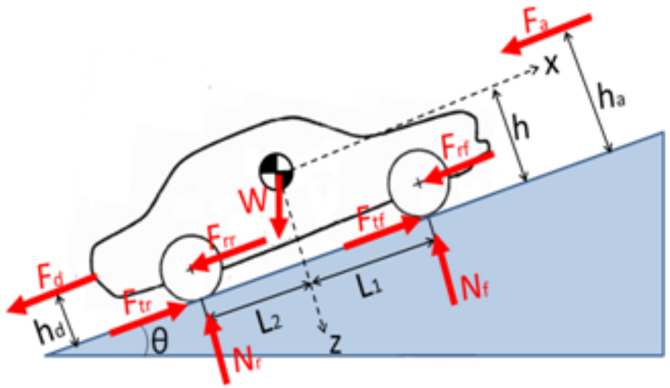
LONGITUDINAL beam DYNAMICS RECAP

Frank Tecker
CERN, BE-OP



Beam Injection, Extraction, and Transfer
Erice, 10-19/3/2017

Summary of the lecture:

- Introduction
- Linac: Phase stability
- Synchrotron:
  - Synchronous Phase
  - Dispersion Effects in Synchrotron
  - Stability and Longitudinal Phase Space Motion
  - Equations of motion
- Injection Matching
- Hamiltonian of Longitudinal Motion

- Appendices: some derivations and details

More related lectures:

- Injection: Electron Beams – M. Aiba
- Longitudinal Aspects I + II – Heiko Damerau

Particle types and acceleration

The accelerating system will depend upon the evolution of the particle velocity:

- electrons reach a constant velocity (~speed of light) at relatively low energy
- heavy particles reach a constant velocity only at very high energy
  - -> we need different types of resonators, optimized for different velocities
  - -> the **revolution frequency will vary**, so the RF frequency will be changing

Particle rest mass:

electron	0.511	MeV
proton	938	MeV
^{239}U	~220000	MeV

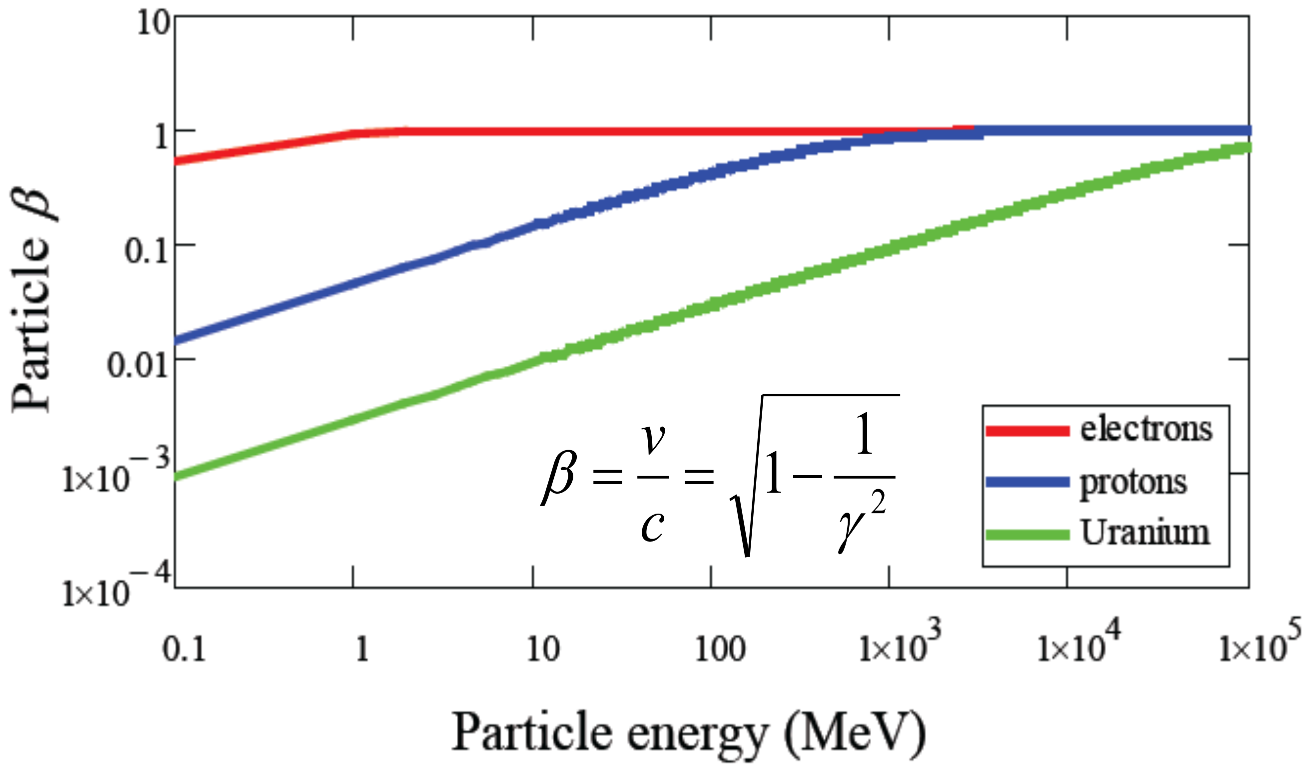
Total Energy: $E = \gamma m_0 c^2$

Relativistic gamma factor:

$$\gamma = \frac{E}{E_0} = \frac{m}{m_0} = \frac{1}{\sqrt{1-\beta^2}}$$

Momentum:

$$p = mv = \frac{E}{c^2}\beta c = \beta\frac{E}{c} = \beta\gamma m_0 c$$



Acceleration + Energy Gain

May the force be with you!

To accelerate, we need a **force in the direction of motion**!

Newton-Lorentz Force on a charged particle:

$$\vec{F} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = e\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

2nd term always perpendicular to motion => no acceleration

Hence, it is necessary to have an electric field E (preferably) along the direction of the initial momentum (z), which changes the momentum p of the particle.

$$\frac{dp}{dt} = eE_z$$

In relativistic dynamics, total energy E and momentum p are linked by

$$E^2 = E_0^2 + p^2c^2 \quad \Rightarrow \quad dE = v\,dp \qquad \left(2E\,dE = 2c^2 p\,dp \Leftrightarrow dE = c^2 mv/E\;dp = v\,dp\right)$$

The rate of energy gain per unit length of acceleration (along z) is then:

$$\frac{dE}{dz} = v\frac{dp}{dz} = \frac{dp}{dt} = eE_z$$

and the kinetic energy gained from the field along the z path is:

$$dW = dE = qE_z\,dz \quad \rightarrow \quad W = q\int E_z\,dz = qV$$

- V is a potential
- q the charge

Summary: Relativity + Energy Gain

Newton-Lorentz Force $\vec{F} = \frac{d\vec{p}}{dt} = e\left(\vec{E} + \vec{v} \times \vec{B}\right)$

2nd term always perpendicular to motion => no acceleration

Relativistics Dynamics

$$\beta = \frac{v}{c} = \sqrt{1 - \frac{1}{\gamma^2}} \qquad \gamma = \frac{E}{E_0} = \frac{m}{m_0} = \frac{1}{\sqrt{1-\beta^2}}$$

$$p = mv = \frac{E}{c^2}\beta c = \beta\frac{E}{c} = \beta\gamma m_0 c$$

$$E^2 = E_0^2 + p^2c^2 \longrightarrow dE = v\,dp$$

$$\frac{dE}{dz} = v\frac{dp}{dz} = \frac{dp}{dt} = eE_z$$

$$dE = dW = eE_z\,dz \rightarrow W = e\int E_z\,dz$$

RF Acceleration

$$E_z = \hat{E}_z \sin\omega_{RF}t = \hat{E}_z \sin\phi(t)$$

$$\int \hat{E}_z\,dz = \hat{V}$$

$$W = e\hat{V}\sin\phi$$

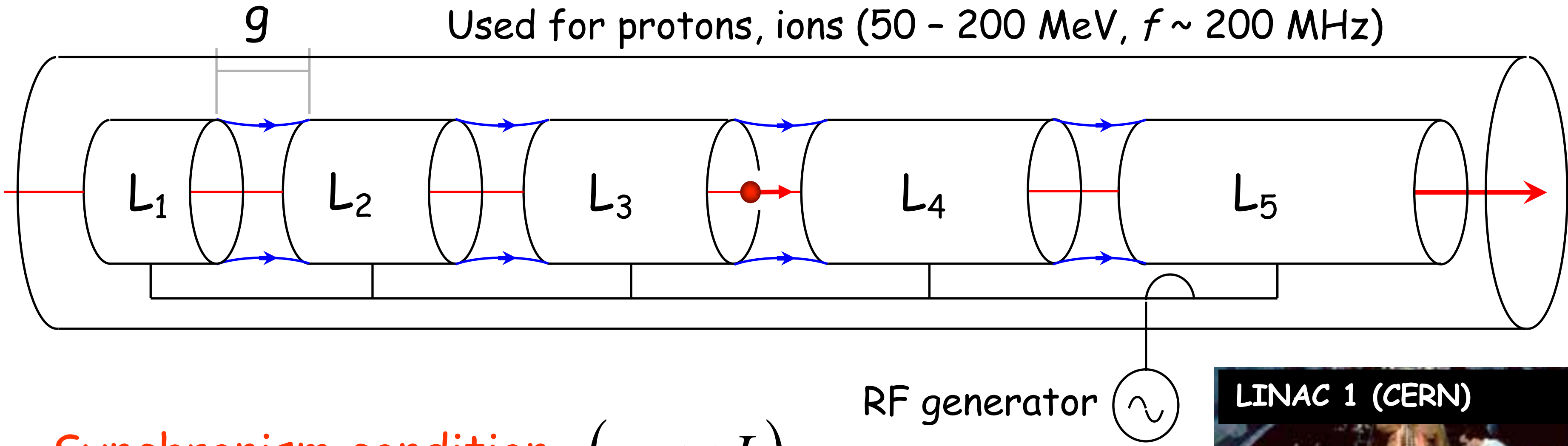
(neglecting transit time factor)

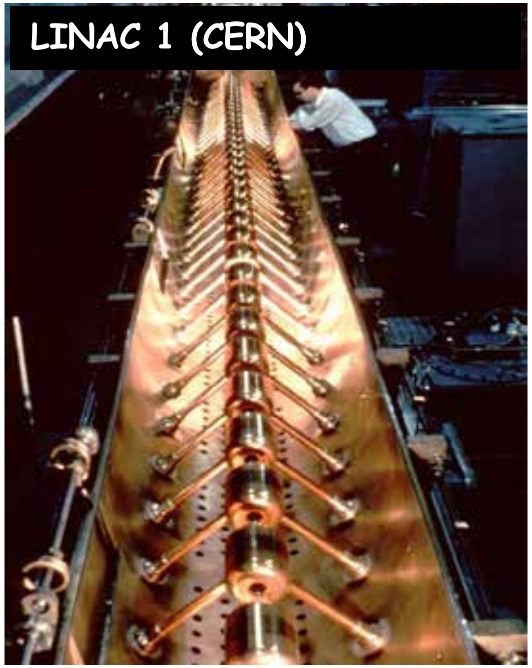
The field will change during the passage of the particle through the cavity
=> effective energy gain is lower

Radio Frequency (RF) acceleration: Alvarez Structure

Electrostatic acceleration limited by insulation possibilities => use RF fields

Used for protons, ions (50 - 200 MeV, $f \sim$ 200 MHz)





Synchronism condition $(g \ll L)$

$$L = v_s T_{RF} = \beta_s \lambda_{RF}$$

$$\omega_{RF} = 2\pi \frac{v_s}{L}$$

Note: - Drift tubes become longer for higher velocity

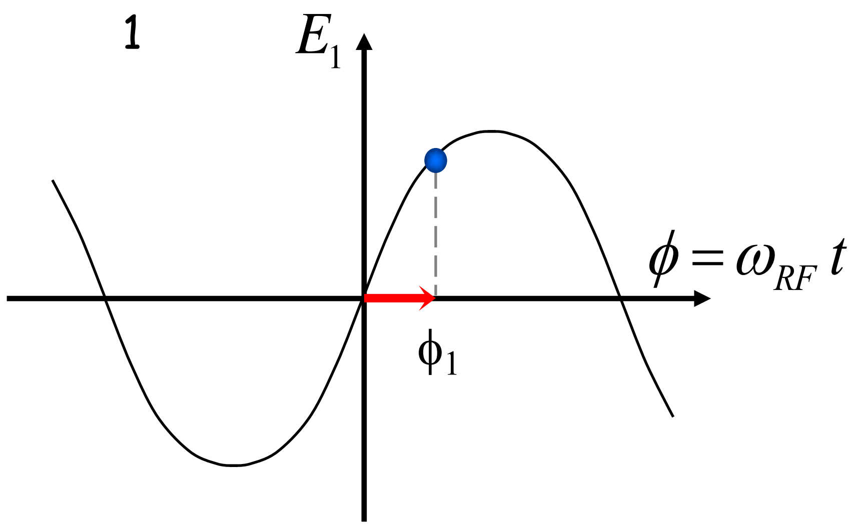
- Acceleration only for bunched beam (not continuous)

Common Phase Conventions

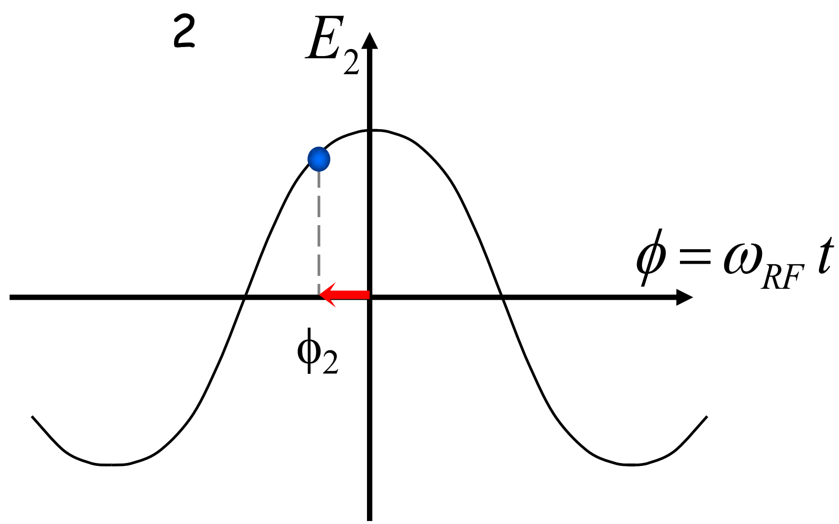
1. For circular accelerators, the origin of time is taken at the zero crossing of the RF voltage with positive slope

2. For linear accelerators, the origin of time is taken at the positive crest of the RF voltage

Time t= 0 chosen such that:



$$E_1(t) = E_0 \sin\left(\omega_{RF}\, t\right)$$



$$E_2(t) = E_0 \cos\left(\omega_{RF}\, t\right)$$

3. I will stick to **convention 1** in the following to avoid confusion

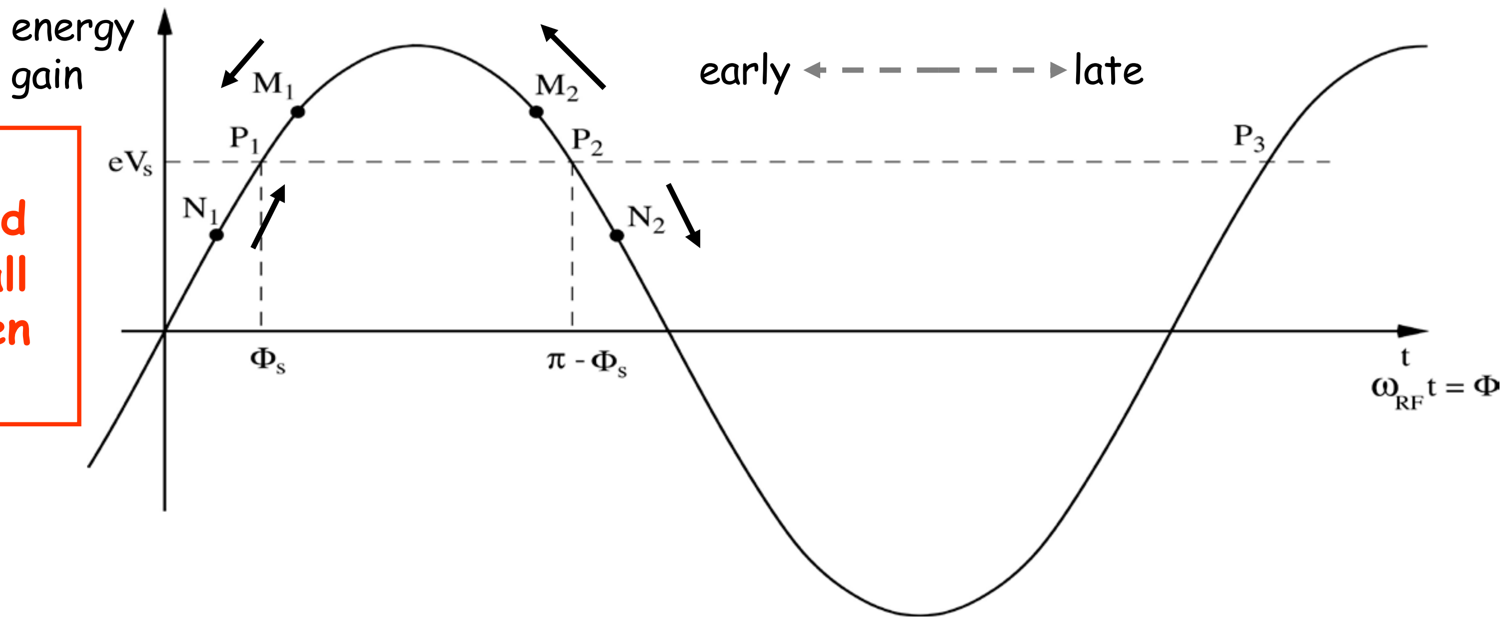
Principle of Phase Stability (Linac)

Let's consider a succession of accelerating gaps, operating in the 2π mode, for which the synchronism condition is fulfilled for a phase Φ_s .

$$eV_s = e\hat{V}\sin\Phi_s$$

is the energy gain in one gap for the particle to reach the next gap with the same RF phase: P_1 ,P_2, …… are fixed points.

For a 2π mode, the electric field is the same in all gaps at any given time.



If an energy increase is transferred into a velocity increase =>

M_1 & N_1 will move towards P_1 => stable

M_2 & N_2 will go away from P_2 => unstable

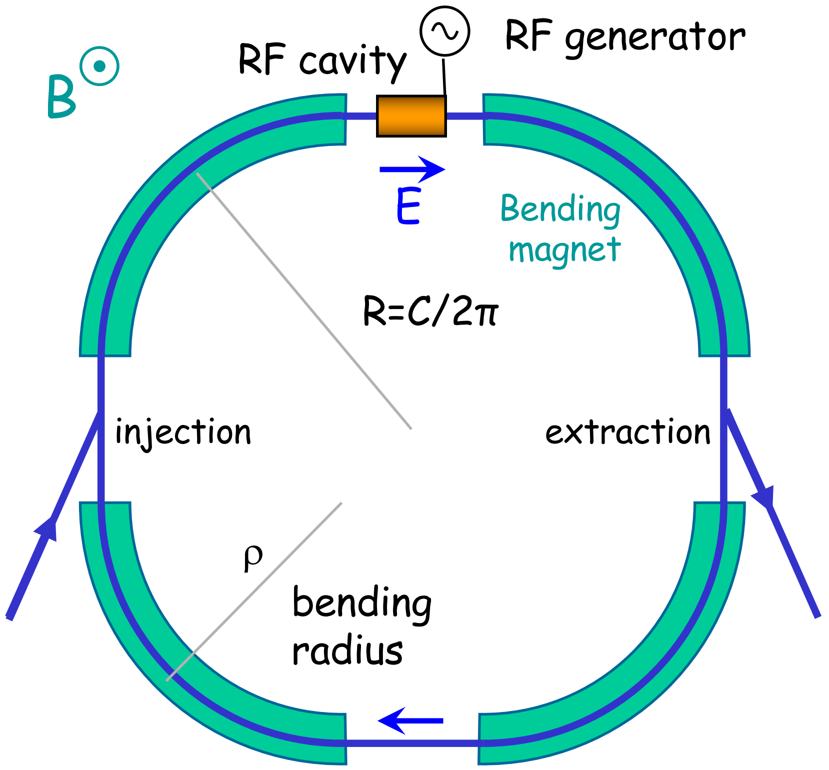
(Highly relativistic particles have no significant velocity change)

Circular accelerators

Cyclotron (not covered here)

Synchrotron

Circular accelerators: The Synchrotron



1. Constant orbit during acceleration
2. To keep particles on the closed orbit, B should increase with time
3. ω and ω_{RF} increase with energy

RF frequency can be multiple of revolution frequency

$$\omega_{RF} = h\omega$$

Synchronism condition

$$T_s = h\,T_{RF}$$

$$\frac{2\pi R}{v_s} = h\,T_{RF}$$

h integer, harmonic number: number of RF cycles per revolution

Circular accelerators: The Synchrotron

LEAR (CERN)
Low Energy Antiproton Ring

EPA (CERN)
Electron Positron Accumulator

PS (CERN)
Proton Synchrotron

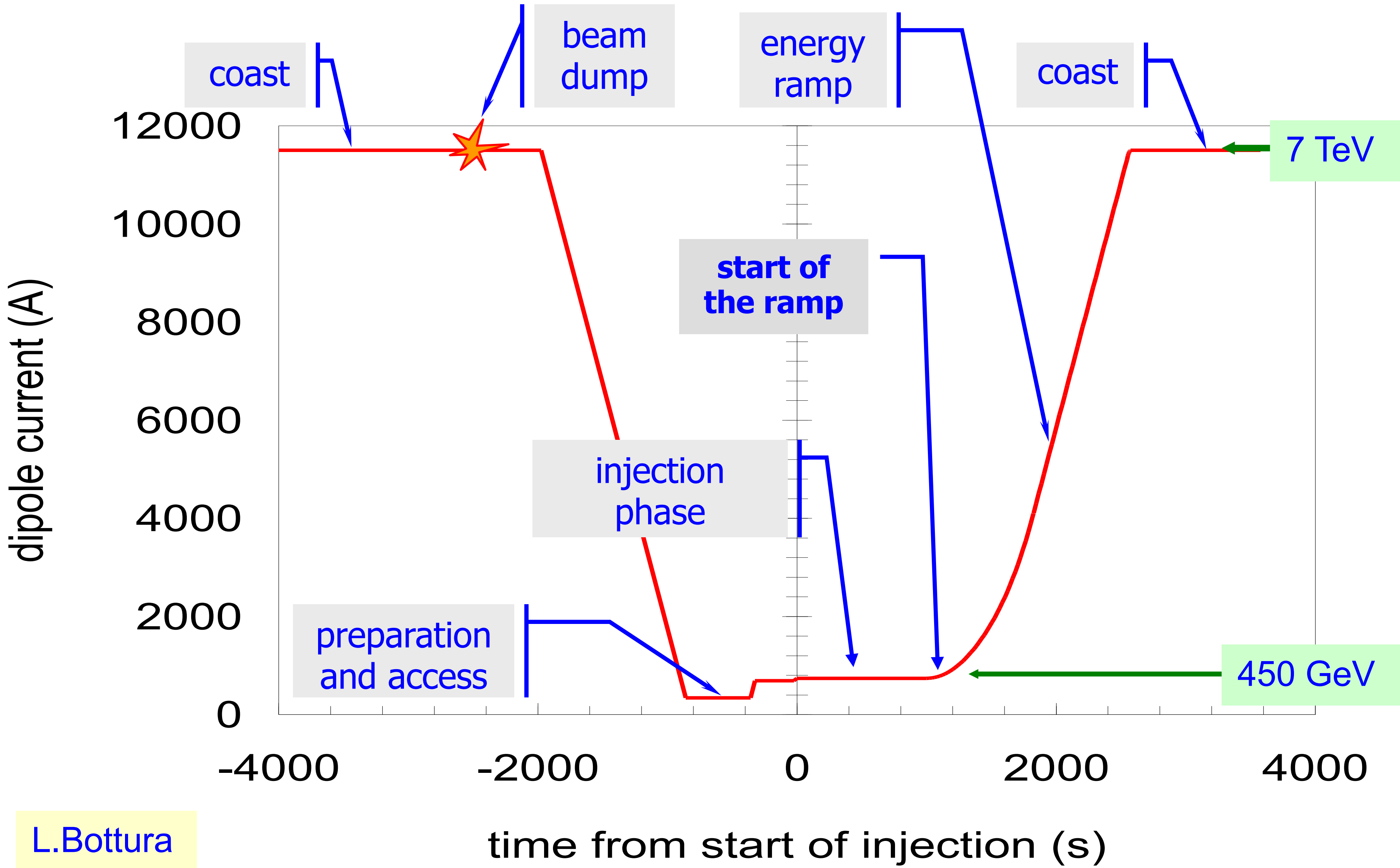
© CERN Geneva

Examples of different proton and electron synchrotrons at CERN

+ LHC (of course!)

The Synchrotron – LHC Operation Cycle

The magnetic field (dipole current) is increased during the acceleration.



L.Bottura

The Synchrotron – Energy ramping

Energy ramping by increasing the B field (frequency has to follow v):

$$p = eB\rho \quad \Rightarrow \quad \frac{dp}{dt} = e\rho\,\dot{B} \quad \Rightarrow (\Delta p)_{turn} = e\rho\,\dot{B}T_r = \frac{2\pi\,e\rho\,R\dot{B}}{v}$$

Since: $$E^2 = E_0^2 + p^2c^2 \quad \Rightarrow \quad \Delta E = v\Delta p$$

$$(\Delta E)_{turn} = (\Delta W)_s = 2\pi e\rho R\dot{B} = e\hat{V}\sin\phi_s$$

Stable phase φ_s changes during energy ramping!

$$\sin\phi_s = 2\pi\,\rho\,R\,\frac{\dot{B}}{\hat{V}_{RF}} \quad \Rightarrow \quad \phi_s = \arcsin\left(2\pi\,\rho\,R\,\frac{\dot{B}}{\hat{V}_{RF}}\right)$$

- The number of stable synchronous particles is equal to the harmonic number h. They are equally spaced along the circumference.
- Each synchronous particle satisfies the relation p=eBρ. They have the nominal energy and follow the nominal trajectory.

The Synchrotron – Frequency change

During the energy ramping, the RF frequency increases to follow the increase of the revolution frequency :

$$\omega = \frac{\omega_{RF}}{h} = \omega(B, R_s)$$

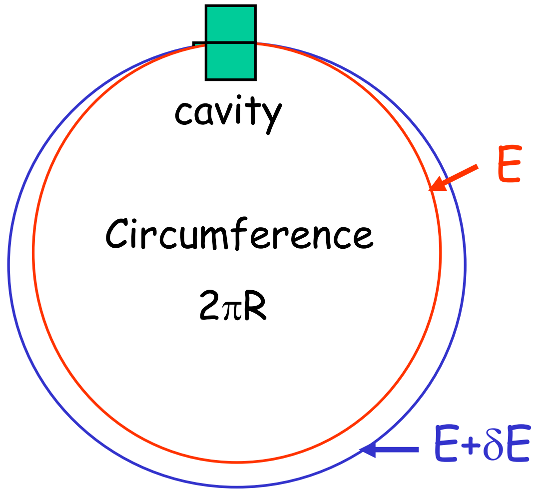
Hence: $\dfrac{f_{RF}(t)}{h} = \dfrac{v(t)}{2\pi R_s} = \dfrac{1}{2\pi}\dfrac{ec^2}{E_s(t)}\dfrac{\rho}{R_s}B(t)$ (using $p(t) = eB(t)\rho, \quad E = mc^2$)

Since $E^2 = (m_0c^2)^2 + p^2c^2$ the RF frequency must follow the variation of the B field with the law

$$\frac{f_{RF}(t)}{h} = \frac{c}{2\pi R_s}\left\{\frac{B(t)^2}{(m_0c^2/ec\rho)^2 + B(t)^2}\right\}^{1/2}$$

This asymptotically tends towards $f_r \rightarrow \dfrac{c}{2\pi R_s}$ when B becomes large compared to $m_0c^2/(ec\rho)$ which corresponds to $v \rightarrow c$

Dispersion Effects in a Synchrotron



A particle slightly shifted in momentum will have a

- dispersion orbit and a different orbit length
- a different velocity.

As a result of both effects the revolution frequency changes with a "slip factor η":

$$\eta = \frac{\mathrm{d} f_r / f_r}{\mathrm{d} p / p} \Rightarrow$$

p=particle momentum

R=synchrotron physical radius

f_r=revolution frequency

Note: you also find η defined with a minus sign!

Effect from orbit defined by Momentum compaction factor: $\alpha_c = \frac{dL/L}{dp/p}$

Property of the beam optics:
(derivation in Appendix)

$$\alpha_c = \frac{1}{L}\int_C \frac{D_x(s)}{\rho(s)} ds_0$$

Dispersion Effects – Revolution Frequency

The two effects of the orbit length and the particle velocity change the revolution frequency as:

$$f_r = \frac{\beta c}{2\pi R} \quad \Rightarrow \quad \frac{df_r}{f_r} = \frac{d\beta}{\beta} - \frac{dR}{R} = \frac{d\beta}{\beta} - \alpha_c \frac{dp}{p}$$

definition of momentum compaction factor

$$\frac{df_r}{f_r} = \left(\frac{1}{\gamma^2} - \alpha_c\right)\frac{dp}{p}$$

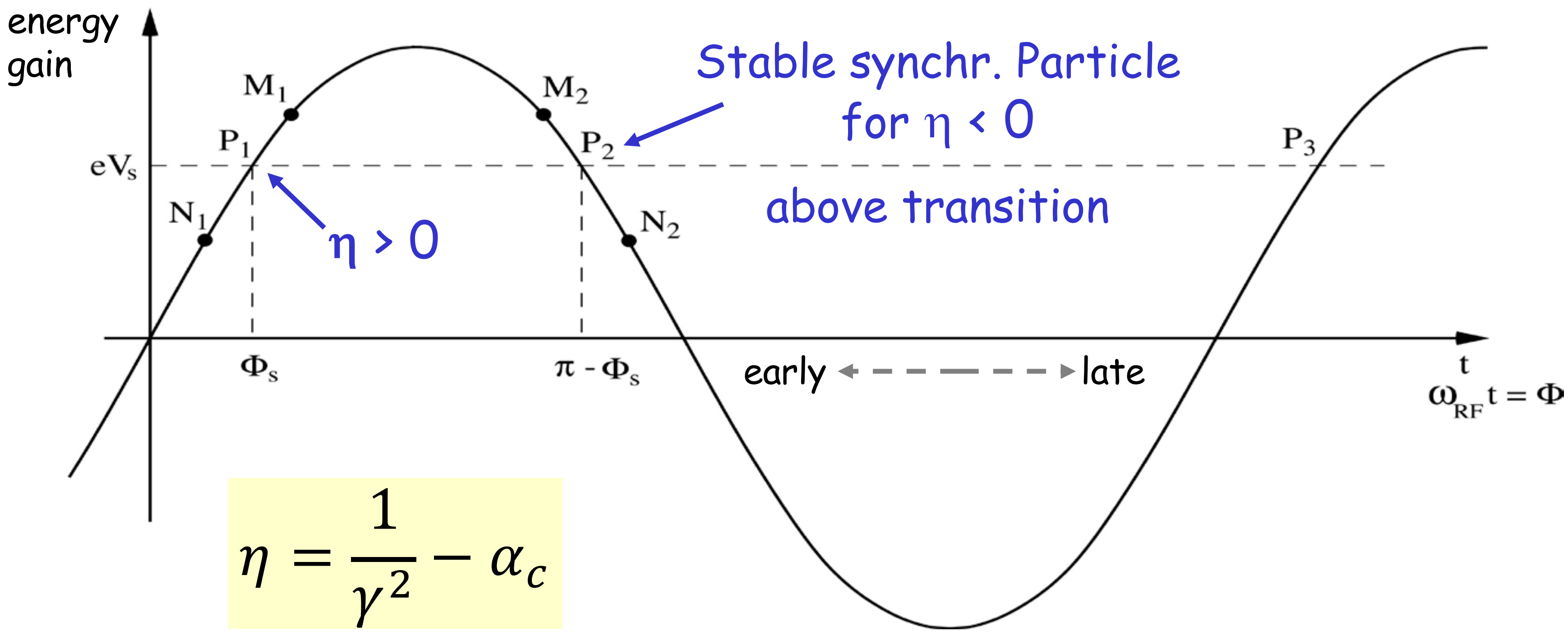
$$p = mv = \beta\gamma \frac{E_0}{c} \quad \Rightarrow \quad \frac{dp}{p} = \frac{d\beta}{\beta} + \frac{d\left(1-\beta^2\right)^{-1/2}}{\left(1-\beta^2\right)^{-1/2}} = \underbrace{\left(1-\beta^2\right)^{-1}}_{\gamma^2}\frac{d\beta}{\beta}$$

Slip factor:
$$\eta = \frac{1}{\gamma^2} - \alpha_c$$
or
$$\eta = \frac{1}{\gamma^2} - \frac{1}{\gamma_t^2}$$
with
$$\gamma_t = \frac{1}{\sqrt{\alpha_c}}$$

At transition energy, $\eta = 0$, the velocity change and the path length change with momentum compensate each other. So the revolution frequency there is independent from the momentum deviation.

Phase Stability in a Synchrotron

From the definition of η it is clear that an increase in momentum gives
- below transition ($\eta > 0$) a higher revolution frequency (increase in velocity dominates) while
- above transition ($\eta < 0$) a lower revolution frequency ($v \approx c$ and longer path) where the momentum compaction (generally > 0) dominates.



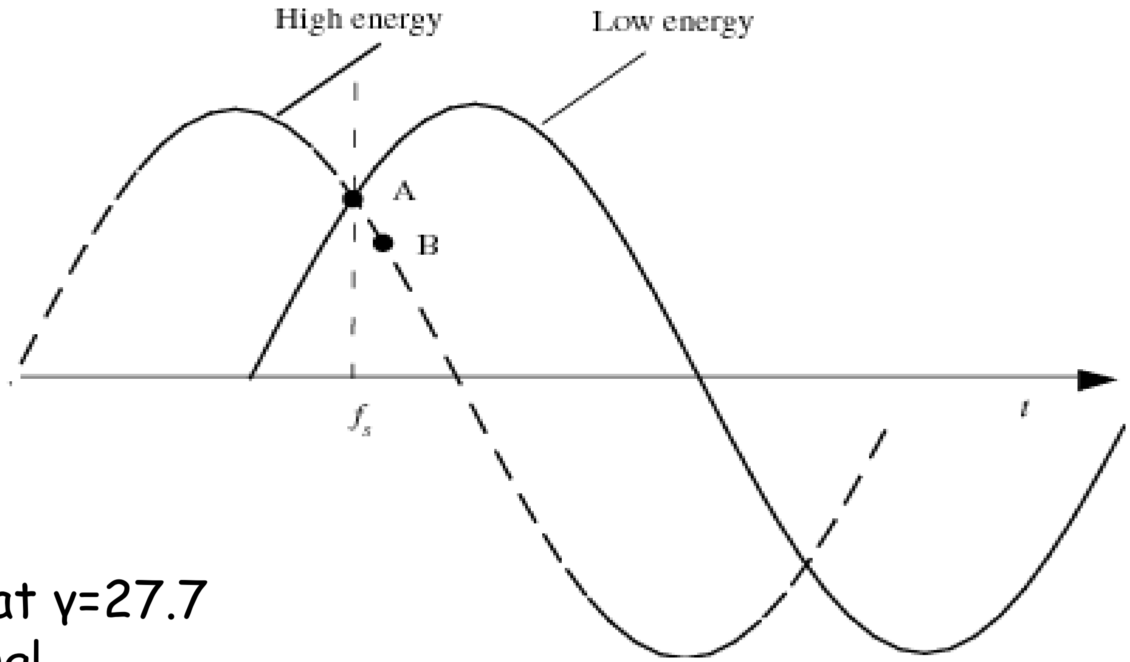
$$\eta = \frac{1}{\gamma^2} - \alpha_c$$

Crossing Transition

At transition, the velocity change and the path length change with momentum compensate each other. So the revolution frequency there is independent from the momentum deviation.

Crossing transition during acceleration makes the previous stable synchronous phase unstable. The RF system needs to make a rapid change of the RF phase, a 'phase jump'.

$$\alpha_c \sim \frac{1}{Q_x^2} \qquad \gamma_t = \frac{1}{\sqrt{\alpha_c}} \sim Q_x$$



In the PS: γ_t is at ~6 GeV
In the SPS: γ_t= 22.8, injection at γ=27.7
=> no transition crossing!
In the LHC: γ_t is at ~55 GeV, also far below injection energy

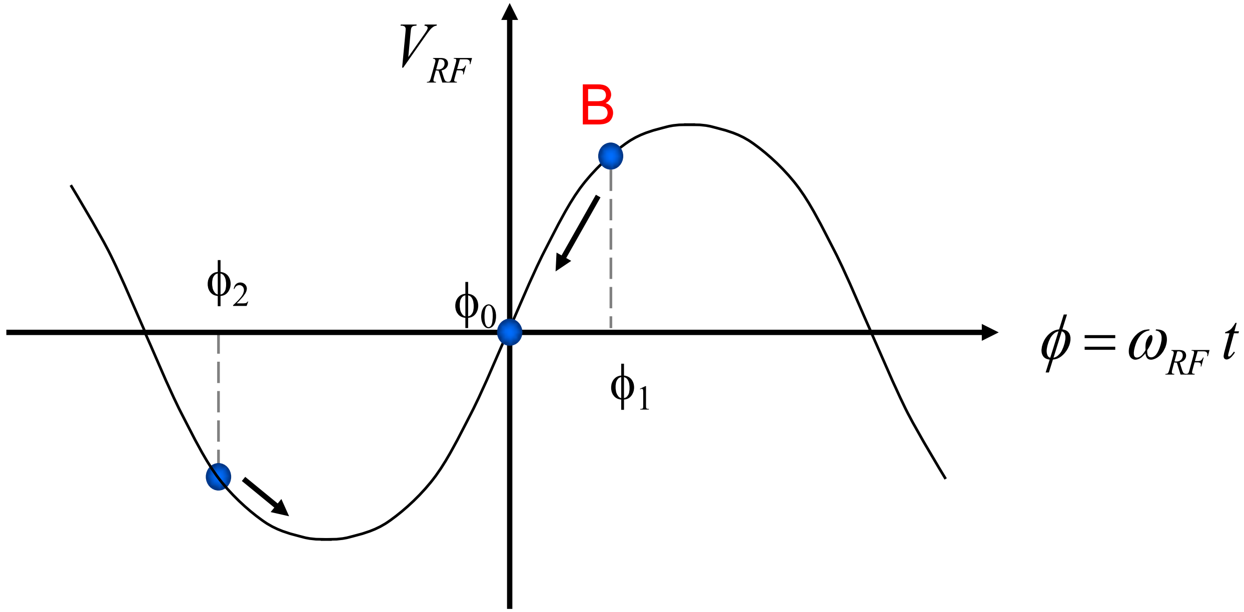
Transition crossing not needed in leptons machines, why?

Dynamics: Synchrotron oscillations

Simple case (no accel.): B = const., below transition $\gamma < \gamma_t$

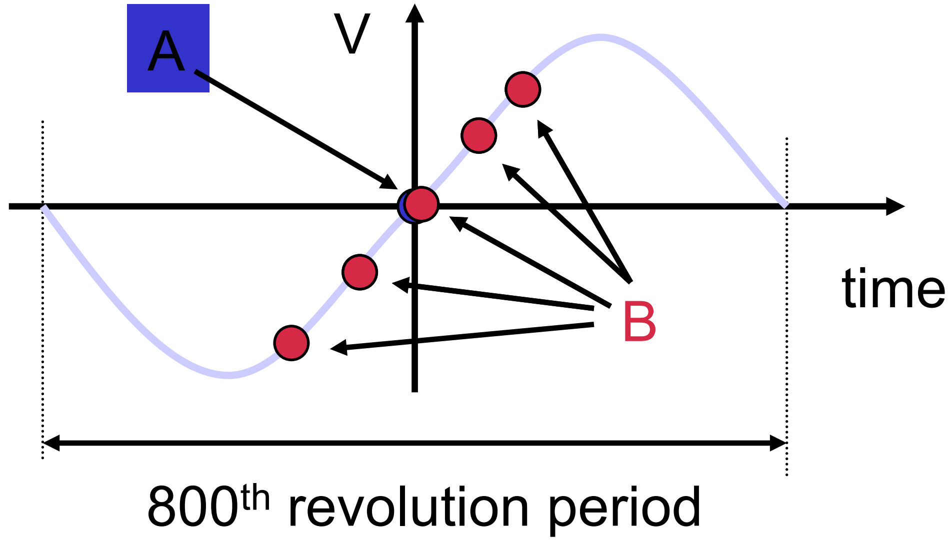
The phase of the synchronous particle must therefore be $\phi_0 = 0$.

Φ_1 - The particle B is accelerated
- Below transition, an energy increase means an increase in revolution frequency
- The particle arrives earlier – tends toward ϕ_0



ϕ_2 - The particle is decelerated
- decrease in energy - decrease in revolution frequency
- The particle arrives later – tends toward ϕ_0

Synchrotron oscillations

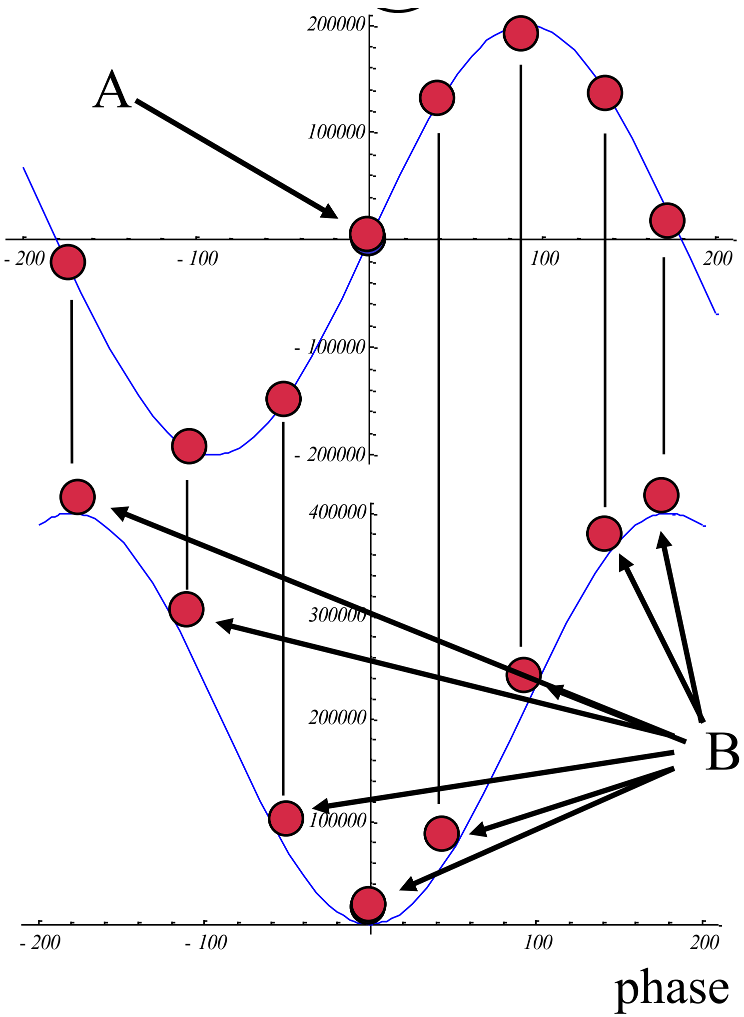


Particle B is performing Synchrotron Oscillations around synchronous particle A.

The amplitude depends on the initial phase and energy.

The oscillation frequency is much slower than in the transverse plane.
It takes a large number of revolutions for one complete oscillation.
Restoring electric force smaller than magnetic force.

The Potential Well

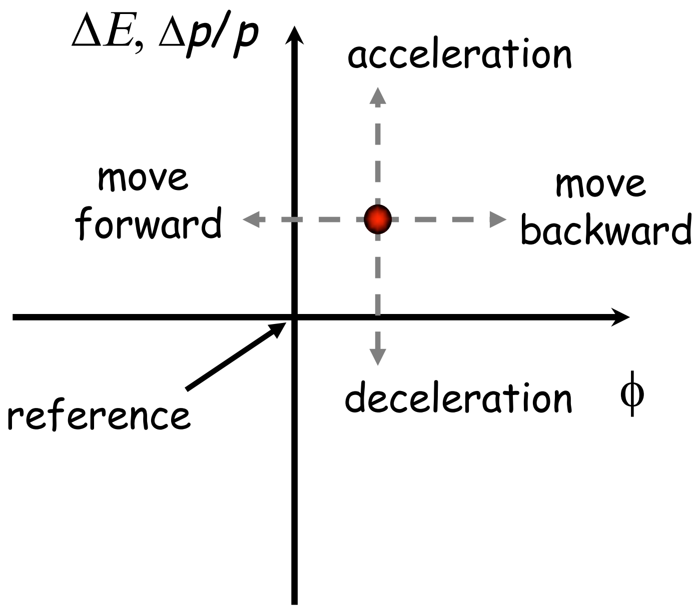


Cavity voltage

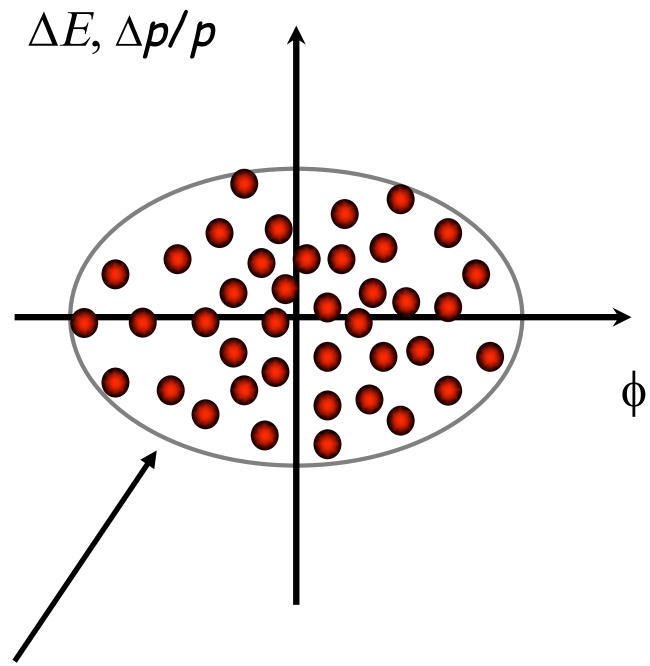
Potential well

Longitudinal phase space

The energy – phase oscillations can be drawn in phase space:



The particle trajectory in the phase space ($\Delta p/p$, ϕ) describes its longitudinal motion.



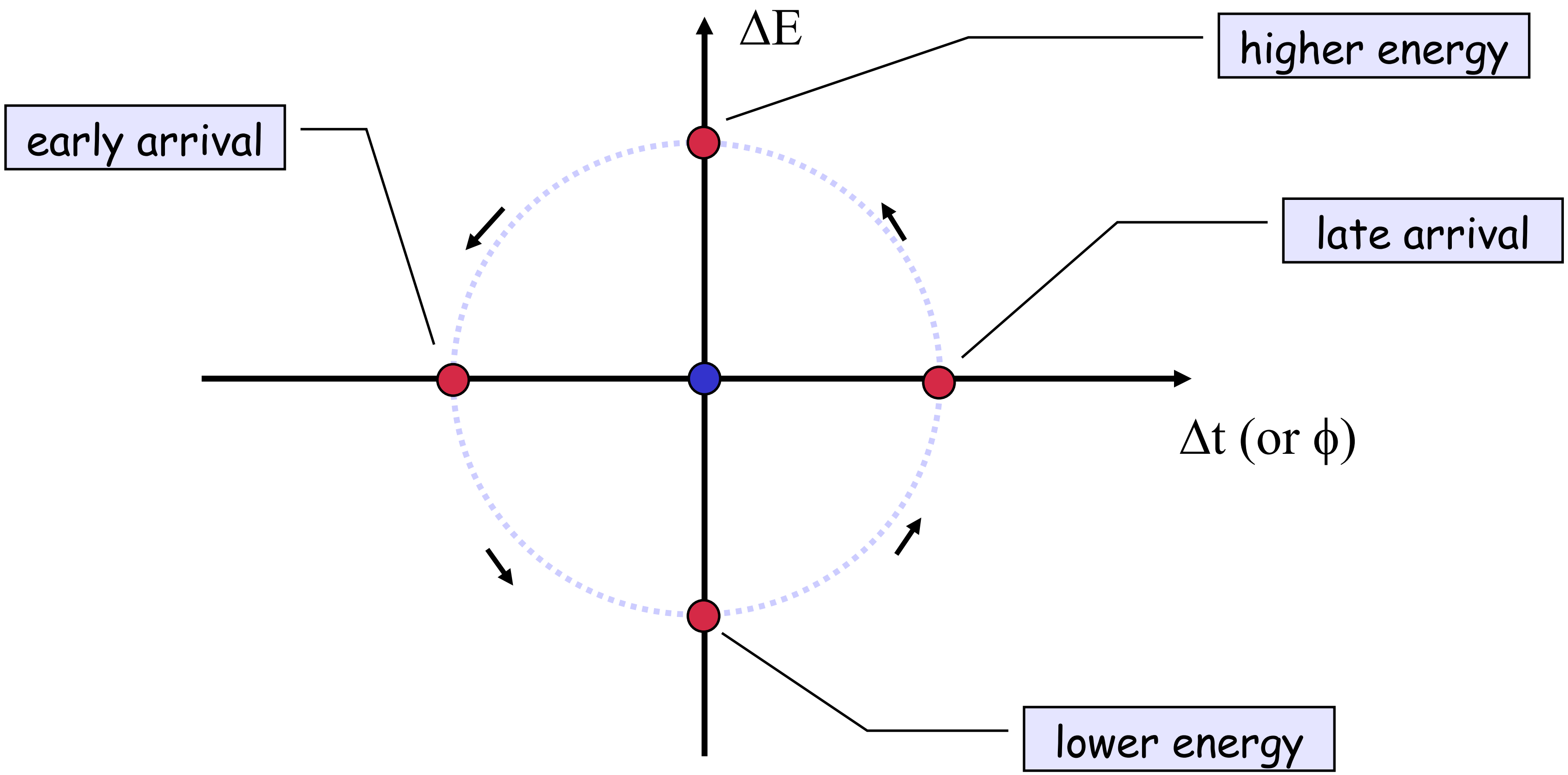
Emittance: phase space area including all the particles

NB: if the emittance contour correspond to a possible orbit in phase space, its shape does not change with time (matched beam)

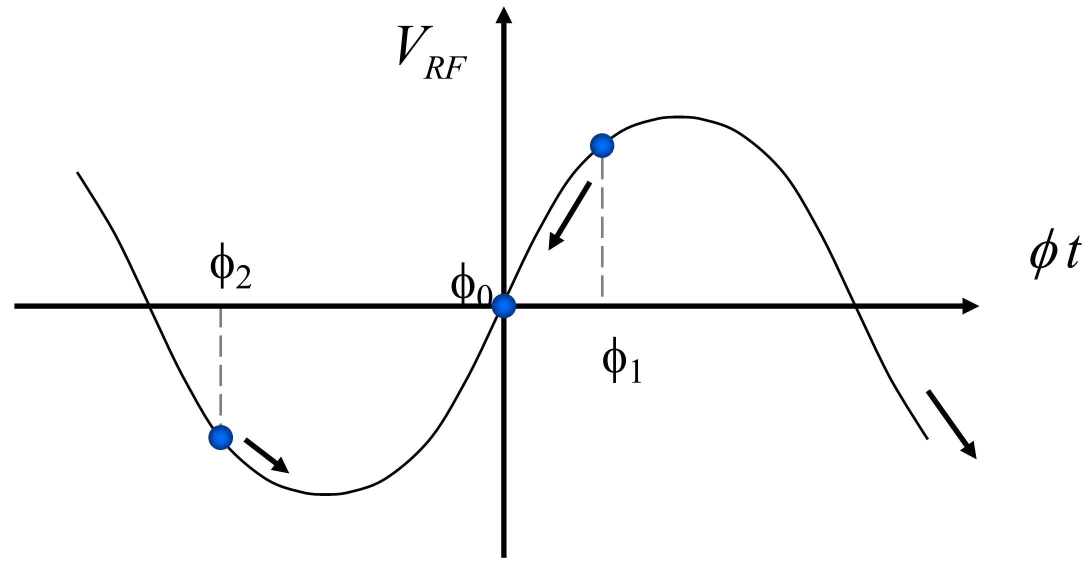
Longitudinal Phase Space Motion

Particle B performs a synchrotron oscillation around synchronous particle A.

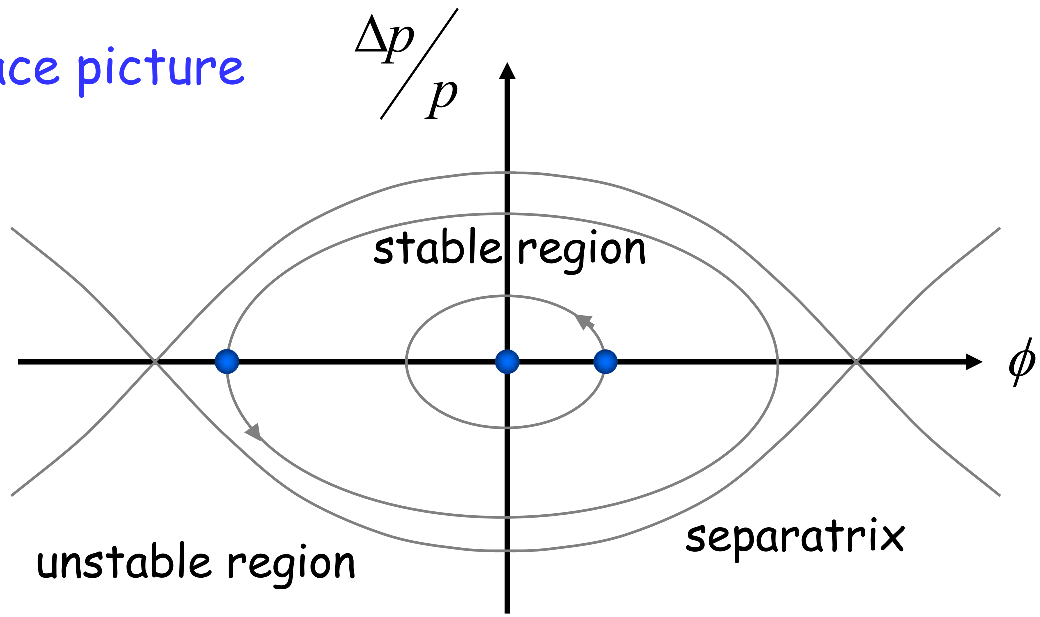
Plotting this motion in longitudinal phase space gives:



Synchrotron oscillations – No acceleration



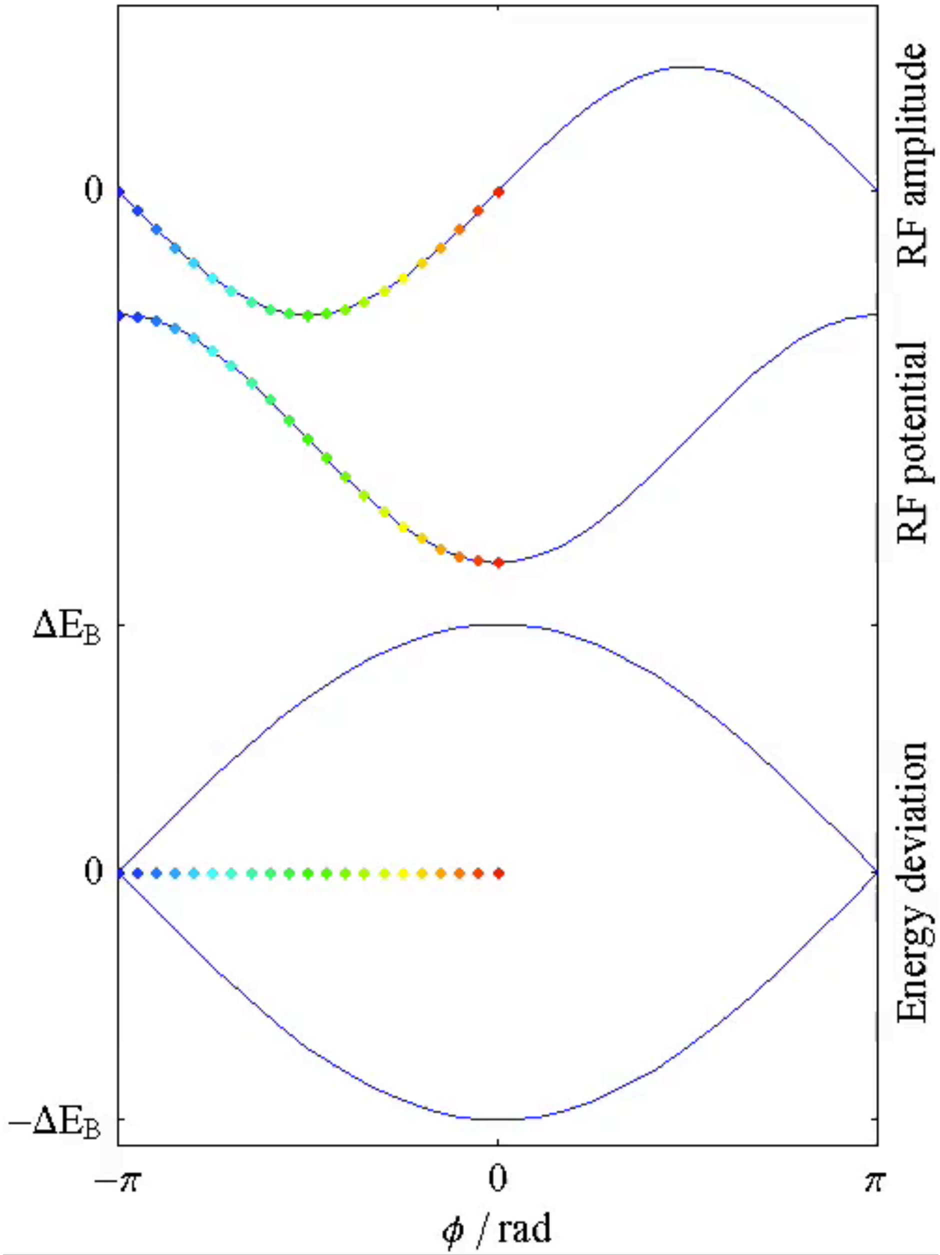
Phase space picture



Synchrotron motion in phase space

The restoring force is non-linear.
⇒ speed of motion depends on position in phase-space

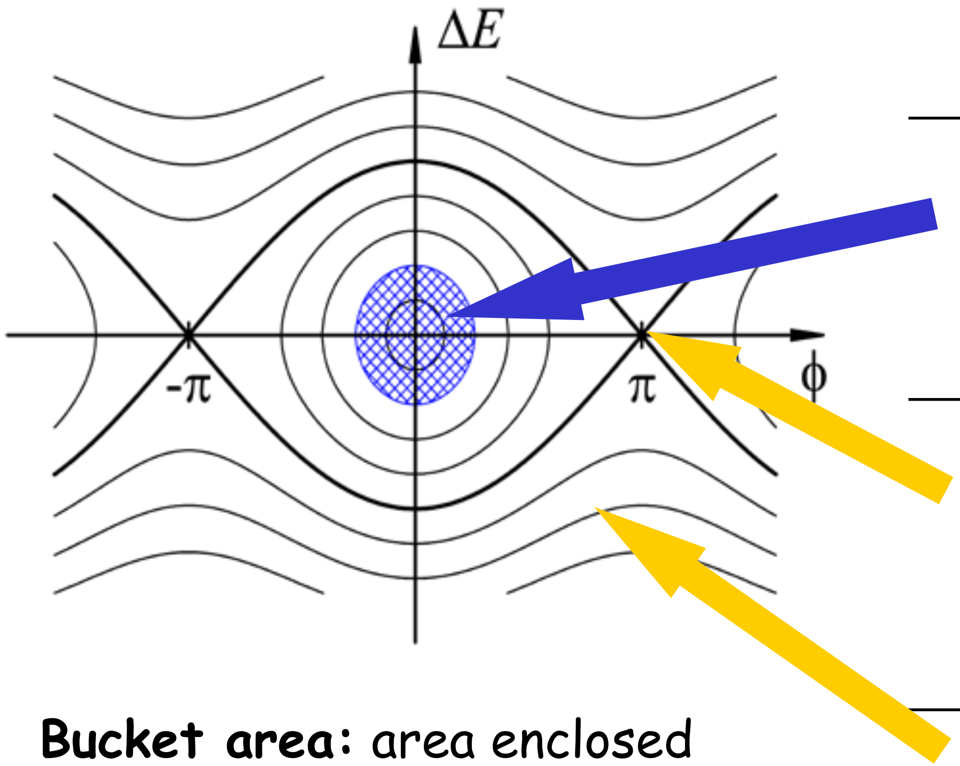
(here shown for a stationary bucket)



Heiko Damerau

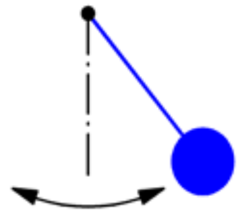
Synchrotron motion in phase space

ΔE-ϕ **phase space** of a stationary bucket
(when there is no acceleration)



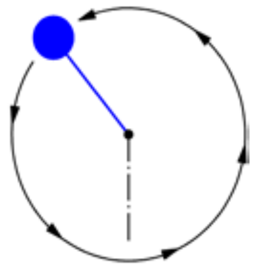
Dynamics of a particle
Non-linear, conservative oscillator → e.g. pendulum

Particle **inside the separatrix:**



Particle at the **unstable fix-point**

Particle **outside the separatrix:**

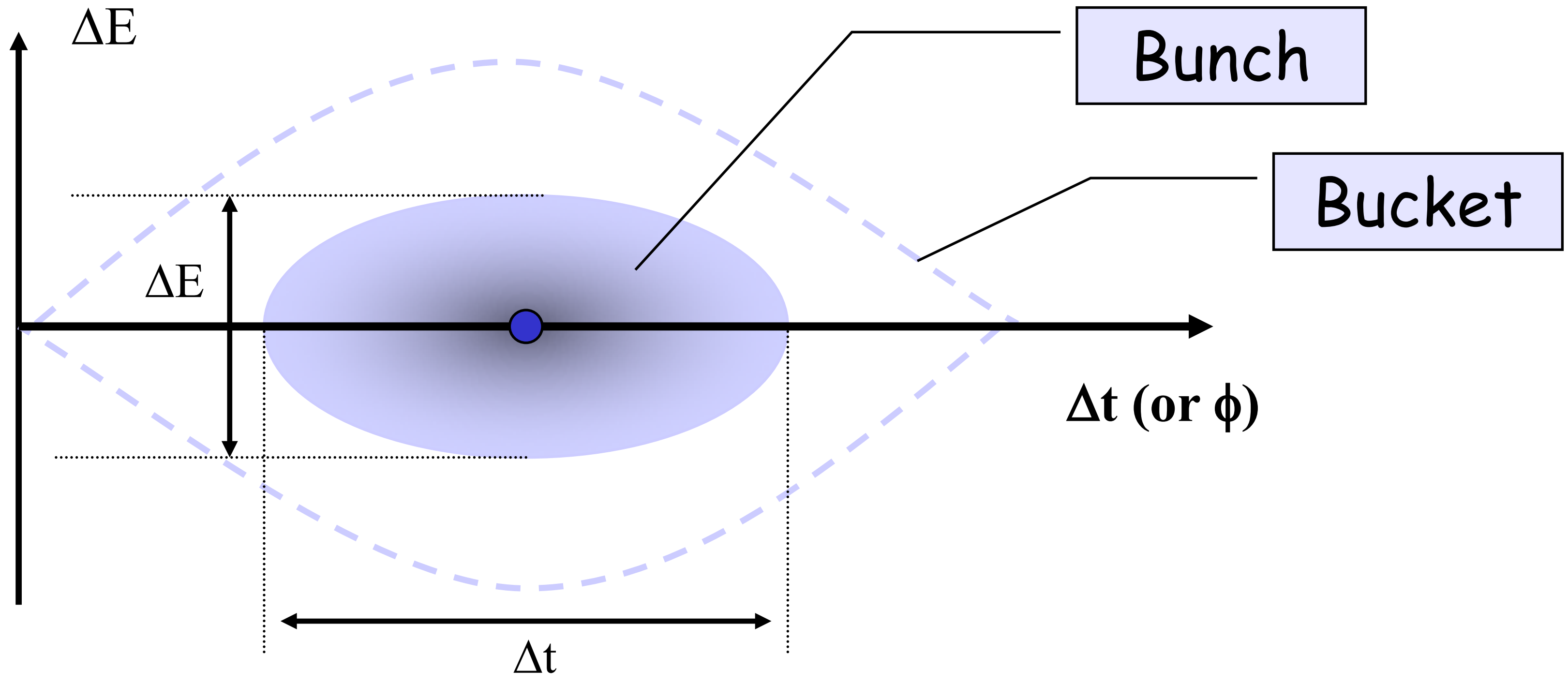


Bucket area: area enclosed by the separatrix

The area covered by particles is the longitudinal emittance.

(Stationary) Bunch & Bucket

The bunches of the beam fill usually a part of the bucket area.



Bucket area = **longitudinal Acceptance** [eVs]

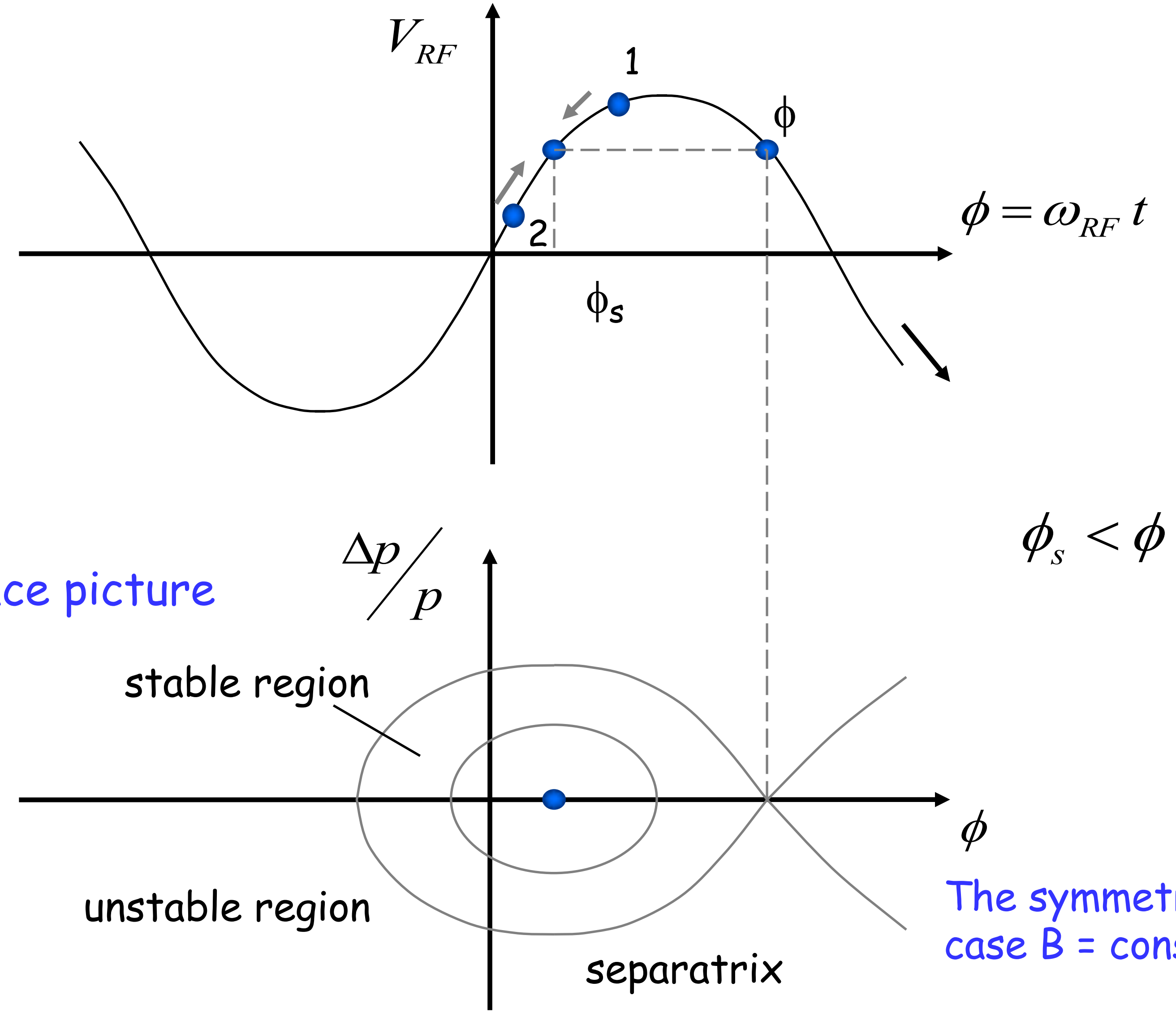
Bunch area = **longitudinal beam emittance** = $4\pi\,\sigma_E\,\sigma_t$ **[eVs]**

Attention: Different definitions are used!

Synchrotron oscillations (with acceleration)

Case with acceleration *B* increasing

$\gamma < \gamma_t$

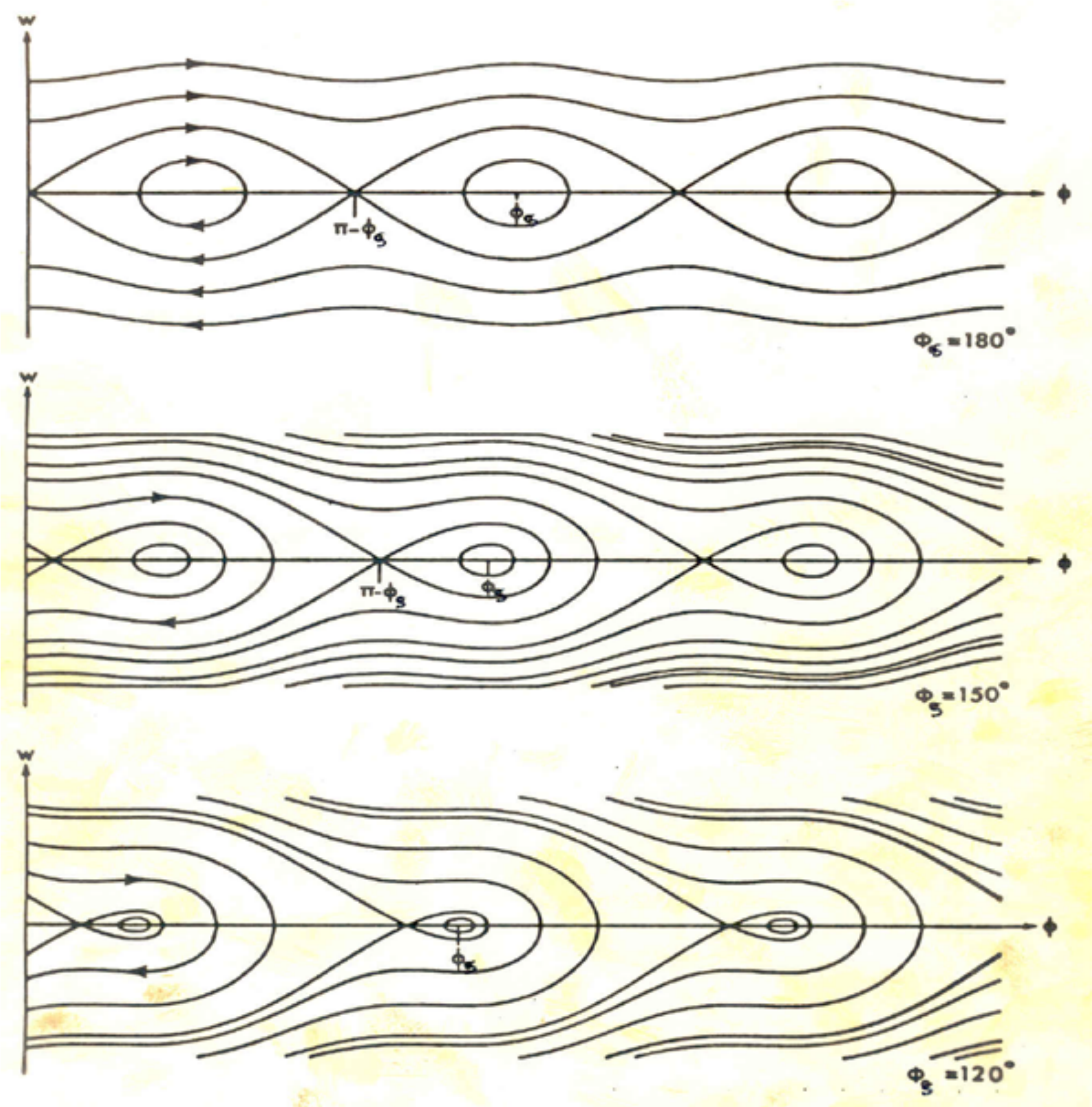


$$\phi_s < \phi < \pi - \phi_s$$

Phase space picture

The symmetry of the case B = const. is lost

RF Acceptance versus Synchronous Phase



The areas of stable motion (closed trajectories) are called "BUCKET". The number of circulating buckets is equal to "h".

The phase extension of the bucket is maximum for ϕ_s =180° (or 0°) which means no acceleration.

During acceleration, the buckets get smaller, both in length and energy acceptance.

=> Injection preferably without acceleration.

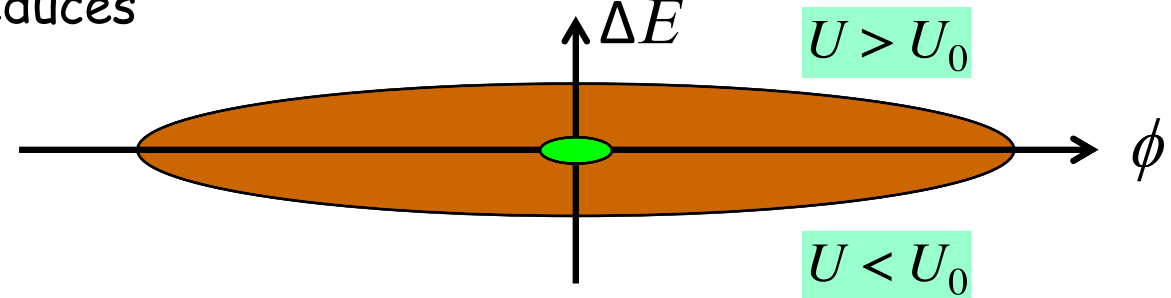
Longitudinal Motion with Synchrotron Radiation

Synchrotron radiation energy-loss energy dependant:

$$U_0 = \frac{4}{3}\pi \frac{r_e}{(m_0 c^2)^3} \frac{E^4}{\rho}$$

During one period of synchrotron oscillation:

- when the particle is in the upper half-plane, it loses more energy per turn, its energy gradually reduces



- when the particle is in the lower half-plane, it loses less energy per turn, but receives U_0 on the average, so its energy deviation gradually reduces

The phase space trajectory spirals towards the origin (limited by quantum excitations)

=> The synchrotron motion is damped toward an equilibrium bunch length and energy spread.

$$\sigma_\tau = \frac{\alpha}{\Omega_s}\left(\frac{\sigma_\varepsilon}{E}\right)$$

More details in *tomorrow's* lecture on *'Injection Electron Beams'*

Longitudinal Dynamics in Synchrotrons

Now we will look more quantitatively at the "synchrotron motion".

The RF acceleration process clearly emphasizes two coupled variables, the energy gained by the particle and the RF phase experienced by the same particle.
Since there is a well defined synchronous particle which has always the same phase ϕ_s, and the nominal energy E_s, it is sufficient to follow other particles with respect to that particle.
So let's introduce the following reduced variables:

revolution frequency	:	$\Delta f_r = f_r - f_{rs}$
particle RF phase	:	$\Delta\phi = \phi - \phi_s$
particle momentum	:	$\Delta p = p - p_s$
particle energy	:	$\Delta E = E - E_s$
azimuth angle	:	$\Delta\theta = \theta - \theta_s$

Equations of Longitudinal Motion

In these reduced variables, the equations of motion are (see Appendix):

$$\frac{\Delta E}{\omega_{rs}} = -\frac{p_s R_s}{h\eta\omega_{rs}}\frac{d(\Delta\phi)}{dt} = -\frac{p_s R_s}{h\eta\omega_{rs}}\dot{\phi}$$

$$2\pi\frac{d}{dt}\left(\frac{\Delta E}{\omega_{rs}}\right) = e\hat{V}(\sin\phi - \sin\phi_s)$$

deriving and combining

$$\frac{d}{dt}\left[\frac{R_s p_s}{h\eta\omega_{rs}}\frac{d\phi}{dt}\right] + \frac{e\hat{V}}{2\pi}(\sin\phi - \sin\phi_s) = 0$$

This second order equation is non linear. Moreover the parameters within the bracket are in general slowly varying with time.

We will simplify in the following…

Small Amplitude Oscillations

Let's assume constant parameters R_s, p_s, ω_s and η:

$$\ddot{\phi} + \frac{\Omega_s^2}{\cos\phi_s}(\sin\phi - \sin\phi_s) = 0$$

with

$$\Omega_s^2 = \frac{h\eta\omega_{rs} e\hat{V}\cos\phi_s}{2\pi R_s p_s}$$

Consider now small phase deviations from the reference particle:

$$\sin\phi - \sin\phi_s = \sin(\phi_s + \Delta\phi) - \sin\phi_s \cong \cos\phi_s \Delta\phi$$

(for small $\Delta\phi$)

and the corresponding linearized motion reduces to a harmonic oscillation:

$$\ddot{\phi} + \Omega_s^2 \Delta\phi = 0$$

where Ω_s is the synchrotron angular frequency.

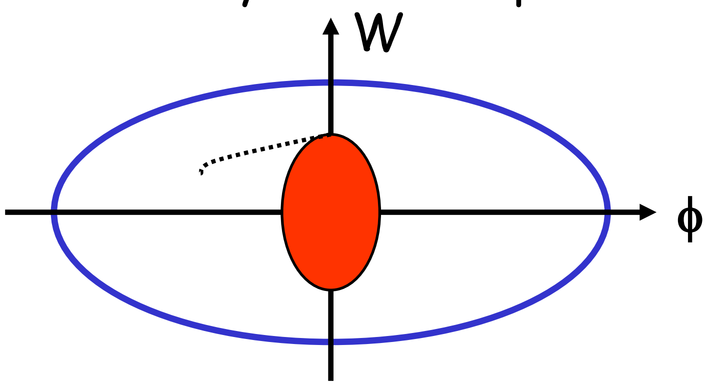
The synchrotron tune ν_s is the number of synchrotron oscillations per revolution:

$$\nu_s = \Omega_s / \omega_r$$

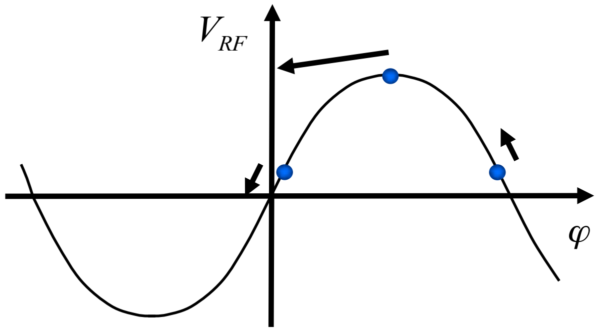
See Appendix for large amplitude treatment and further details.

Injection: Effect of a Mismatch

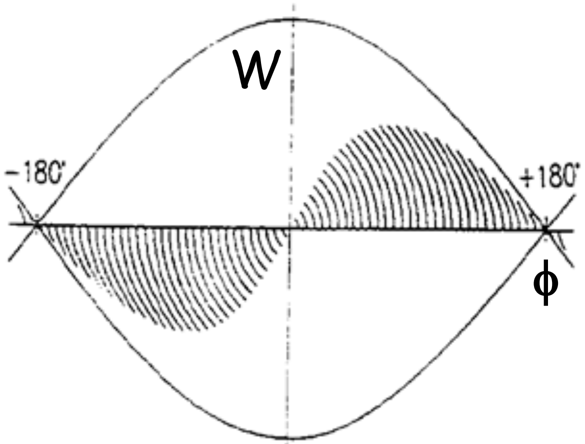
Injected bunch: short length and large energy spread
after 1/4 synchrotron period: longer bunch with a smaller energy spread.



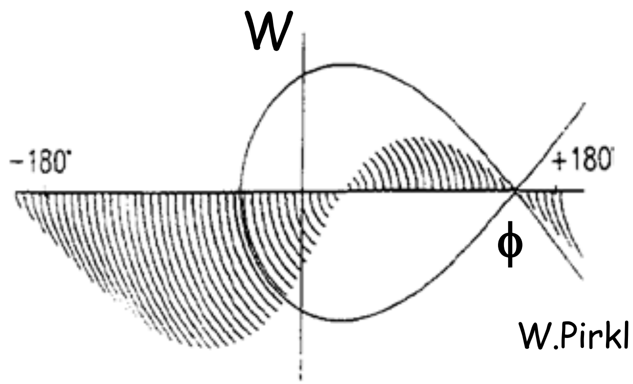
For larger amplitudes, the angular phase space motion is slower
(1/8 period shown below) => can lead to filamentation and emittance growth



restoring force is
non-linear



stationary bucket

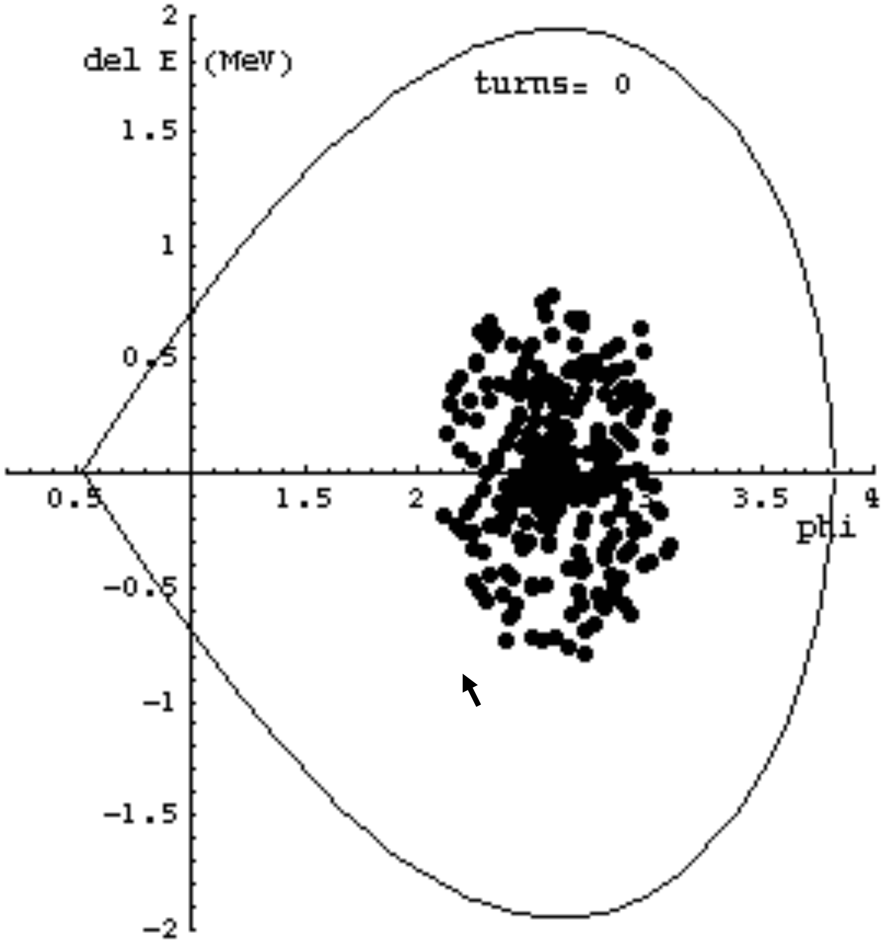


accelerating bucket

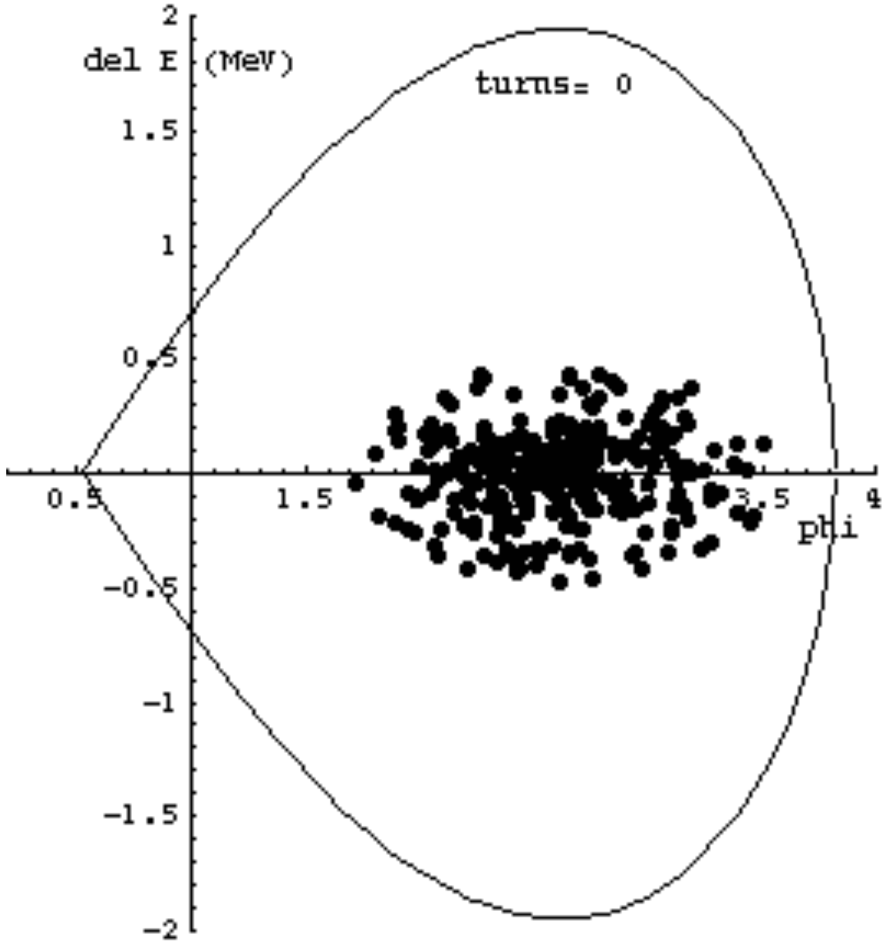
Effect of a Mismatch (2)

Evolution of an injected beam for the first 100 turns.

For a matched transfer, the emittance does not grow (left).



matched beam

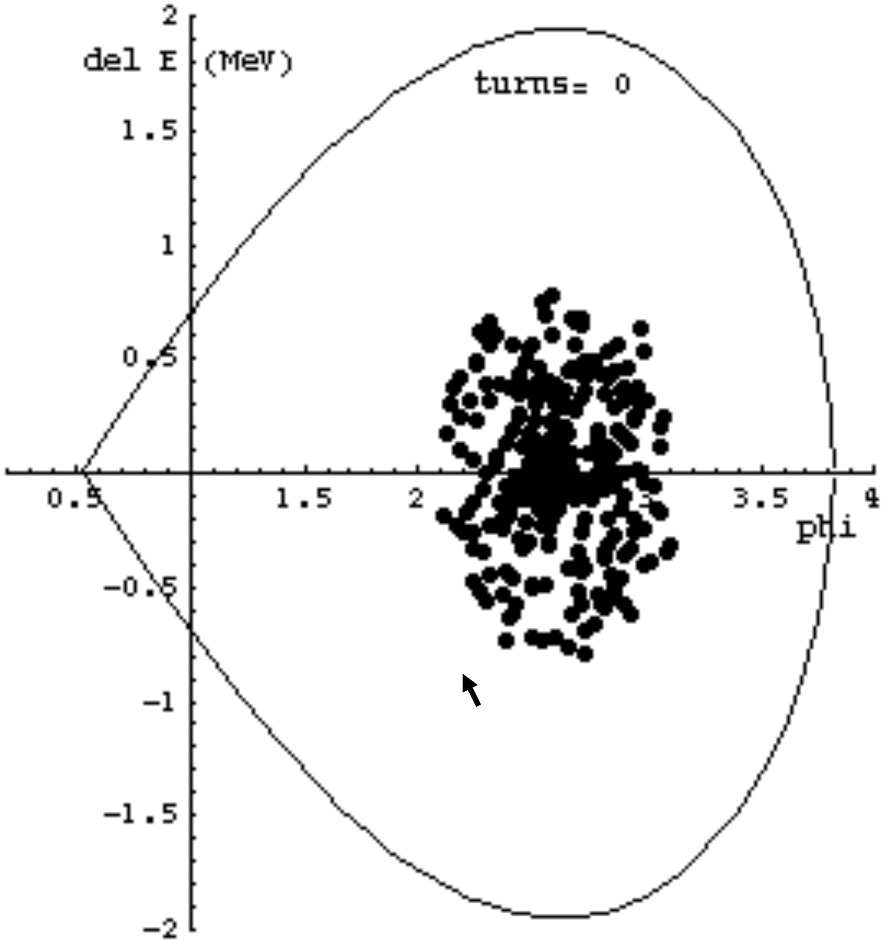


mismatched beam – **bunch length**

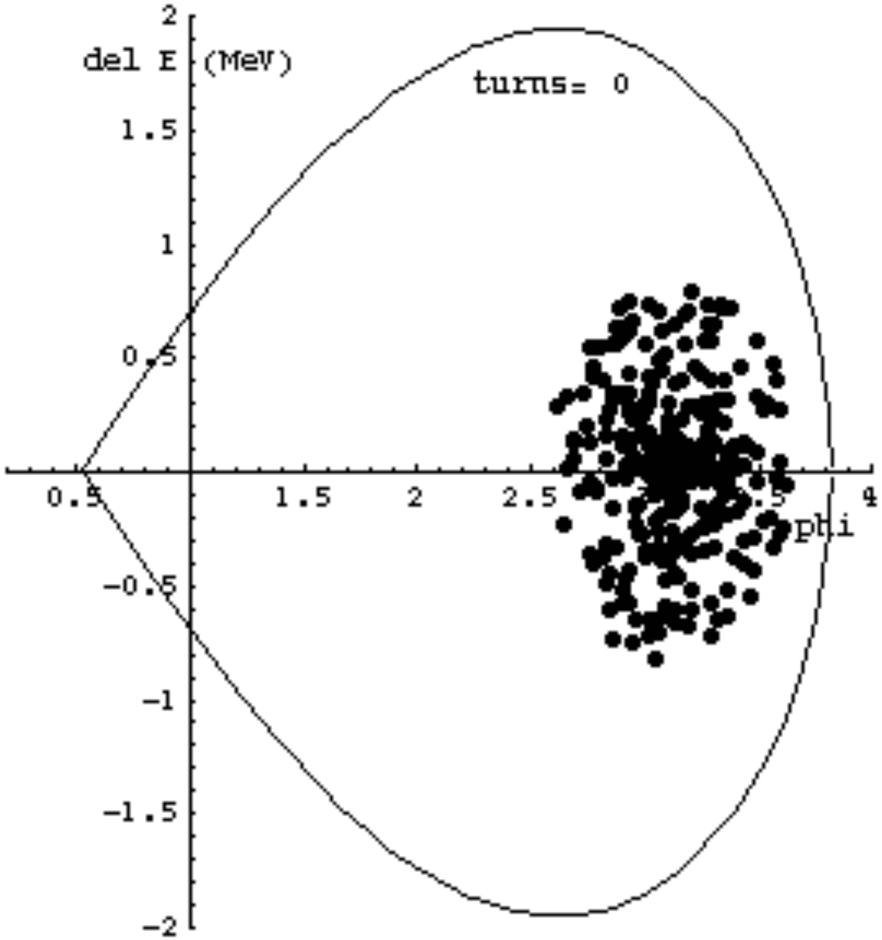
Effect of a Mismatch (3)

Evolution of an injected beam for the first 100 turns.

For a mismatched transfer, the emittance increases (right).



matched beam

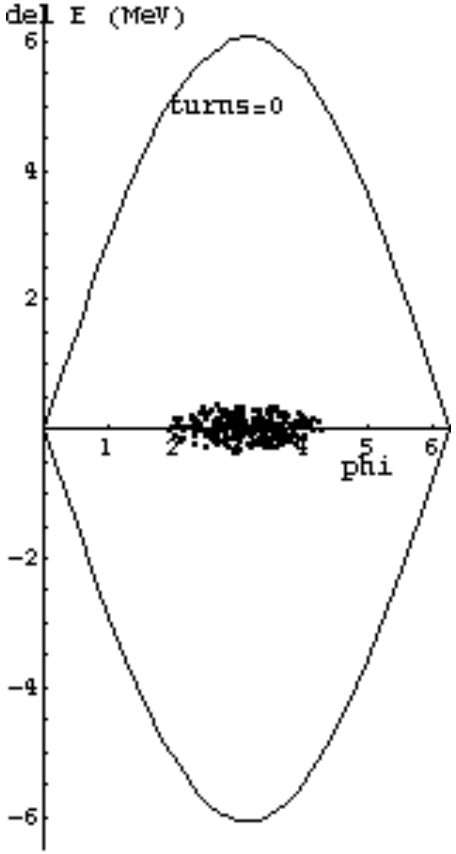


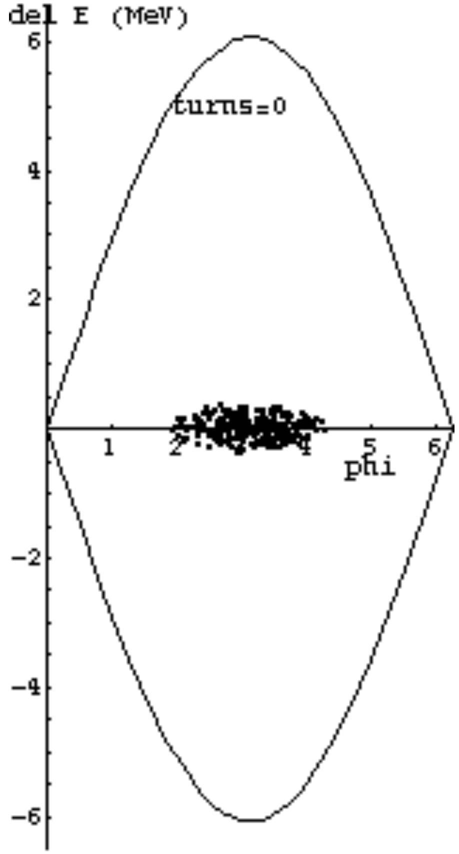
mismatched beam – **phase error**

Bunch Rotation

Phase space motion can be used to make short bunches.

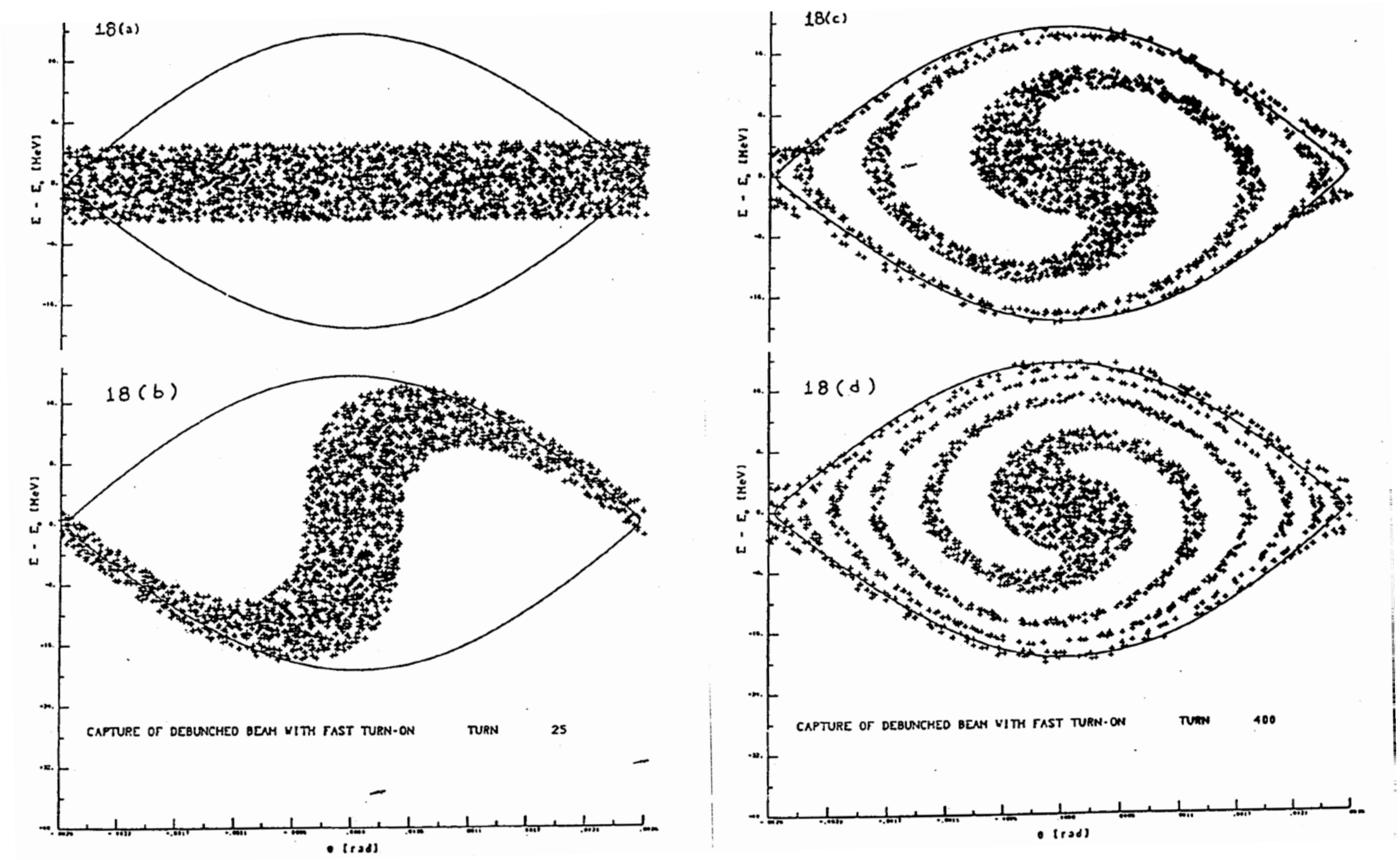
Start with a long bunch and extract or recapture when it's short.



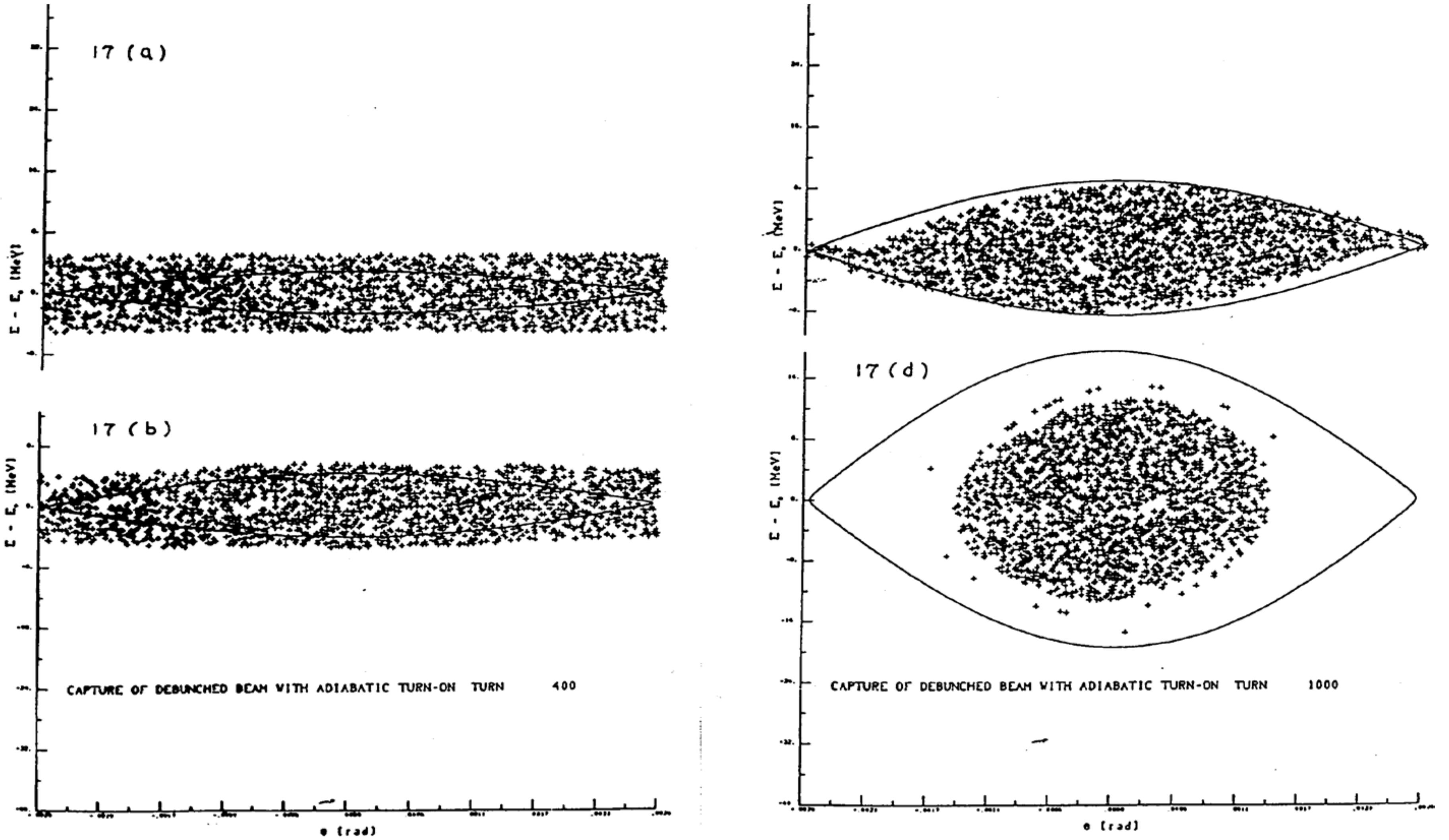


initial beam

Capture of a Debunched Beam with Fast Turn-On



Capture of a Debunched Beam with Adiabatic Turn-On

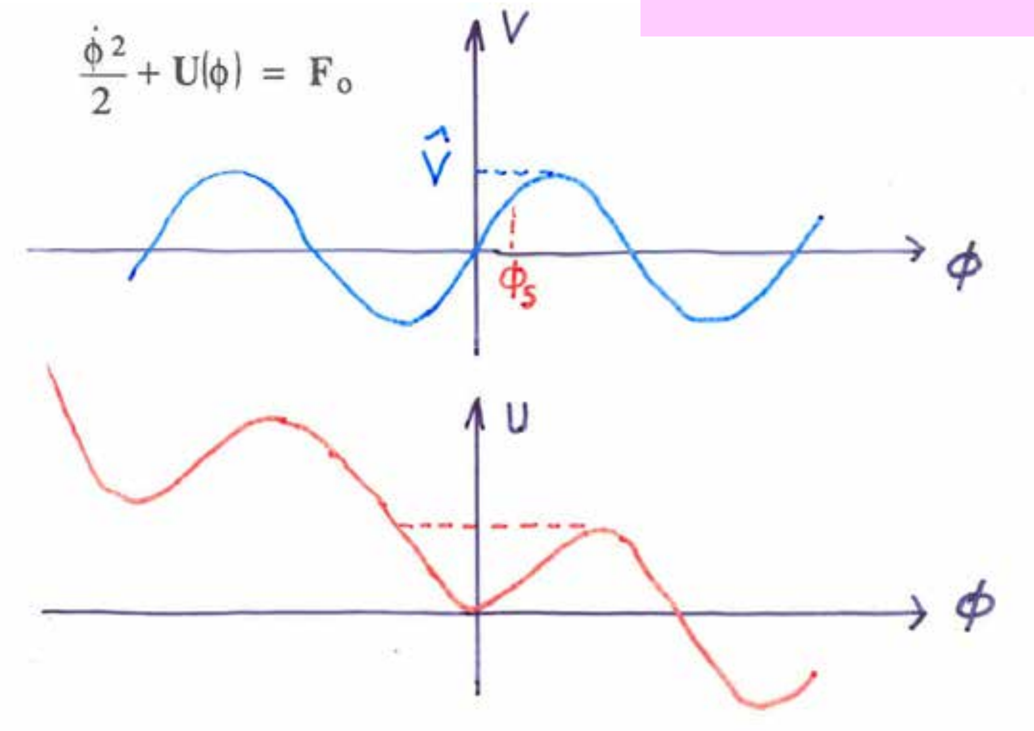


Potential Energy Function

The longitudinal motion is produced by a force that can be derived from a scalar potential:

$$\frac{d^2\phi}{dt^2}=F(\phi) \qquad F(\phi)=-\frac{\partial U}{\partial \phi}$$

$$U=-\int_0^{\phi} F(\phi)d\phi=-\frac{\Omega_s^2}{\cos\phi_s}(\cos\phi+\phi\sin\phi_s)-F_0$$



The sum of the potential energy and kinetic energy is constant and by analogy represents the total energy of a non-dissipative system.

Hamiltonian of Longitudinal Motion

Introducing a new convenient variable, W, leads to the 1st order equations:

$$W = \frac{\Delta E}{\omega_{rs}} \longrightarrow \begin{aligned} \frac{d\phi}{dt} &= -\frac{h\eta\omega_{rs}}{pR} W \\ \frac{dW}{dt} &= \frac{e\hat{V}}{2\pi}(\sin\phi - \sin\phi_s) \end{aligned}$$

The two variables ϕ,W are canonical since these equations of motion can be derived from a Hamiltonian H(ϕ,W,t):

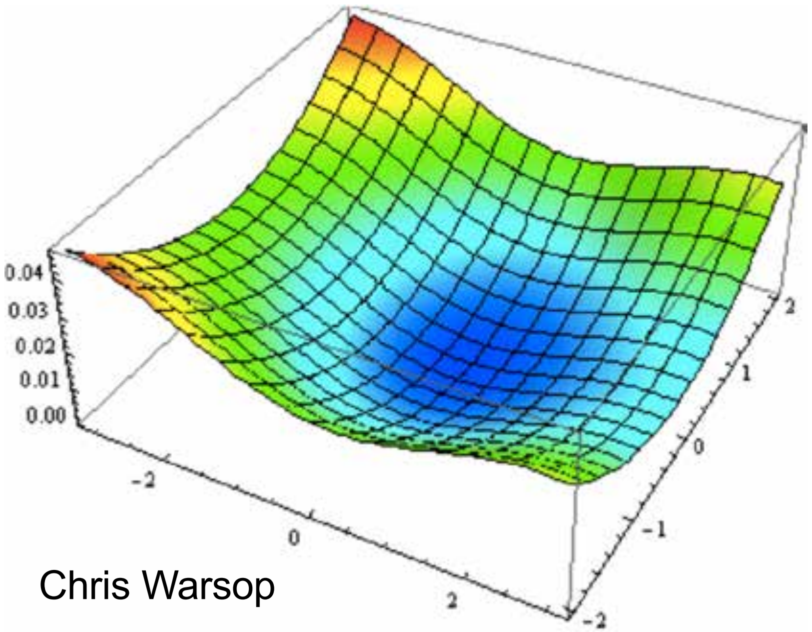
$$\frac{d\phi}{dt} = \frac{\partial H}{\partial W} \qquad \frac{dW}{dt} = -\frac{\partial H}{\partial \phi}$$

$$H(\phi, W) = -\frac{1}{2}\frac{h\eta\omega_{rs}}{pR}W^2 + \frac{e\hat{V}}{2\pi}[\cos\phi - \cos\phi_s + (\phi - \phi_s)\sin\phi_s]$$

Hamiltonian of Longitudinal Motion

What does it represent? The total energy of the system!

Surface of H (φ,W)



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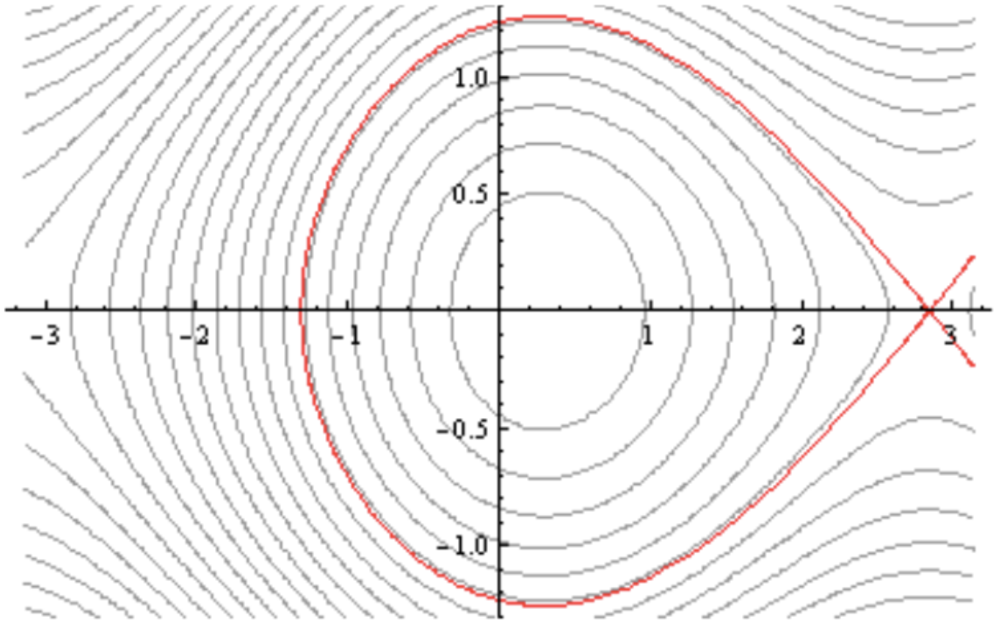
Contours of H (φ,W)



Contours of constant H are particle trajectories in phase space!
(H is conserved)

Hamiltonian Mechanics can help us understand some fairly complicated dynamics (multiple harmonics, bunch splitting, …)

Summary

- Synchrotron oscillations in the longitudinal phase space (E, ϕ)
- Particles perform oscillations around synchronous phase
  - synchronous phase depending on acceleration
  - below or above transition
- Bucket is the region in phase space for stable oscillations
  - Bucket size is the largest without acceleration

- to avoid filamentation and emittance increase it is important to
  - match the shape of the bunch to the bucket and
  - inject with the correct phase and energy

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➡ And CERN Accelerator Schools (CAS) Proceedings

Acknowledgements

I would like to thank everyone for the material that I have used.

In particular (hope I don't forget anyone):

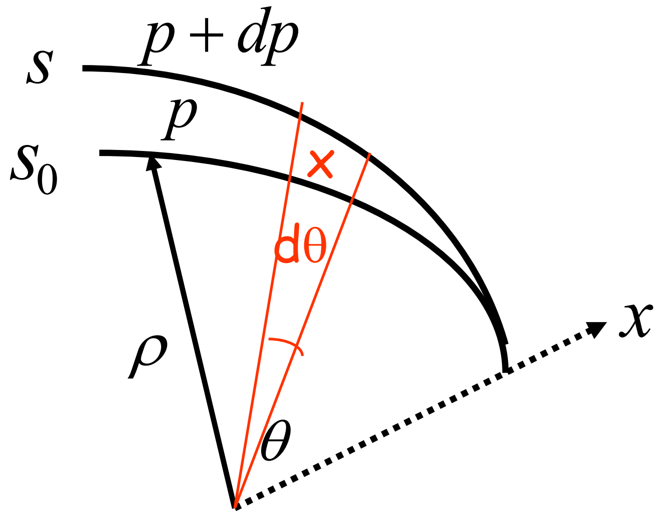
- **Joël Le Duff (from whom I inherited the course)**
- **Rende Steerenberg**
- **Gerald Dugan**
- **Heiko Damerau**
- **Werner Pirkl**
- **Mike Syphers**
- **Roberto Corsini**
- **Roland Garoby**
- **Luca Bottura**
- **Chris Warsop**

Appendix: Momentum Compaction Factor

$$\alpha_c = \frac{p}{L}\frac{dL}{dp}$$

$$ds_0 = \rho d\theta$$

$$ds = (\rho + x)d\theta$$



The elementary path difference from the two orbits is:

definition of dispersion D_x

$$\frac{dl}{ds_0} = \frac{ds - ds_0}{ds_0} = \frac{x}{\rho} = \frac{D_x}{\rho}\frac{dp}{p}$$

leading to the total change in the circumference:

$$dL = \int_C dl = \int \frac{x}{\rho} ds_0 = \int \frac{D_x}{\rho}\frac{dp}{p} ds_0$$

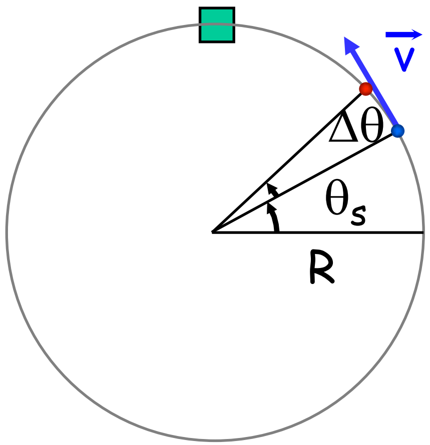
$$\alpha_c = \frac{1}{L}\int_C \frac{D_x(s)}{\rho(s)} ds_0$$

With ρ=∞ in straight sections we get:

$$\alpha_c = \frac{\langle D_x \rangle_m}{R}$$

$\langle\ \rangle_m$ means that the average is considered over the bending magnet only

Appendix: First Energy-Phase Equation



$$f_{RF} = h f_r \quad \Rightarrow \quad \Delta\phi = -h\,\Delta\theta \quad with \quad \theta = \int \omega \; dt$$

particle ahead arrives earlier
=> smaller RF phase

For a given particle with respect to the reference one:

$$\Delta\omega = \frac{d}{dt}(\Delta\theta) = -\frac{1}{h}\frac{d}{dt}(\Delta\phi) = -\frac{1}{h}\frac{d\phi}{dt}$$

Since: $\eta = \frac{p_s}{\omega_{rs}}\left(\frac{d\omega}{dp}\right)_s$ and $E^2 = E_0^2 + p^2c^2$

$$\Delta E = v_s \Delta p = \omega_{rs} R_s \Delta p$$

one gets:

$$\frac{\Delta E}{\omega_{rs}} = -\frac{p_s R_s}{h\eta\omega_{rs}}\frac{d(\Delta\phi)}{dt} = -\frac{p_s R_s}{h\eta\omega_{rs}}\dot{\phi}$$

Appendix: Second Energy-Phase Equation

The rate of energy gained by a particle is: $$\frac{dE}{dt} = e\hat{V}\sin\phi \frac{\omega_r}{2\pi}$$

The rate of relative energy gain with respect to the reference particle is then:

$$2\pi\Delta\left(\frac{\dot{E}}{\omega_r}\right) = e\hat{V}(\sin\phi - \sin\phi_s)$$

Expanding the left-hand side to first order:

$$\Delta\left(\dot{E}T_r\right) \cong \dot{E}\Delta T_r + T_{rs}\Delta\dot{E} = \Delta E\,\dot{T}_r + T_{rs}\Delta\dot{E} = \frac{d}{dt}\left(T_{rs}\Delta E\right)$$

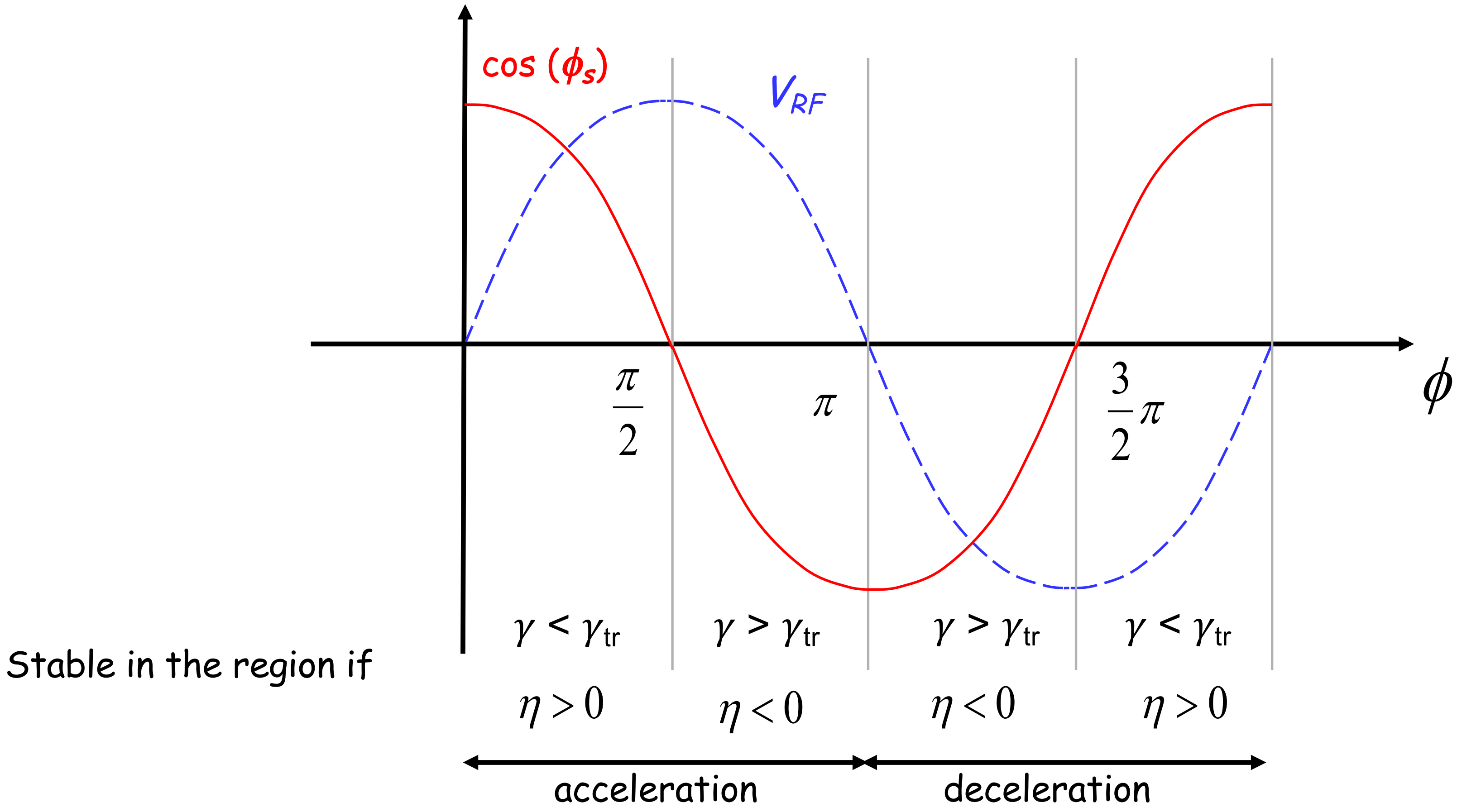
leads to the second energy-phase equation:

$$2\pi\frac{d}{dt}\left(\frac{\Delta E}{\omega_{rs}}\right) = e\hat{V}\left(\sin\phi - \sin\phi_s\right)$$

Appendix: Stability condition for ϕ_s

Stability is obtained when Ω_s is real and so Ω_s^2 positive:

$$\Omega_s^2 = \frac{e\hat{V}_{RF}\,\eta\, h\,\omega_s}{2\pi R_s p_s}\cos\phi_s \quad\Rightarrow\quad \Omega_s^2 > 0 \quad\Leftrightarrow\quad \eta\cos\phi_s > 0$$

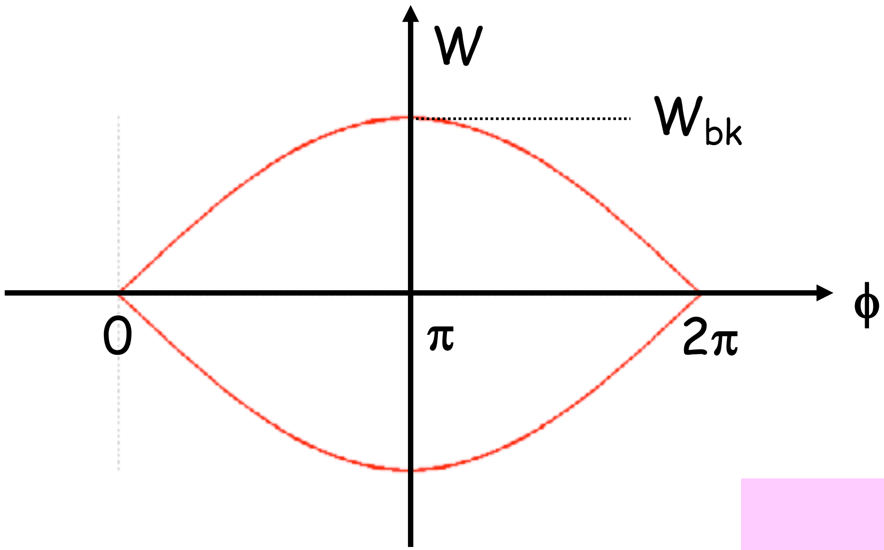


Appendix: Stationary Bucket - Separatrix

This is the case $\sin\phi_s=0$ (no acceleration) which means $\phi_s=0$ or π . The equation of the separatrix for $\phi_s= \pi$ (above transition) becomes:

$$\frac{\dot{\phi}^2}{2}+\Omega_s^2\cos\phi=\Omega_s^2 \qquad \frac{\dot{\phi}^2}{2}=2\Omega_s^2\sin^2\frac{\phi}{2}$$

Replacing the phase derivative by the (canonical) variable W:



$$W=\frac{\Delta E}{\omega_{rf}}=-\frac{p_s R_s}{h\eta\omega_{rf}}\dot{\varphi}$$

and introducing the expression for Ω_s leads to the following equation for the separatrix:

$$W=\pm\frac{C}{\pi hc}\sqrt{\frac{-e\hat{V}E_s}{2\pi h\eta}}\sin\frac{\phi}{2}=\pm W_{bk}\sin\frac{\phi}{2}$$

with $C=2\pi R_s$

Stationary Bucket (2)

Setting ϕ=π in the previous equation gives the height of the bucket:

$$W_{bk} = \frac{C}{\pi hc}\sqrt{\frac{-e\hat{V}E_s}{2\pi h\eta}}$$

This results in the maximum energy acceptance:

$$\Delta E_{\max} = \omega_{rf} W_{bk} = \beta_s \sqrt{2\frac{-e\hat{V}_{RF}E_s}{\pi\eta h}}$$

The area of the bucket is: $A_{bk} = 2\int_0^{2\pi} W\,d\phi$

Since: $\int_0^{2\pi} \sin\frac{\phi}{2}\,d\phi = 4$

one gets: $A_{bk} = 8W_{bk} = 8\frac{C}{\pi hc}\sqrt{\frac{-e\hat{V}E_s}{2\pi h\eta}}$ → $W_{bk} = \frac{A_{bk}}{8}$

Appendix: Large Amplitude Oscillations

For larger phase (or energy) deviations from the reference the second order differential equation is non-linear:

$$\ddot{\phi} + \frac{\Omega_s^2}{\cos\phi_s}\left(\sin\phi - \sin\phi_s\right) = 0$$

(Ω_s as previously defined)

Multiplying by $\dot{\phi}$ and integrating gives an invariant of the motion:

$$\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos\phi_s}\left(\cos\phi + \phi\sin\phi_s\right) = I$$

which for small amplitudes reduces to:

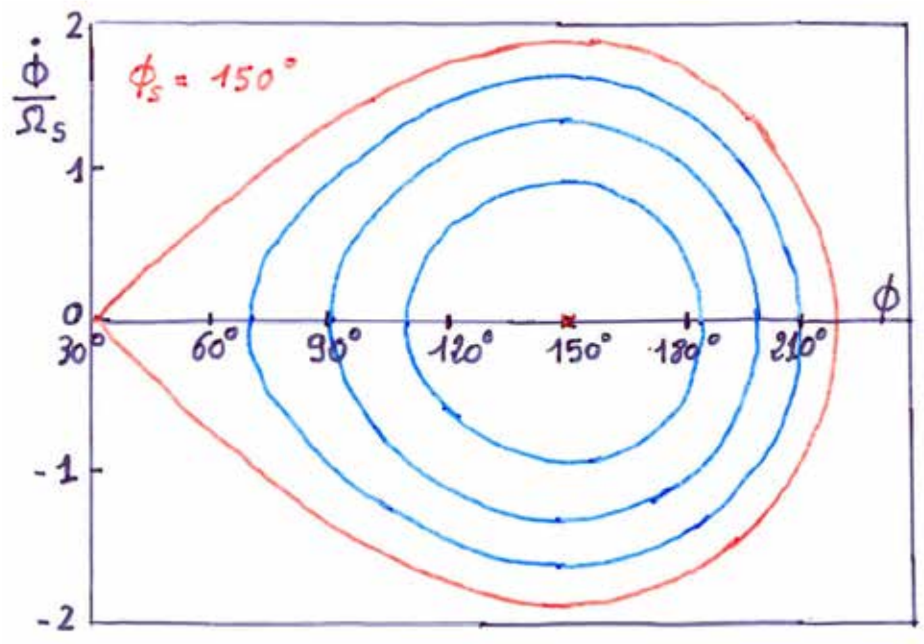
$$\frac{\dot{\phi}^2}{2} + \Omega_s^2\frac{\left(\Delta\phi\right)^2}{2} = I'$$

(the variable is $\Delta\phi$, and ϕ_s is constant)

Similar equations exist for the second variable : $\Delta E \propto d\phi/dt$

Large Amplitude Oscillations (2)

When ϕ reaches π-ϕ_s the force goes to zero and beyond it becomes non restoring.
Hence π-ϕ_s is an extreme amplitude for a stable motion which in the phase space($\frac{\dot{\phi}}{\Omega_s}, \Delta\phi$) is shown as closed trajectories.



Equation of the separatrix:

$$\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos\phi_s}(\cos\phi + \phi\sin\phi_s) = -\frac{\Omega_s^2}{\cos\phi_s}(\cos(\pi-\phi_s) + (\pi-\phi_s)\sin\phi_s)$$

Second value ϕ_m where the separatrix crosses the horizontal axis:

$$\cos\phi_m + \phi_m\sin\phi_s = \cos(\pi-\phi_s) + (\pi-\phi_s)\sin\phi_s$$

Energy Acceptance

From the equation of motion it is seen that $\dot{\phi}$ reaches an extreme when $\ddot{\phi} = 0$, hence corresponding to $\phi = \phi_s$.

Introducing this value into the equation of the separatrix gives:

$$\dot{\phi}^2_{\max} = 2\Omega_s^2 \left\{2 + \left(2\phi_s - \pi\right)\tan\phi_s\right\}$$

That translates into an acceptance in energy:

$$\left(\frac{\Delta E}{E_s}\right)_{\max} = \mp\beta\sqrt{-\frac{e\hat{V}}{\pi h \eta E_s}G\left(\phi_s\right)}$$

$$G\left(\phi_s\right) = \left[2\cos\phi_s + \left(2\phi_s - \pi\right)\sin\phi_s\right]$$

This "RF acceptance" depends strongly on ϕ_s and plays an important role for the capture at injection, and the stored beam lifetime.

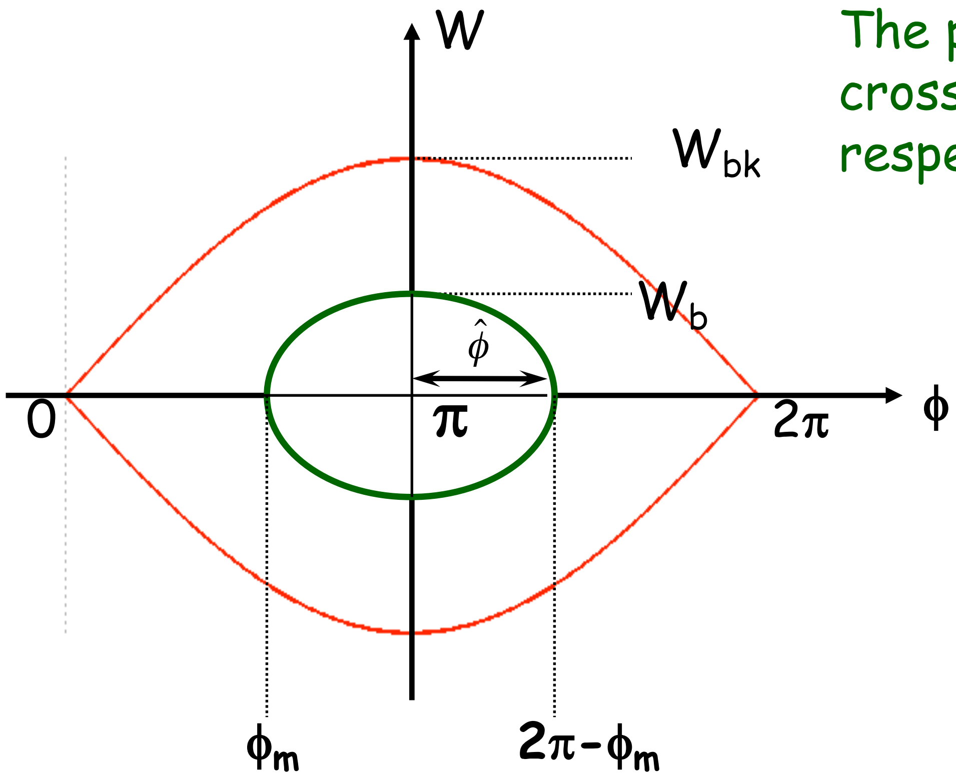
It's largest for ϕ_s=0 and ϕ_s=π (no acceleration, depending on η).

Need a higher RF voltage for higher acceptance.

Bunch Matching into a Stationary Bucket

A particle trajectory inside the separatrix is described by the equation:

$$\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos\phi_s}\left(\cos\phi + \phi\sin\phi_s\right) = I \quad \xrightarrow{\phi_s = \pi} \quad \frac{\dot{\phi}^2}{2} + \Omega_s^2\cos\phi = I$$



The points where the trajectory crosses the axis are symmetric with respect to $\phi_s = \pi$

$$\frac{\dot{\phi}^2}{2} + \Omega_s^2\cos\phi = \Omega_s^2\cos\phi_m$$

$$\dot{\phi} = \pm\Omega_s\sqrt{2\left(\cos\phi_m - \cos\phi\right)}$$

$$W = \pm W_{bk}\sqrt{\cos^2\frac{\varphi_m}{2} - \cos^2\frac{\varphi}{2}}$$

$$\cos(\phi) = 2\cos^2\frac{\phi}{2} - 1$$

Bunch Matching into a Stationary Bucket (2)

Setting $\phi = \pi$ in the previous formula allows to calculate the bunch height:

$$W_b = W_{bk} \cos\frac{\phi_m}{2} = W_{bk} \sin\frac{\hat{\phi}}{2}$$

or:

$$W_b = \frac{A_{bk}}{8} \cos\frac{\phi_m}{2}$$

$$\longrightarrow \quad \left(\frac{\Delta E}{E_s}\right)_b = \left(\frac{\Delta E}{E_s}\right)_{RF} \cos\frac{\phi_m}{2} = \left(\frac{\Delta E}{E_s}\right)_{RF} \sin\frac{\hat{\phi}}{2}$$

This formula shows that for a given bunch energy spread the proper matching of a shorter bunch (ϕ_m close to π, $\hat{\phi}$ small) will require a bigger RF acceptance, hence a higher voltage

For small oscillation amplitudes the equation of the ellipse reduces to:

$$W = \frac{A_{bk}}{16}\sqrt{\hat{\phi}^2 - (\Delta\phi)^2} \quad \longrightarrow \quad \left(\frac{16W}{A_{bk}\hat{\phi}}\right)^2 + \left(\frac{\Delta\phi}{\hat{\phi}}\right)^2 = 1$$

Ellipse area is called longitudinal emittance

$$A_b = \frac{\pi}{16} A_{bk} \hat{\phi}^2$$