The CERN Accelerator School

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Recapitulation of Electromagnetism

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Overview

1. Maxwell's equations
2. Electromagnetic fields in different materials - material equations
3. Electrostatic fields
4. Magnetostatic fields
5. Electromagnetic waves
6. Field attenuation in conductors

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Maxwell‘s Equations

Maxwell‘s Equations in their Integral Representation

$$\oiint_{\partial\boldsymbol{\Omega}} \mathbf{D}(\mathbf{r},t)\cdot \mathrm{d}\mathbf{A} = \iiint_{\boldsymbol{\Omega}} \rho(\mathbf{r},t)\,\mathrm{d}V$$

$$\oiint_{\partial\boldsymbol{\Omega}} \mathbf{B}(\mathbf{r},t)\cdot \mathrm{d}\mathbf{A} = 0$$

$$\oint_{\partial\boldsymbol{\Gamma}} \mathbf{E}(\mathbf{r},t)\cdot \mathrm{d}\mathbf{s} = -\iint_{\boldsymbol{\Gamma}} \frac{\partial}{\partial t}\mathbf{B}(\mathbf{r},t)\cdot \mathrm{d}\mathbf{A}$$

$$\oint_{\partial\boldsymbol{\Gamma}} \mathbf{H}(\mathbf{r},t)\cdot \mathrm{d}\mathbf{s} = \iint_{\boldsymbol{\Gamma}} \left(\frac{\partial}{\partial t}\mathbf{D}(\mathbf{r},t) + \mathbf{J}(\mathbf{r},t)\right)\cdot \mathrm{d}\mathbf{A}$$

James Clerk Maxwell (1831-1879)

Figure: https://upload.wikimedia.org/wikipedia/commons/thumb/1/1e/James_Clerk_Maxwell_big.jpg/390px-James_Clerk_Maxwell_big.jpg

Gauss‘ Law (for Electricity) in Integral Form

Electric charges Q or electric charge densities $\rho(\mathbf{r},t)$ generate electric flux densities $\mathbf{D}(\mathbf{r},t)$.

$$\boxed{\underbrace{\oint\!\!\!\oint_{\partial\boldsymbol{\Omega}} \mathbf{D}(\mathbf{r},t)\cdot \mathrm{d}\mathbf{A}}_{\text{total electric flux through Gaussian surface}} = Q = \underbrace{\iiint_{\boldsymbol{\Omega}} \rho(\mathbf{r},t)\,\mathrm{d}V}_{\text{total electric charge enclosed in Gaussian surface}}}$$

Quick Quiz (I / II) – Value of Net Flux through Surface?

$$\underbrace{\oiint_{\partial\boldsymbol{\Omega}} \mathbf{D}(\mathbf{r},t) \cdot \mathrm{d}\mathbf{A}}_{\text{total electric flux through Gaussian surface}} = ???$$

Quick Quiz (II/II) – Value of Net Flux through Surface?

$$\underbrace{\oiint_{\partial\boldsymbol{\Omega}} \mathbf{D}(\mathbf{r},t)\cdot \mathrm{d}\mathbf{A}}_{\text{total electric flux through Gaussian surface}} = 0 = \iiint_{\boldsymbol{\Omega}} \underbrace{\rho(\mathbf{r},t)}_{=0!}\,\mathrm{d}V$$

- Total electric flux through the Gaussian surface equals zero since no charges are contained in the volume!
- Total amount of flux flowing into the Gaussian surface is equal to total amount of flux flowing out of the surface
- Absence of charges in the volume does not mean that the electric displacement fields are zero in the volume

Gauss‘ Law (for Electricity) –
from Integral to Differential Form

$$\oiint_{\partial\boldsymbol{\Omega}} \mathbf{D}(\mathbf{r},t)\cdot \mathrm{d}\mathbf{A} = \iiint_{\boldsymbol{\Omega}} \rho(\mathbf{r},t)\,\mathrm{d}V$$

for infinitely small volumes $\boldsymbol{\Omega}$

$$\nabla\cdot\mathbf{D}(\mathbf{r},t) = \rho(\mathbf{r},t)$$

Gauss‘ Law for Magnetism in Integral Form

Magnetic flux densities $\mathbf{B}(\mathbf{r},t)$ do not have sources, i.e. they are solely curl fields.

$$\underbrace{\oiint_{\partial\boldsymbol{\Omega}} \mathbf{B}(\mathbf{r},t)\cdot \mathrm{d}\mathbf{A}}_{\text{total magnetic flux through Gaussian surface}} = 0$$

Gauss' Law for Magnetism – from Integral to Differential Form

$$\oiint_{\partial\boldsymbol{\Omega}} \mathbf{B}(\mathbf{r},t)\cdot \mathrm{d}\mathbf{A} = 0$$

↓

for infinitely small volumes $\boldsymbol{\Omega}$

↓

$$\nabla \cdot \mathbf{B}(\mathbf{r},t) = 0$$

Faraday's Law of Induction

Time-dependent magnetic flux densities $\mathbf{B}(\mathbf{r},t)$ generate curled electric field strength $\mathbf{E}(\mathbf{r},t)$.

$$\boxed{\oint\limits_{\partial\mathbf{\Gamma}} \mathbf{E}(\mathbf{r},t)\cdot \mathrm{d}\mathbf{s} = -\iint\limits_{\mathbf{\Gamma}} \frac{\partial}{\partial t}\mathbf{B}(\mathbf{r},t)\cdot \mathrm{d}\mathbf{A}}$$

Faraday's Law of Induction – The Minus Sign (I / II)

The polarity of the induced electric field strength is such that it tends to produce a current that creates a magnetic flux to oppose the change in magnetic flux through the area enclosed by the current loop. This is known as Lenz's Law.

$$\boxed{\oint_{\partial\mathbf{\Gamma}} \mathbf{E}(\mathbf{r},t)\cdot\mathrm{d}\mathbf{s} = -\iint_{\mathbf{\Gamma}} \frac{\partial}{\partial t}\mathbf{B}(\mathbf{r},t)\cdot\mathrm{d}\mathbf{A}}$$

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- The minus sign in the induction law is also required for Maxwell's equation to be energy conserving!

Faraday's Law of Induction – from Integral to Differential Form

$$\oint\limits_{\partial\boldsymbol{\Gamma}} \mathbf{E}(\mathbf{r},t)\cdot \mathrm{d}\mathbf{s} = -\iint\limits_{\boldsymbol{\Gamma}} \frac{\partial}{\partial t}\mathbf{B}(\mathbf{r},t)\cdot \mathrm{d}\mathbf{A}$$

for infinitely small areas $\boldsymbol{\Gamma}$

$$\nabla \times \mathbf{E}(\mathbf{r},t) = -\frac{\partial}{\partial t}\mathbf{B}(\mathbf{r},t)$$

Ampère's Law with Maxwell's Extension

Electric currents $\mathbf{J}(\mathbf{r},t)$ and time-dependent electric displacement currents $\dfrac{\partial}{\partial t}\mathbf{D}(\mathbf{r},t)$ generate curled magnetic field strengths $\mathbf{H}(\mathbf{r},t)$.

$$\boxed{\oint\limits_{\partial\mathbf{\Gamma}} \mathbf{H}(\mathbf{r},t)\cdot \mathrm{d}\mathbf{s} = \iint\limits_{\mathbf{\Gamma}} \left(\mathbf{J}(\mathbf{r},t) + \frac{\partial}{\partial t}\mathbf{D}(\mathbf{r},t)\right)\cdot \mathrm{d}\mathbf{A}}$$

Ampère's Law with Maxwell's Extension – from Integral to Differential Form

$$\oint_{\partial\boldsymbol{\Gamma}} \mathbf{H}(\mathbf{r},t)\cdot \mathrm{d}\mathbf{s} = \iint_{\boldsymbol{\Gamma}} \left(\mathbf{J}(\mathbf{r},t) + \frac{\partial}{\partial t}\mathbf{D}(\mathbf{r},t)\right)\cdot \mathrm{d}\mathbf{A}$$

for infinitely small areas $\boldsymbol{\Gamma}$

$$\nabla \times \mathbf{H}(\mathbf{r},t) = \mathbf{J}(\mathbf{r},t) + \frac{\partial}{\partial t}\mathbf{D}(\mathbf{r},t)$$

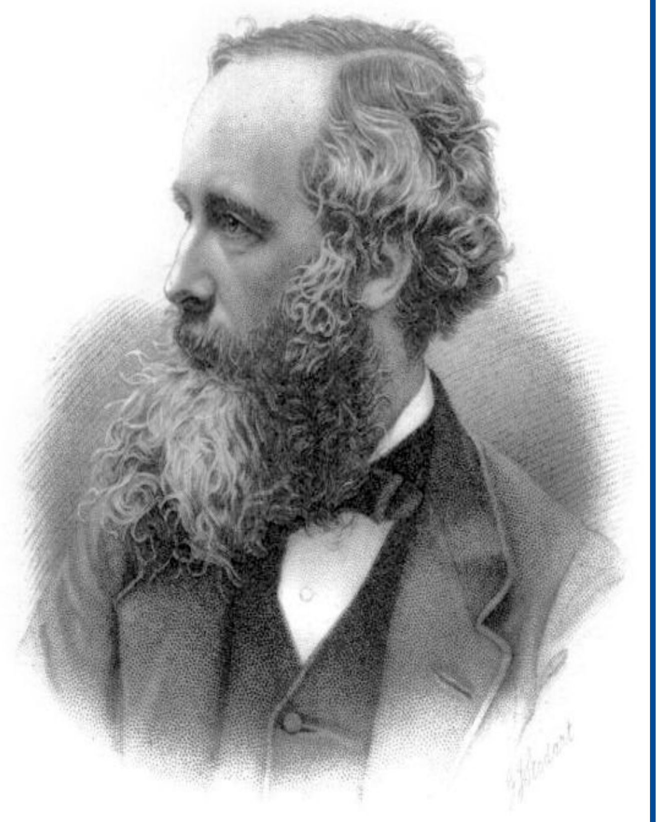
Maxwell's Equations in their Differential Representation

$$\nabla \cdot \mathbf{D}(\mathbf{r},t) = \rho(\mathbf{r},t)$$

$$\nabla \cdot \mathbf{B}(\mathbf{r},t) = 0$$

$$\nabla \times \mathbf{E}(\mathbf{r},t) = -\frac{\partial}{\partial t}\mathbf{B}(\mathbf{r},t)$$

$$\nabla \times \mathbf{H}(\mathbf{r},t) = \frac{\partial}{\partial t}\mathbf{D}(\mathbf{r},t) + \mathbf{J}(\mathbf{r},t)$$



James Clerk Maxwell (1831-1879)

Figure: https://upload.wikimedia.org/wikipedia/commons/thumb/1/1e/James_Clerk_Maxwell_big.jpg/390px-James_Clerk_Maxwell_big.jpg

The Divergence Operator

- The divergence operator $\nabla \cdot \mathbf{F}(x,y,z)$ measures the source strength of the vector field $\mathbf{F}(x,y,z)$ in that point

- In some textbooks the divergence is denoted by $\operatorname{div} \mathbf{F}(x,y,z)$

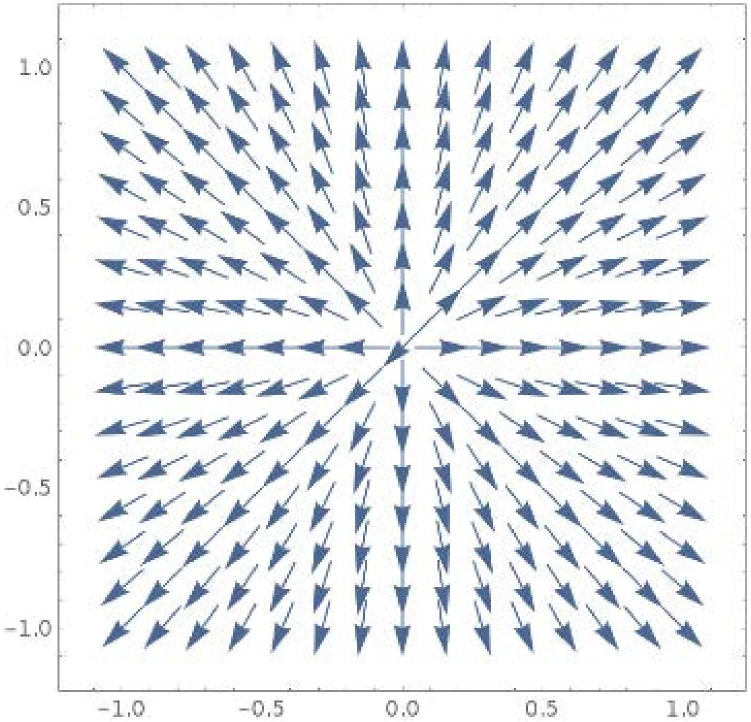
- The divergence acts on a vector field and gives back a scalar field, i.e. the source strength!

- In Cartesian coordinates, the divergence is defined in terms of:

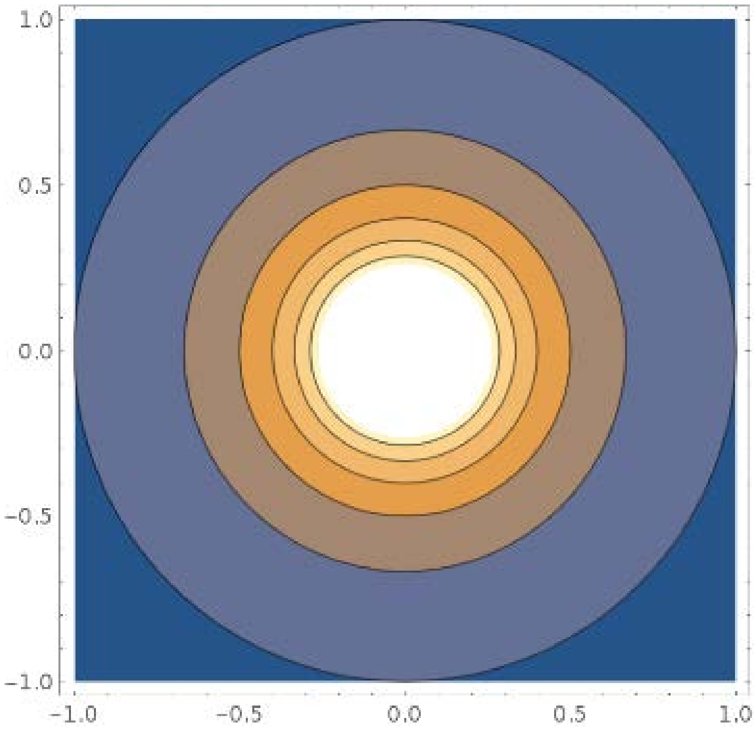
$$\nabla \cdot \mathbf{F}(x,y,z) = \operatorname{div} \mathbf{F}(x,y,z) = \frac{\partial}{\partial x} F_x(x,y,z) + \frac{\partial}{\partial y} F_y(x,y,z) + \frac{\partial}{\partial z} F_z(x,y,z)$$

The Divergence Operator – A 2D Example

vector field $\mathbf{F}(x, y)$



scalar field $\nabla \cdot \mathbf{F}(x, y)$



The Curl Operator

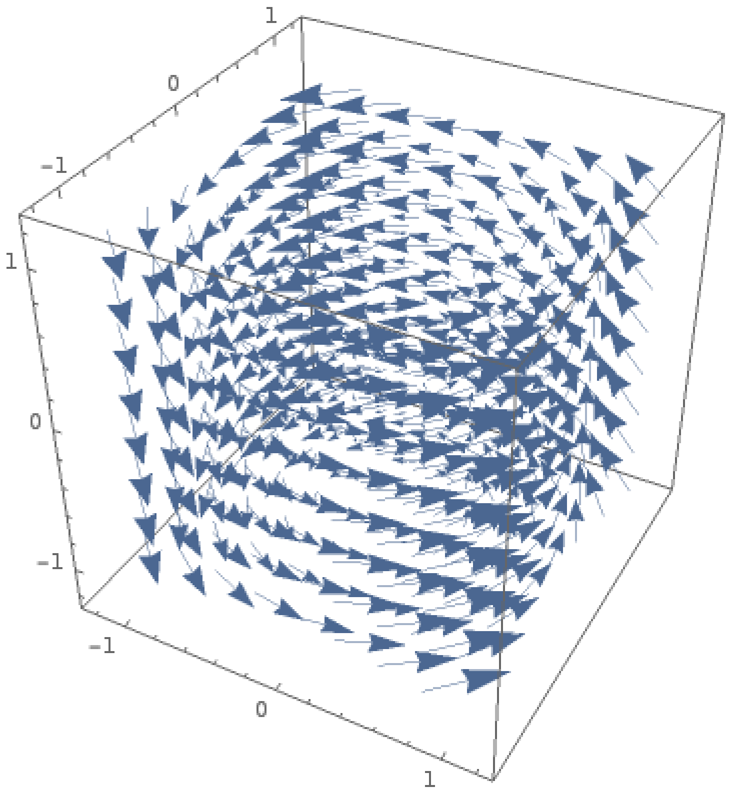
- The curl operator $\nabla \times \mathbf{F}(x,y,z)$ measures the rotation of a vector field $\mathbf{F}(x,y,z)$ in that point
- In some textbooks the curl (or rotation) is denoted by $\operatorname{curl}\mathbf{F}(x,y,z)$
- The curl operator acts on a vector field and gives back a vector field, i.e. the curl strenght!
- In Cartesian coordinates, the curl is defined in terms of:

$$\nabla \times \mathbf{F}(x,y,z) = \operatorname{curl}\mathbf{F}(x,y,z) = \left| \begin{pmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x(x,y,z) & F_y(x,y,z) & F_z(x,y,z) \end{pmatrix} \right|$$

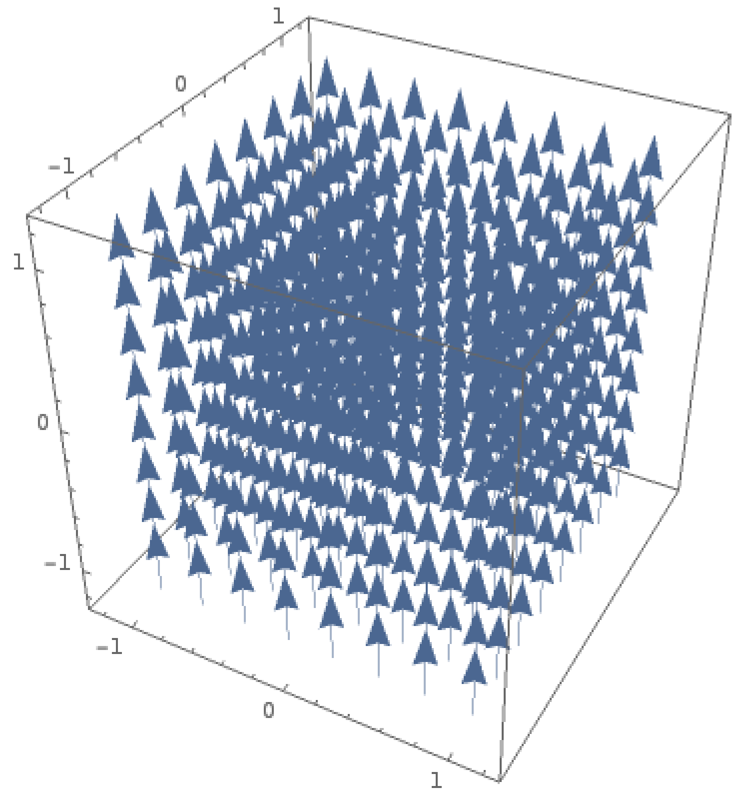
$$= \left| \begin{pmatrix} \frac{\partial}{\partial y} F_z(x,y,z) - \frac{\partial}{\partial z} F_y(x,y,z) \\ \frac{\partial}{\partial z} F_x(x,y,z) - \frac{\partial}{\partial x} F_z(x,y,z) \\ \frac{\partial}{\partial x} F_y(x,y,z) - \frac{\partial}{\partial y} F_x(x,y,z) \end{pmatrix} \right|$$

The Curl Operator – A 3D Example

vector field $\mathbf{F}(x, y, z)$



vector field $\nabla \times \mathbf{F}(x, y, z)$



Electromagnetic Fields in Materials

Electric Fields in Matter

- Materials can be polarized by applied electric fields $\mathbf{E}(\mathbf{r})$
- The polarization $\mathbf{P}(\mathbf{r})$ is in fact a displacement $\mathbf{D}(\mathbf{r})$ of electric charges
- Permittivity of free space: $\varepsilon_0 = 8.85 \cdot 10^{-12} \mathrm{As/Vm}$
- Relative permittivity: $\varepsilon_r = 1 \ldots 10^5$

$$\boxed{\begin{aligned} \mathbf{D}(\mathbf{r}) &= \varepsilon_0 \mathbf{E}(\mathbf{r}) + \mathbf{P}(\mathbf{r}) \\ \mathbf{D}(\mathbf{r}) &= \varepsilon_0 \varepsilon_r \mathbf{E}(\mathbf{r}) \end{aligned}}$$

Magnetic Fields in Matter

- Materials can be magnetized by applied magnetic fields $\mathbf{B}(\mathbf{r})$
- The magnetization $\mathbf{M}(\mathbf{r})$ is in fact a change of the orientation of magnetic dipoles
- Permeability of free space: $\mu_0 = 4\pi \cdot 10^{-7}\mathrm{Vs/Am}$
- Relative permeability: $\mu_r = 0 \ldots 10^3$

$$\mathbf{B}(\mathbf{r}) = \mu_0 \mathbf{H}(\mathbf{r}) + \mathbf{M}(\mathbf{r})$$
$$\mathbf{B}(\mathbf{r}) = \mu_0 \mu_r \mathbf{H}(\mathbf{r})$$

Some Remarks in Material Modelling

Often it is not sufficient to consider the material parameters as constants, because matter can be

- inhomogeneous

$$\varepsilon_r = \varepsilon_r(\mathbf{r}) \qquad \mu_r = \mu_r(\mathbf{r})$$

- dispersive, so that the material parameters are complex-valued and frequency-dependent:

$$\varepsilon_r = \underline{\varepsilon}_r(j\omega) \qquad \mu_r = \underline{\mu}_r(j\omega)$$

- anisotropic (directional dependent), so that the material parameters become tensors

$$\varepsilon_r = \begin{pmatrix} \varepsilon_{xx,r} & \varepsilon_{xy,r} & \varepsilon_{xz,r} \\ \varepsilon_{yx,r} & \varepsilon_{yy,r} & \varepsilon_{yz,r} \\ \varepsilon_{zx,r} & \varepsilon_{zy,r} & \varepsilon_{zz,r} \end{pmatrix} \qquad \mu_r = \begin{pmatrix} \mu_{xx,r} & \mu_{xy,r} & \mu_{xz,r} \\ \mu_{yx,r} & \mu_{yy,r} & \mu_{yz,r} \\ \mu_{zx,r} & \mu_{zy,r} & \mu_{zz,r} \end{pmatrix}$$

- non-linear (and can have a hysteresis in addition), so that the material parameters are functions on the field strength itself

$$\varepsilon_r = \varepsilon_r(\mathbf{E}) \qquad \mu_r = \mu_r(\mathbf{H})$$

Electrostatics

Electrostatics – Maxwell Simplifications

$$\nabla \times \mathbf{E}(\mathbf{r}) = \underbrace{-\frac{\partial}{\partial t}\mathbf{B}(\mathbf{r})}_{\mathbf{0}}$$

$$\nabla \cdot \mathbf{D}(\mathbf{r}) = \rho(\mathbf{r})$$

The electric field is curl-free

Gauss‘ law of electricity:
The divergence of the electric flux density is equal to the charge density

Due to the electric field being curl-free it can be expressed as negative gradient of an arbitrary scalar potential

$$\mathbf{E}(\mathbf{r}) = -\nabla\phi(\mathbf{r})$$

With this approach, we can ensure that Faraday’s law of induction holds for the static electric field:

$$\nabla \times \mathbf{E}(\mathbf{r}) = -\nabla \times \nabla\phi(\mathbf{r}) = \mathbf{0}$$

Electrostatics – Derivation of Poisson‘s equation

Starting with Gauss’ law of electricity

$$\nabla \cdot \mathbf{D}(\mathbf{r}) = \rho(\mathbf{r})$$

we employ the material equation for electric fields and assume that the permittivity $\varepsilon = \varepsilon_0 \varepsilon_r$ is homogeneous:

$$\nabla \cdot \mathbf{E}(\mathbf{r}) = \frac{\rho(\mathbf{r})}{\varepsilon}$$

Next, we express the electric field in terms of the gradient of an arbitrary scalar potential

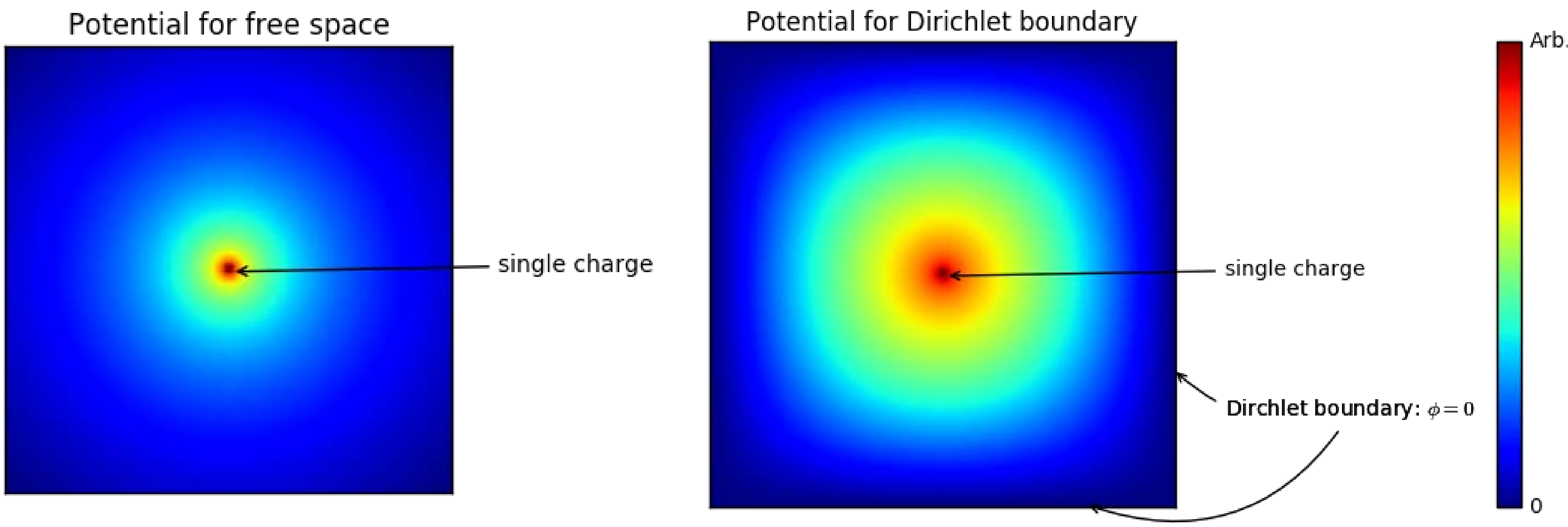
$$\mathbf{E}(\mathbf{r}) = -\nabla \phi(\mathbf{r})$$

Combining both equations delivers the so-called Poisson equation (or potential equation)

$$\Delta \phi(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\varepsilon}$$

Electrostatics – Poisson‘s equation

In its simplest case, Poisson’s equation $\Delta\phi(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\varepsilon}$ describes the electric potential of a point charge:



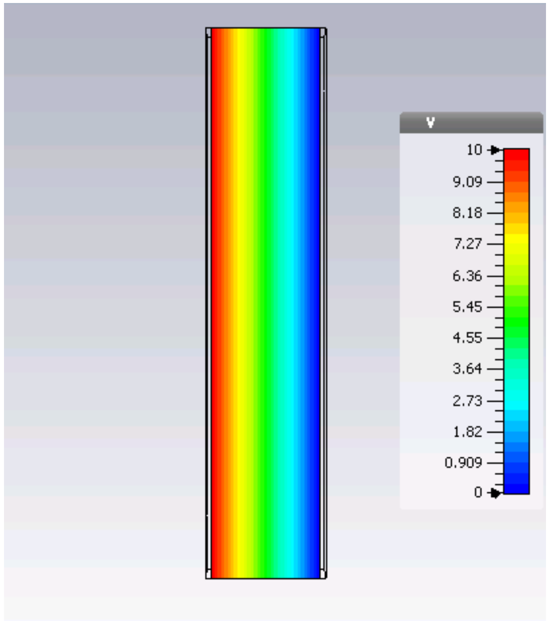
Electrostatics – A simple example: Capacitor

A capacitor is free of charges between its plates:

$$\Delta\phi(\mathbf{r}) = 0$$

Assuming that the potential does only depend on one spatial direction, the Laplace-operator can be simplified to

$$\Delta\phi(\mathbf{r}) = \frac{\partial^2}{\partial x^2}\phi(\mathbf{r})$$



Magnetostatics

Magnetostatics – Maxwell Simplifications

$$\nabla \times \mathbf{H}(\mathbf{r}) = \underbrace{\frac{\partial}{\partial t}\mathbf{D}(\mathbf{r},t)}_{\mathbf{0}} + \mathbf{J}(\mathbf{r},t)$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0$$

Simplified Ampère‘s law: The curl of the magnetic field equals the current-density

Gauss' law of magnetism:
The magnetic flux density is divergence-free

The magnetic flux density is divergence-free so that it can be expressed as curl of an arbitrary vector-potential

$$\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r})$$

With this approach, we can ensure that Gauss’ law for magnetism holds:

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = \nabla \cdot (\nabla \times \mathbf{A}(\mathbf{r})) = 0$$

Magnetostatics – Derivation of Poisson‘s equation

Starting with Ampère‘s law

$$\nabla \times \mathbf{H}(\mathbf{r}) = \mathbf{J}(\mathbf{r})$$

we employ the material equation for magnetic fields and assume that the permeability $\mu = \mu_0 \mu_r$ is homogeneous:

$$\nabla \times \mathbf{B}(\mathbf{r}) = \mu \mathbf{J}(\mathbf{r})$$

Next, we express the magnetic flux density in terms of the gradient of a vector potential

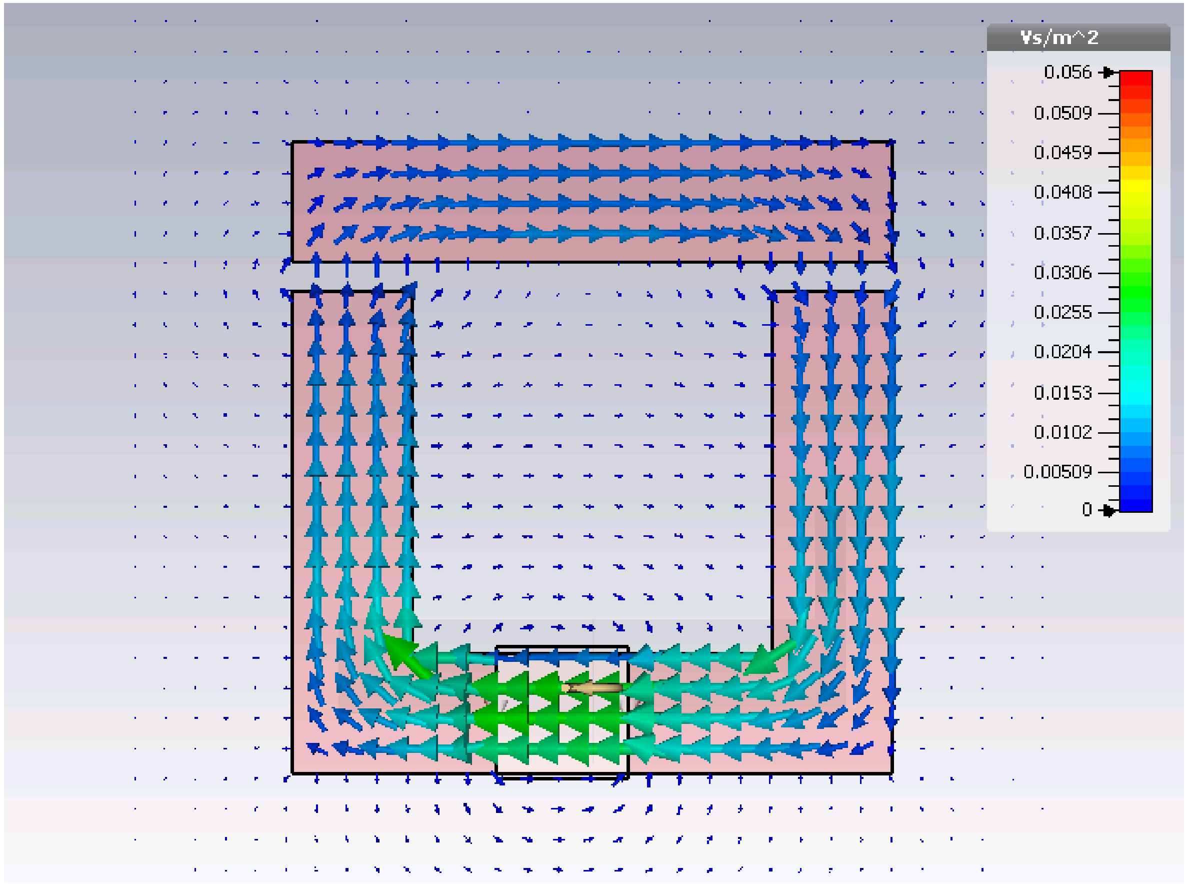
$$\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r})$$

Combining both equations delivers Poisson’s equation for the magnetic vector potential

$$\nabla \times \nabla \times \mathbf{A}(\mathbf{r}) = -\Delta \mathbf{A}(\mathbf{r}) + \nabla \left(\nabla \cdot \mathbf{A}(\mathbf{r}) \right) = \mu \mathbf{J}(\mathbf{r})$$

Magnetostatics – Electromagnet

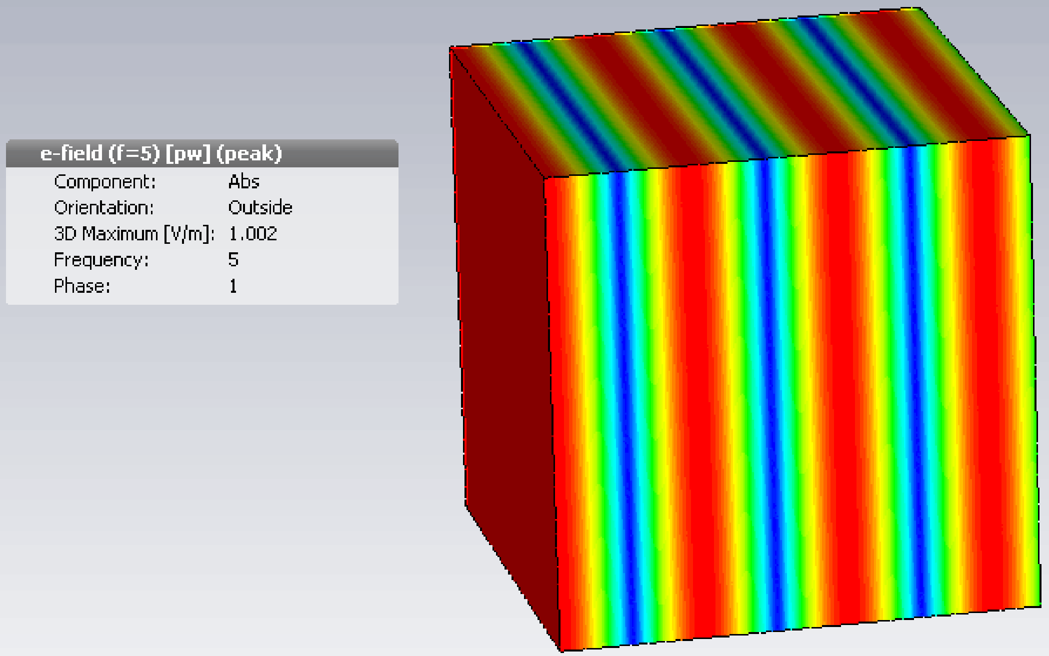
A possible application is the computation of magnetic fields in an electromagnet



Electromagnetic Waves

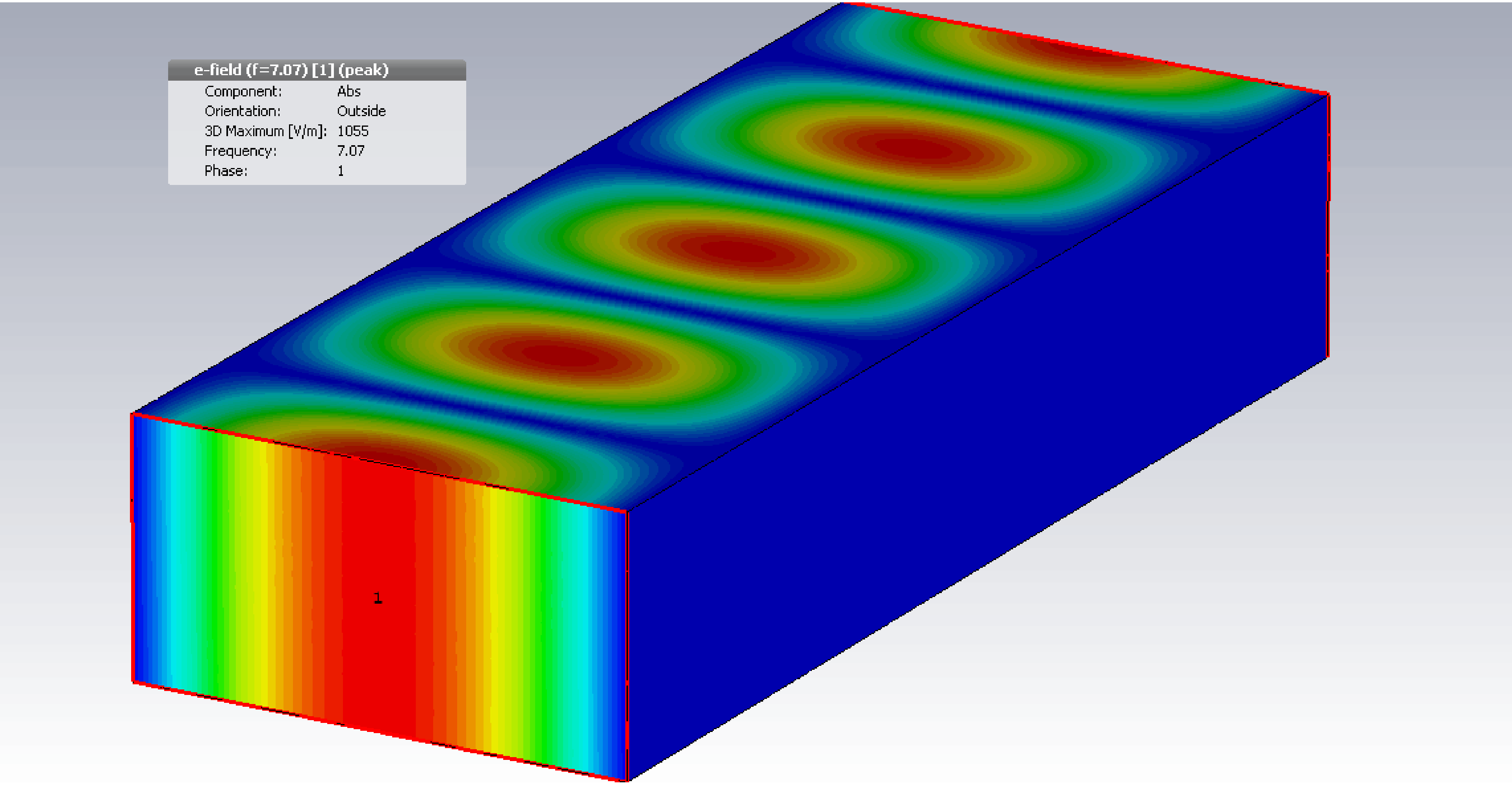
Electromagnetic Waves

Electromagnetic waves exist with different properties - such as waves in free space



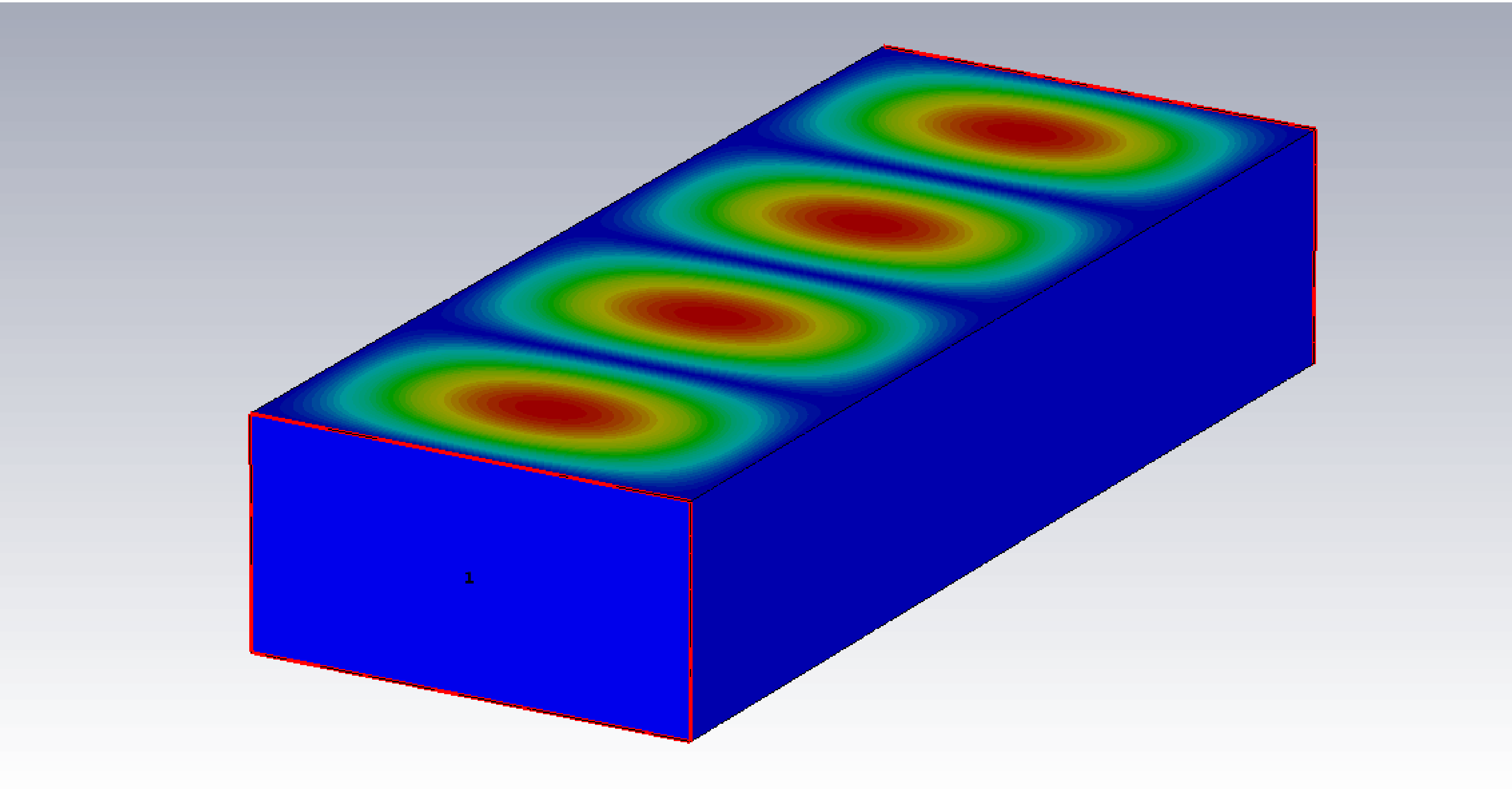
Electromagnetic Waves

Electromagnetic waves with different properties exist - such as guided waves



Electromagnetic Waves

Electromagnetic waves with different properties exist - such as standing waves



Wave Equation arising from Maxwell's Equations

All electromagnetic waves in homogeneous media satisfy Maxwell's equation, in particular, the wave equation that we will derive here:

$$\nabla \times \mathbf{E}(\mathbf{r},t) = -\frac{\partial}{\partial t}\mathbf{B}(\mathbf{r},t) \quad |\nabla\times$$

$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r},t) = \nabla \times \left(-\frac{\partial}{\partial t}\mathbf{B}(\mathbf{r},t)\right)$$

$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r},t) = -\frac{\partial}{\partial t}\left(\nabla \times \mathbf{B}(\mathbf{r},t)\right)$$

$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r},t) = -\frac{\partial}{\partial t}\left(\nabla \times \mu\mathbf{H}(\mathbf{r},t)\right)$$

$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r},t) = -\mu\frac{\partial}{\partial t}\nabla \times \mathbf{H}(\mathbf{r},t)$$

$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r},t) = -\mu\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}\mathbf{D}(\mathbf{r},t) + \mathbf{J}(\mathbf{r},t)\right)$$

$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r},t) = -\mu\frac{\partial^2}{\partial t^2}\mathbf{D}(\mathbf{r},t) - \mu\frac{\partial}{\partial t}\mathbf{J}(\mathbf{r},t)$$

Wave Equation arising from Maxwell's Equations

All electromagnetic waves in homogeneous media satisfy Maxwell's equation, in particular, the wave equation – here we continue its derivation:

$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r},t) = -\mu \frac{\partial^2}{\partial t^2}\mathbf{D}(\mathbf{r},t) - \mu \frac{\partial}{\partial t}\mathbf{J}(\mathbf{r},t)$$

$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r},t) = -\varepsilon\mu \frac{\partial^2}{\partial t^2}\mathbf{E}(\mathbf{r},t) - \mu \frac{\partial}{\partial t}\mathbf{J}(\mathbf{r},t)$$

$$\boxed{\nabla \times \nabla \times \mathbf{E}(\mathbf{r},t) + \varepsilon\mu \frac{\partial^2}{\partial t^2}\mathbf{E}(\mathbf{r},t) = -\mu \frac{\partial}{\partial t}\mathbf{J}(\mathbf{r},t)}$$

Curl-Curl Equation

$$\nabla\big(\underbrace{\nabla \cdot \mathbf{E}(\mathbf{r},t)}_{\frac{\rho(\mathbf{r},t)}{\varepsilon}}\big) - \Delta\mathbf{E}(\mathbf{r},t) + \varepsilon\mu \frac{\partial^2}{\partial t^2}\mathbf{E}(\mathbf{r},t) = -\mu \frac{\partial}{\partial t}\mathbf{J}(\mathbf{r},t) \quad |\text{if } \rho(\mathbf{r},t) = 0$$

$$\boxed{\Delta\mathbf{E}(\mathbf{r},t) - \varepsilon\mu \frac{\partial^2}{\partial t^2}\mathbf{E}(\mathbf{r},t) = \mu \frac{\partial}{\partial t}\mathbf{J}(\mathbf{r},t)}$$

Wave Equation (with excitation)

Eigenmodes – Solutions of the Homogeneous Wave Equations

Eigenmodes are solutions of the wave equation for the non-excited, loss free and charge-free case:

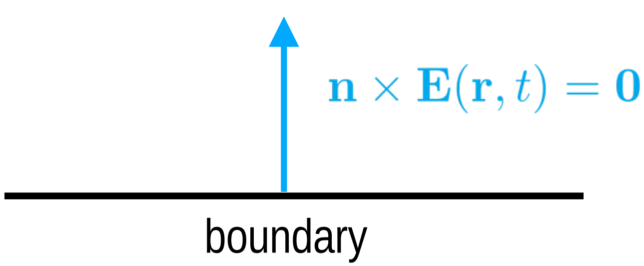
$$\Delta \mathbf{E}(\mathbf{r},t) - \varepsilon\mu \frac{\partial^2}{\partial t^2}\mathbf{E}(\mathbf{r},t) = \mathbf{0}$$

$$\Delta \mathbf{E}(\mathbf{r}) \cos(\omega t - \varphi) + \varepsilon\mu\omega^2 \mathbf{E}(\mathbf{r}) \cos(\omega t - \varphi) = \mathbf{0}$$

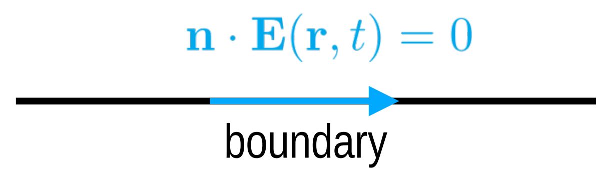
$$\Delta \mathbf{E}(\mathbf{r}) + \underbrace{\varepsilon\mu\omega^2}_{k^2} \mathbf{E}(\mathbf{r}) = \mathbf{0}$$

The partial differential equation comes with either of these boundary conditions:

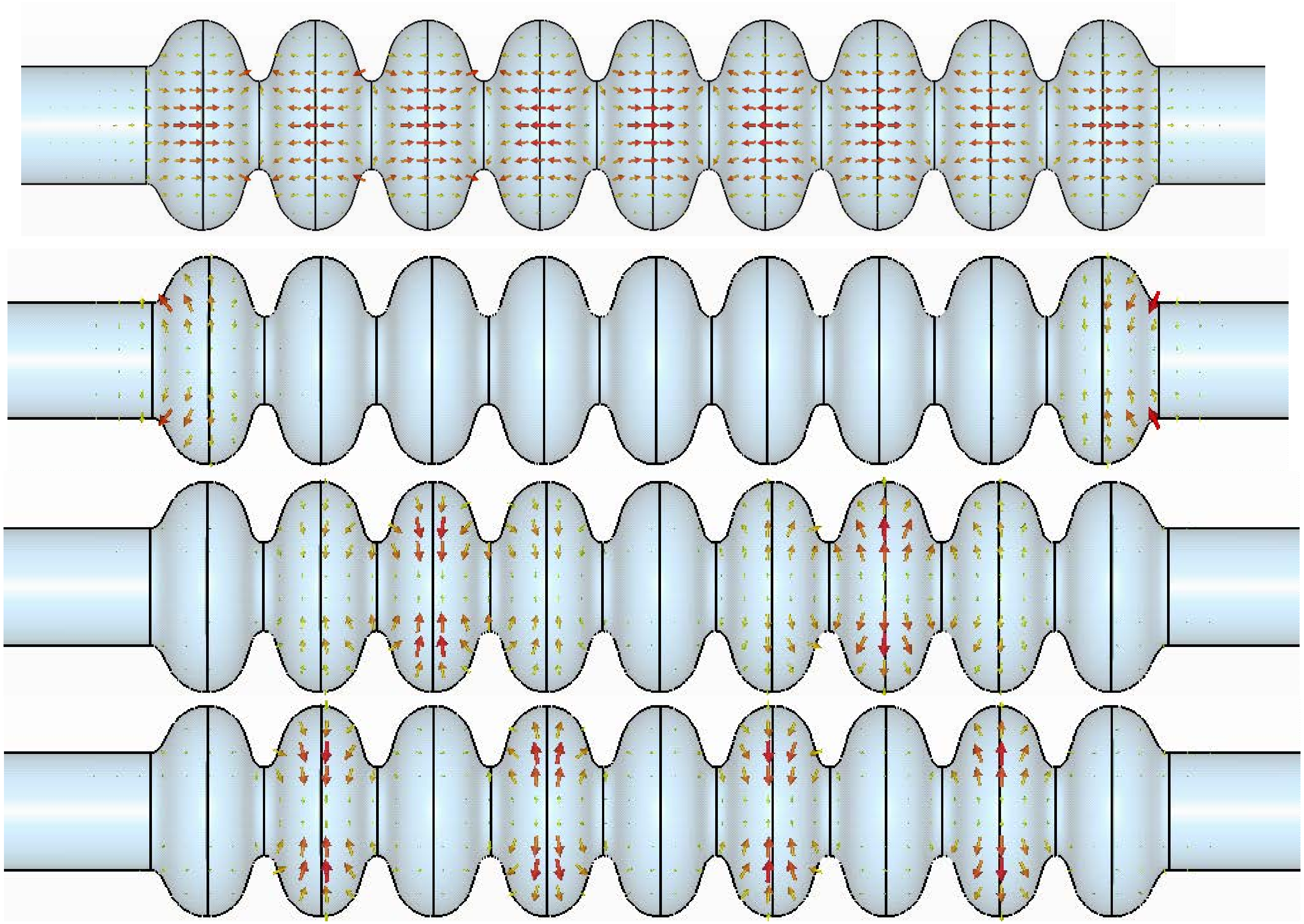
Perfect Electric Conducting



Perfect Magnetic Conducting



Electric Field of some Eigenmodes in a Resonator



Field Attenuation in Conductors

Influence on Conducting Matter on Waves (I / II)

In conducting matter, Ohmic electric current densities will flow. They are proportional to the electric field strength with the conductivity σ as constant:

$$\mathbf{J}(\mathbf{r},t) = \sigma \mathbf{E}(\mathbf{r},t)$$

Replacing the electric current density in the wave equation with the upper relation gives

$$\Delta \mathbf{E}(\mathbf{r},t) - \varepsilon\mu \frac{\partial^2}{\partial t^2}\mathbf{E}(\mathbf{r},t) = \mu\sigma \frac{\partial}{\partial t}\mathbf{E}(\mathbf{r},t)$$

Transforming this equation into frequency domain delivers

$$\Delta \underline{\mathbf{E}}(\mathbf{r}) + \varepsilon\mu\omega^2 \underline{\mathbf{E}}(\mathbf{r}) = j\omega\mu\sigma \underline{\mathbf{E}}(\mathbf{r})$$

Now, consider a plane wave propagation in +z – direction:

$$\underline{\mathbf{E}}(\mathbf{r}) = \mathbf{e}_x E_0 \mathrm{e}^{-j\underline{k}z}$$

Plugging this into the frequency-domain representation of the wave equation gives

$$\underline{k}^2 = \varepsilon\mu\omega^2 - j\omega\mu\sigma$$

Influence on Conducting Matter on Waves (II / II)

The wave number is complex valued

$$\underline{k} = k' - jk''$$

with the following real and imaginary parts

$$k' = \frac{\mu\sigma\omega}{2\sqrt{-\frac{1}{2}\varepsilon\mu\omega^2 + \frac{1}{2}\sqrt{\mu^2\sigma^2\omega^2 + \varepsilon^2\mu^2\omega^4}}}$$

$$k'' = \sqrt{-\frac{1}{2}\varepsilon\mu\omega^2 + \frac{1}{2}\sqrt{\mu^2\sigma^2\omega^2 + \varepsilon^2\mu^2\omega^4}}$$

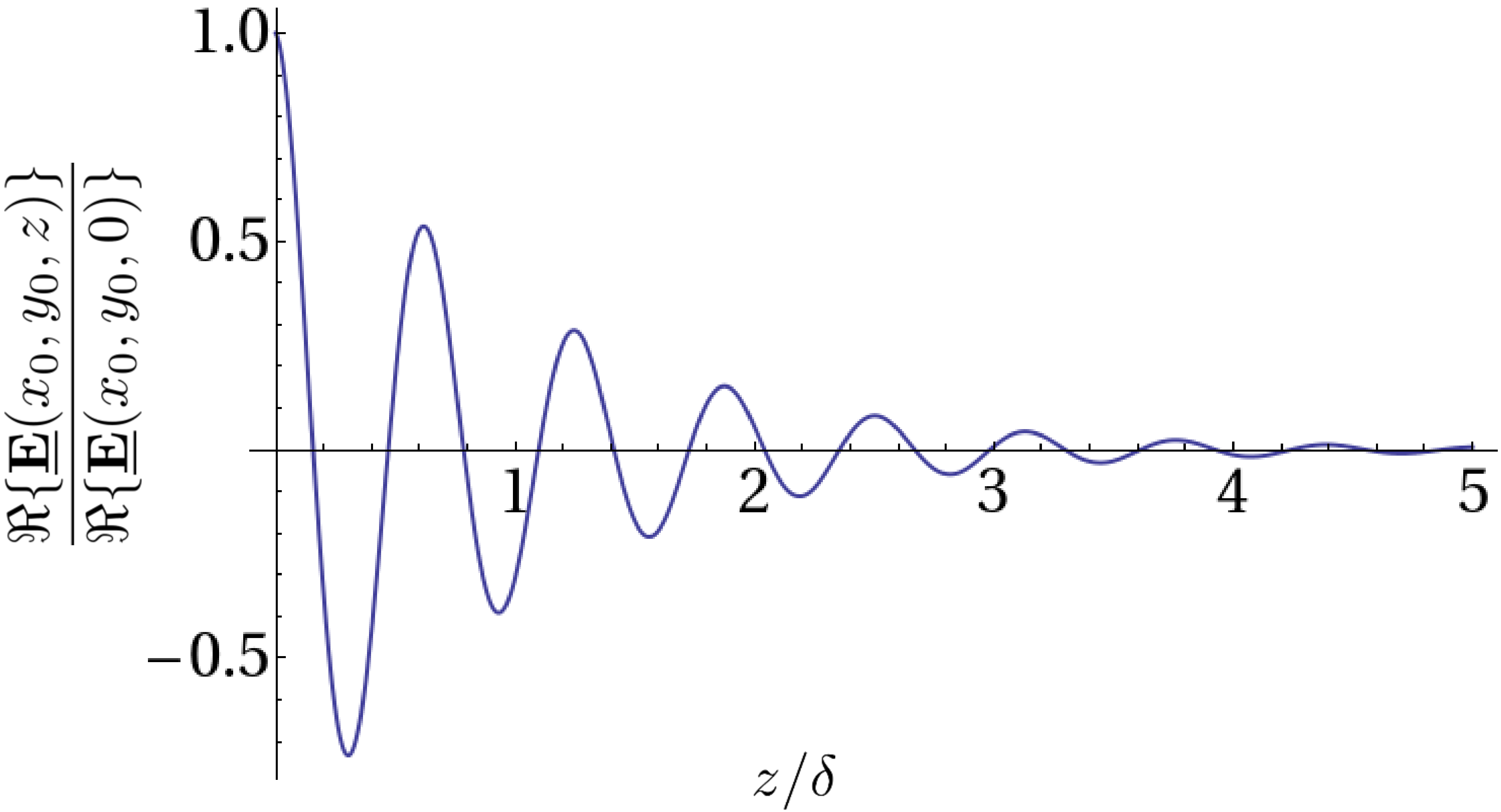
The real part describes the propagation of the wave while the imaginary part describes the exponential decay of the field strengt in the conductor

$$\underline{\mathbf{E}}(\mathbf{r}) = \mathbf{e}_x E_0 \mathrm{e}^{-j\underline{k}z} = \mathbf{e}_x E_0 \mathrm{e}^{-jk'z}\, \mathrm{e}^{-k''z}$$

The distance which is required for the fields to drop by a factor of e^{-1} is called penetration depth

$$\delta = \frac{1}{k''} = \frac{\sqrt{2}}{\sqrt{-\varepsilon\mu\omega^2 + \sqrt{\mu^2\omega^2\left(\sigma^2 + \varepsilon^2\omega^2\right)}}} \approx \sqrt{\frac{2}{\mu\omega\sigma}}$$

Exponential Decay of Amplitudes in Conductors



What we have done

1. Maxwell’s equations
2. Electromagnetic fields in different materials - material equations
3. Electrostatic fields
4. Magnetostatic fields
5. Electromagnetic waves
6. Field attenuation in conductors