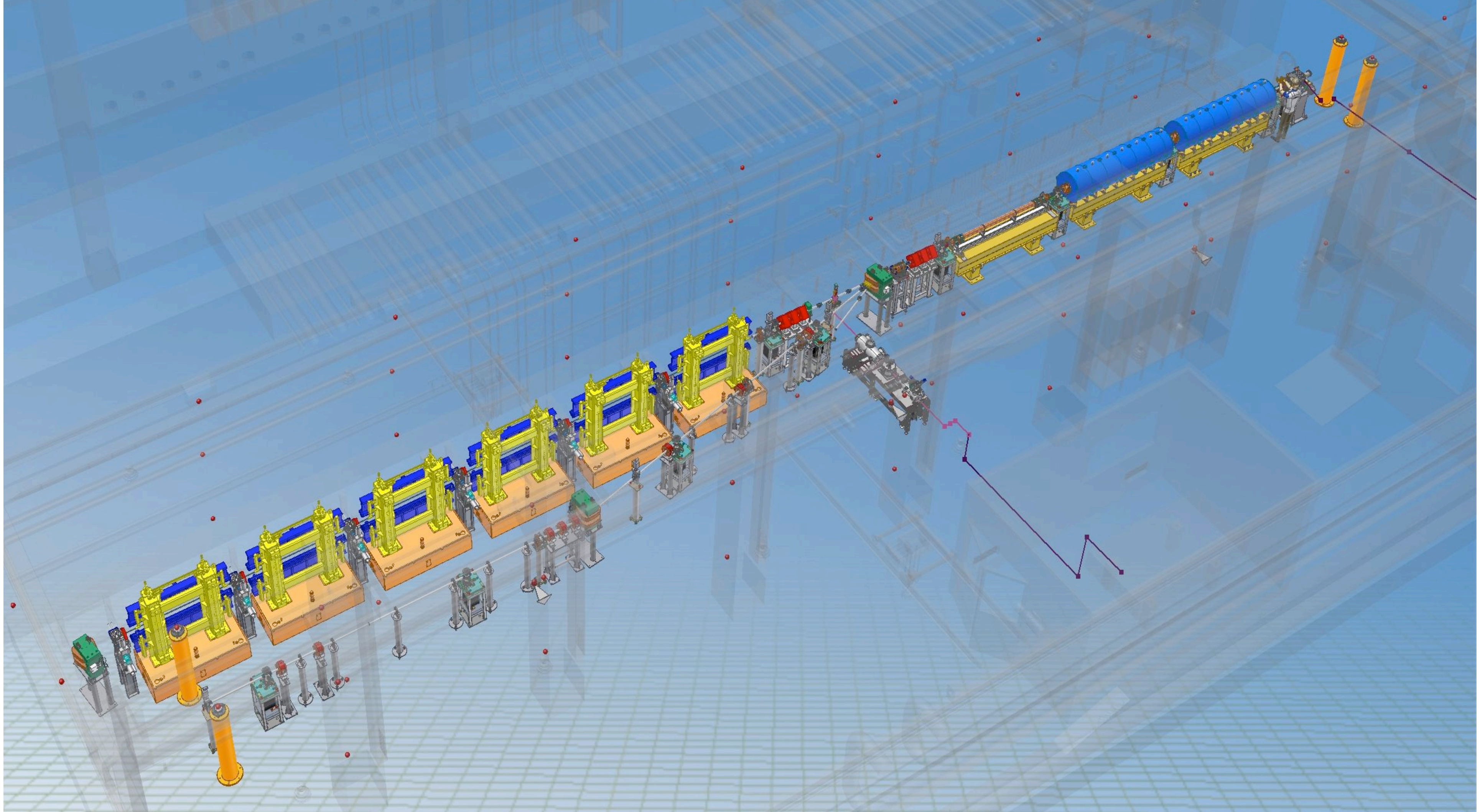
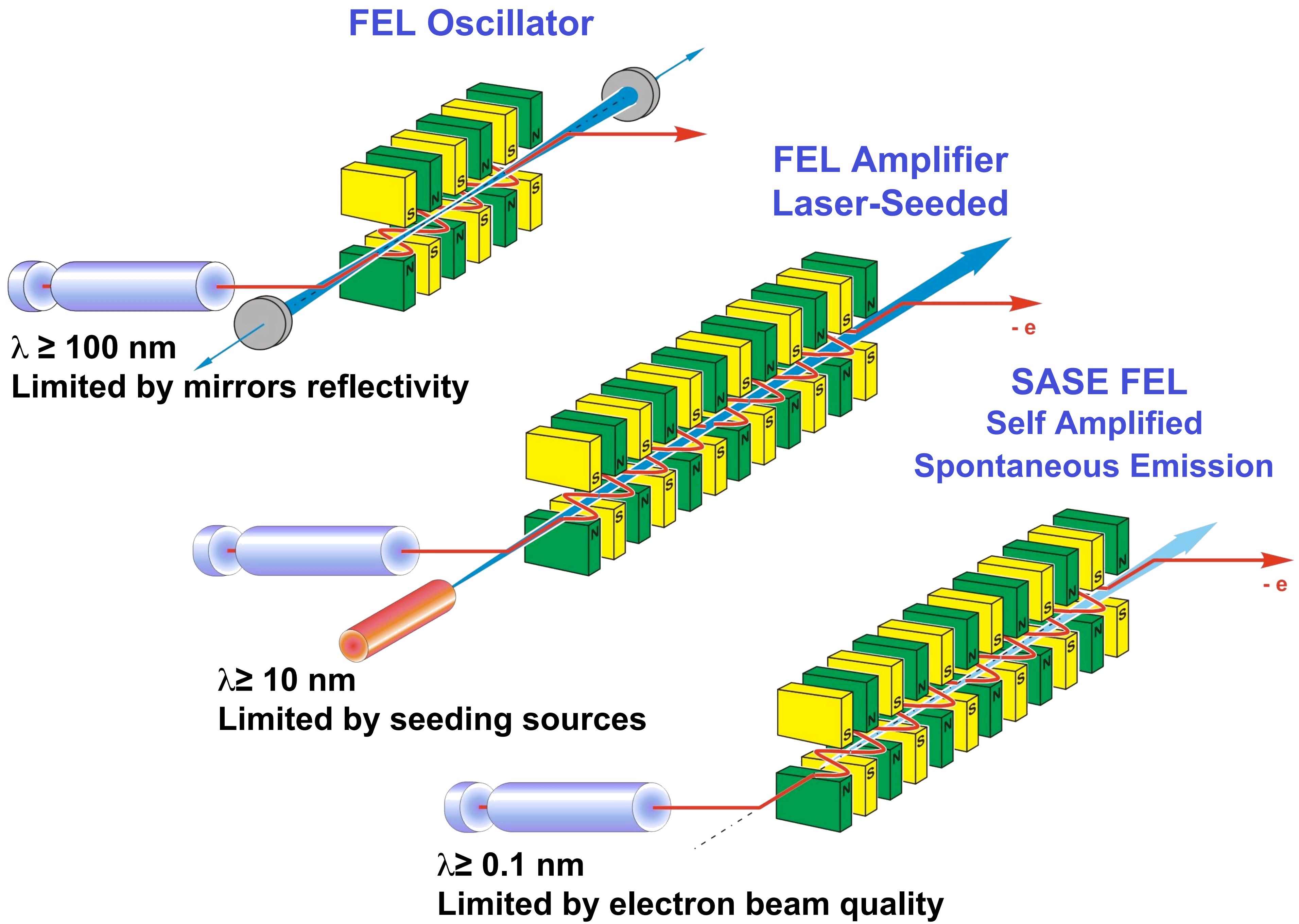


CAS
THE CERN ACCELERATOR SCHOOL



FEL Oscillator
λ ≥ 100 nm
Limited by mirrors reflectivity
FEL Amplifier
Laser-Seeded
- e
λ≥ 10 nm
Limited by seeding sources
SASE FEL
Self Amplified
Spontaneous Emission
- e
λ≥ 0.1 nm
Limited by electron beam quality

SASE FEL Electron Beam Requirements: High Brightness B_n

$$\lambda_r^{MIN} \propto \sigma_\delta \sqrt{\frac{\left(1+K^2/2\right)}{\gamma B_n K^2}}$$

minimum radiation wavelength

energy spread

undulator parameter

$$B_n = \frac{2I}{\varepsilon_n^2}$$

$$L_g \propto \frac{\gamma^{3/2}}{K\sqrt{B_n\left(1+K^2/2\right)}}$$

gain length

R. Saldin et al. in *Conceptual Design of a 500 GeV e+e- Linear Collider with Integrated X-ray Laser Facility*, DESY-1997-048

Short Wavelength SASE FEL Electron Beam Requirement: High Brightness $B_n > 10^{15}$ A/m^2

$$\uparrow B_n \approx \frac{2I \uparrow}{\varepsilon_n^2 \downarrow}$$

Bunch compressors

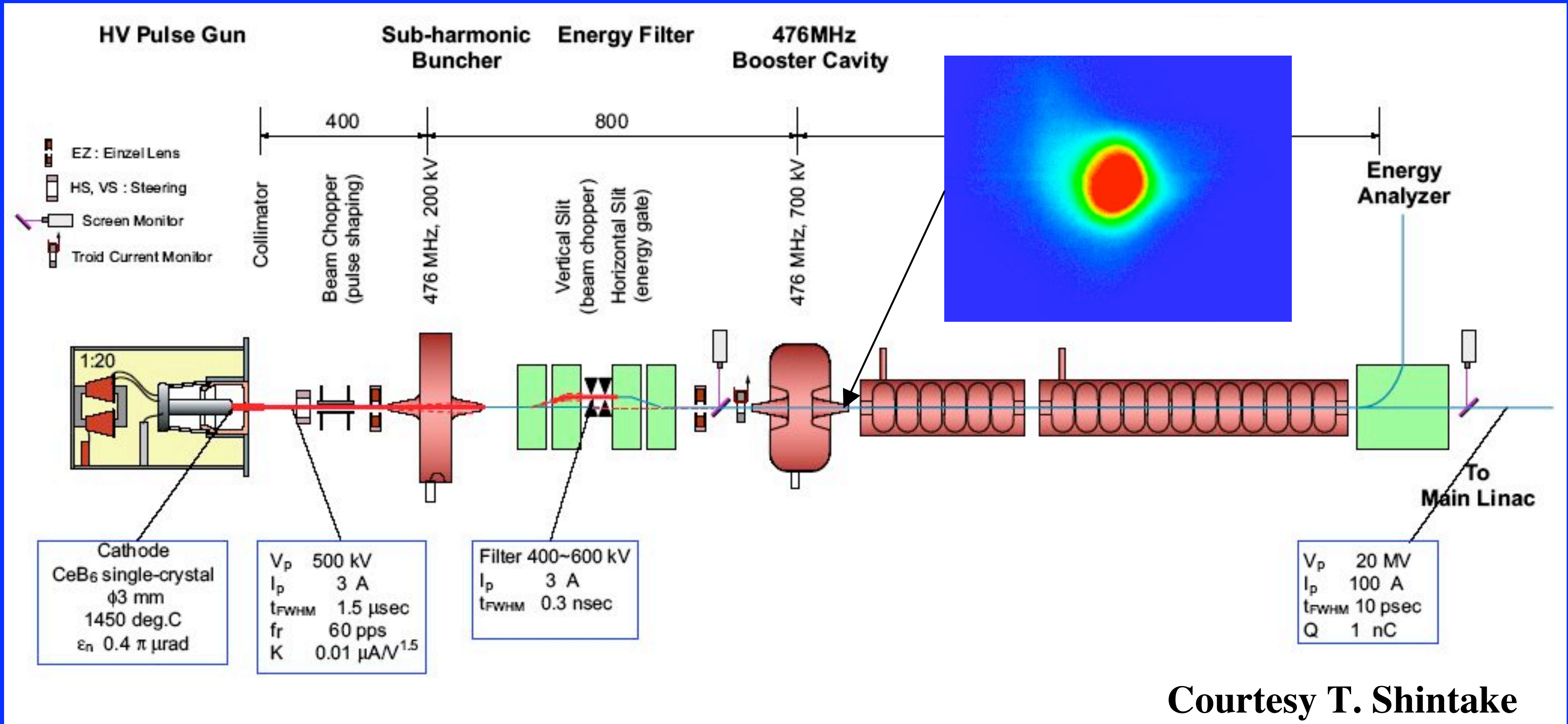
(RF & magnetic)

Laser Pulse shaping

Emittance compensation

Cathode emittance

500 kV pulsed thermionic gun for SCSS



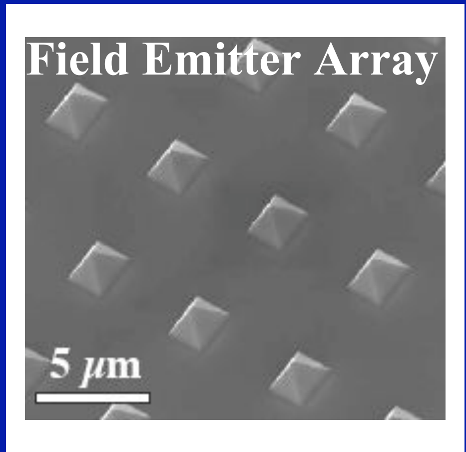
Stable operation with uniform beam quality

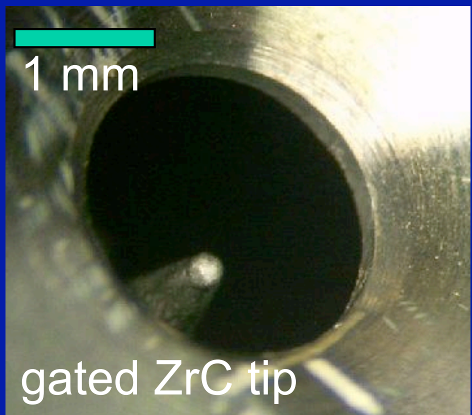
Low thermal emittance single crystal CeB_6 (Cerium Hexaborite)

Low accelerating gradient (10 MV/m) ==> Low charge density

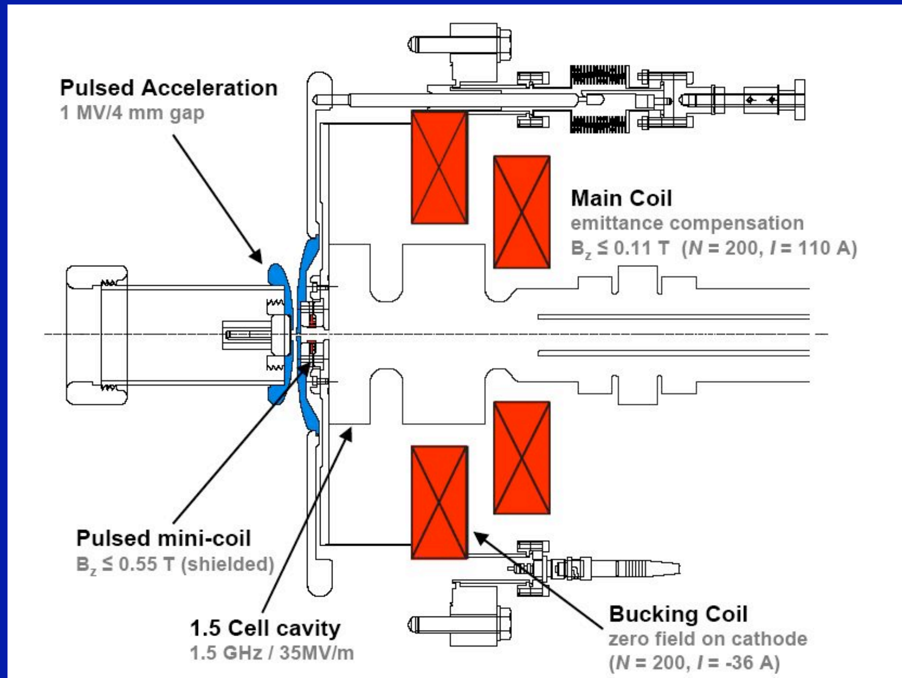
==> Free from dark current

Ulta-Low thermal emittance gun



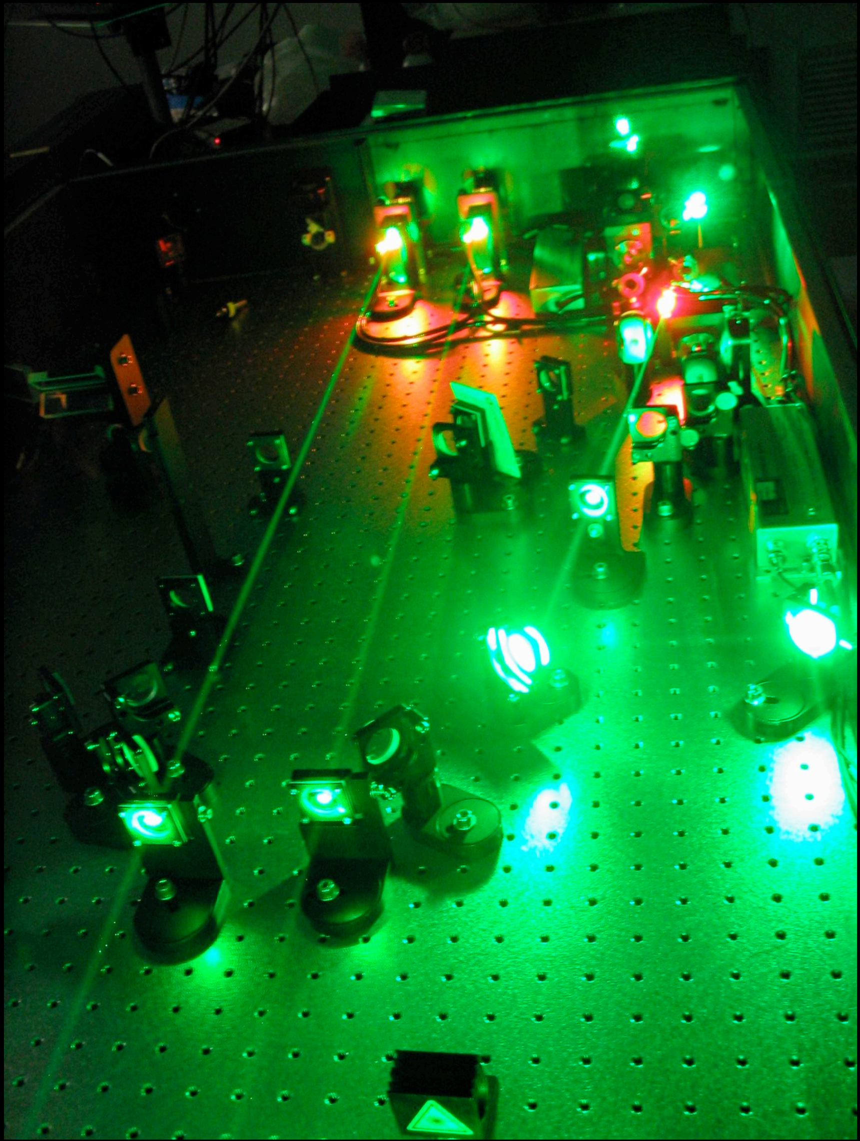


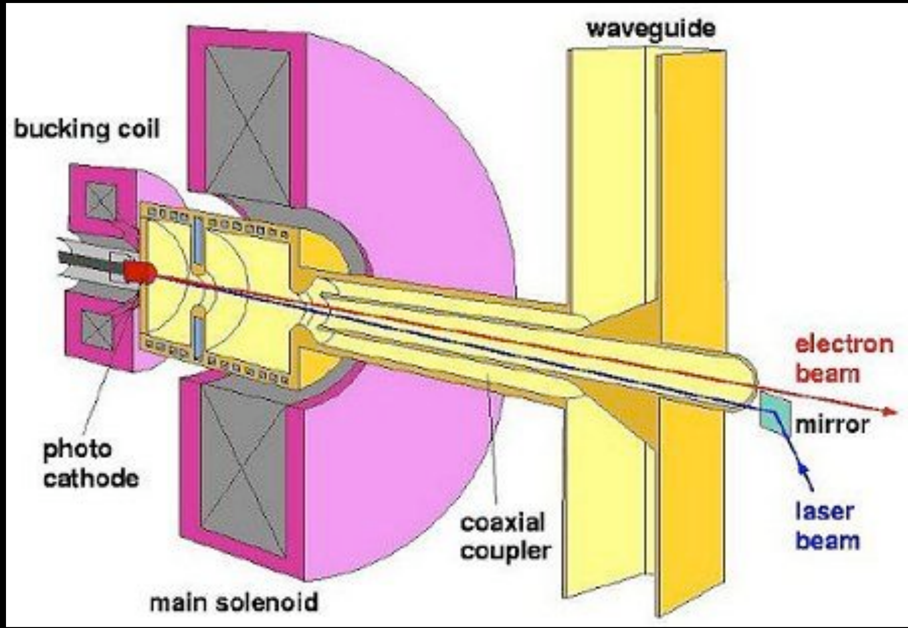
Courtesy R. Backer

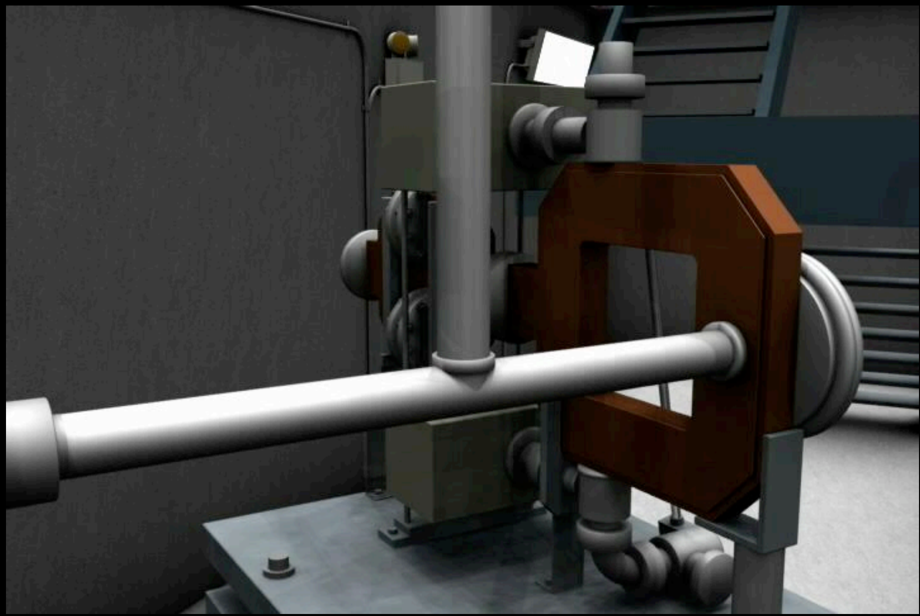


Frequency	1x10 Hz (macro pulse)	-
Peak field at cathode	6 GeV/m	> 6 GV/m
Charge per bunch	200 pC	> 1 nC (long pulses)
Rms norm. emittance	0.05 mm mrad (at cathode)	< 0.1 mm mrad (I < 0.6 A)
Peak Current	5.5 A (at cathode)	0.6 A
Average Current	2 nA	> 5 μA (long pulses)

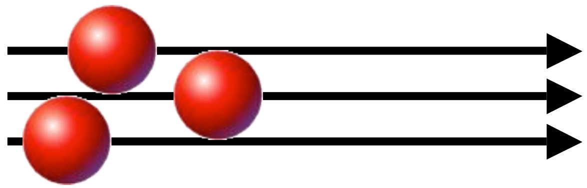
RF photoinjectors

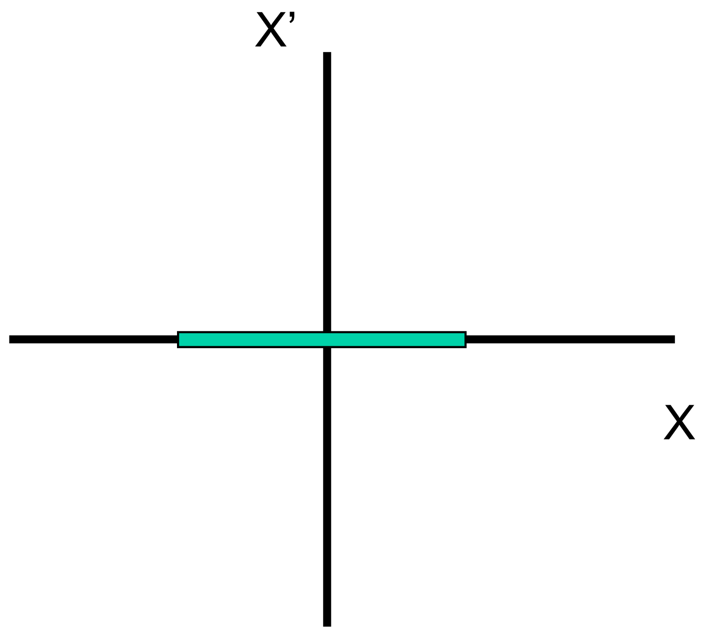




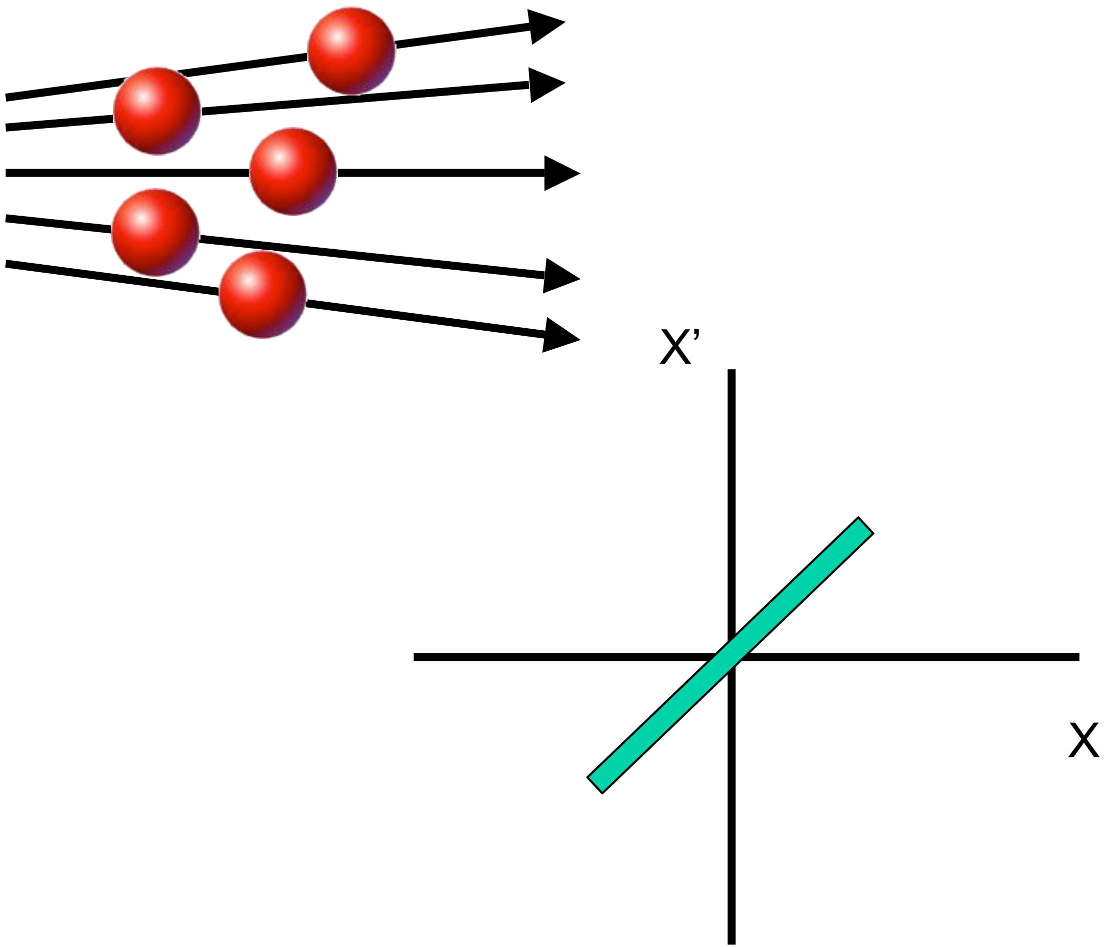


Phase space of a parallel laminar beam

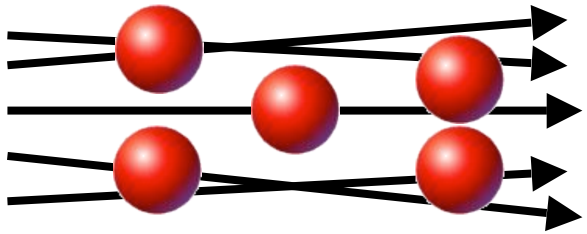


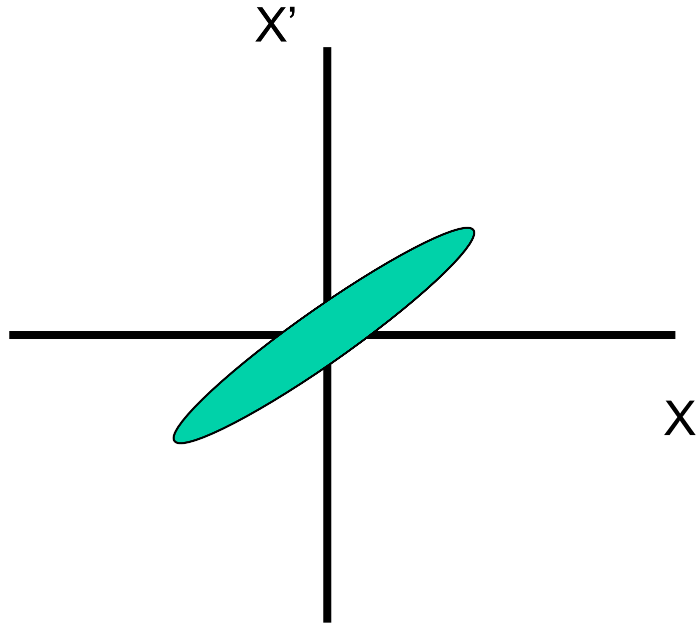


Phase space laminar beam



Phase space of non laminar beam



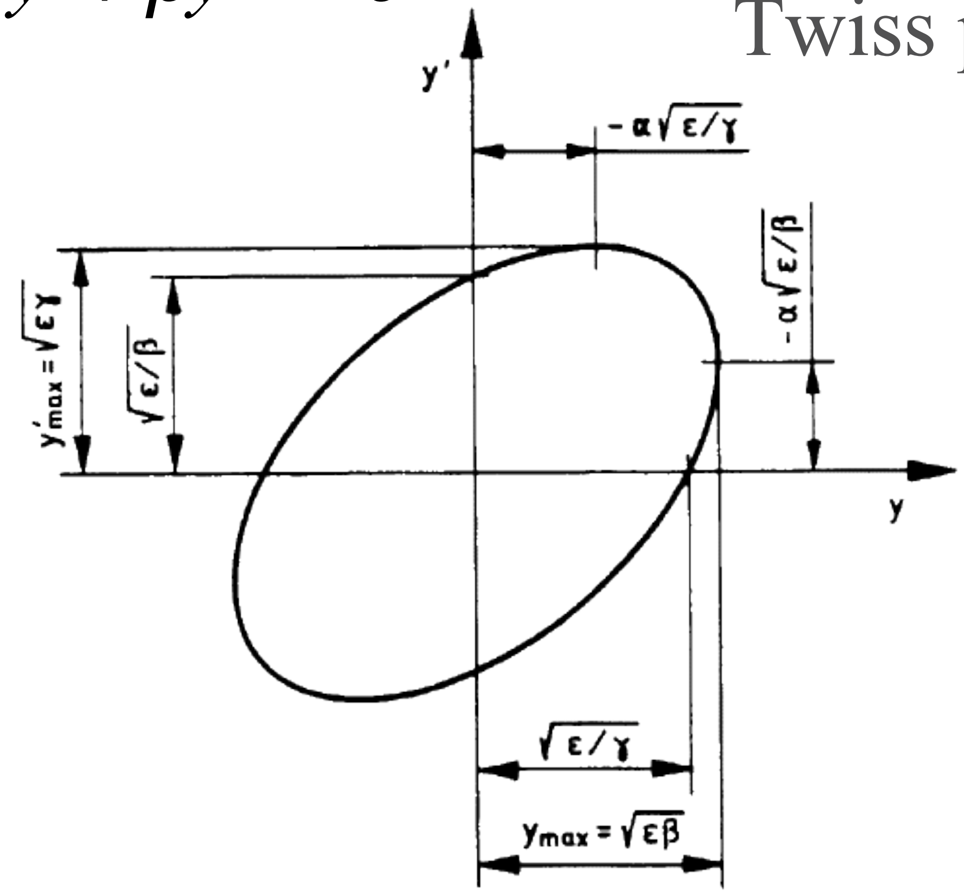


Ellipse equation

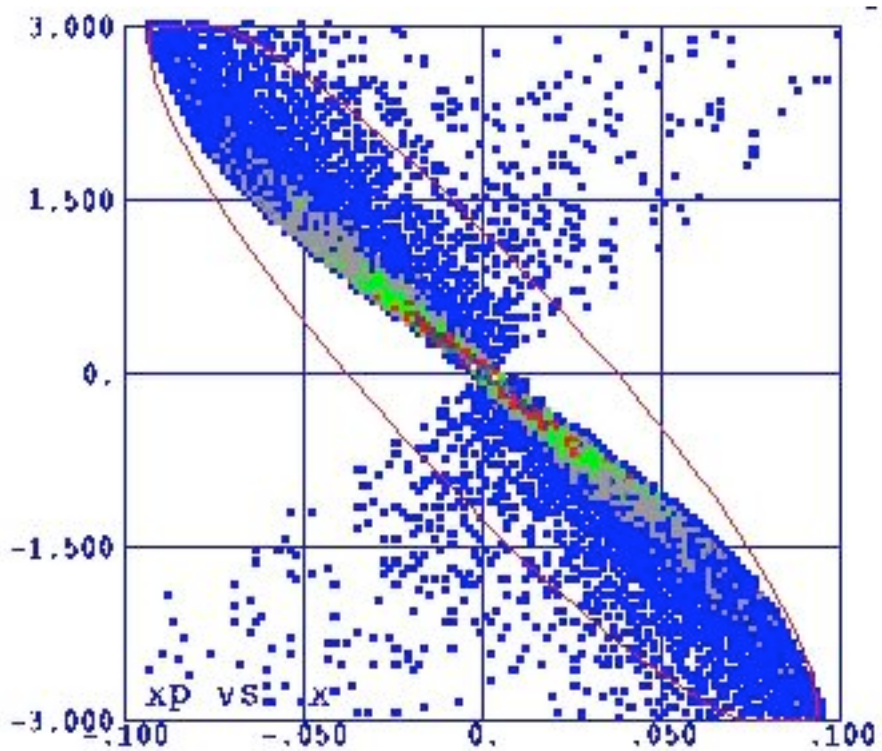
$$\gamma y^2 + 2\alpha y y' + \beta y'^2 = \varepsilon$$

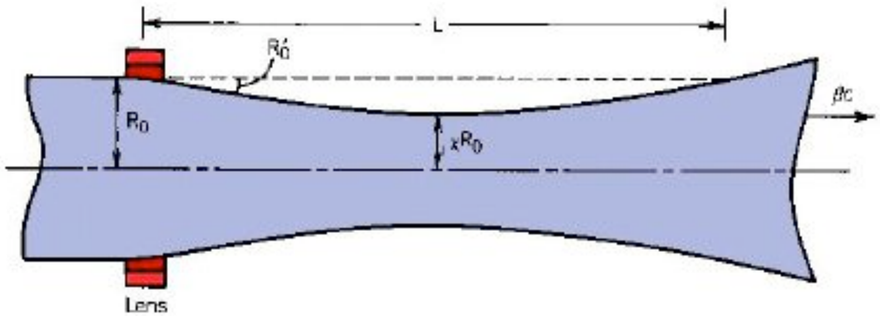
Twiss parameters

$$\beta\gamma - \alpha^2 = 1$$

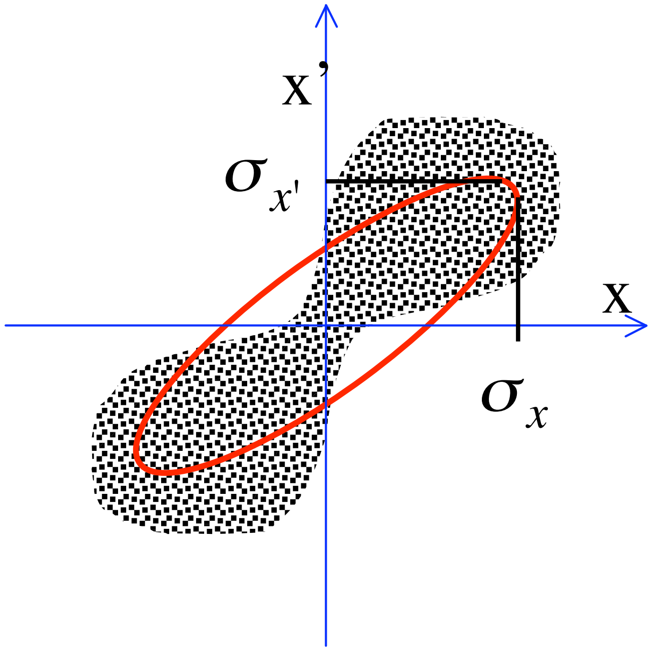


Phase space evolution at injector exit





rms Envelope Equations and rms Emittance



rms beam envelope:

$$\sigma_x^2 = \langle x^2 \rangle = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x^2 f(x,x')\,dx\,dx'$$

Define rms emittance:

$$\boxed{\gamma x^2 + 2\alpha x x' + \beta x'^2 = \varepsilon_{rms}}$$

such that:

$$\sigma_x = \sqrt{\langle x^2 \rangle} = \sqrt{\beta \varepsilon_{rms}}$$

$$\sigma_{x'} = \sqrt{\langle x'^2 \rangle} = \sqrt{\gamma \varepsilon_{rms}}$$

Since: $\alpha = -\dfrac{\beta'}{2}$

it follows: $\alpha = -\dfrac{1}{2\varepsilon_{rms}} \dfrac{d}{dz}\langle x^2 \rangle = -\dfrac{\langle x x' \rangle}{\varepsilon_{rms}} = -\dfrac{\sigma_{xx'}}{\varepsilon_{rms}}$

$$\boxed{\begin{aligned}\sigma_x &= \sqrt{\langle x^2\rangle} = \sqrt{\beta\varepsilon_{rms}}\\ \sigma'_x &= \sqrt{\langle x'^2\rangle} = \sqrt{\gamma\varepsilon_{rms}}\\ \sigma_{xx'} &= \langle xx'\rangle = \alpha\varepsilon_{rms}\end{aligned}}$$

It holds also the relation: $\gamma\beta - \alpha^2 = 1$

Substituting α, β, γ we get $\dfrac{\sigma_{x'}^2}{\varepsilon_{rms}}\dfrac{\sigma_x^2}{\varepsilon_{rms}} - \left(\dfrac{\sigma_{xx'}}{\varepsilon_{rms}}\right)^2 = 1$

We end up with the definition of rms emittance in terms of the second moments of the distribution:

$$\boxed{\varepsilon_{rms} = \sqrt{\sigma_x^2\sigma_{x'}^2 - \sigma_{xx'}{}^2} = \sqrt{\left(\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2\right)}}$$

$$\varepsilon_n = \langle\beta\gamma\rangle\varepsilon_{rms}$$

Envelope Equation without Acceleration

Now take the derivatives:

$$\frac{d\sigma_x}{dz} = \frac{d}{dz}\sqrt{\left\langle x^2 \right\rangle} = \frac{1}{2\sigma_x}\frac{d}{dz}\left\langle x^2 \right\rangle = \frac{1}{2\sigma_x}2\left\langle xx' \right\rangle = \frac{\sigma_{xx'}}{\sigma_x}$$

$$\frac{d^2\sigma_x}{dz^2} = \frac{d}{dz}\frac{\sigma_{xx'}}{\sigma_x} = \frac{1}{\sigma_x}\frac{d\sigma_{xx'}}{dz} - \frac{\sigma_{xx'}^2}{\sigma_x^3} = \frac{1}{\sigma_x}\left(\left\langle x'^2 \right\rangle - \left\langle xx'' \right\rangle\right) - \frac{\sigma_{xx'}^2}{\sigma_x^3} = \frac{\sigma_{x'}^2 + \left\langle xx'' \right\rangle}{\sigma_x} - \frac{\sigma_{xx'}^2}{\sigma_x^3}$$

And simplify:

$$\boxed{\sigma_x'' = \frac{\sigma_x^2\sigma_{x'}^2 - \sigma_{xx'}^2}{\sigma_x^3} - \frac{\left\langle xx'' \right\rangle}{\sigma_x} = \frac{\varepsilon_{rms}^2}{\sigma_x^3} - \frac{\left\langle xx'' \right\rangle}{\sigma_x}}$$

Assuming that each particle is subject only to a linear focusing force, without acceleration: $x'' + k_x^2(z)x = 0$

take the average over the entire particle ensemble $\left\langle xx'' \right\rangle = -k^2\left\langle x^2 \right\rangle$

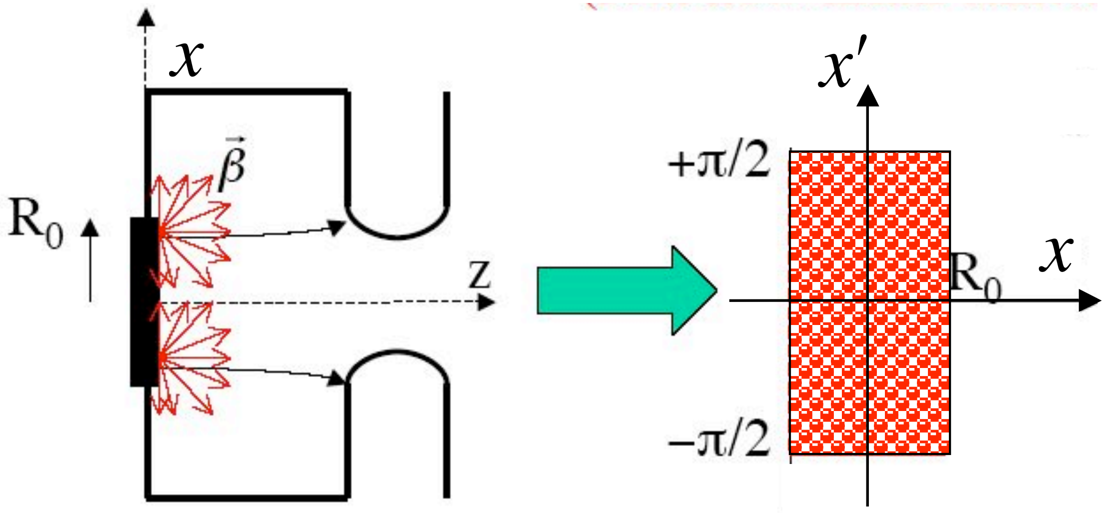
$$\boxed{\sigma_x'' + k_x^2\sigma_x = \frac{\varepsilon_{rms}^2}{\sigma_x^3}}$$

We obtain the rms envelope equation in which the rms emittance enters as defocusing pressure like term

Thermal emittance

$$\boxed{\varepsilon_{rms} = \langle\beta\gamma\rangle\sqrt{\sigma_x^2\sigma_{x'}^2 - \sigma_{xx'}{}^2}}$$

Emittance evaluation close to the cathode sourface:



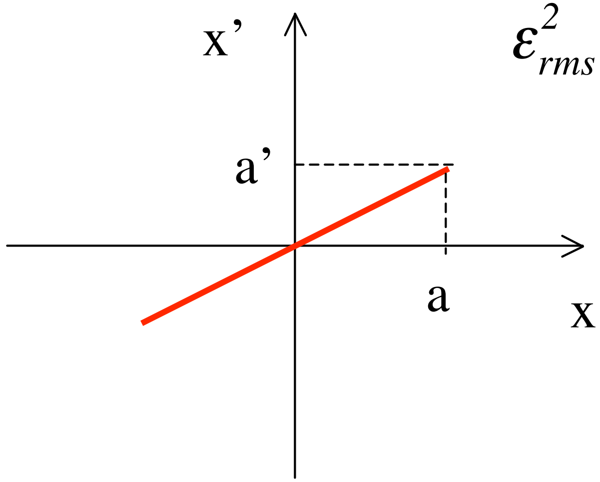
$$\sigma_{xx'} = 0$$

$$\sigma_{x'} = \frac{\pi}{4}$$

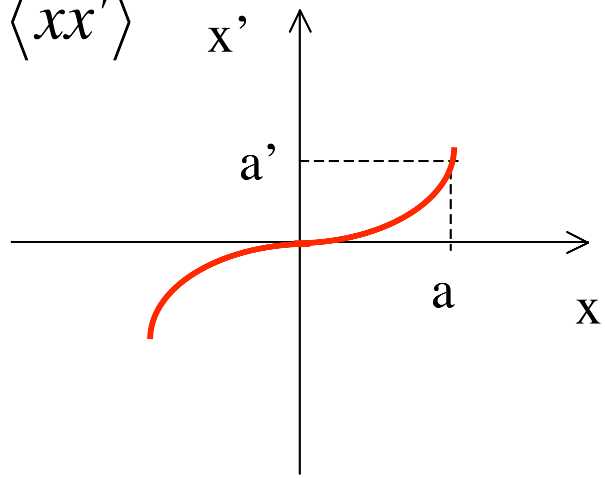
$$\langle\beta\gamma\rangle = \langle\beta\rangle = \sqrt{\frac{2E_{kin}}{m_o c^2}}$$

$$\boxed{\varepsilon_{rms}^{th} = \langle\beta\rangle\sigma_x\sigma_{x'} = \sigma_x\frac{\pi\langle\beta\rangle}{4}}$$

What does rms emittance tell us about phase space distributions under linear or non-linear forces acting on the beam?



$$\varepsilon_{rms}^2 = \langle x^2 \rangle \langle x'^2 \rangle - \langle x x' \rangle^2$$



Assuming a generic x, x' correlation of the type: $x' = Cx^n$

$$\varepsilon_{rms}^2 = C^2 \left(\langle x^2 \rangle \langle x^{2n} \rangle - \langle x^{n+1} \rangle^2 \right)$$

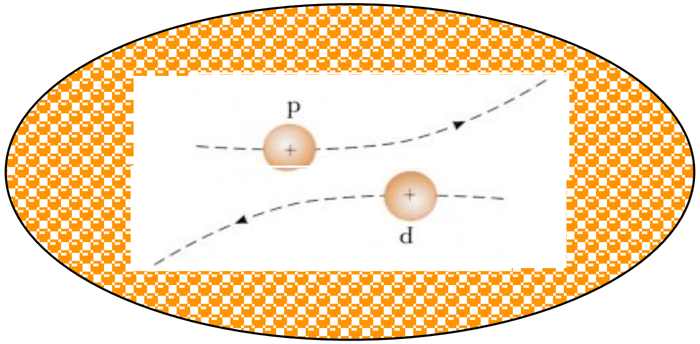
When $n = 1$ ==> $\varepsilon_{rms} = 0$

When $n \neq 1$ ==> $\varepsilon_{rms} \neq 0$

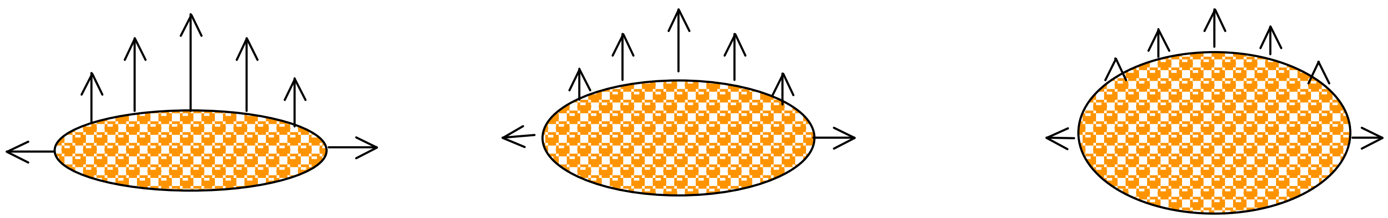
Space Charge: What does it mean?

The net effect of the **Coulomb** interactions in a multi-particle system can be classified into two regimes:

1) **Collisional Regime** ==> dominated by **binary collisions** caused by close particle encounters ==> **Single Particle Effects**



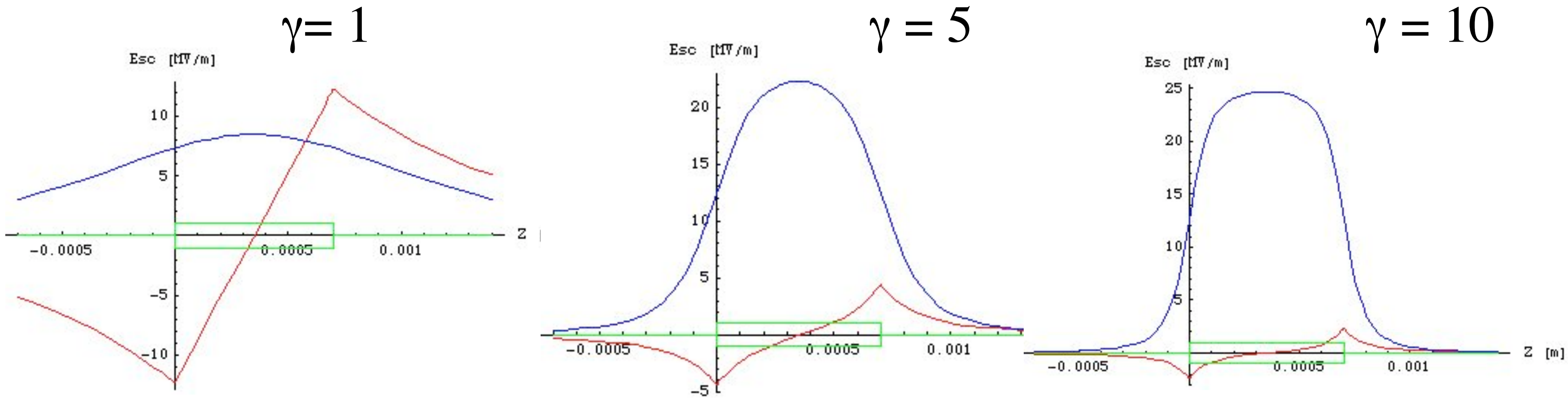
2) **Space Charge Regime** ==> dominated by the **self field** produced by the particle distribution, which varies appreciably only over large distances compare to the average separation of the particles ==> **Collective Effects, Single Component Cold Plasma**

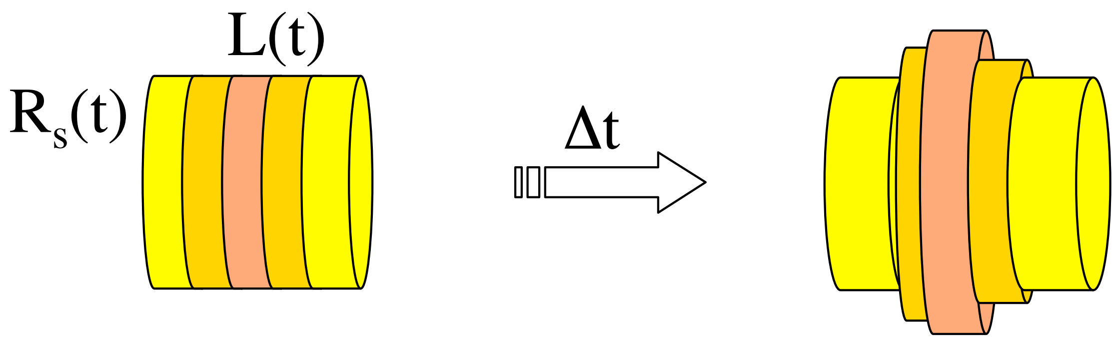


Longitudinal and Transverse Space charge Fields In a uniform charged cilindrical bunch

$$\boxed{E_z(0,s,\gamma) = \frac{I}{2\pi\gamma\varepsilon_0 R^2 \beta c} h(s,\gamma)}$$

$$\boxed{E_r(r,s,\gamma) = \frac{Ir}{2\pi\varepsilon_0 R^2 \beta c} g(s,\gamma)}$$





$$\boxed{B_\vartheta = \frac{\beta}{c} E_r}$$

Lorentz Force

$$\boxed{E_r(r,s,\gamma) = \frac{Ir}{2\pi\varepsilon_0 R^2 \beta c} g(s,\gamma)}$$

$$\boxed{F_r = e\left(E_r - \beta c B_\vartheta\right) = e\left(1-\beta^2\right)E_r = \frac{eE_r}{\gamma^2}}$$

is a **linear** function of the transverse coordinate

$$\boxed{\frac{dp_r}{dt} = F_r = \frac{eE_r}{\gamma^2} = \frac{eIr}{2\pi\gamma^2\varepsilon_0 R^2 \beta c} g(s,\gamma)}$$

The attractive magnetic force , which becomes significant at high velocities, tends to compensate for the repulsive electric force. Therefore space charge defocusing is primarily a non-relativistic effect.

$$\boxed{F_x = \frac{eIx}{2\pi\gamma^2\varepsilon_0\sigma_x^2 \beta c} g(s,\gamma)}$$

Envelope Equation with Space Charge

Single particle transverse motion:

$$\frac{dp_x}{dt} = F_x \qquad p_x = p_o x' = \beta\gamma m_o c x'$$

$$\frac{d}{dt}\left(p_o x'\right) = \beta c \frac{d}{dz}\left(p_o x'\right) = F_x$$

$$x'' = \frac{F_x}{\beta c p_o}$$

$$\boxed{x'' = \frac{k_{sc}\left(s,\gamma\right)}{\sigma_x^2}\, x}$$

Space Charge de-focusing force

Generalized perveance

$$k_{sc}\left(s,\gamma\right) = \frac{2I}{I_A\left(\beta\gamma\right)^3}\, g\left(s,\gamma\right)$$

$$I_A = \frac{4\pi\varepsilon_o m_o c^3}{e} = 17kA$$

Now we can calculate the term $\langle xx'' \rangle$ that enters in the envelope equation

$$\langle xx'' \rangle = \frac{k_{sc}}{\sigma_x^2} \langle x^2 \rangle = k_{sc}$$

$$\boxed{\sigma_x'' = \frac{\sigma_x^2 \sigma_{x'}^2 - \sigma_{xx'}^2}{\sigma_x^3} - \frac{\langle xx'' \rangle}{\sigma_x} = \frac{\varepsilon_{rms}^2}{\sigma_x^3} - \frac{\langle xx'' \rangle}{\sigma_x}}$$

Including all the other terms the envelope equation reads:

Space Charge De-focusing Force

$$\boxed{\sigma_x'' + k^2 \sigma_x = \frac{\varepsilon_n^2}{(\beta\gamma)^2 \sigma_x^3} + \frac{k_{sc}}{\sigma_x}}$$

Emittance Pressure

External Focusing Forces

Laminarity Parameter: $\boxed{\rho = \frac{(\beta\gamma)^2 k_{sc} \sigma_x^2}{\varepsilon_n^2}}$

The beam undergoes two regimes along the accelerator

$$\sigma_x'' + k^2\sigma_x = \cancel{\frac{\varepsilon_n^2}{(\beta\gamma)^2\sigma_x^3}} + \frac{k_{sc}}{\sigma_x}$$

ρ>>1 Laminar Beam

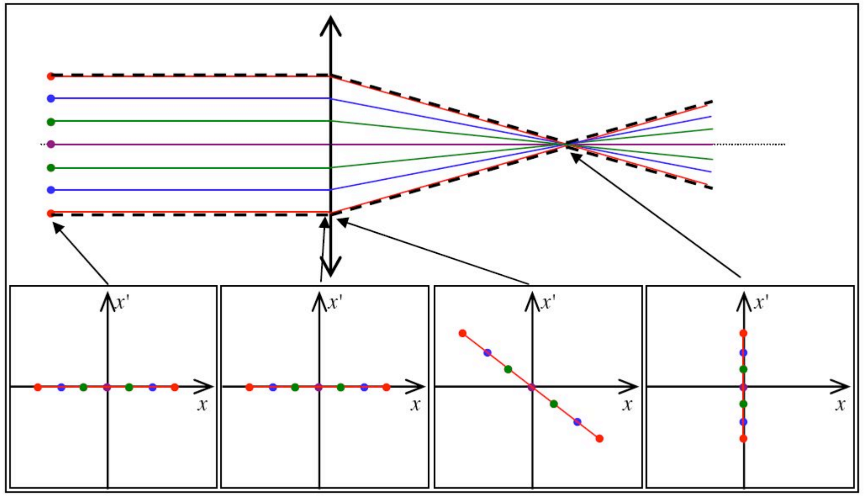


Fig. 10: Particle trajectories in laminar beam

$$\sigma_x'' + k^2\sigma_x = \frac{\varepsilon_n^2}{(\beta\gamma)^2\sigma_x^3} + \cancel{\frac{k_{sc}}{\sigma_x}}$$

ρ<<1 Thermal Beam

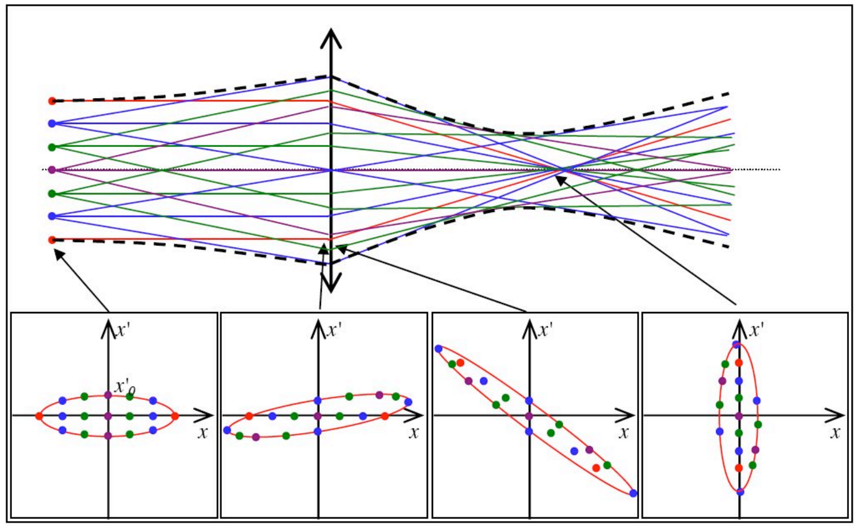


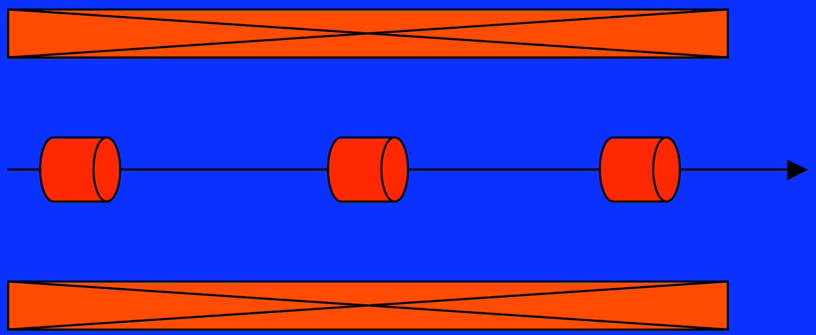
Fig. 11: Particle trajectories in non-zero emittance beam

Space Charge induced emittance oscillations

in a laminar beam

Simple Case: Transport in a Long Solenoid

$$k_s = \frac{qB}{2mc\beta\gamma}$$



$$\sigma'' + k_s^2\sigma = \frac{k_{sc}(s,\gamma)}{\sigma}$$

$\sigma = \sigma_{eq}$ ==> Equilibrium solution ? ==> $\boxed{\sigma_{eq}(s,\gamma) = \frac{\sqrt{k_{sc}(s,\gamma)}}{k_s}}$

Small perturbations around the equilibrium solution

$$\boxed{\sigma'' + k_s^2 \sigma = \frac{k_{sc}(s,\gamma)}{\sigma}}$$

$$\boxed{\sigma(\zeta) = \sigma_{eq}(s) + \delta\sigma(s)}$$

$$\delta\sigma'' + k_s^2\left(\sigma_{eq} + \delta\sigma\right) = \frac{k_{sc}(s,\gamma)}{\sigma_{eq}}\left(1 - \frac{\delta\sigma}{\sigma_{eq}}\right)$$

$$\boxed{\sigma_{eq}(\zeta) = \frac{\sqrt{k_{sc}(s,\gamma)}}{k_s}}$$

$$\boxed{\delta\sigma''(s) + 2k_s^2\,\delta\sigma(s) = 0}$$

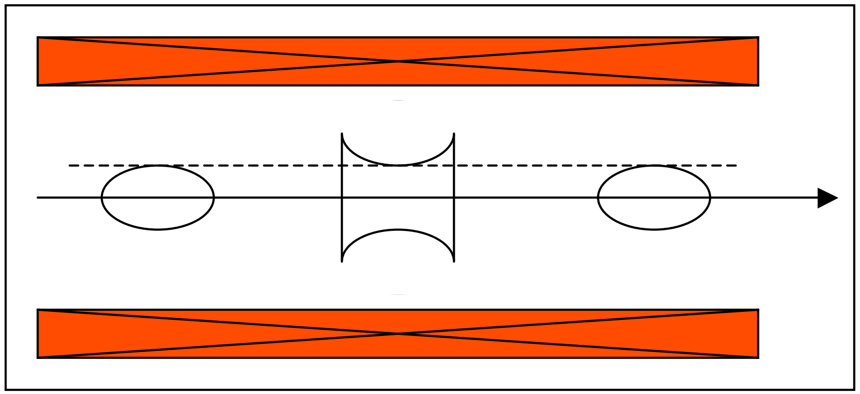
$$\boxed{\delta\sigma'' + 2k_s^2\delta\sigma = 0}$$

$$\sigma = \sigma_{eq} + \delta\sigma$$

Perturbed trajectories oscillate around the equilibrium with the same frequency but with different amplitudes

$$\sigma(s) = \sigma_{eq}(s) + \left(\sigma(s) - \sigma_{eq}(s)\right) cos\left(\sqrt{2}k_s z\right)$$

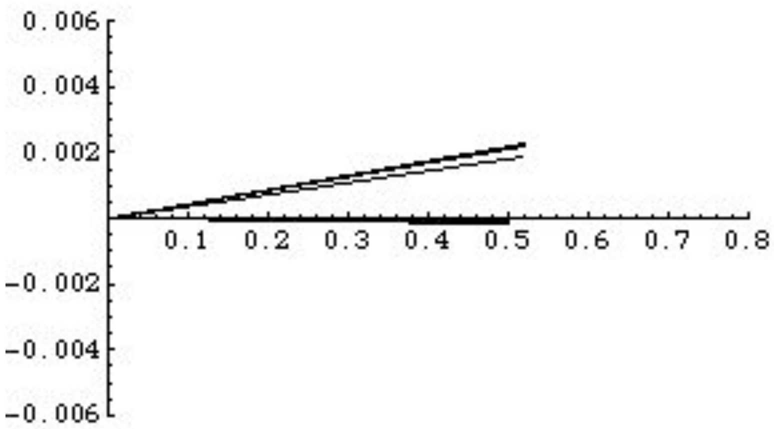
$$\sigma'(s) = -\sqrt{2}k\left(\sigma(s) - \sigma_{eq}(s)\right) sin\left(\sqrt{2}k_s z\right)$$

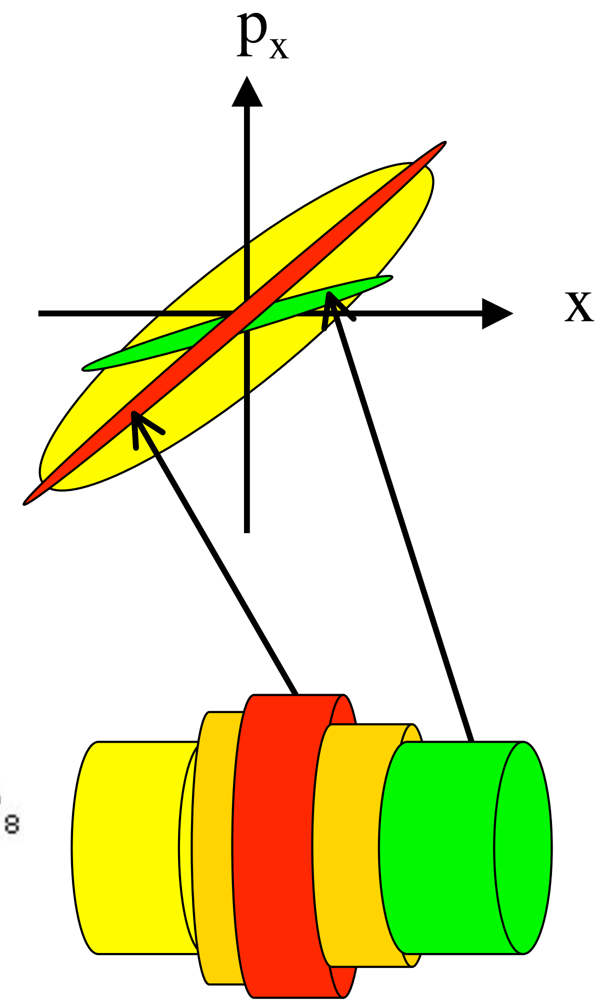


Plasma frequency

Emittance Oscillations are driven by space charge differential defocusing in core and tails of the beam

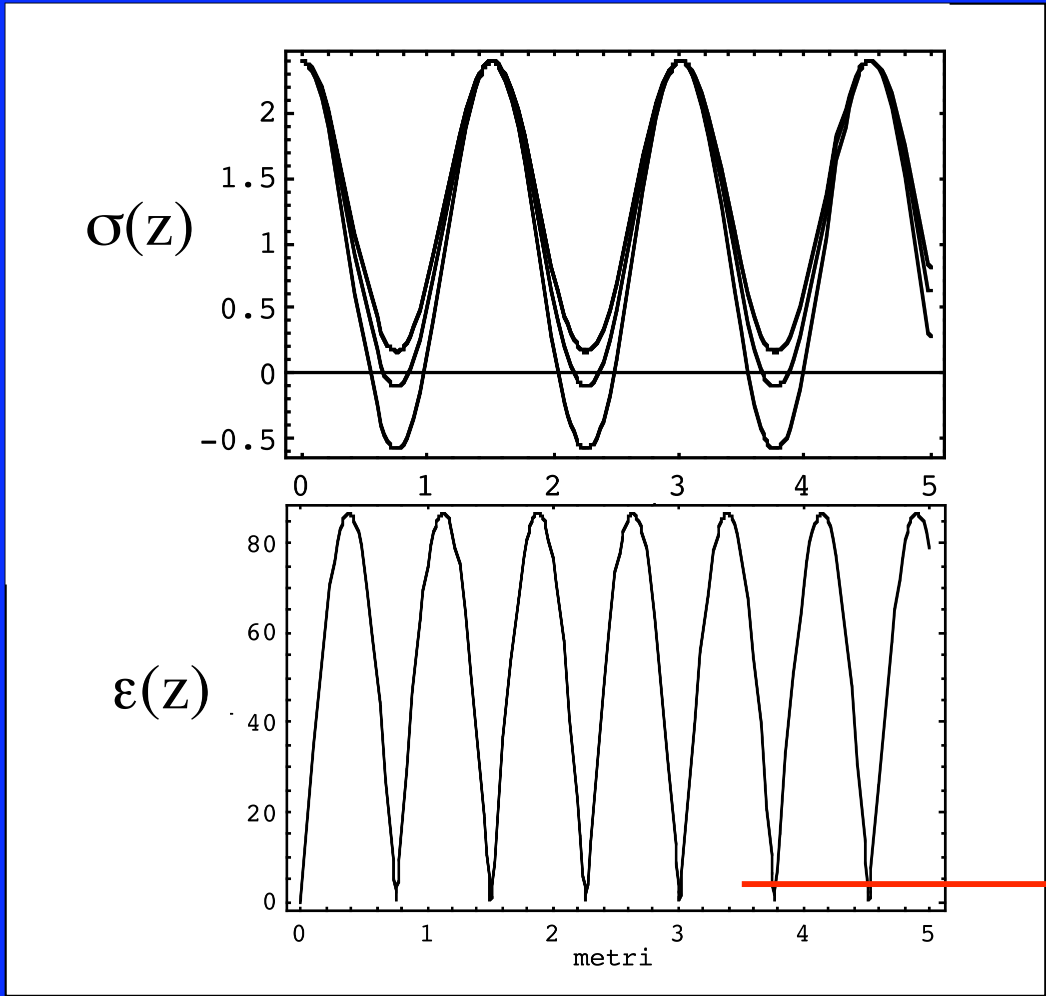
Projected Phase Space





Slice Phase Spaces

Envelope oscillations drive Emittance oscillations

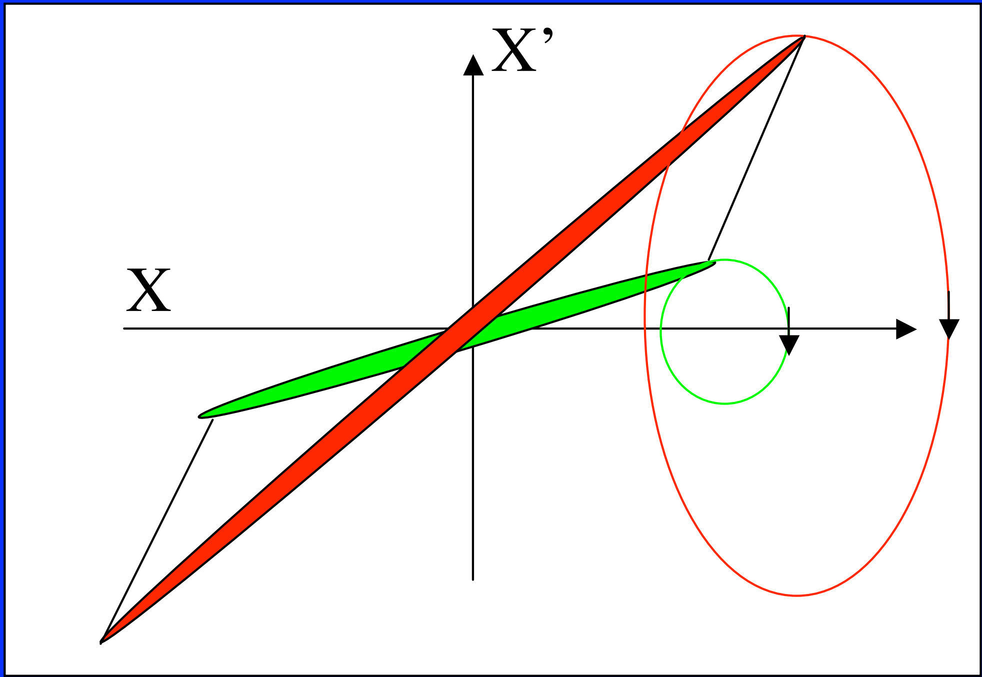


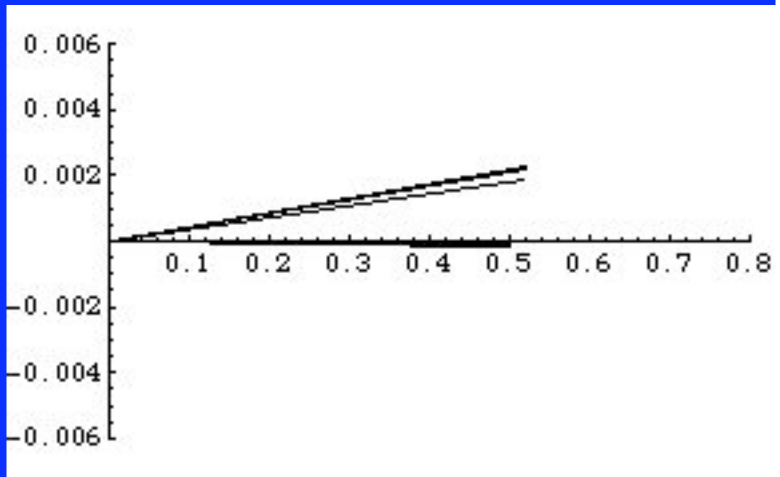
$$\frac{\delta\gamma}{\gamma} = 0$$

$$\sigma' = 0$$

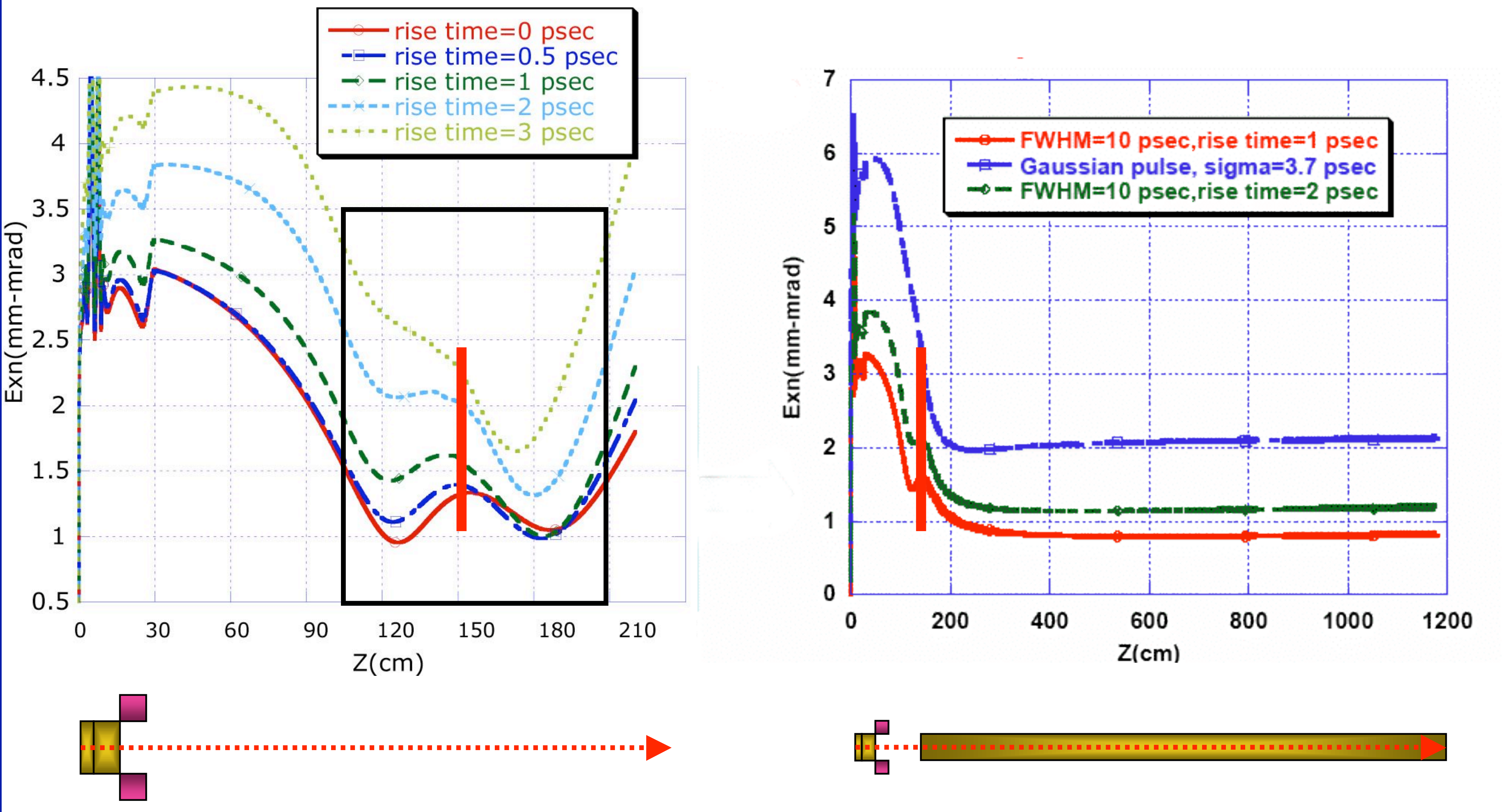
$$\varepsilon_{rms} = \sqrt{\sigma_x^2 \sigma_{x'}^2 - \sigma_{xx'}{}^2} = \sqrt{\left(\left\langle x^2 \right\rangle \left\langle x'^2 \right\rangle - \left\langle xx' \right\rangle^2\right)} \approx \left| sin\left(\sqrt{2} k_s z\right) \right|$$

Perturbed trajectories oscillate around the equilibrium with the same frequency but with different amplitudes





Emittance evolution for different pulse shapes

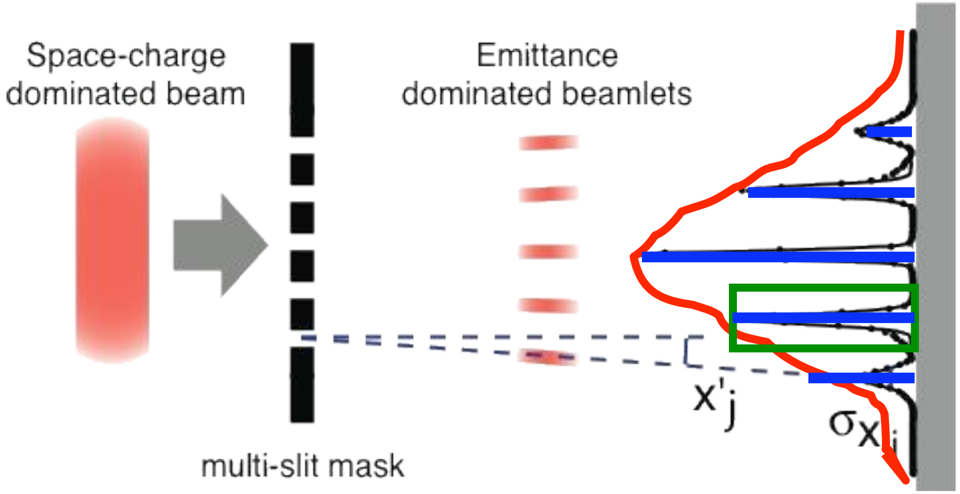


Optimum injection in to the linac with:

$$\sigma' = 0$$

$$\gamma' = \frac{eE_{acc}}{mc^2} = \frac{2}{\sigma}\sqrt{\frac{I}{2\gamma I_A}}$$

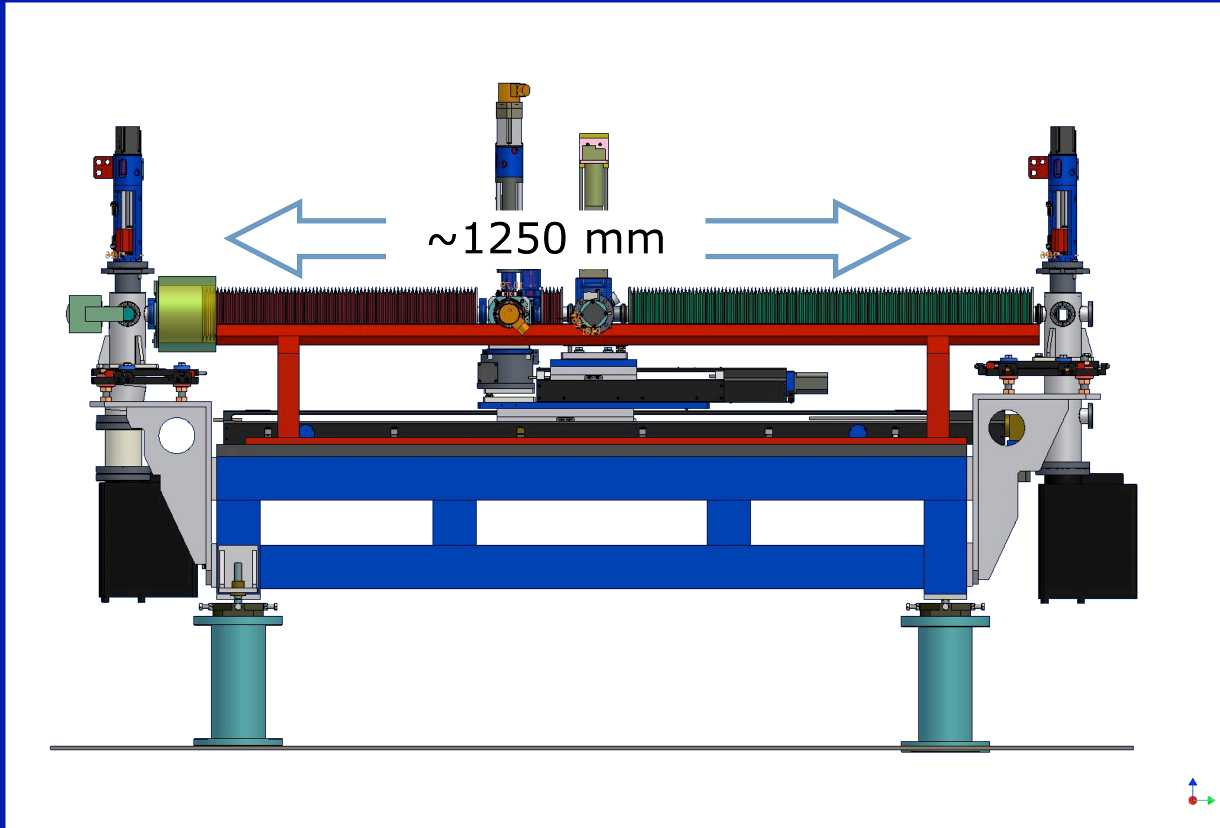
Emittance measurements for space charge dominated beams

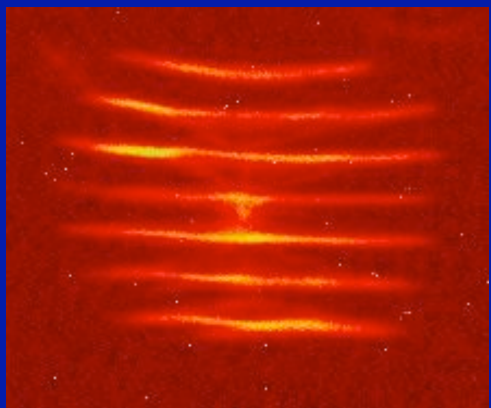


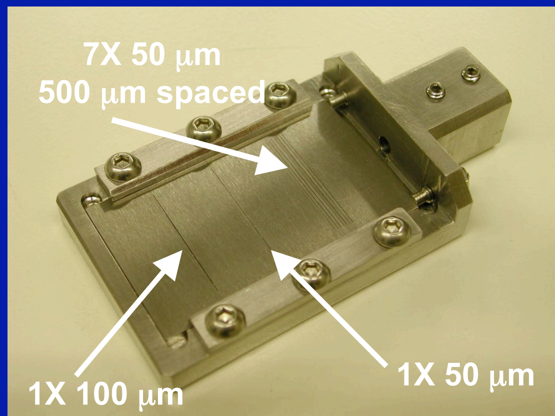
The emittance can be reconstructed from the second momentum of the distribution

$$\varepsilon = \sqrt{\boxed{< x^2 >}\boxed{< x'^2 >} - \boxed{< xx' >}^2}$$

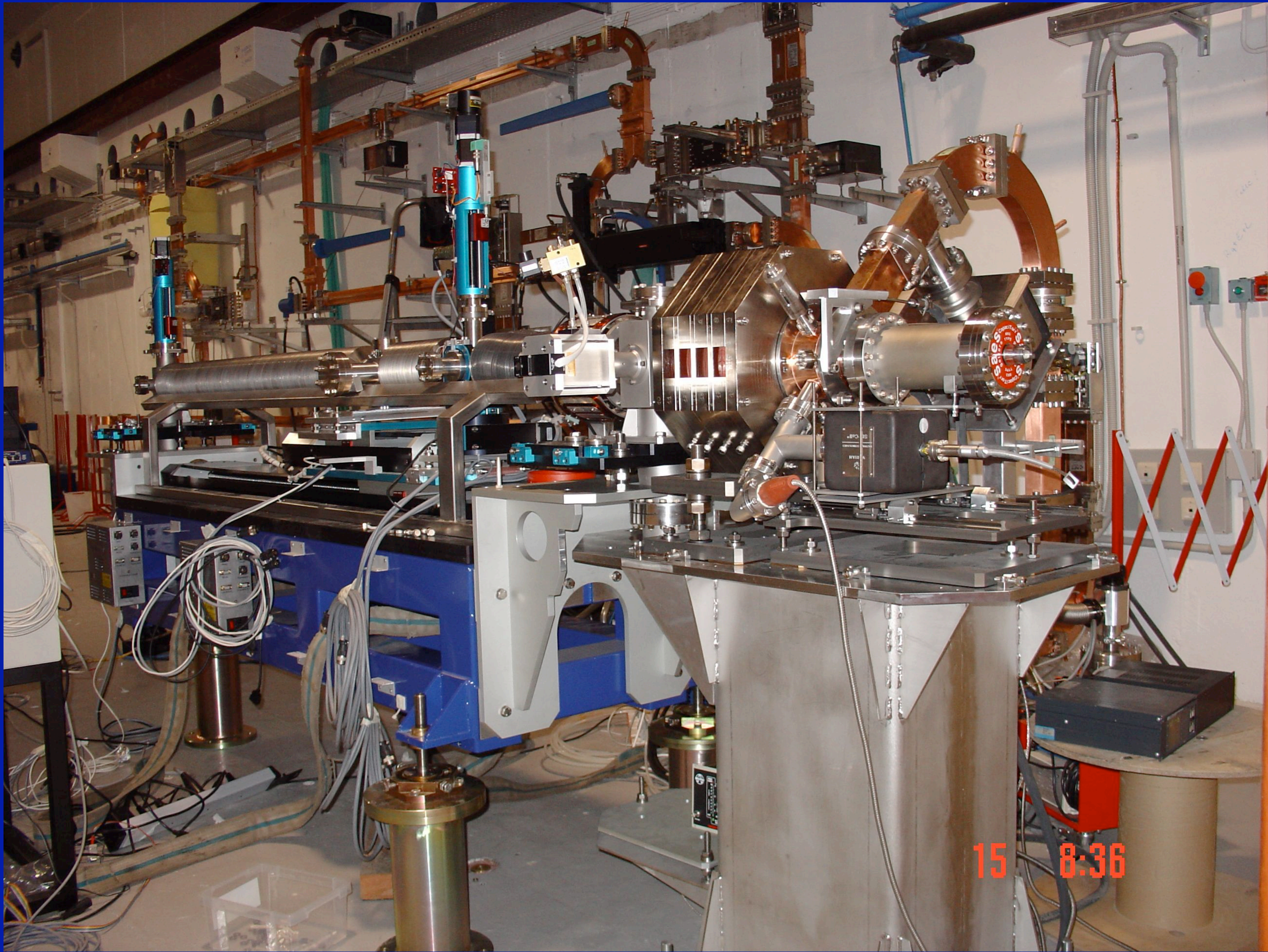
The SPARC Emittance Meter

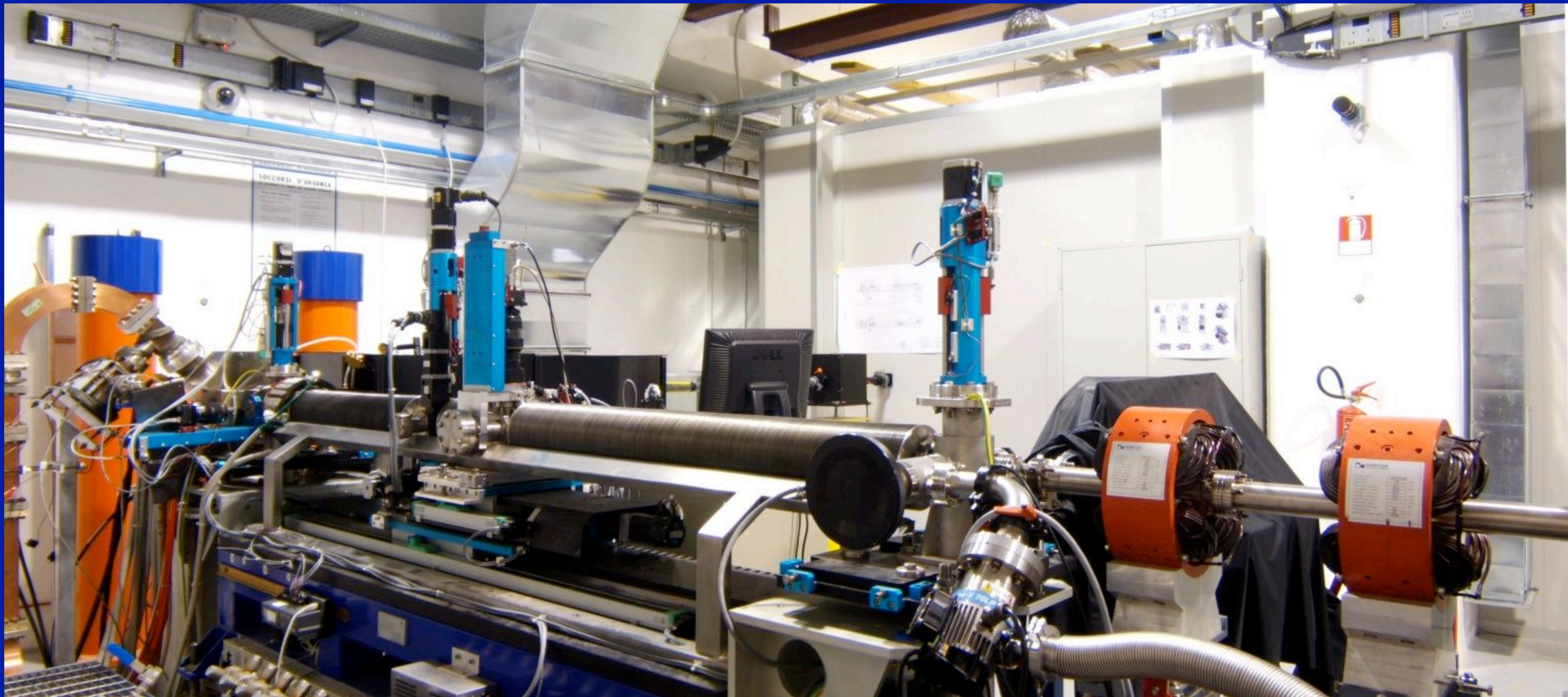




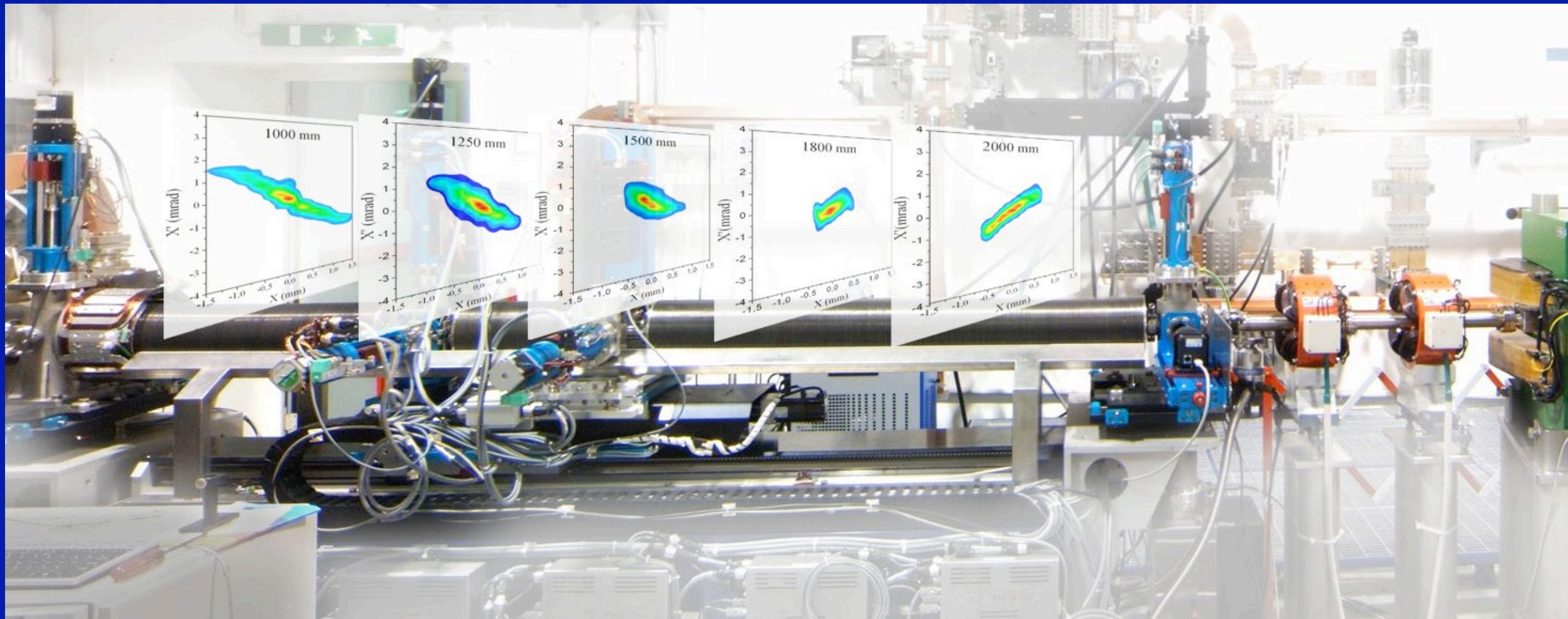


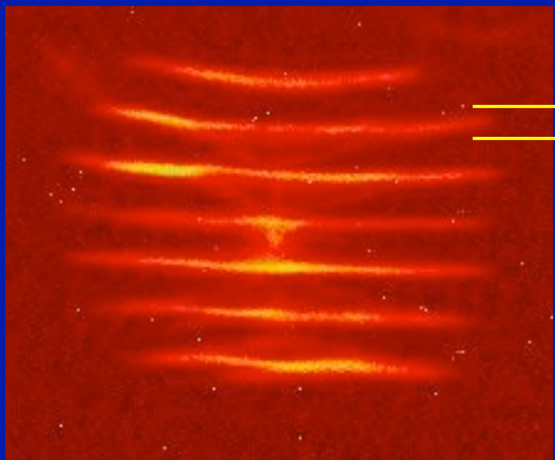
Gun and emittance meter in the SPARC bunker

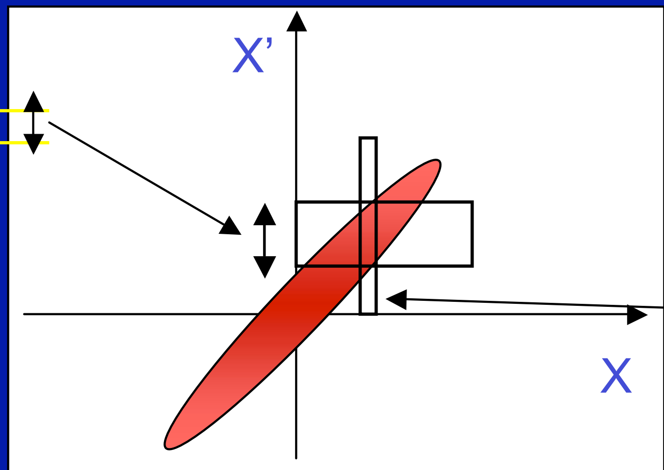


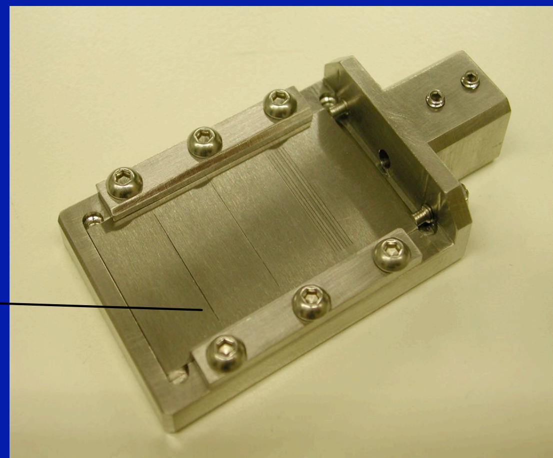


Phase space reconstruction

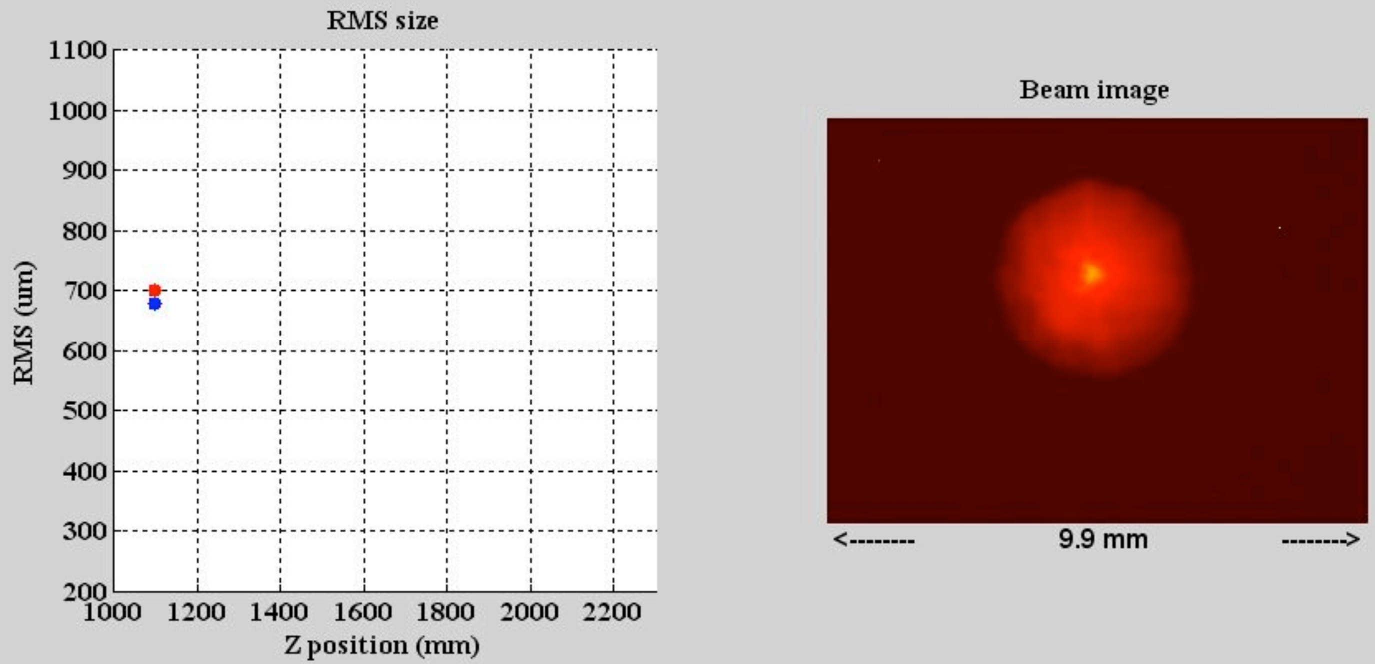




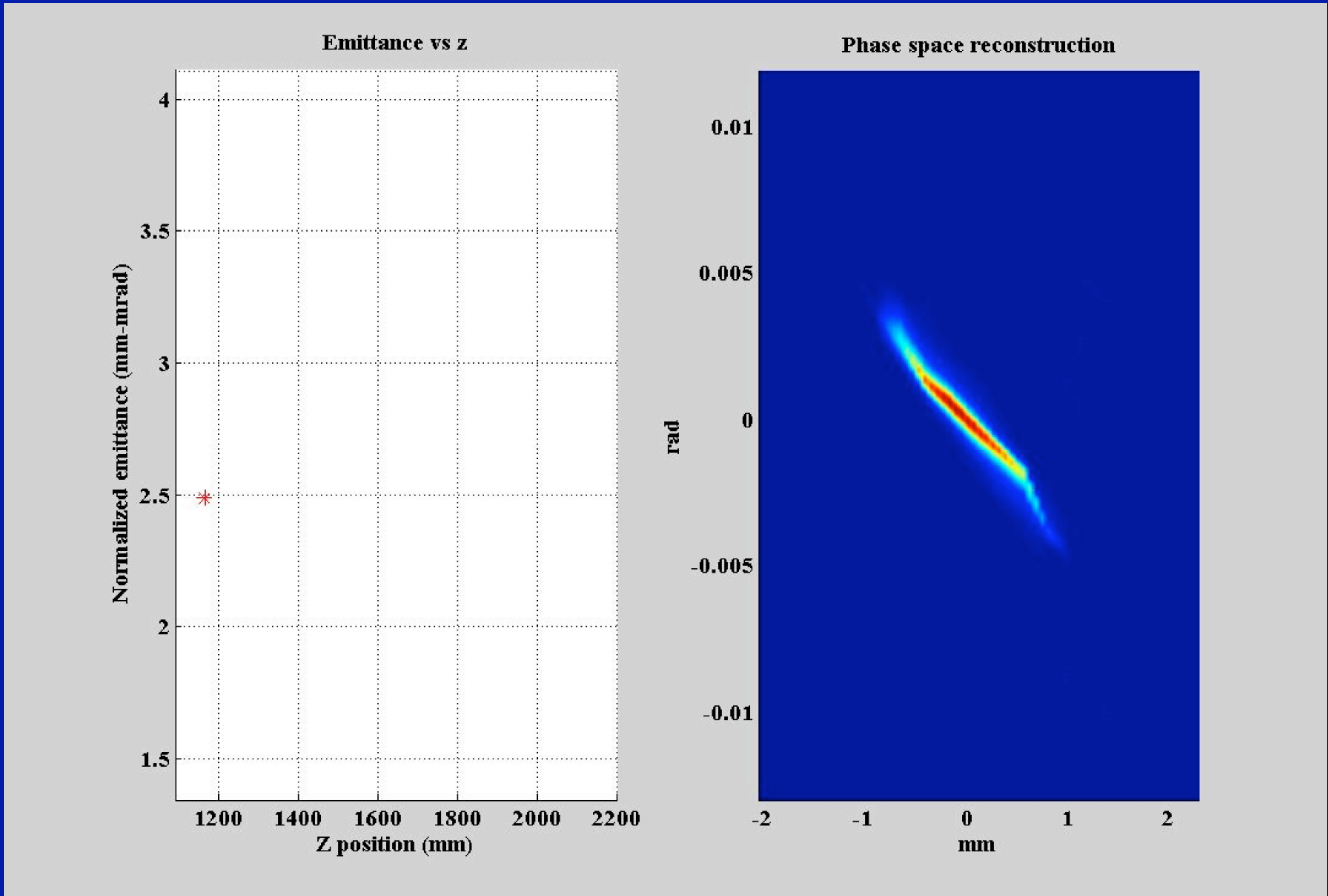




Beam Envelope automatic measurement

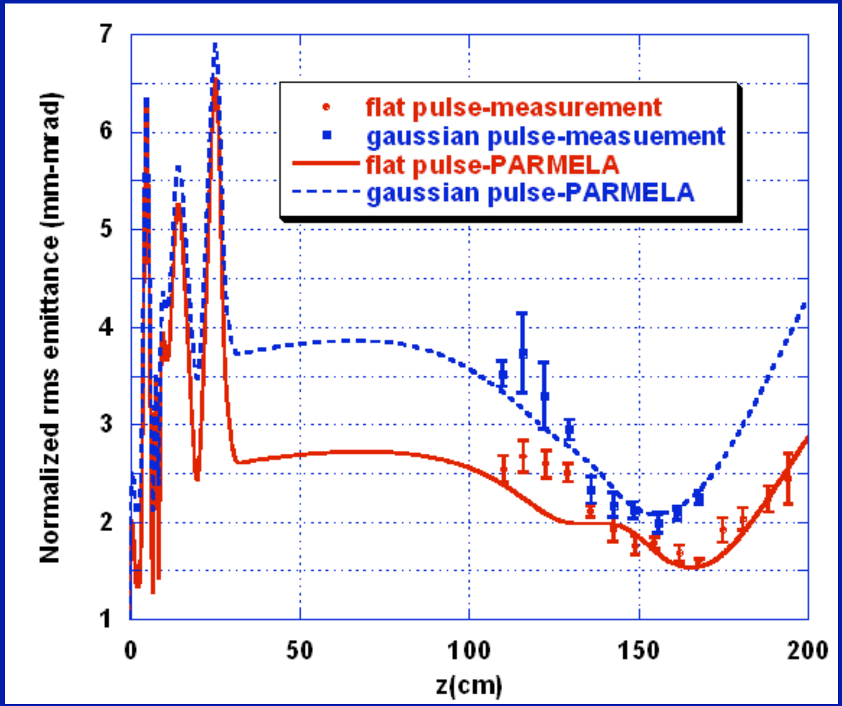


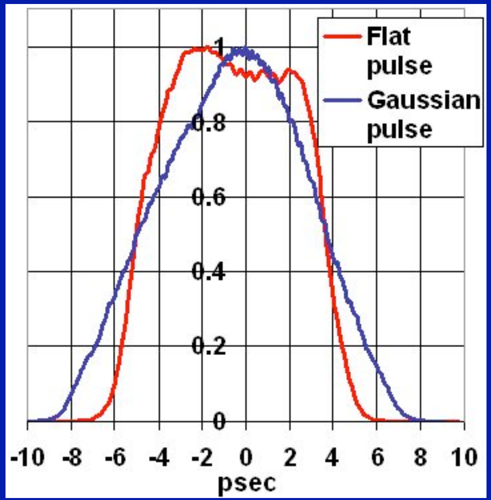
Beam Emittance automatic measurement



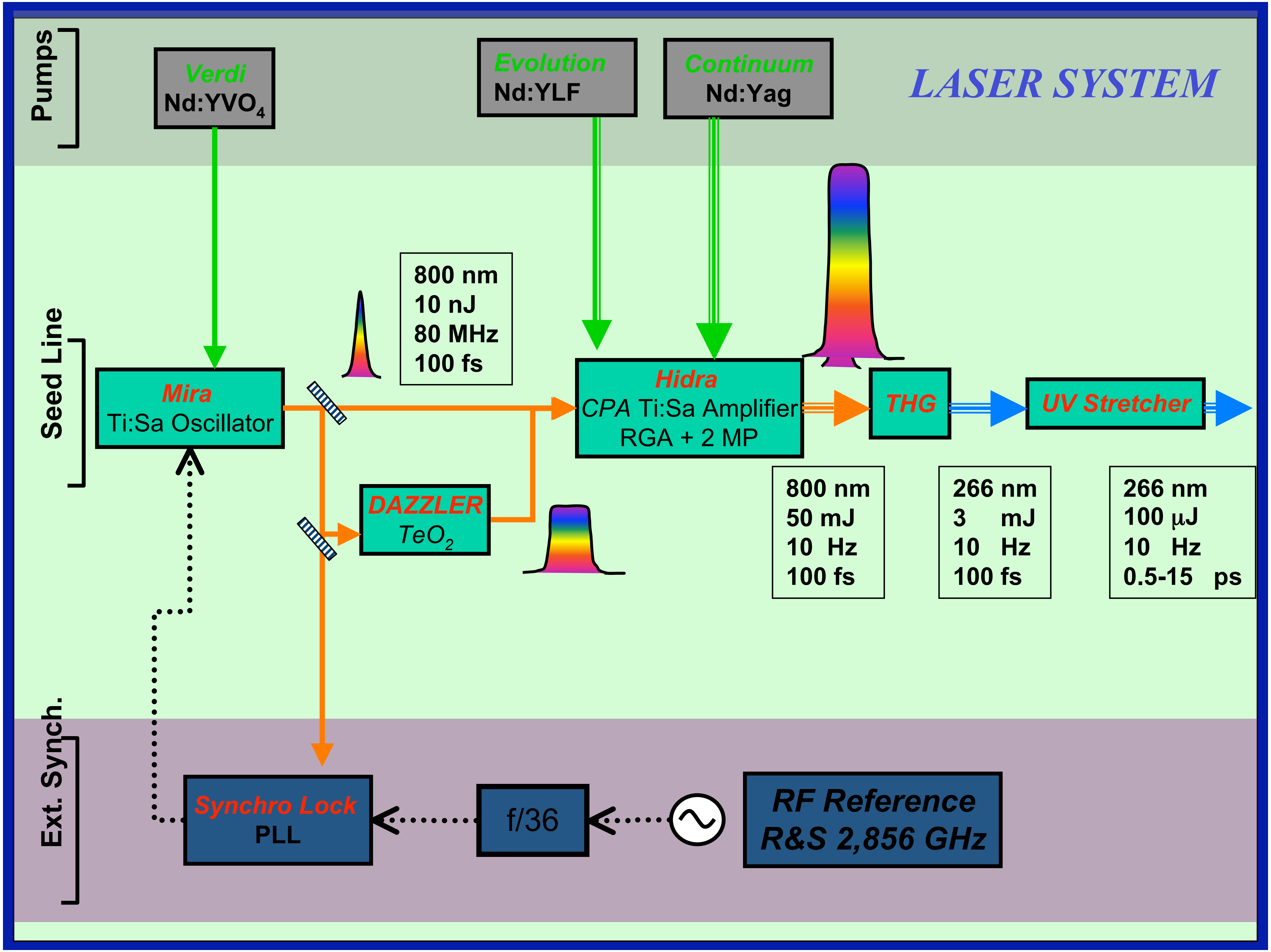
Result highlights

Flat top vs gaussian pulse shape



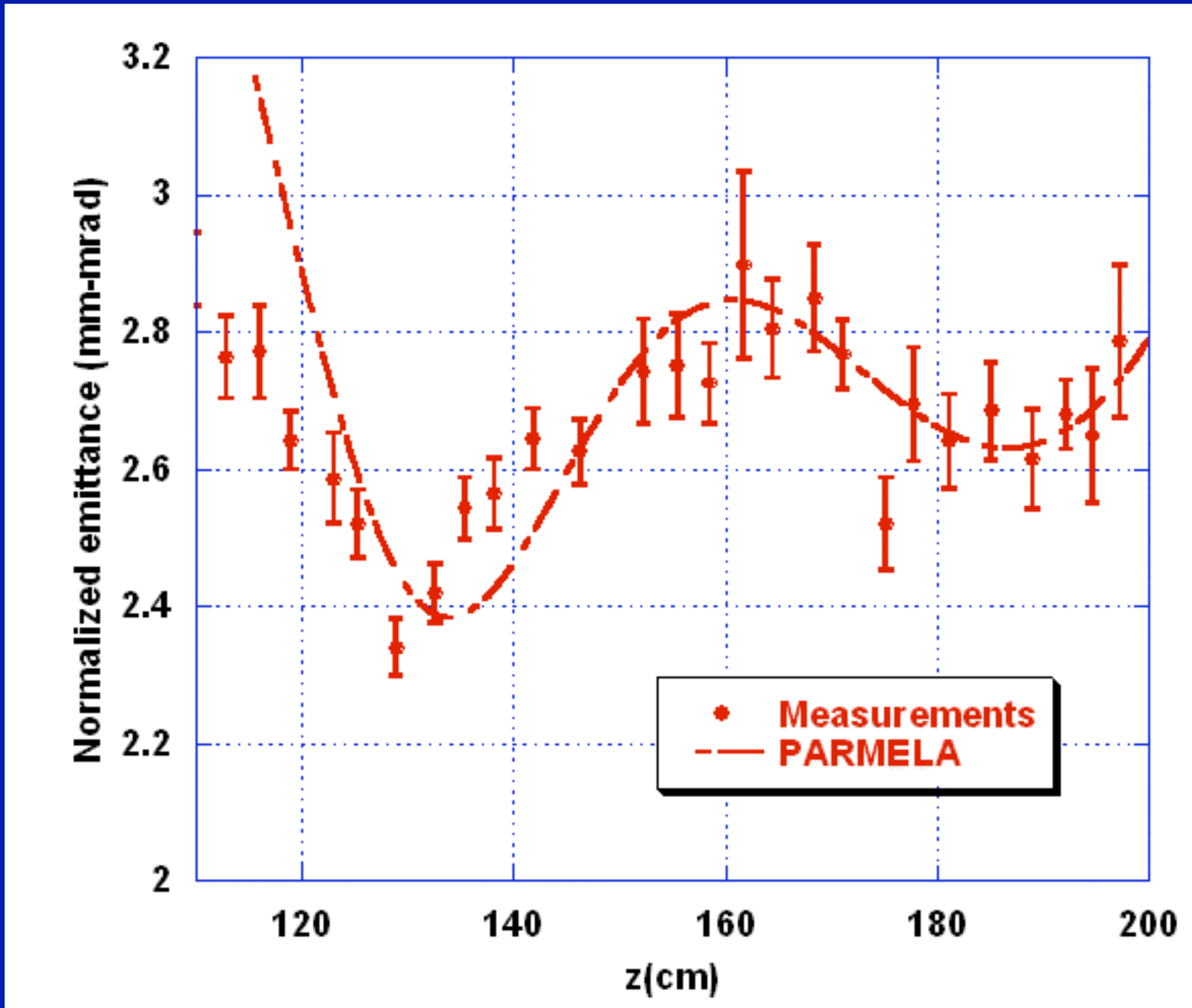


charge	0.74 nC
pulse length (FWHM)	8.7 ps
rise time	2.6 ps
rms spot size	0.31 mm
RF phase ($\varphi-\varphi_{max}$)	-8°



LASER SYSTEM
Pumps
Seed Line
Ext. Synch.
Verdi
Nd:YVO4
Evolution
Nd:YLF
Continuum
Nd:Yag
800 nm
10 nJ
80 MHz
100 fs
Mira
Ti:Sa Oscillator
Hidra
CPA Ti:Sa Amplifier
RGA + 2 MP
THG
UV Stretcher
DAZZLER
TeO2
800 nm
50 mJ
10 Hz
100 fs
266 nm
3 mJ
10 Hz
100 fs
266 nm
100 µJ
10 Hz
0.5-15 ps
Synchro Lock
PLL
f/36
RF Reference
R&S 2,856 GHz

Emittance measurements showing the double minimum



Envelope Equation with Longitudinal Acceleration

$$\boxed{\begin{aligned} p_o &= \gamma_o m_o \beta_o c \\ p_x &<< p_o \\ p &= p_o + p'z \\ p' &= (\beta\gamma)' m_o c \end{aligned}}$$

$$\frac{dp_x}{dt} = 0$$

$$\frac{d}{dt}(px') = \beta c \frac{d}{dz}(px') = 0$$

$$x'' + \frac{p'}{p} x' = 0$$

$$\boxed{x'' + \frac{(\beta\gamma)'}{\beta\gamma} x' = 0}$$

$$\langle xx'' \rangle = \frac{(\beta\gamma)'}{\beta\gamma} \langle xx' \rangle$$

Space Charge De-focusing Force

$$\boxed{\sigma_x'' + \frac{(\beta\gamma)'}{\beta\gamma}\sigma_x' + k^2\sigma_x = \frac{\varepsilon_n^2}{(\beta\gamma)^2\sigma_x^3} + \frac{k_{sc}}{\sigma_x}}$$

Adiabatic Damping

Emittance Pressure

Other External Focusing Forces

$$\boxed{\varepsilon_n = \beta\gamma\varepsilon_{rms}}$$

Beam subject to strong acceleration

$$\boxed{\sigma_x'' + \frac{(\beta\gamma)'}{\beta\gamma}\sigma_x' + k_{RF}^2\sigma_x = \frac{\varepsilon_n^2}{(\beta\gamma)^2\sigma_x^3} + \frac{k_{sc}^o}{(\beta\gamma)^3\sigma_x}}$$

We must include also the RF focusing force $k_{RF} = \frac{1}{4}\left(\frac{\gamma'}{\gamma}\right)$

$$k_{sc}^o = \frac{2I}{I_A} g(s,\gamma)$$

$$\sigma''_x + \frac{(\beta\gamma)'}{\beta\gamma}\sigma'_x + k_{RF}^2\sigma_x = \frac{\varepsilon_n^2}{(\beta\gamma)^2\sigma_x^3} + \frac{k_{sc}^o}{(\beta\gamma)^3\sigma_x}$$

$\gamma = 1 + \alpha z$ ==> $\gamma'' = 0$

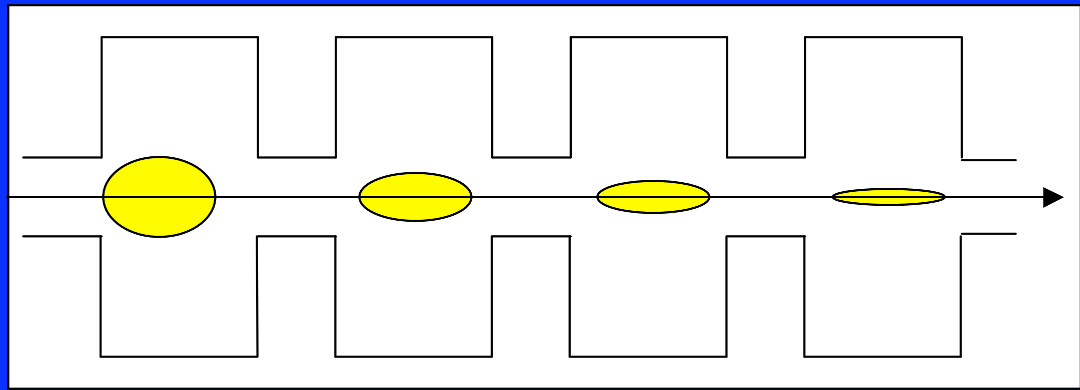
Looking for an "equilibrium" solution $\sigma_{inv} = \sigma_o \gamma^n$
==> all terms must have the same dependence on γ

Laminar beam $\rho >> 1 \Rightarrow n = -\frac{1}{2}$ $\sigma_q = \frac{\sigma_o}{\sqrt{\gamma}}$

Thermal beam $\rho << 1 \Rightarrow n = 0$ $\sigma_\varepsilon = \sigma_o$

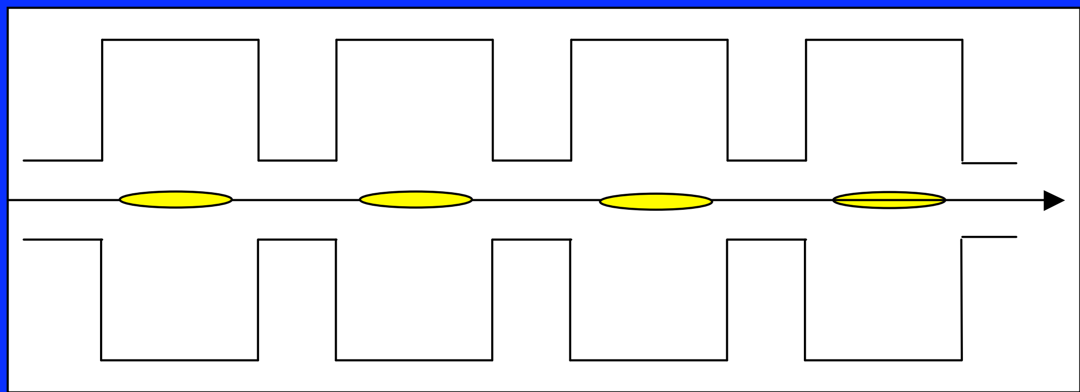
Space charge dominated beam (Laminar)

$$\sigma_q = \frac{1}{\gamma'}\sqrt{\frac{2I}{I_A\gamma}}$$

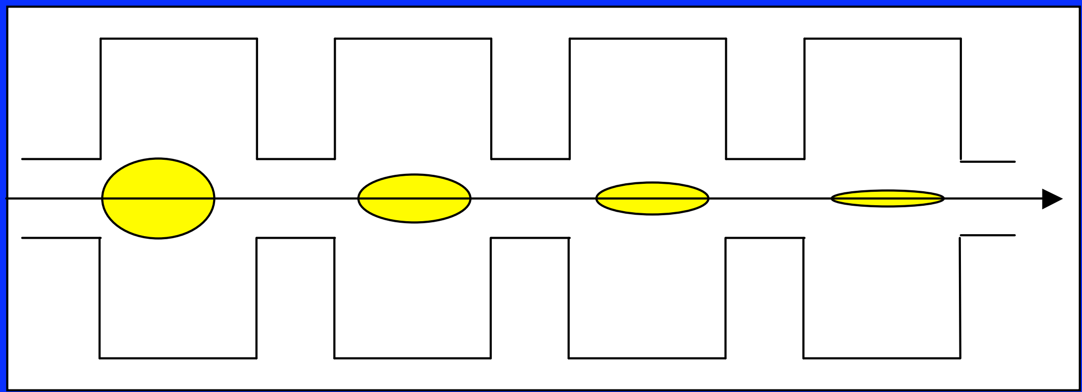


Emittance dominated beam (Thermal)

$$\sigma_\varepsilon = \sqrt{\frac{2\varepsilon_n}{\gamma'}}$$



$$\sigma_q = \frac{1}{\gamma'}\sqrt{\frac{2I}{I_A \gamma}}$$



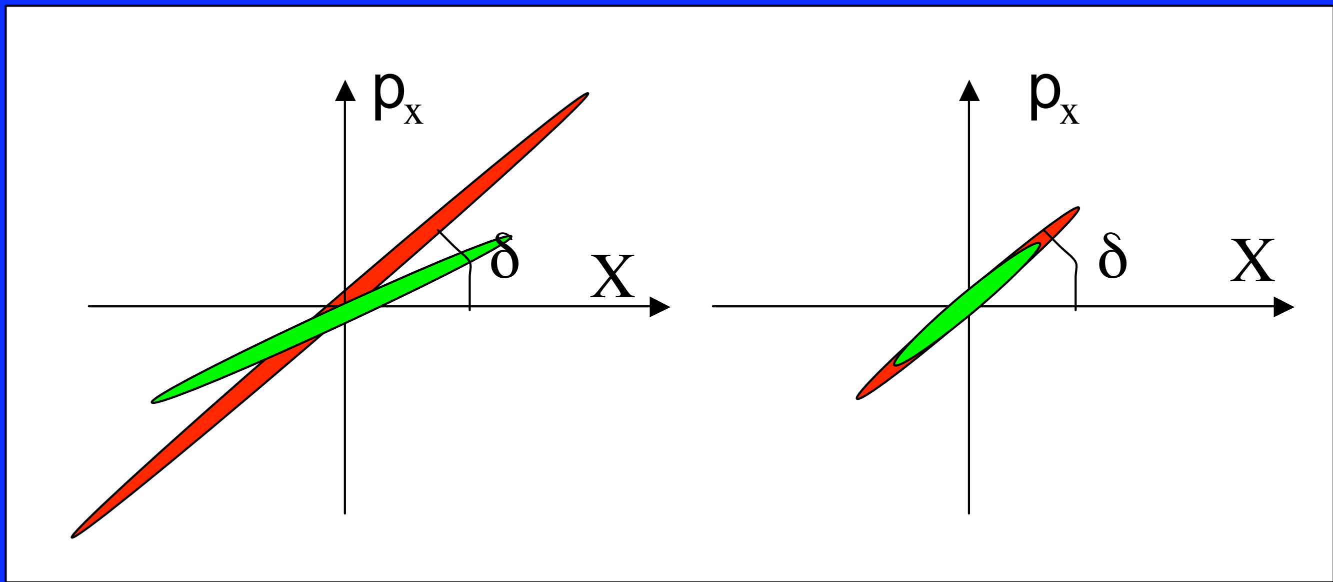
This solution represents a **beam equilibrium mode** that turns out to be the transport mode for achieving minimum emittance at the end of the **emittance correction process**

An important property of the laminar beam

$$\sigma_q = \frac{1}{\gamma'}\sqrt{\frac{2I}{I_A\gamma}}$$

$$\sigma_q' = -\sqrt{\frac{2I}{I_A\gamma^3}}$$

Constant phase space angle: $\delta = \dfrac{\gamma\sigma_q'}{\sigma_q} = -\dfrac{\gamma'}{2}$

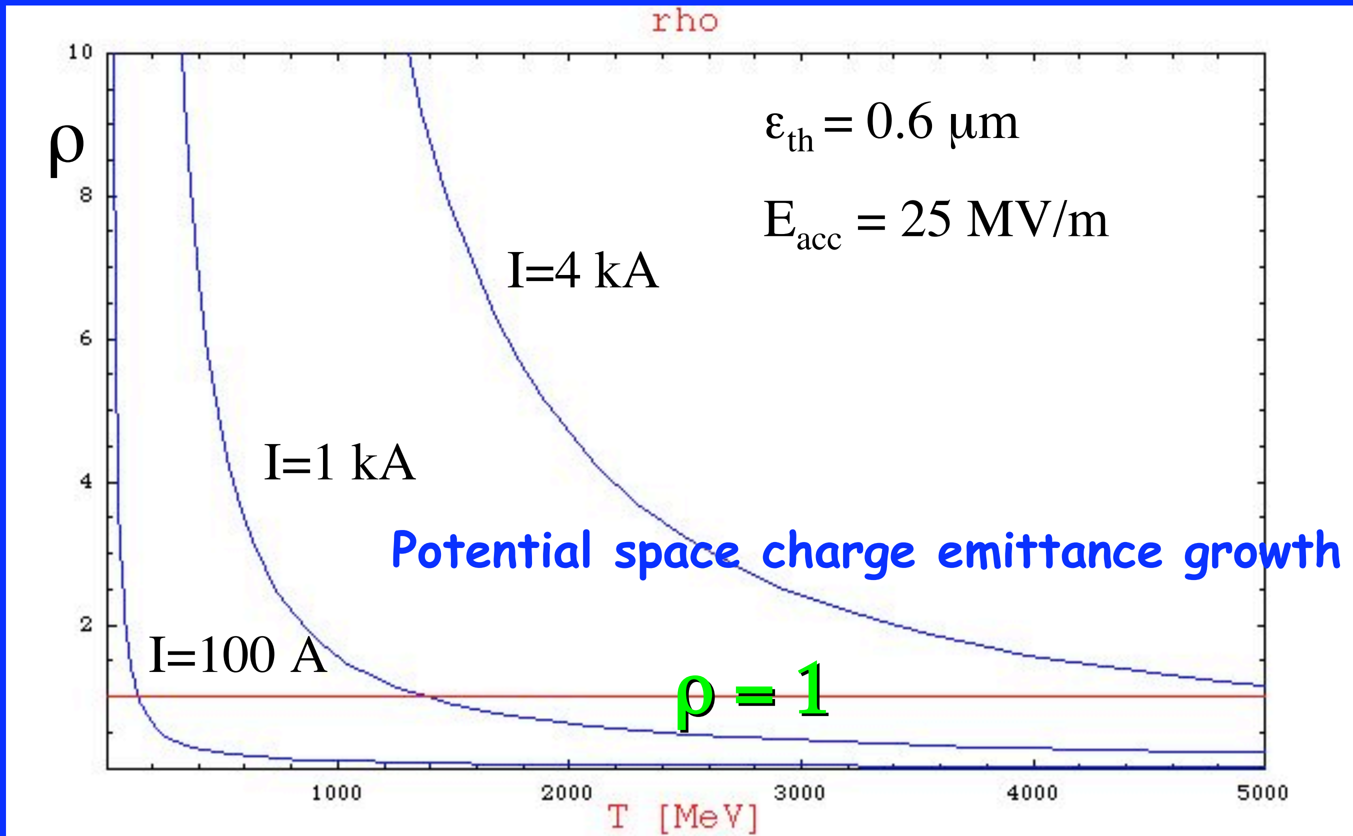


Laminarity parameter

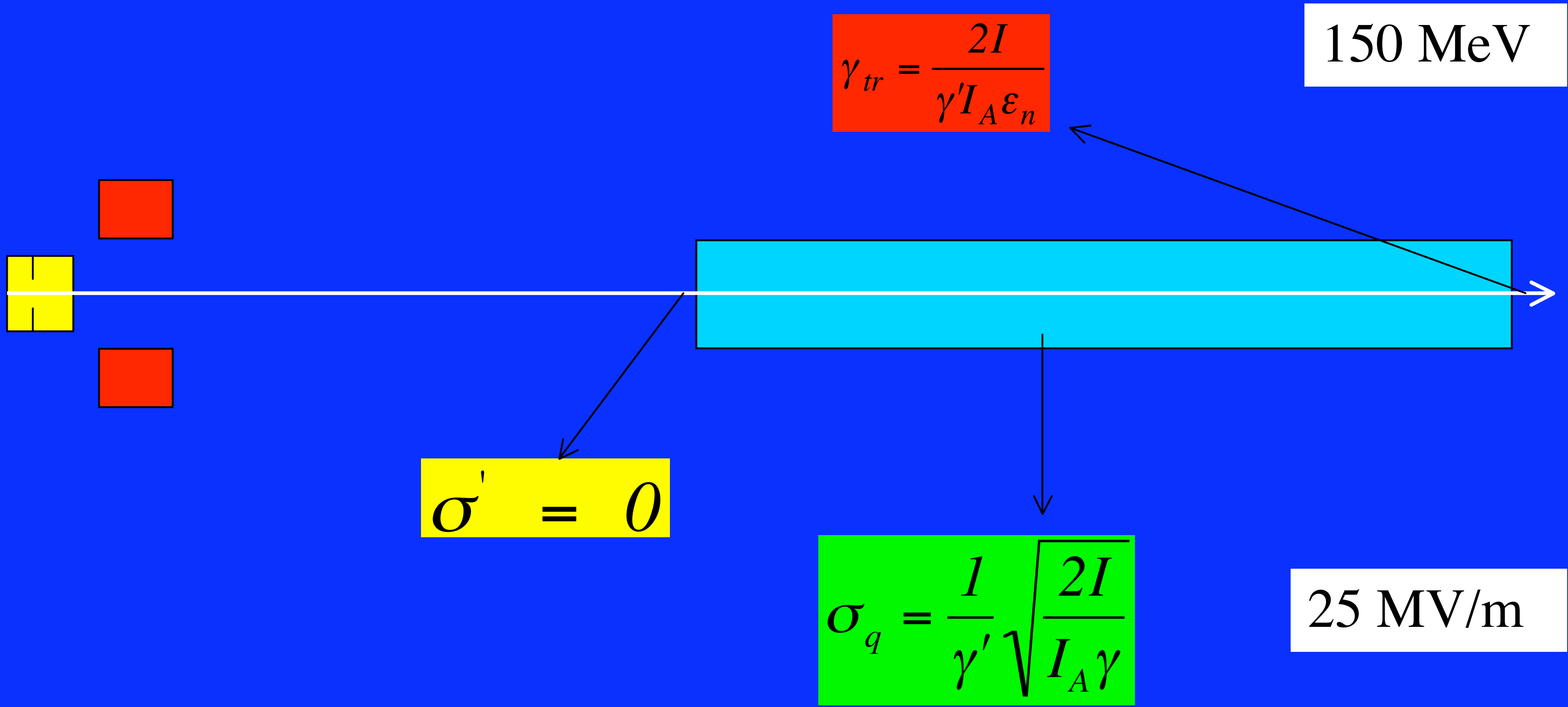
$$\rho = \frac{2I\sigma^2}{\gamma I_A \varepsilon_n^2} \equiv \frac{2I\sigma_q^2}{\gamma I_A \varepsilon_n^2} = \frac{4I^2}{\gamma'^2 I_A^2 \varepsilon_n^2 \gamma^2}$$

Transition Energy (ρ=1)

$$\gamma_{tr} = \frac{2I}{\gamma' I_A \varepsilon_n}$$



Matching Conditions with a TW Linac

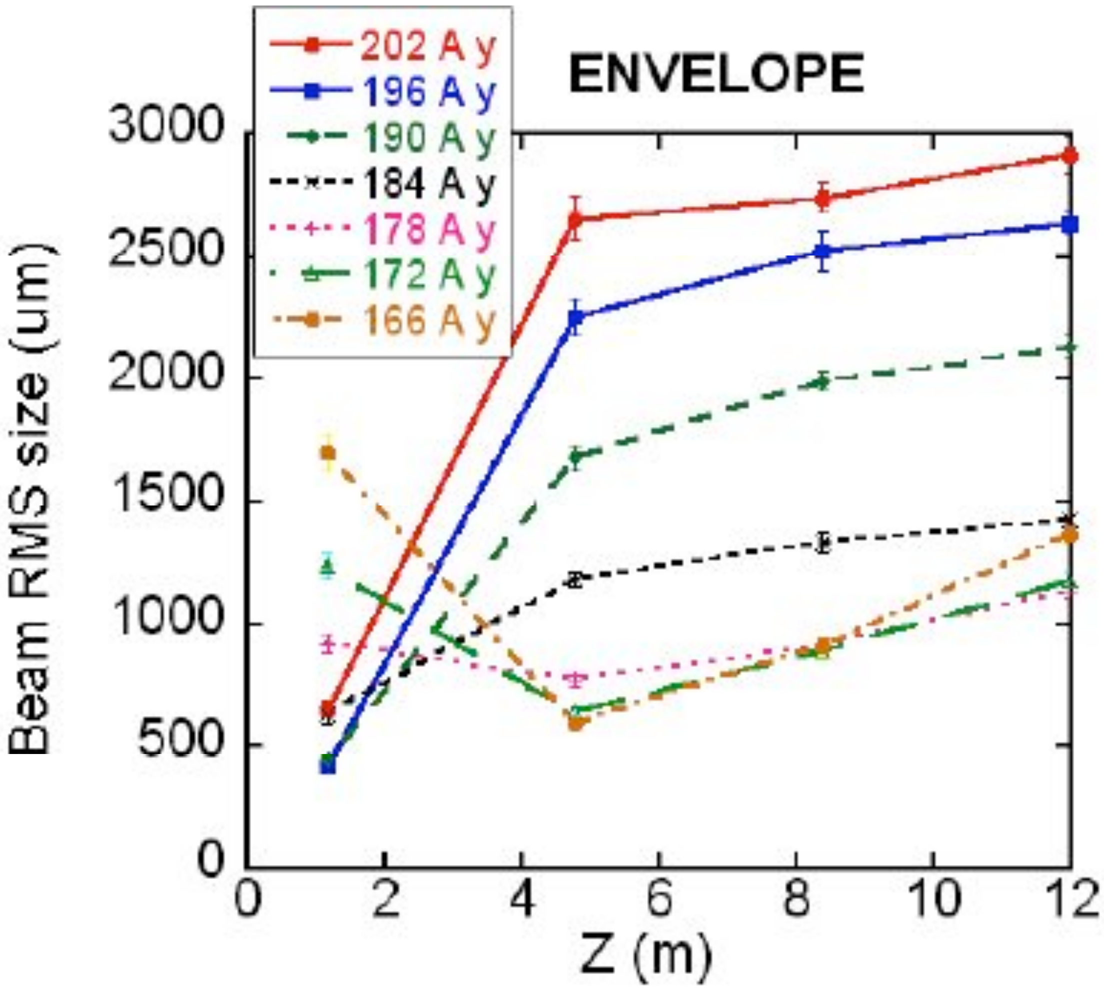


Emittance Compensation in a Photoinjector: Controlled Damping of Plasma Oscillations

- ε_n oscillations are driven by Space Charge

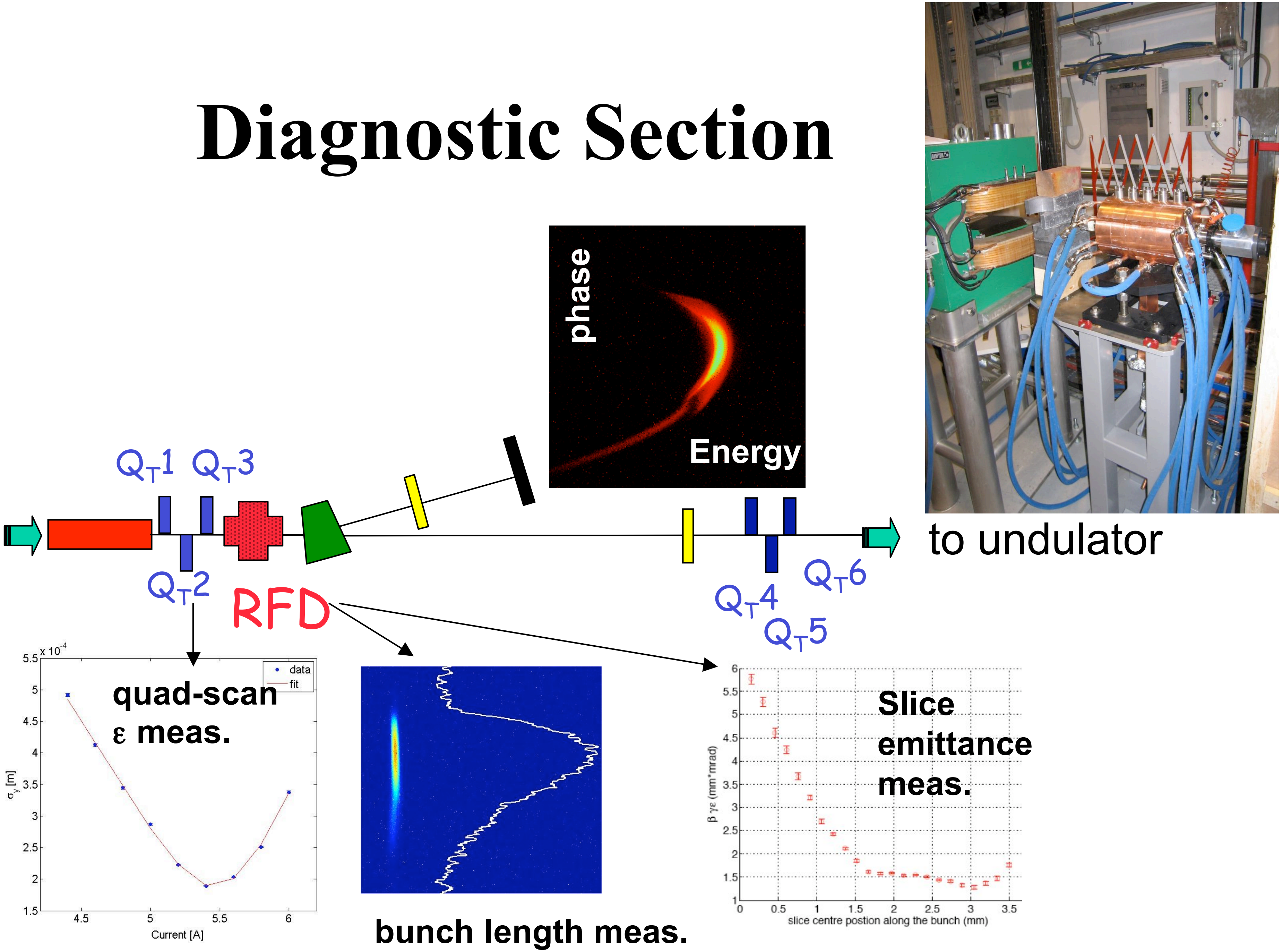
- propagation close to the laminar solution allows control of ε_n oscillation "phase"

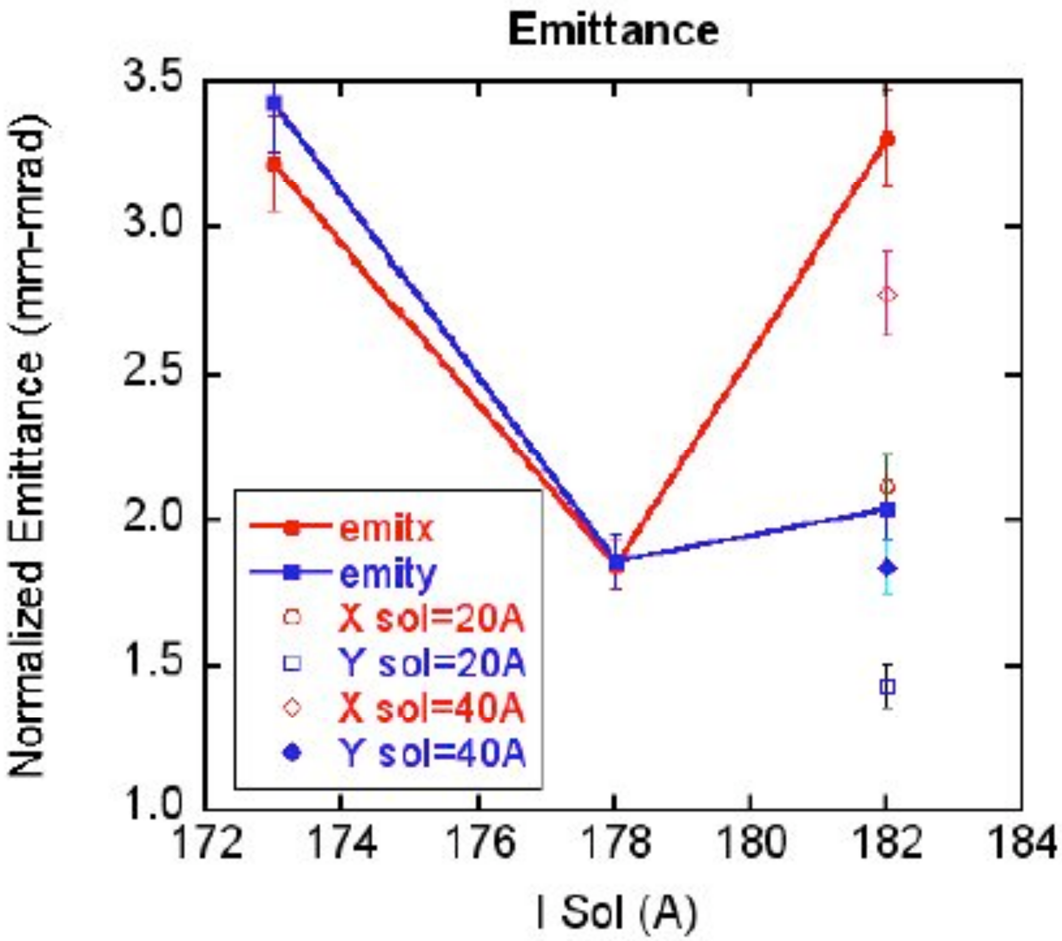
- ε_n sensitive to SC up to the transition energy



Envelope measurements along the linac for different current settings of the Gun Solenoid. The bunch length was 6.5 ps, with 340 pC of charge and a rms size on the virtual cathode of about 340 μm.

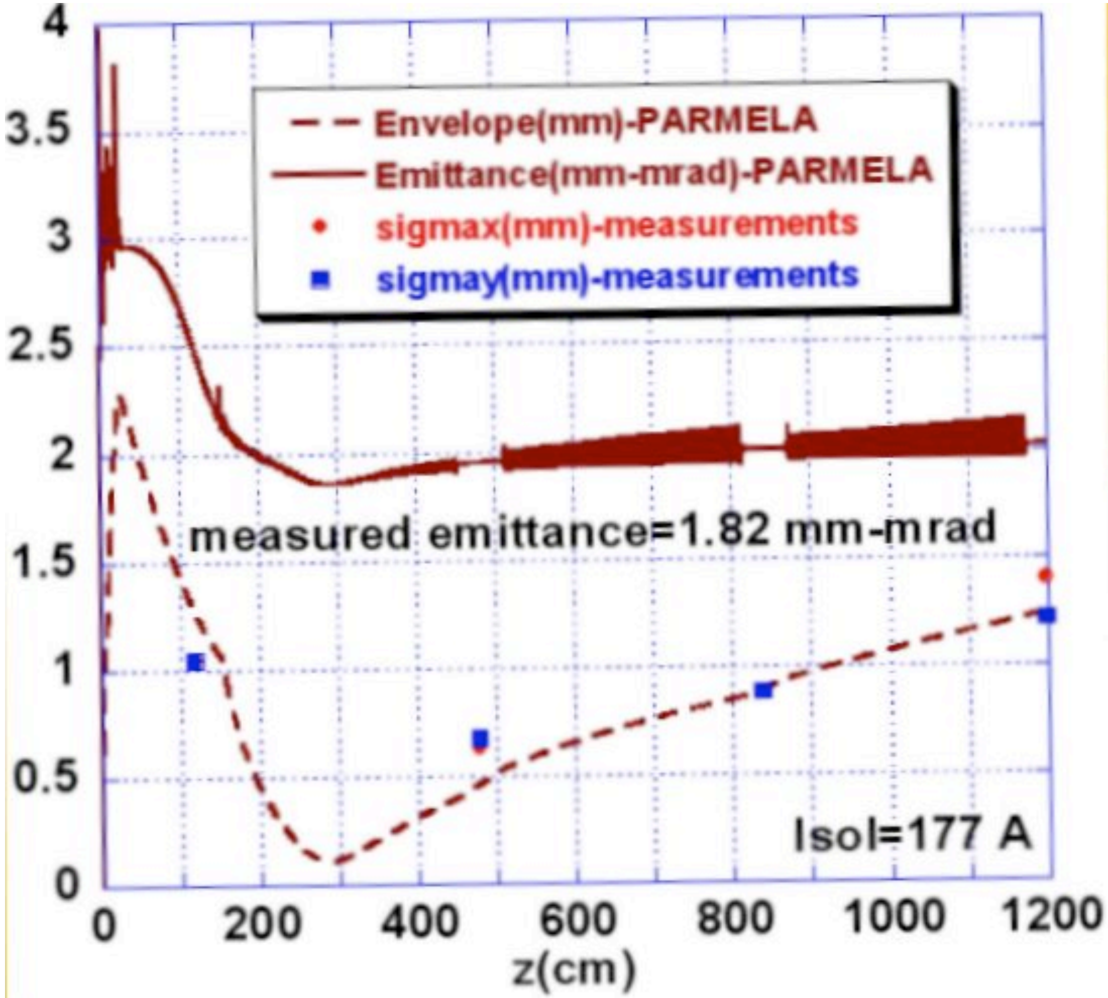
Diagnostic Section



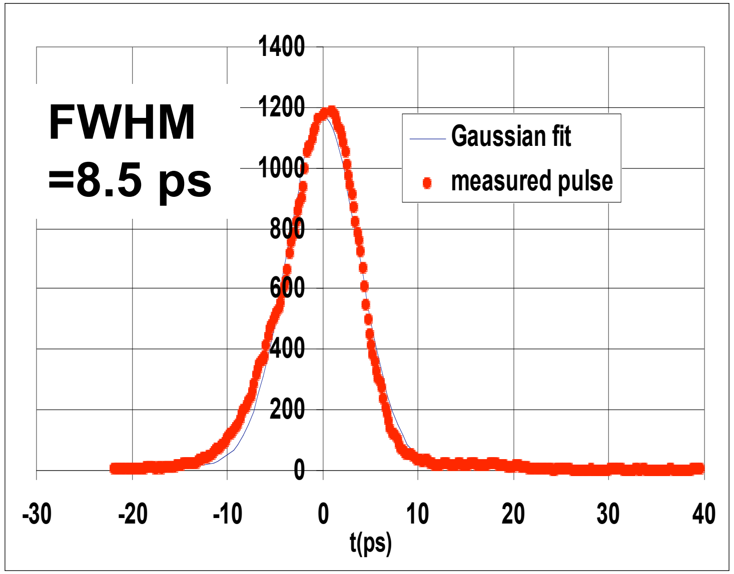


Emittance measurement for different current settings of the gun solenoid for two values (20 A and 40 A) of the coils around the accelerating structures.

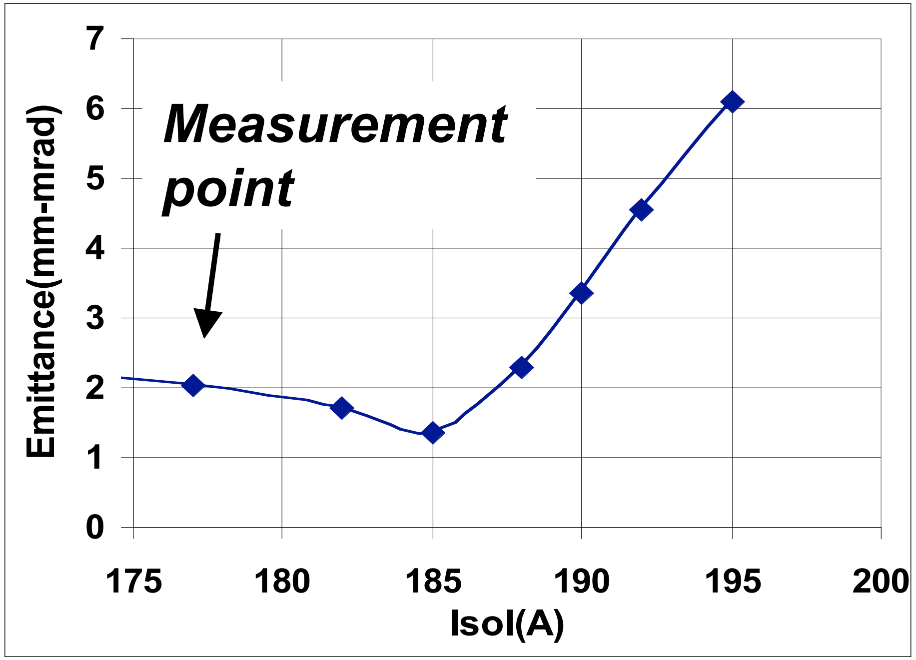
Achieved beam matching with the linac

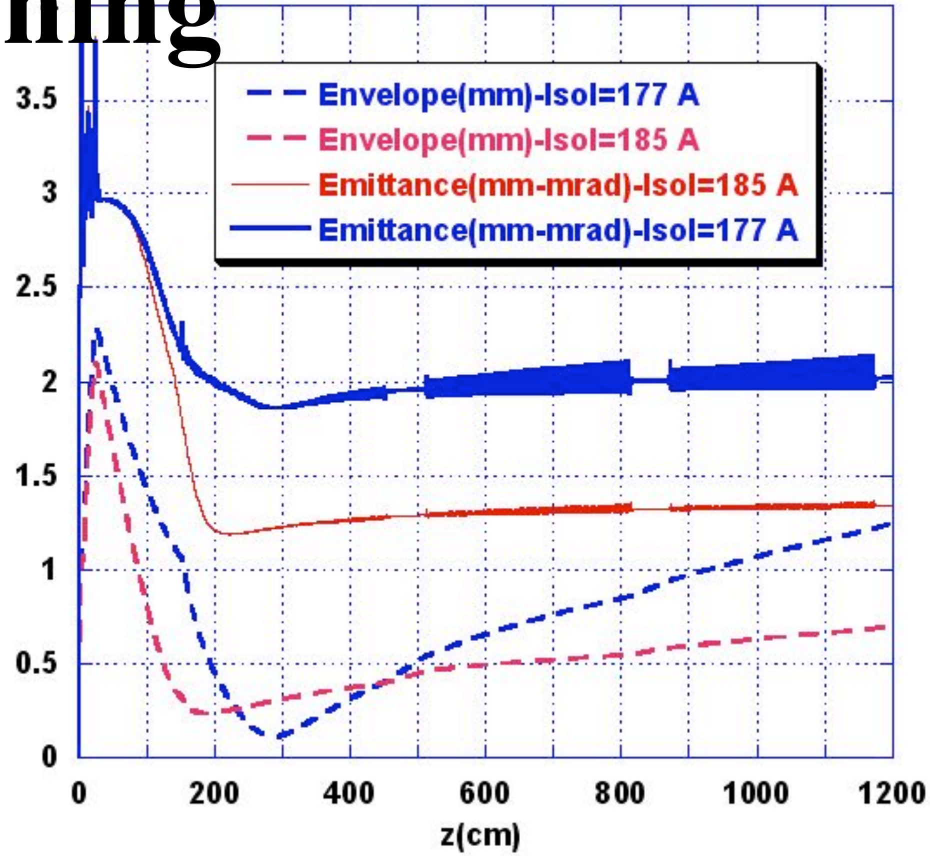


Q=500 pC
Energy=147 MeV



Expected optimized beam matching





THE END

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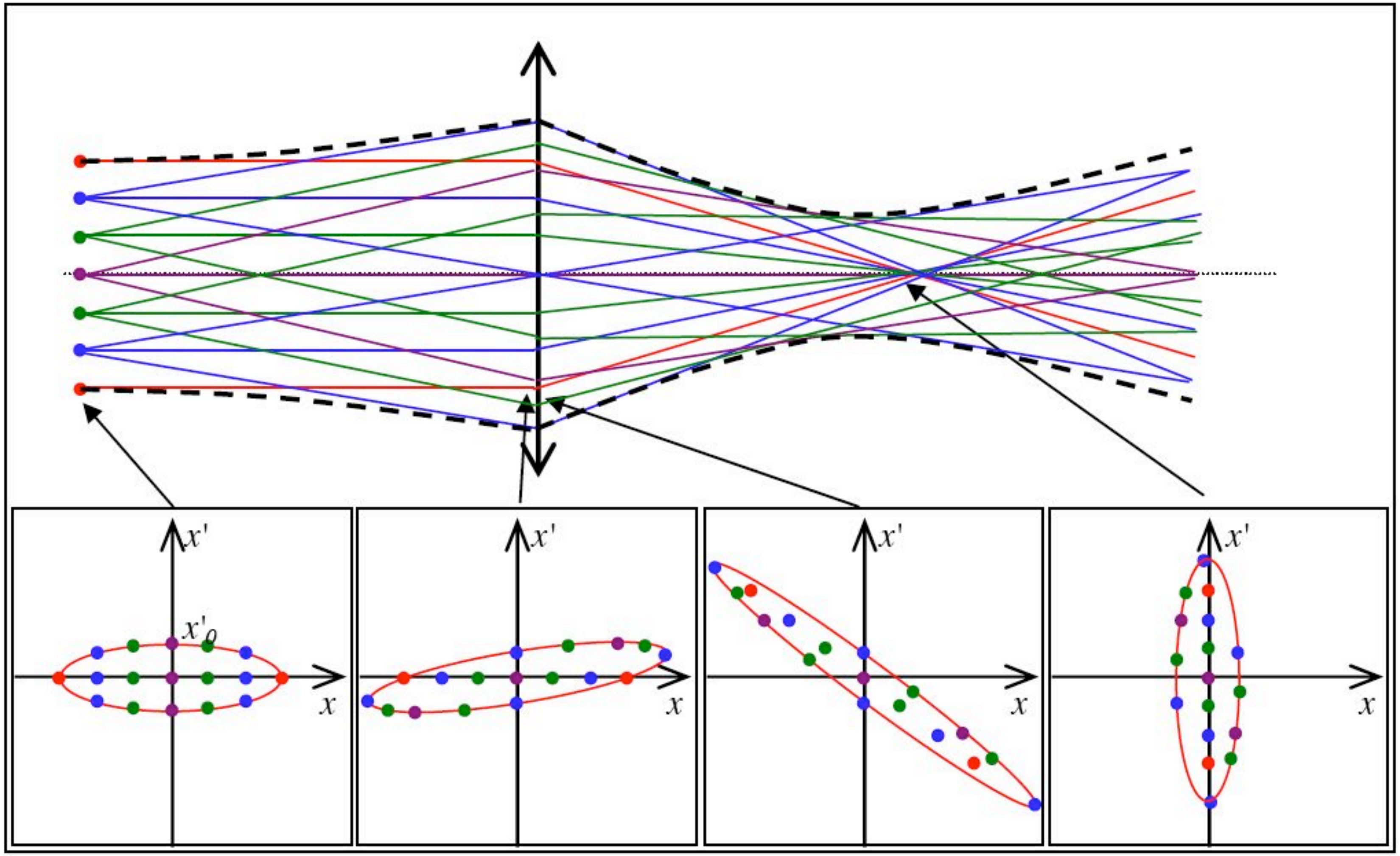

Fig. 11: Particle trajectories in non-zero emittance beam

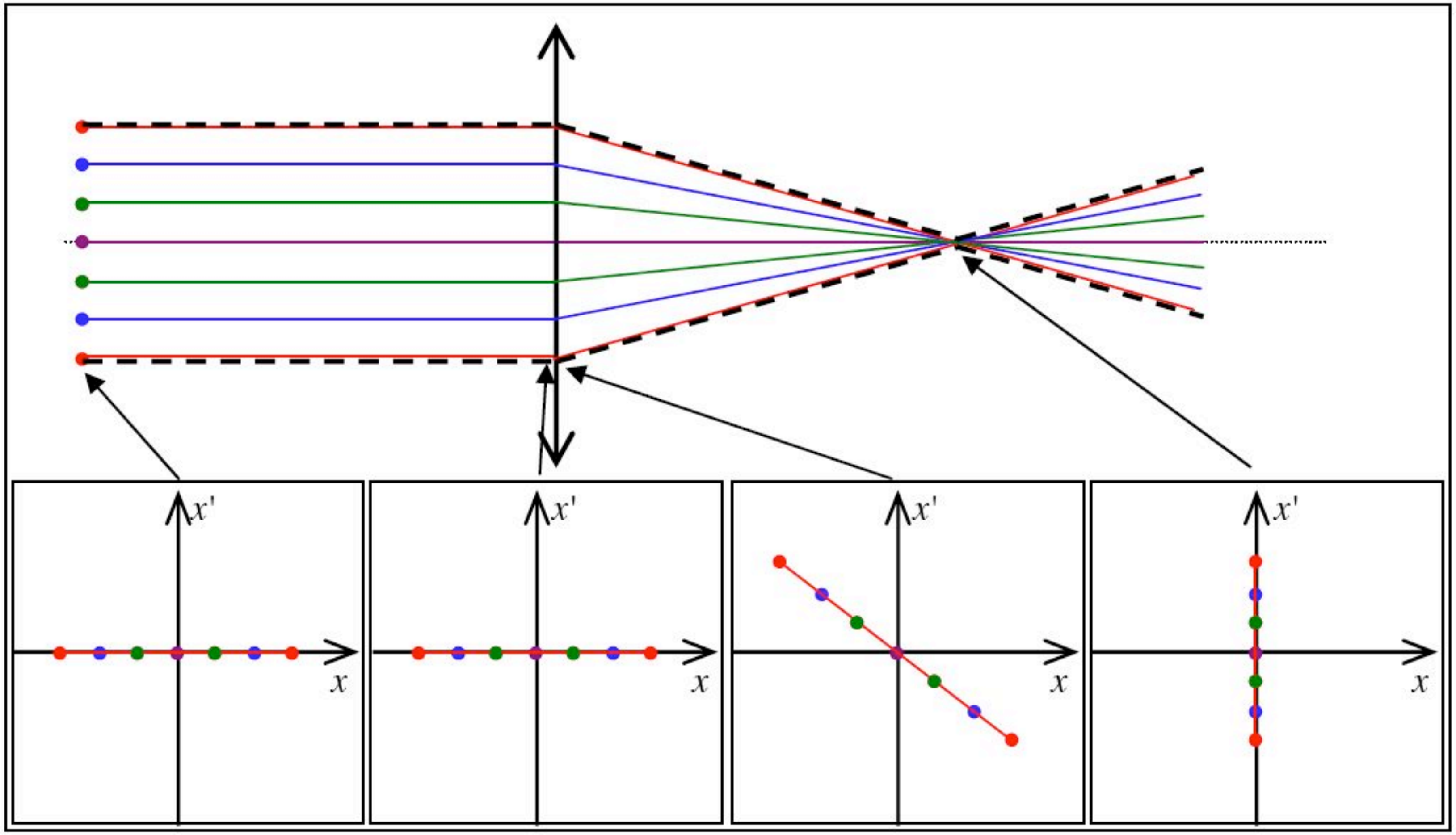

Fig. 10: Particle trajectories in laminar beam

Transport Matrix

$$\begin{pmatrix} x \\ x' \end{pmatrix} = M \begin{pmatrix} x \\ x' \end{pmatrix}_0 \qquad M(s_1 s_2) = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix}$$

$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = \varepsilon = \gamma_0 x_0{}^2 + 2\alpha_0 x_0 x_0' + \beta_0 x_0'^2$$

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix} = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & S'C + SC' & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \alpha_0 \\ \gamma_0 \end{pmatrix}$$

The σ matrix

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix} = \varepsilon \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}$$

$\det\sigma = \varepsilon^2$

$$\sigma_{11}x^2 + 2\sigma_{12}xx' + \sigma_{22}x'^2 = 1$$

$$\sigma_1 = M\sigma_0 M^T$$

Quadrupole Scan

$$\sigma_{11} = C^2(k)\sigma_{11} + 2C(k)S(k)\sigma_{12} + S^2(k)\sigma_{22}$$

- It is possible to measure in the same position changing the optical functions

Emittance measurements for emittance dominated beams

Diagnostic Section