Introduction to Transverse Beam Dynamics

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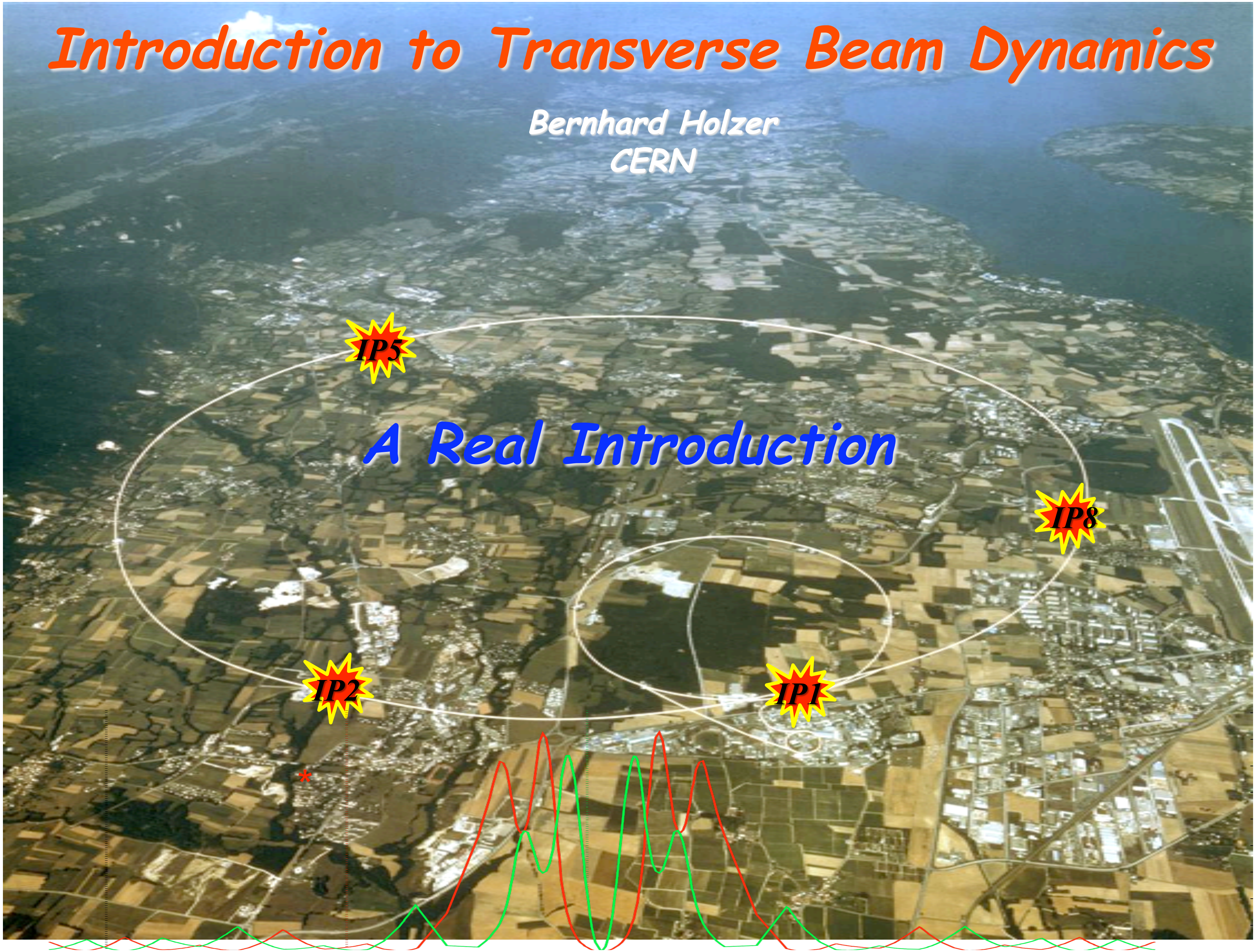
CERN



Introduction and Basic Ideas

„ ... in the end and after all it should be a kind of circular machine"

→ need transverse deflecting force

Lorentz force $\vec{F} = q * (\cancel{\vec{E}} + \vec{v} \times \vec{B})$

typical velocity in high energy machines: $v \approx c \approx 3 * 10^8 \, m/s$

Example:

$$B = 1\,T \quad \rightarrow \quad F = q * 3 * 10^8 \frac{m}{s} * 1 \frac{Vs}{m^2}$$

$$F = q * \underbrace{300 \frac{MV}{m}}_{E}$$

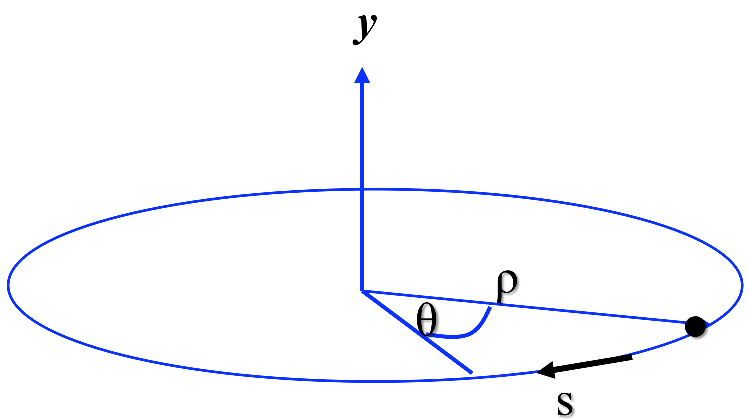
equivalent el. field ... E

technical limit for el. field:

$$E \leq 1 \frac{MV}{m}$$

old greek dictum of wisdom:
if you are clever, you use magnetic fields in an accelerator wherever it is possible.

The ideal circular orbit



circular coordinate system

condition for circular orbit:

Lorentz force $$F_L = e\,v\,B$$

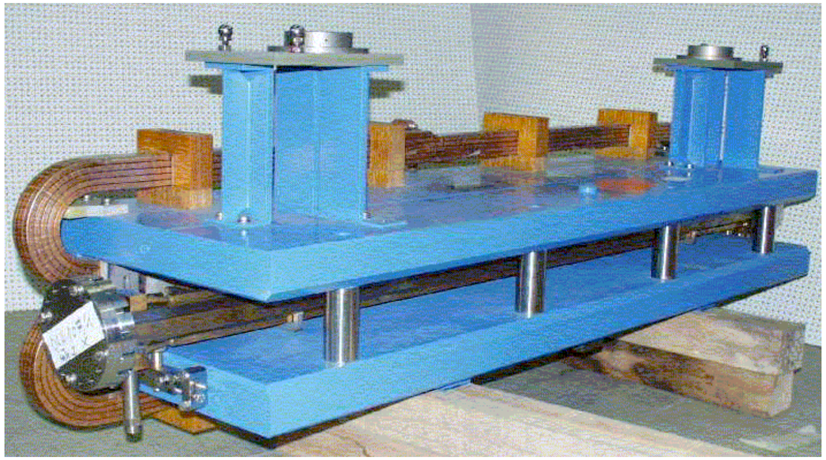
centrifugal force $$F_{centr} = \frac{\gamma\, m_0\, v^2}{\rho}$$

$$\frac{\gamma\, m_0\, \cancel{v^2}}{\rho} = e\,\cancel{v}\,B$$

$$\boxed{\frac{p}{e} = B\,\rho}$$

$B\rho$ = "beam rigidity"

The Magnetic Guide Field



Dipole Magnets:

define the ideal orbit
homogeneous field created
by two flat pole shoes

$$B = \frac{\mu_0 \, n \, I}{h}$$

Normalise magnetic field to momentum:

$$\frac{p}{e} = B\rho \longrightarrow \frac{1}{\rho} = \frac{e\,B}{p}$$

convenient units:

$$B = [T] = \left[\frac{Vs}{m^2}\right] \qquad p = \left[\frac{GeV}{c}\right]$$

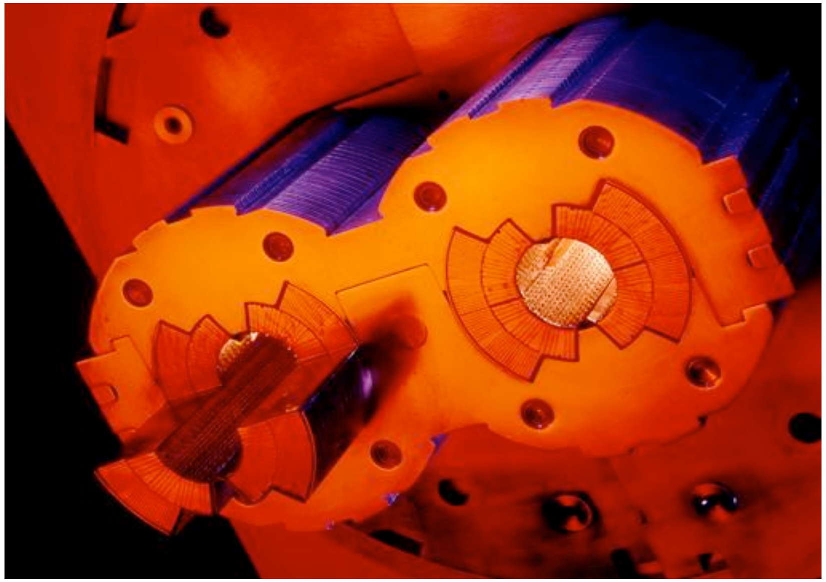
Example LHC:

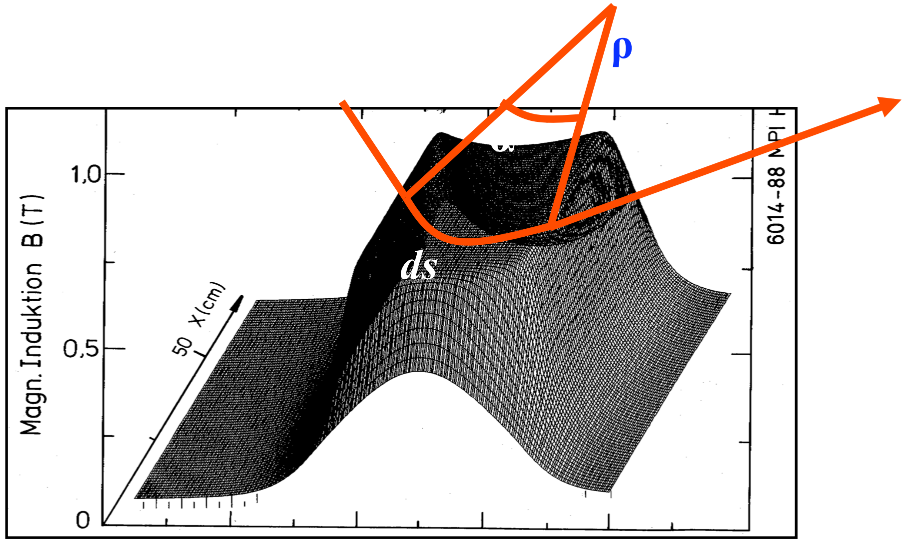
$$\left.\begin{array}{l} B = 8.3\,T \\ \\ p = 7000\,\dfrac{GeV}{c} \end{array}\right\}$$

$$\frac{1}{\rho} = e\,\frac{8.3\,{}^{Vs}\!/_{m^2}}{7000 * 10^9\,{}^{eV}\!/_{c}} = \frac{8.3\,s\,3*10^8\,{}^{m}\!/_{s}}{7000*10^9\,m^2}$$

$$\frac{1}{\rho} = 0.3\,\frac{8.3}{7000}\,{}^{1}\!/_{m}$$

The Magnetic Guide Field





field map of a storage ring dipole magnet

$\rho = 2.53\ \boldsymbol{km}$ → $\boldsymbol{2\pi\rho = 17.6\ km}$
$\boldsymbol{\approx 66\%}$

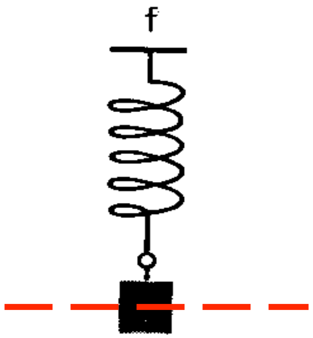
$\boldsymbol{B} \approx 1 \ldots 8\ \boldsymbol{T}$

rule of thumb: $$\frac{1}{\rho} \approx 0.3\, \frac{\boldsymbol{B[T]}}{\boldsymbol{p[GeV/c]}}$$

„normalised bending strength“

Focusing Properties – Transverse Beam Optics

Classical Mechanics:
pendulum



there is a restoring force, proportional to the elongation x:

$$F = m * \frac{d^2x}{dt^2} = -k * x$$

Ansatz $x(t) = A * \cos(\omega t + \varphi)$

general solution: free harmonic oszillation

Storage Ring: we need a Lorentz force that rises as a function of the distance to the design orbit

$$F(x) = q * v * B(x)$$

Quadrupole Magnets:

required: ***focusing forces to keep trajectories in vicinity of the ideal orbit***

linear increasing Lorentz force

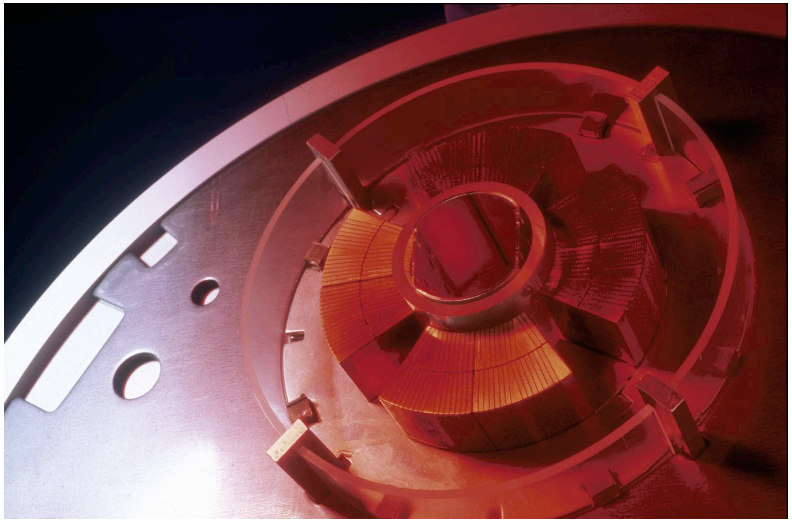
linear increasing magnetic field $\boldsymbol{B_y = g\ x}$ $\boldsymbol{B_x = g\ y}$

Quadrupole Magnets:

normalised quadrupole field:

gradient of a quadrupole magnet:

$$g = \frac{2\mu_0 n I}{r^2}$$



LHC main quadrupole magnet

$$k = \frac{g}{p/e}$$

$$g \approx 25 \ldots 220 \;\; T/m$$

simple rule:

$$\boxed{k = 0.3 \frac{g(T/m)}{p(GeV/c)}}$$

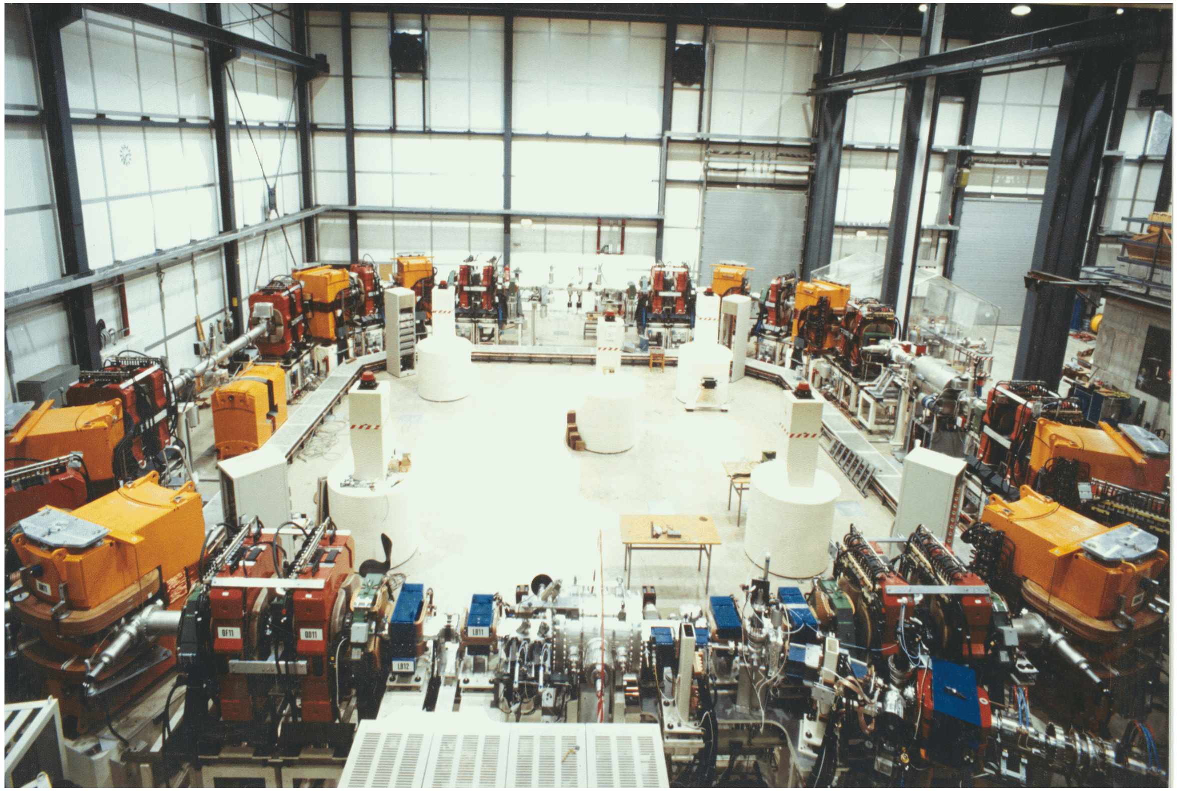
what about the vertical plane:
... Maxwell

$$\vec{\nabla} \times \vec{B} = \cancel{\vec{j}} + \cancel{\frac{\partial \vec{E}}{\partial t}} = 0 \qquad \Rightarrow \qquad \frac{\partial B_y}{\partial x} = \frac{\partial B_x}{\partial y}$$

The Equation of Motion:

$$\frac{B(x)}{p/e} = \frac{1}{\rho} + k\,x + \cancel{\frac{1}{2!} m\,x^2} + \cancel{\frac{1}{3!} n\,x^3} + \dots$$

only terms linear in x, y taken into account *dipole fields*
quadrupole fields



Separate Function Machines:

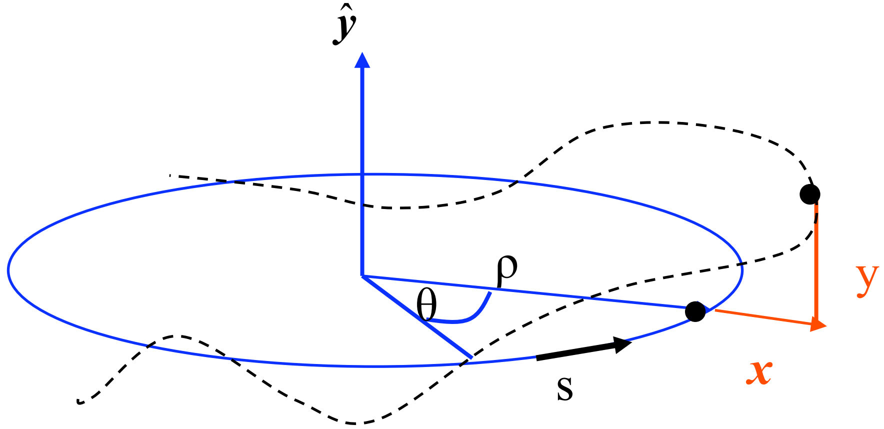
Split the magnets and optimise them according to their job:

bending, focusing etc

Example:
heavy ion storage ring TSR

* *man sieht nur*
dipole und quads → linear

The Equation of Motion:



Equation for the ***horizontal motion:***

$$\boldsymbol{x}'' + \boldsymbol{x}\left(\frac{1}{\rho^2} - \boldsymbol{k}\right) = 0$$

Equation for the ***vertical motion:***

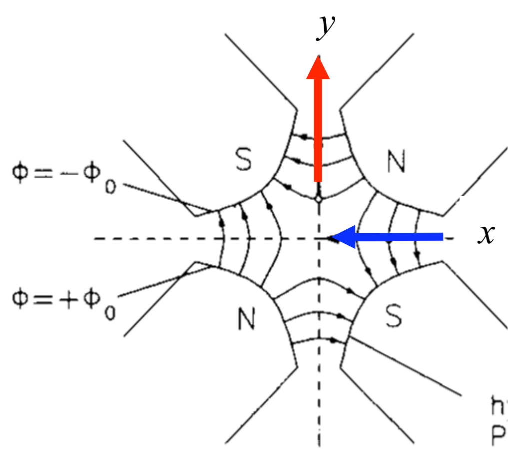
$$\frac{1}{\rho^2} = 0$$

no dipoles … in general …

$$\boldsymbol{k} \quad \leftrightarrow \quad -\boldsymbol{k}$$

quadrupole field changes sign

$$\boldsymbol{y}'' + \boldsymbol{k}\ \boldsymbol{y} = 0$$



* Hard Edge Model:

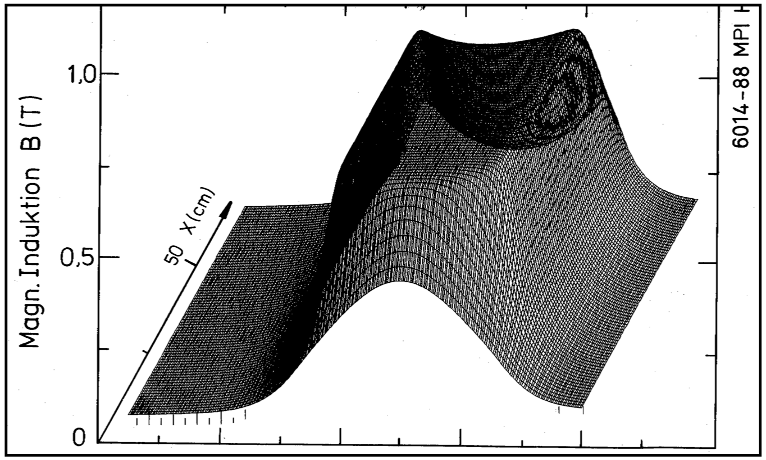
$$x'' + \left\{ \frac{1}{\rho^2} - k \right\} x = 0$$

... this equation is not correct !!!

$$x''(s) + \left\{ \frac{1}{\rho^2(s)} - k(s) \right\} x(s) = 0$$

bending and focusing fields ... are functions of the independent variable „s“

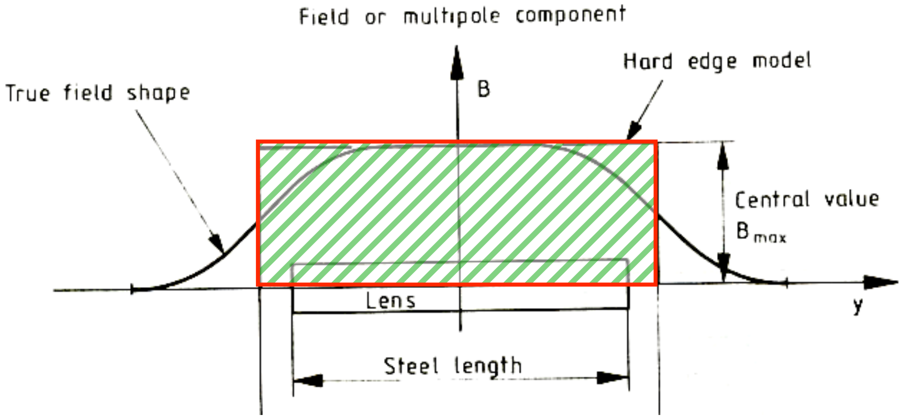
!



Inside a magnet we assume constant focusing properties !

$$\frac{1}{\rho} = const \qquad k = const$$

$$B\, l_{eff} = \int_0^{l_{mag}} B\, ds$$



Solution of Trajectory Equations

Define … hor. plane: $K = 1/\rho^2 - k$

… vert. Plane: $K = k$

$$\boxed{x'' + K\, x = 0}$$

Differential Equation of harmonic oscillator … with spring constant K

Ansatz: $$\boxed{x(s) = a_1 \cdot \cos(\omega s) + a_2 \cdot \sin(\omega s)}$$

general solution: linear combination of two independent solutions

$$x'(s) = -a_1\omega \sin(\omega s) + a_2\omega \cos(\omega s)$$

$$x''(s) = -a_1\omega^2 \cos(\omega s) - a_2\omega^2 \sin(\omega s) = -\omega^2 x(s) \longrightarrow \omega = \sqrt{K}$$

general solution:

$$x(s) = a_1 \cos(\sqrt{K} s) + a_2 \sin(\sqrt{K} s)$$

determine a_1 , a_2 by boundary conditions:

$$s = 0 \longrightarrow \begin{cases} x(0) = x_0 \quad , \quad a_1 = x_0 \\ x'(0) = x'_0 \quad , \quad a_2 = \dfrac{x'_0}{\sqrt{K}} \end{cases}$$

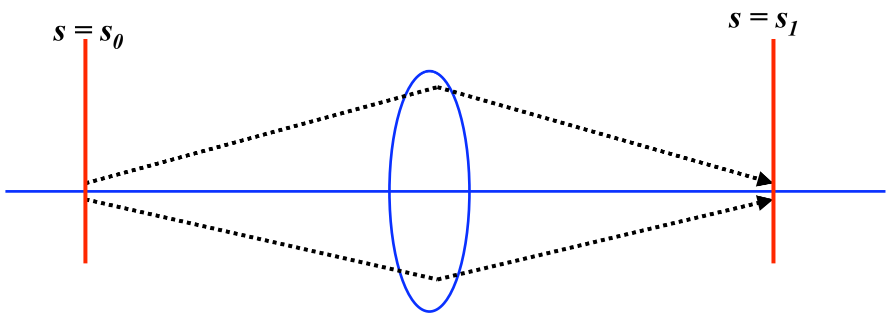
Hor. Focusing Quadrupole $K > 0$:

$$x(s) = x_0 \cdot \cos(\sqrt{|K|}s) + x'_0 \cdot \frac{1}{\sqrt{|K|}} \sin(\sqrt{|K|}s)$$

$$x'(s) = -x_0 \cdot \sqrt{|K|} \cdot \sin(\sqrt{|K|}s) + x'_0 \cdot \cos(\sqrt{|K|}s)$$

For convenience expressed in matrix formalism:

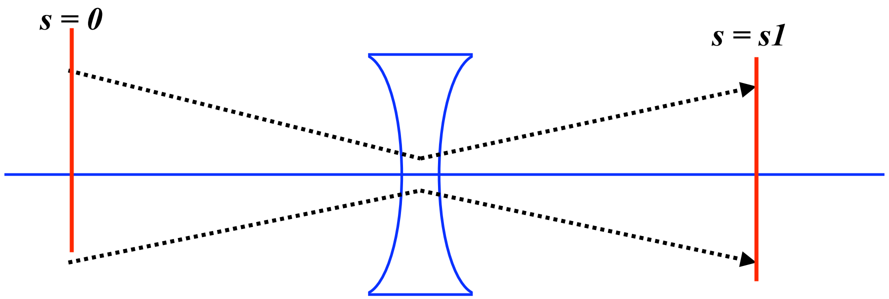
$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s1} = M_{foc} * \begin{pmatrix} x \\ x' \end{pmatrix}_{s0}$$



$$M_{foc} = \begin{pmatrix} \cos(\sqrt{|K|}s) & \dfrac{1}{\sqrt{|K|}} \sin(\sqrt{|K|}s \\ -\sqrt{|K|} \sin(\sqrt{|K|}s) & \cos(\sqrt{|K|}s) \end{pmatrix}_0$$

hor. defocusing quadrupole:

$$x'' - K\,x = 0$$



Remember from school:

$$f(s) = \cosh(s) \quad , \qquad f'(s) = \sinh(s)$$

Ansatz: $$x(s) = a_1 \cdot \cosh(\omega s) + a_2 \cdot \sinh(\omega s)$$

$$M_{defoc} = \begin{pmatrix} \cosh\sqrt{|K|}l & \frac{1}{\sqrt{|K|}}\sinh\sqrt{|K|}l \\ \sqrt{|K|}\sinh\sqrt{|K|}l & \cosh\sqrt{|K|}l \end{pmatrix}$$

drift space:

$K = 0$

$$M_{drift} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$$

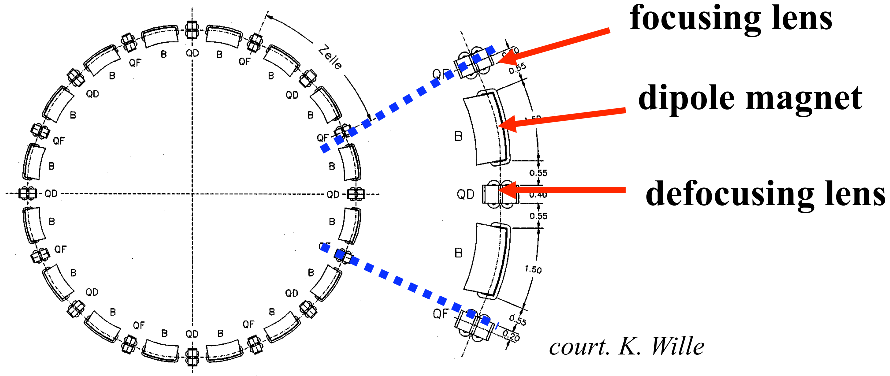
! with the assumptions made, the motion in the horizontal and vertical planes are independent „ ... the particle motion in x & y is uncoupled“

Transformation through a system of lattice elements

combine the single element solutions by multiplication of the matrices

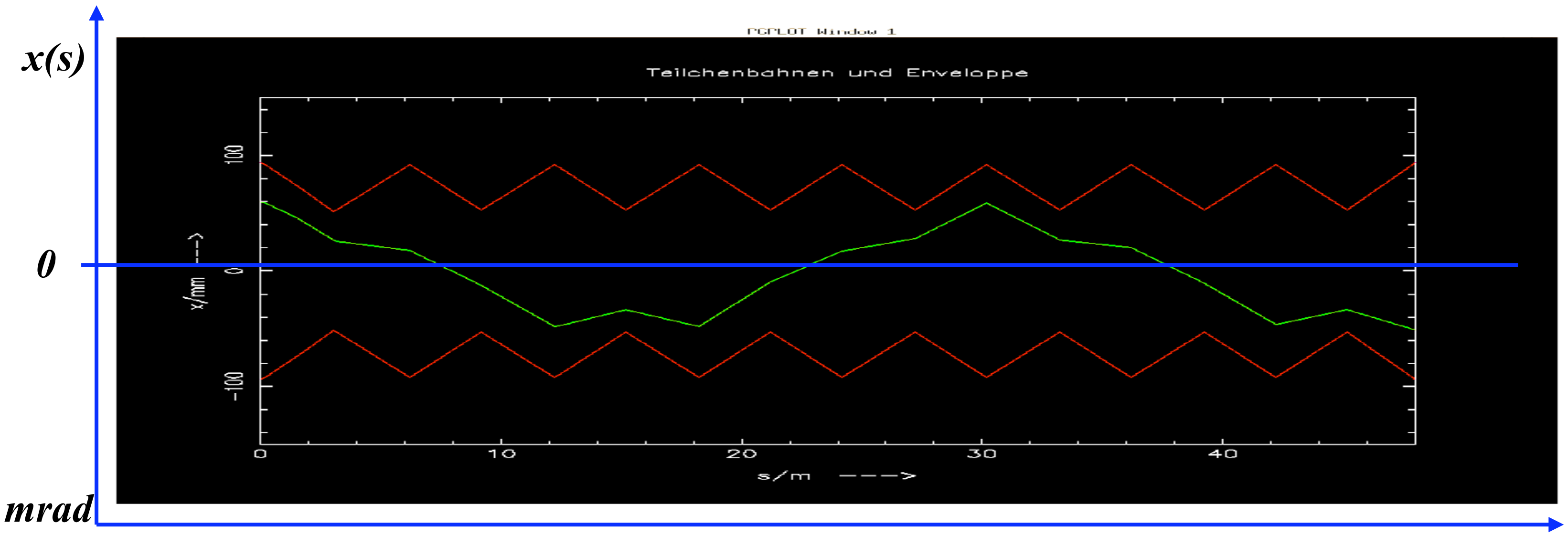
$$M_{total} = M_{QF} * M_D * M_{QD} * M_{Bend} * M_{D*.....}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s2} = M(s_2, s_1) * \begin{pmatrix} x \\ x' \end{pmatrix}_{s1}$$



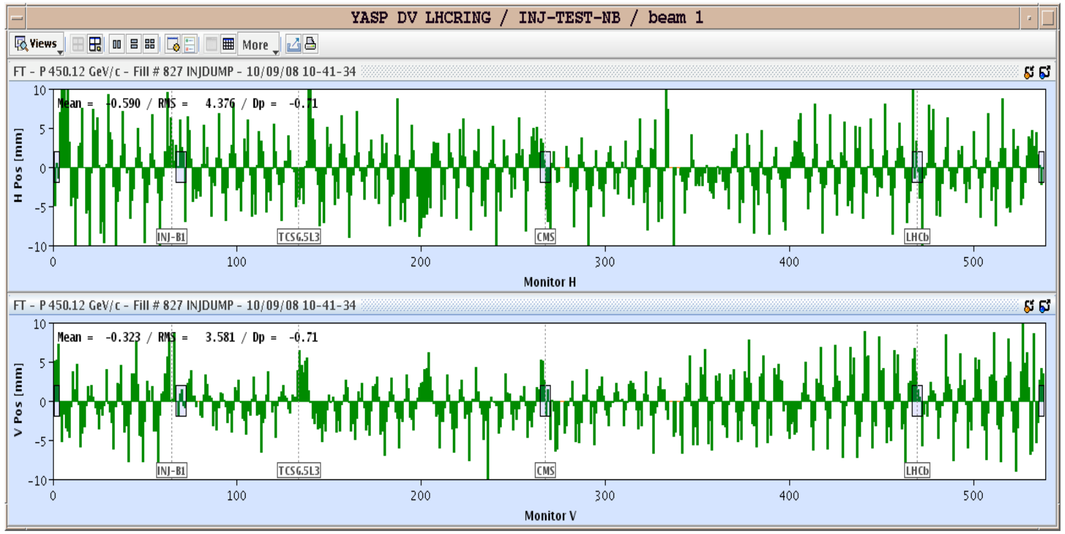
court. K. Wille

in each accelerator element the particle trajectory corresponds to the movement of a harmonic oscillator „



typical values in a strong foc. machine:
x ≈ mm, x´ ≤ mrad

Orbit & Tune:



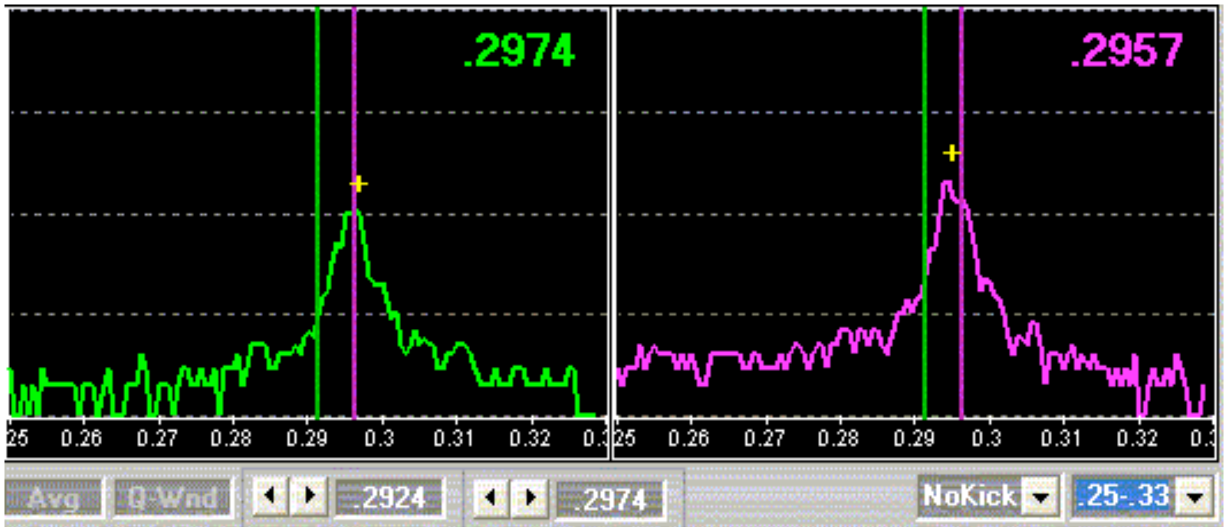
Tune: number of oscillations per turn

64.31
59.32

Relevant for beam stability:
non integer part

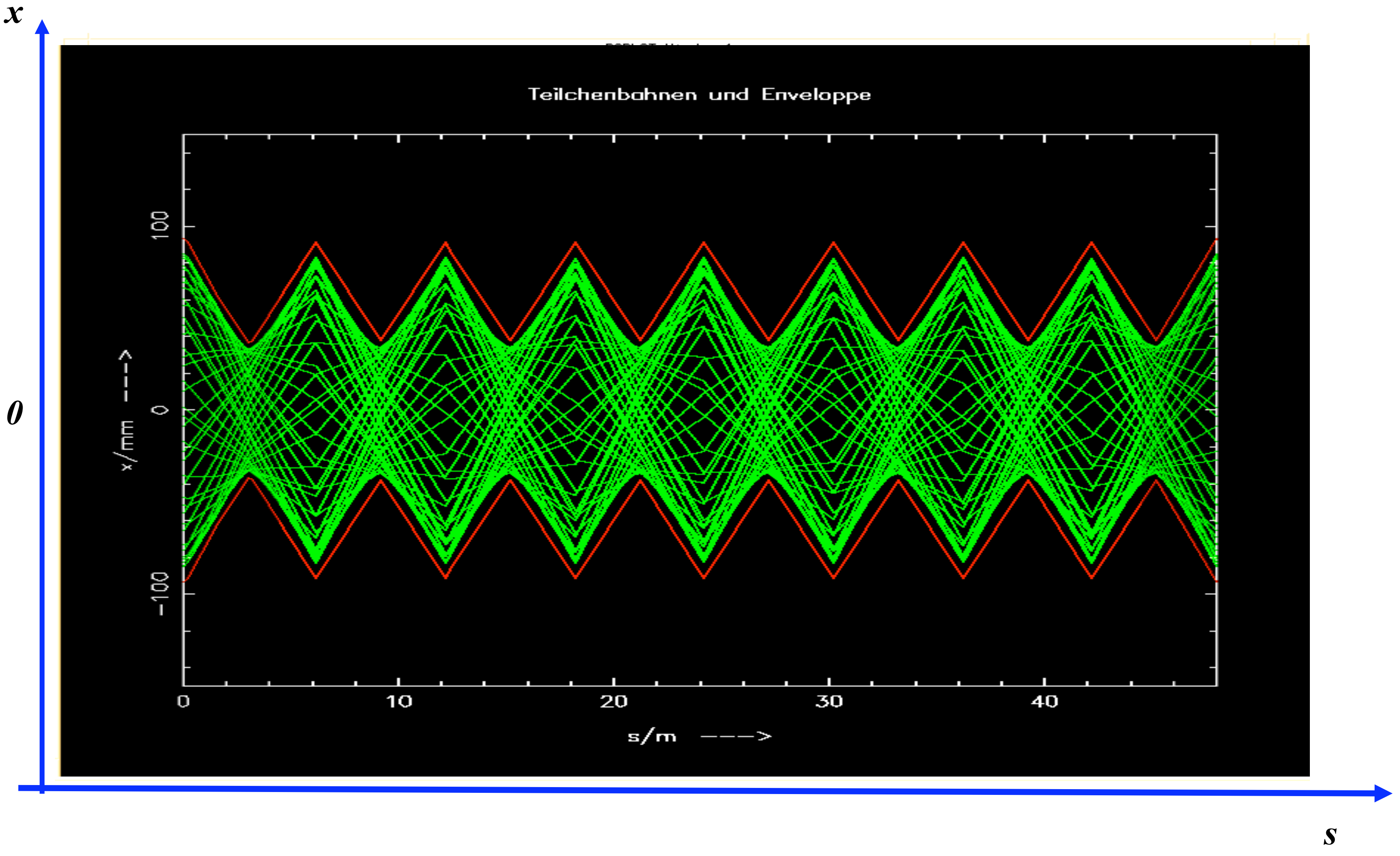
LHC revolution frequency: 11.3 kHz

$$0.31 * 11.3 = 3.5 kHz$$



Question: ***what will happen, if the particle performs a second turn ?***

... or a third one or ... 10^{10} turns



Astronomer Hill:

differential equation for motions with periodic focusing properties
„Hill's equation"

Example: particle motion with periodic coefficient

equation of motion: $$x''(s) - k(s)x(s) = 0$$

restoring force $\neq$ const,
k(s) = depending on the position s
k(s+L) = k(s), periodic function

we expect a kind of quasi harmonic oscillation: amplitude & phase will depend on the position s in the ring.

The Beta Function

General solution of Hill´s equation:

(i) $$x(s) = \sqrt{\varepsilon}\,\sqrt{\beta(s)} \cdot \cos(\psi(s) + \phi)$$

ε, Φ = integration constants determined by initial conditions

β(s) periodic function given by focusing properties of the lattice ↔ quadrupoles

$$\beta(s + L) = \beta(s)$$

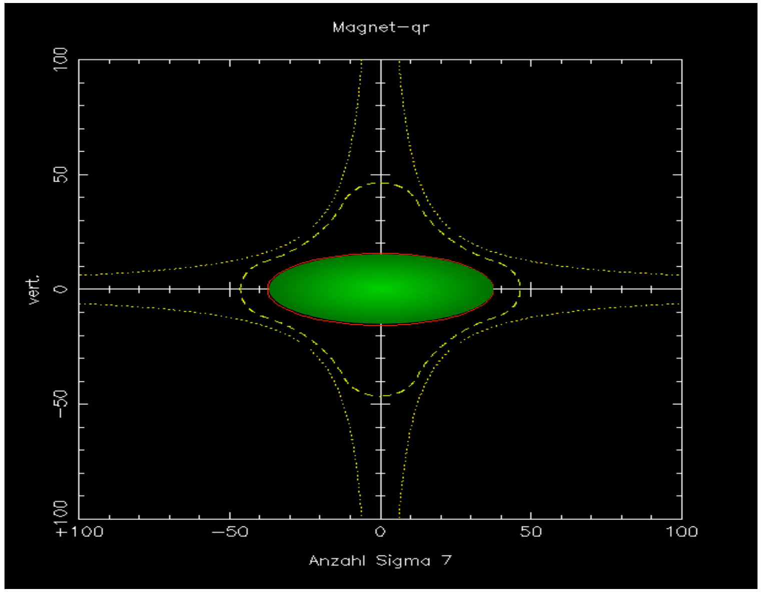
Inserting (i) into the equation of motion …

$$\psi(s) = \int_0^s \frac{ds}{\beta(s)}$$

Ψ(s) = „phase advance“ of the oscillation between point „0“ and „s“ in the lattice.
For one complete revolution: number of oscillations per turn „Tune“

$$Q_y = \frac{1}{2\pi} \oint \frac{ds}{\beta(s)}$$

The Beta Function

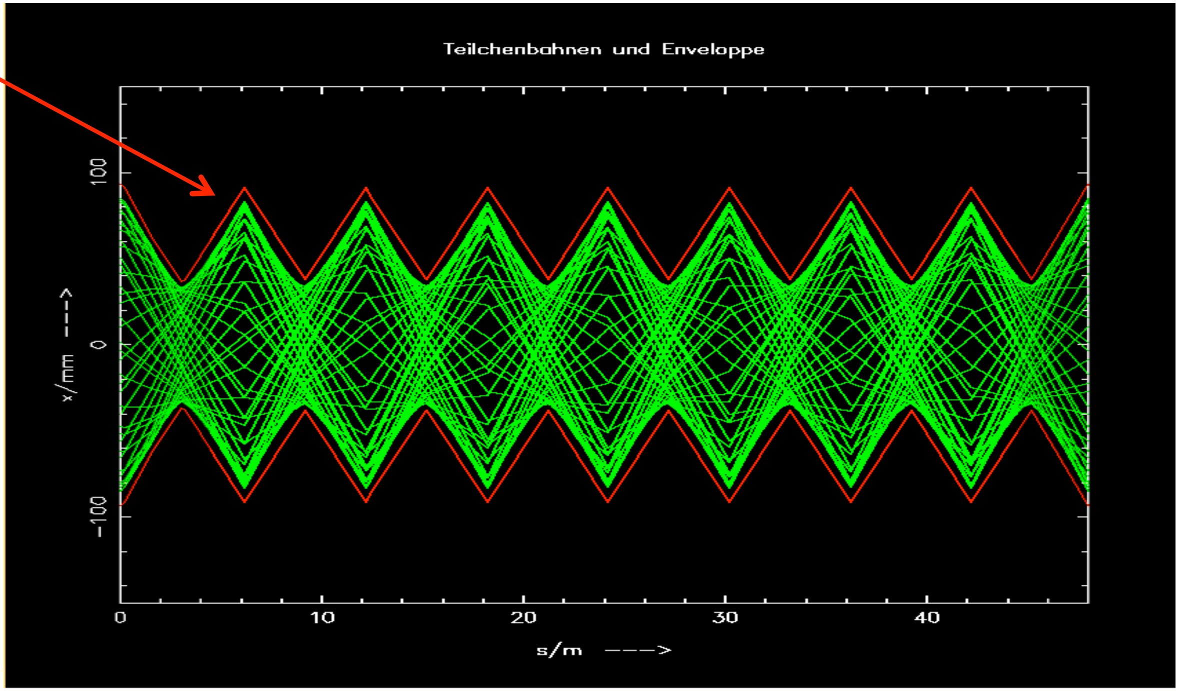


Amplitude of a particle trajectory:

$$x(s) = \sqrt{\varepsilon} * \sqrt{\beta(s)} * \cos(\psi(s) + \varphi)$$

Maximum size of a particle amplitude

$$\hat{x}(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}$$



β determines the beam size
(... the envelope of all particle trajectories at a given position "s" in the storage ring.

It reflects the periodicity of the magnet structure.

Beam Emittance and Phase Space Ellipse

general solution of Hill equation

$$\begin{cases} (1) \quad x(s) = \sqrt{\varepsilon}\,\sqrt{\beta(s)}\,\cos(\psi(s)+\phi) \\ (2) \quad x'(s) = -\dfrac{\sqrt{\varepsilon}}{\sqrt{\beta(s)}}\,\{\alpha(s)\cos(\psi(s)+\phi)+\sin(\psi(s)+\phi)\} \end{cases}$$

from (1) we get

$$\cos(\psi(s)+\phi) = \frac{x(s)}{\sqrt{\varepsilon}\,\sqrt{\beta(s)}}$$

$$\alpha(s) = \frac{-1}{2}\beta'(s)$$

$$\gamma(s) = \frac{1+\alpha(s)^2}{\beta(s)}$$

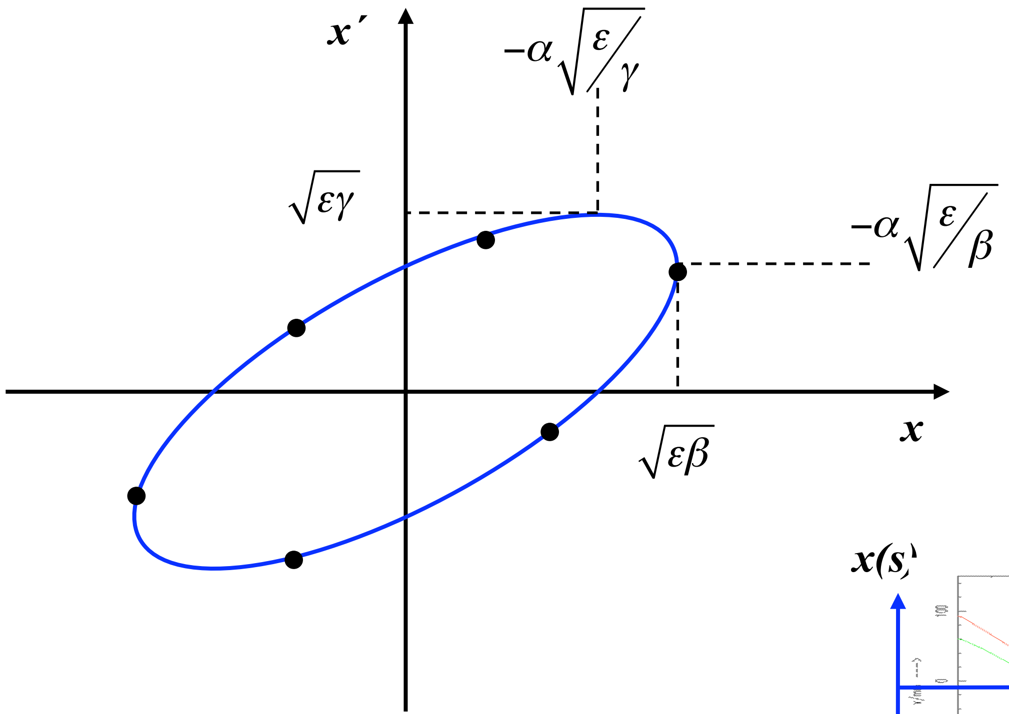
Insert into (2) and solve for ε

$$\varepsilon = \gamma(s)\,x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)\,x'^2(s)$$

**** ε is a constant of the motion … it is independent of „s“***
**** parametric representation of an ellipse in the x x‘ space***
**** shape and orientation of ellipse are given by α, β, γ***

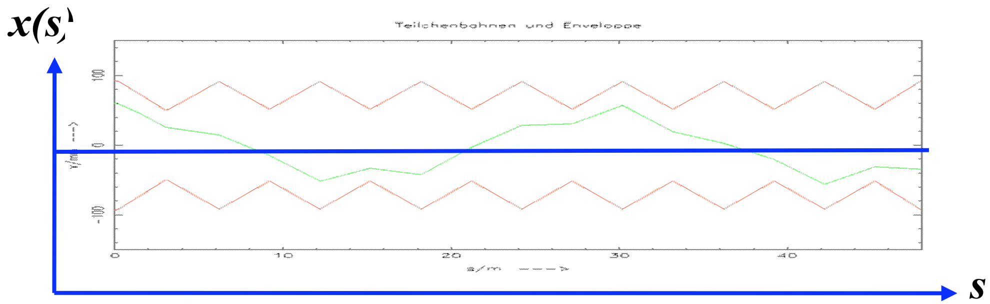
Beam Emittance and Phase Space Ellipse

$$\varepsilon = \gamma(s)\, x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)\, x'^2(s)$$



***Liouville:* in reasonable storage rings**
area in phase space is constant.

A = π*ε=const

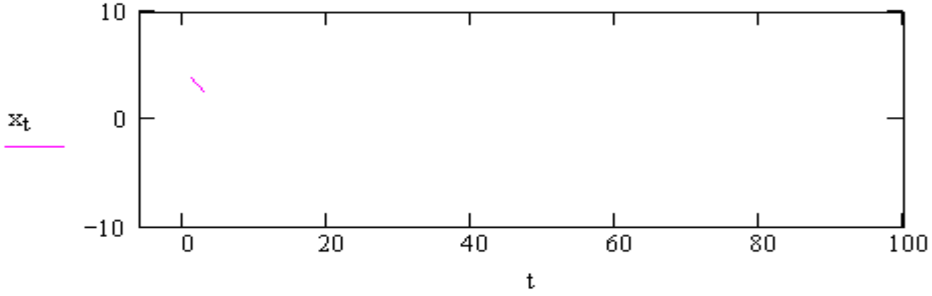


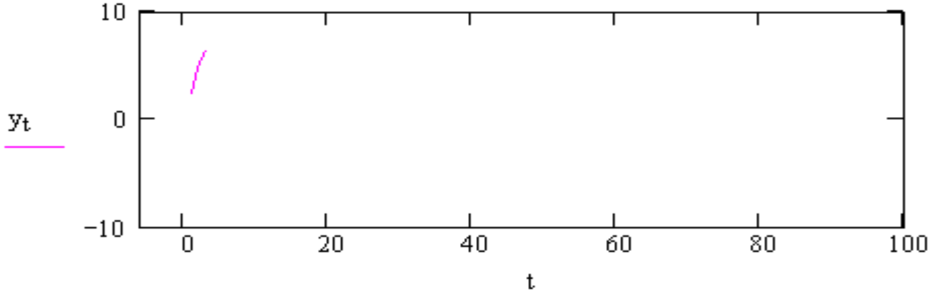
ε beam emittance = woozilycity of the particle ensemble, intrinsic beam parameter,
cannot be changed by the foc. properties.
Scientifiquely speaking: area covered in transverse x, x´ phase space … and it is constant !!!

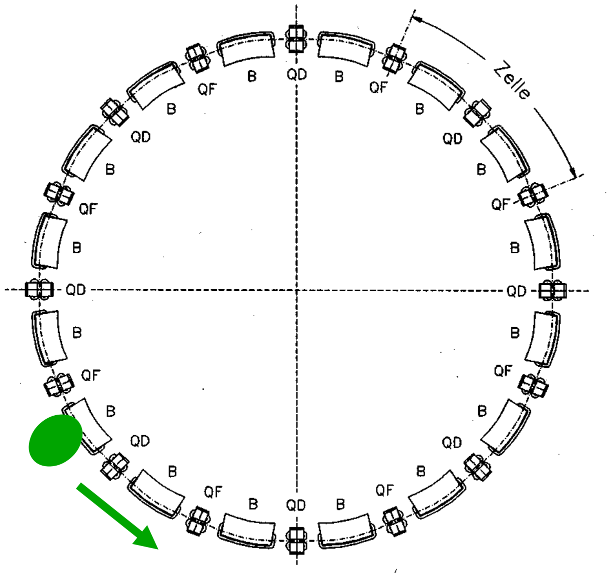
Particle Tracking in a Storage Ring

Calculate x, x´ for each linear accelerator element according to matrix formalism

plot x, x´as a function of „s"

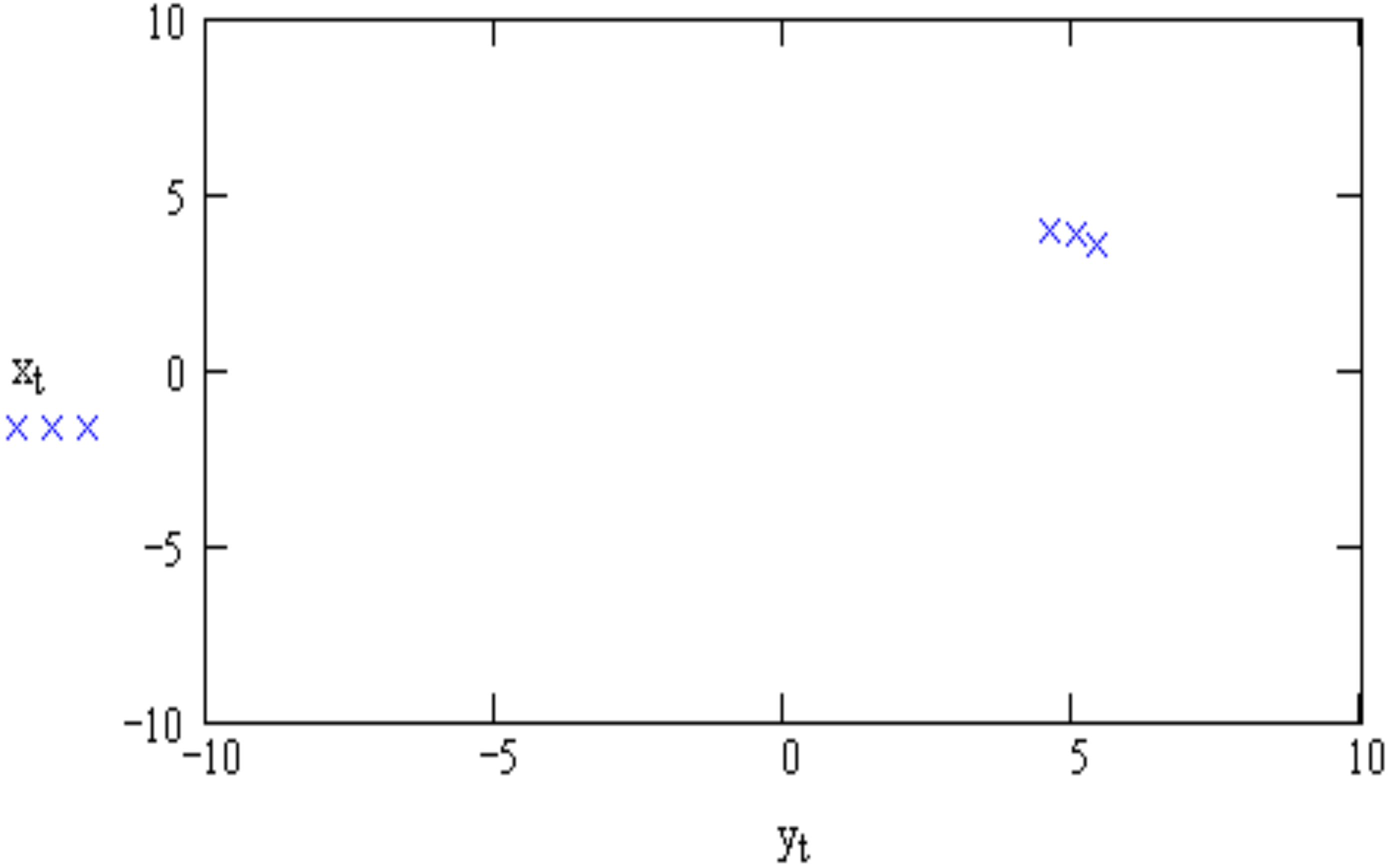






… and now the ellipse:

note for each turn x, x´at a given position „s_1" and plot in the phase space diagram



Phase Space Ellipse

particel trajectory: $x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}\cos\{\psi(s)+\phi\}$

max. Amplitude: $\hat{x}(s) = \sqrt{\varepsilon\beta}$ → ***x´ at that position …?***

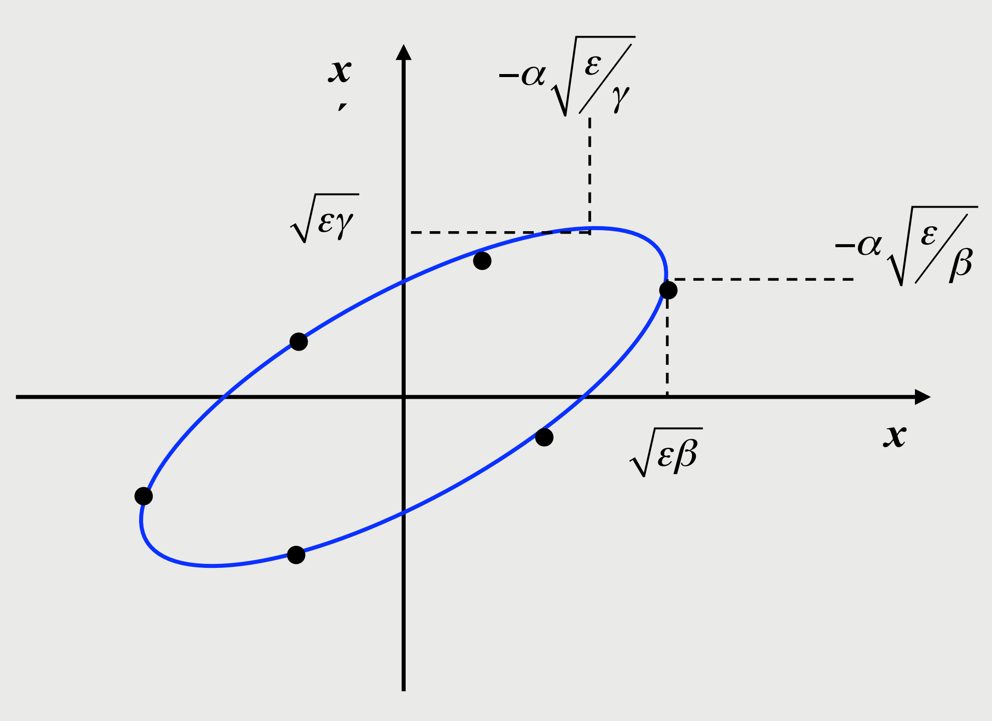
… put $\hat{x}(s)$ ***into***

$$\varepsilon = \gamma(s)\, x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)\, x'^2(s)$$

and solve for x´

$$\varepsilon = \gamma\cdot\varepsilon\beta + 2\alpha\sqrt{\varepsilon\beta}\cdot x' + \beta x'^2$$

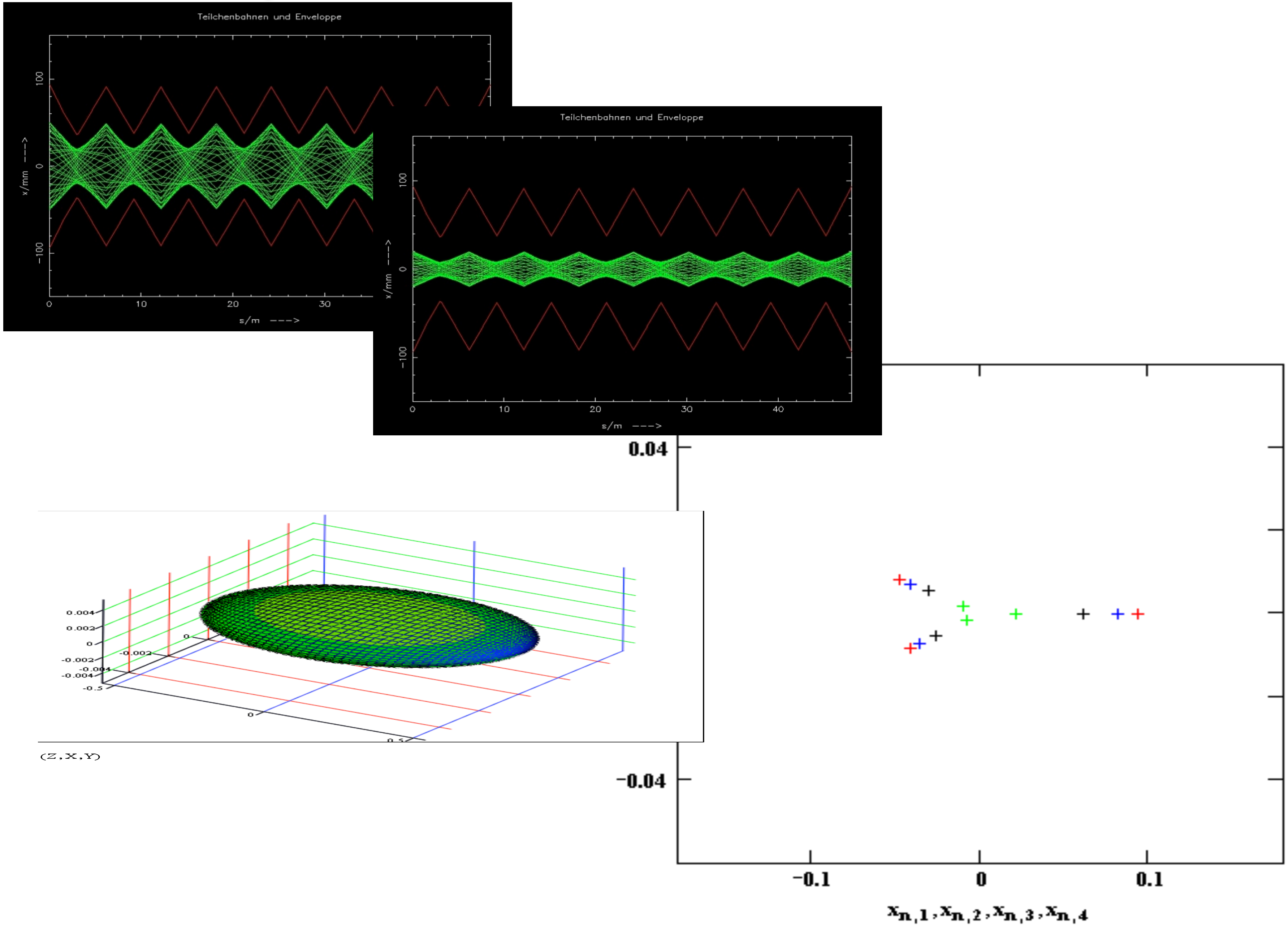
→ $x' = -\alpha\cdot\sqrt{\varepsilon/\beta}$



* ***The optical functions determine the shape and orientation of the phase space ellipse.***

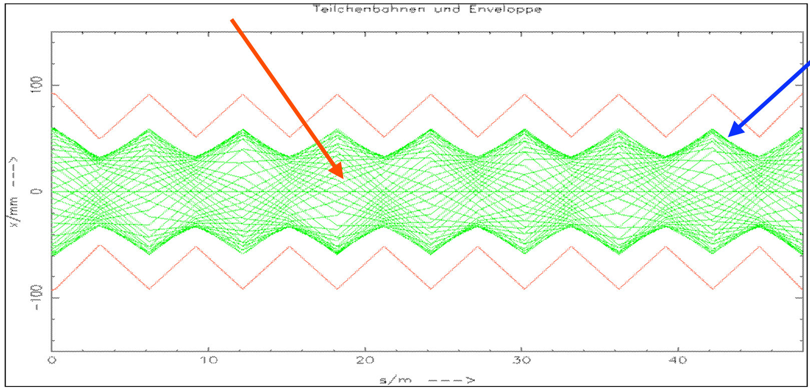
* ***A high β-function means a large beam size and a small beam divergence.***
  … et vice versa !!!

Emittance of the Particle Ensemble:



Emittance of the Particle Ensemble:

$$x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)} \cdot \cos(\Psi(s) + \phi) \qquad \hat{x}(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}$$



single particle trajectories, N ≈ 10 ^{11} per bunch

Gauß Particle Distribution:

$$\rho(x) = \frac{N \cdot e}{\sqrt{2\pi}\sigma_x} \cdot e^{-\frac{1}{2}\frac{x^2}{\sigma_x^2}}$$

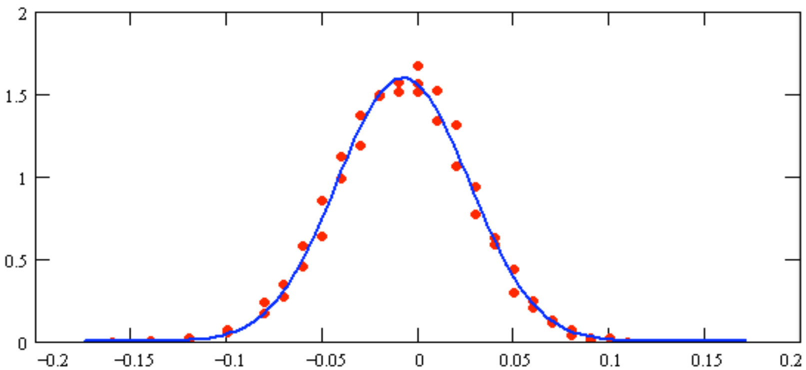
particle at distance 1 σ from centre
↔ 68.3 % of all beam particles

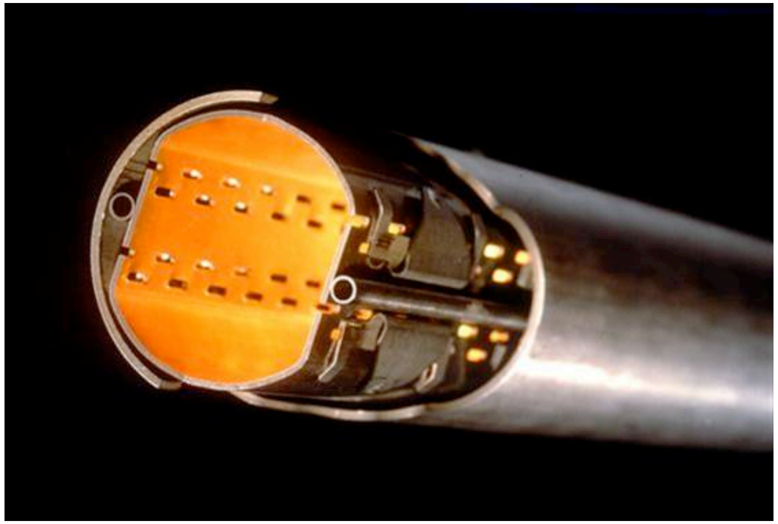
LHC:

$$\beta = 180\,m$$

$$\varepsilon = 5 * 10^{-10}\,m\,rad$$

$$\sigma = \sqrt{\varepsilon * \beta} = \sqrt{5 * 10^{-10} m * 180\,m} = 0.3\,mm$$

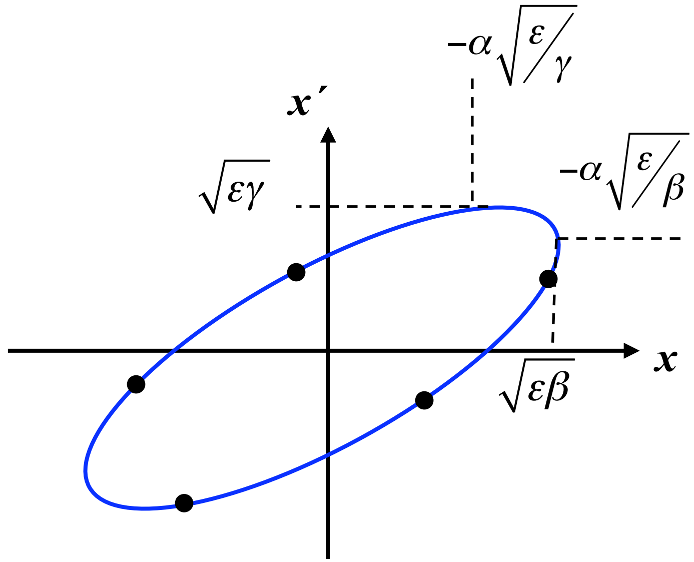




aperture requirements: r_0 = 12 * σ

Liouville during Acceleration

$$\varepsilon = \gamma(s)\,x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)\,x'^2(s)$$



***Beam Emittance** corresponds to the area covered in the x, x' Phase Space Ellipse*

But so sorry ... $\varepsilon \neq const$!

***Liouville:** Area in phase space is constant.* $\int p\,dq = const$

for convenience (i.e. because we are lazy bones) we use in accelerator theory:

$$x' = \frac{dx}{ds} = \frac{dx}{dt}\frac{dt}{ds} = \frac{\beta_x}{\beta}$$

where $\beta_x = v_x / c$ and $\gamma = 1\Big/\sqrt{1-\frac{v^2}{c^2}}$

$$\int pdq = mc\int \gamma\beta_x dx = mc\gamma\beta\underbrace{\int x'dx}_{\varepsilon}$$

$$\boxed{\Rightarrow \quad \varepsilon = \int x'dx \propto \frac{1}{\beta\gamma}}$$

the beam emittance shrinks during acceleration $\varepsilon \sim 1/\gamma$

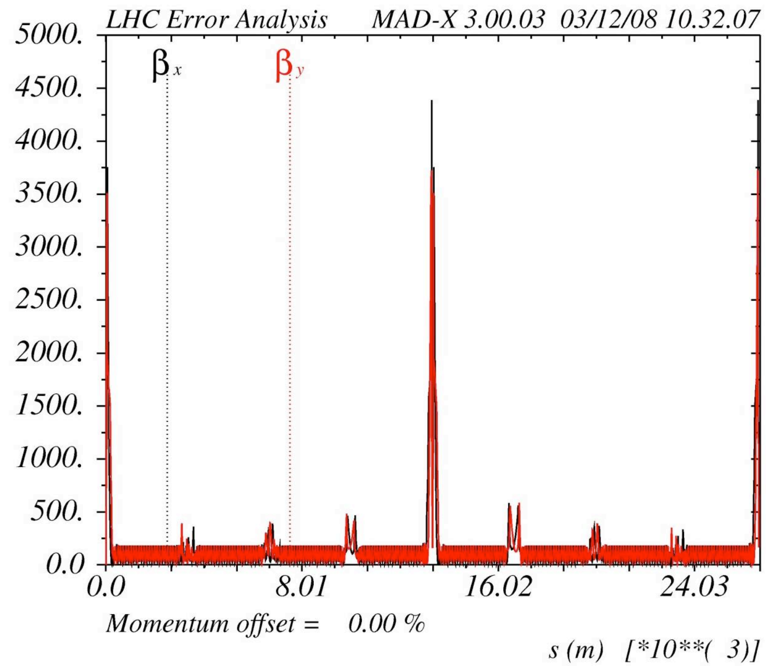
Nota bene:

1.) A proton machine … or an electron linac … needs the highest aperture at injection energy !!!
as soon as we start to accelerate the beam size shrinks as $\gamma^{-1/2}$ in both planes.

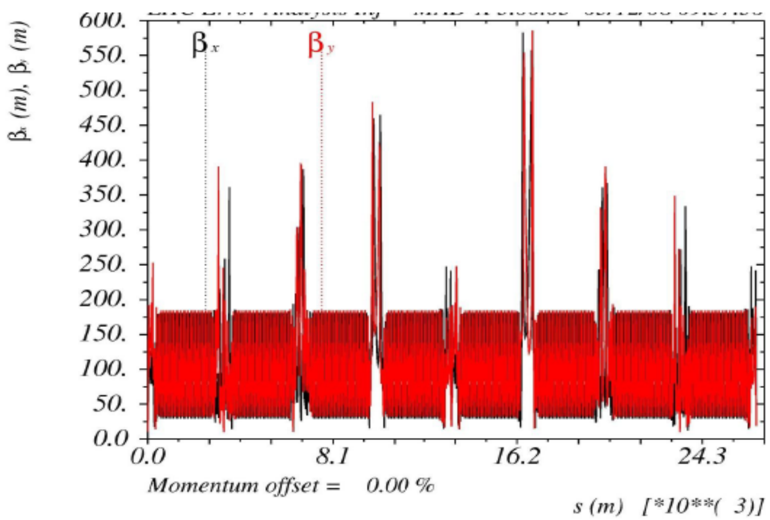
$$\sigma = \sqrt{\varepsilon\beta}$$

2.) At lowest energy the machine will have the major aperture problems,
→ here we have to minimise $\hat{\beta}$

3.) we need different beam optics adopted to the energy:
A Mini Beta concept will only be adequate at flat top.



LHC mini beta optics at 7000 GeV

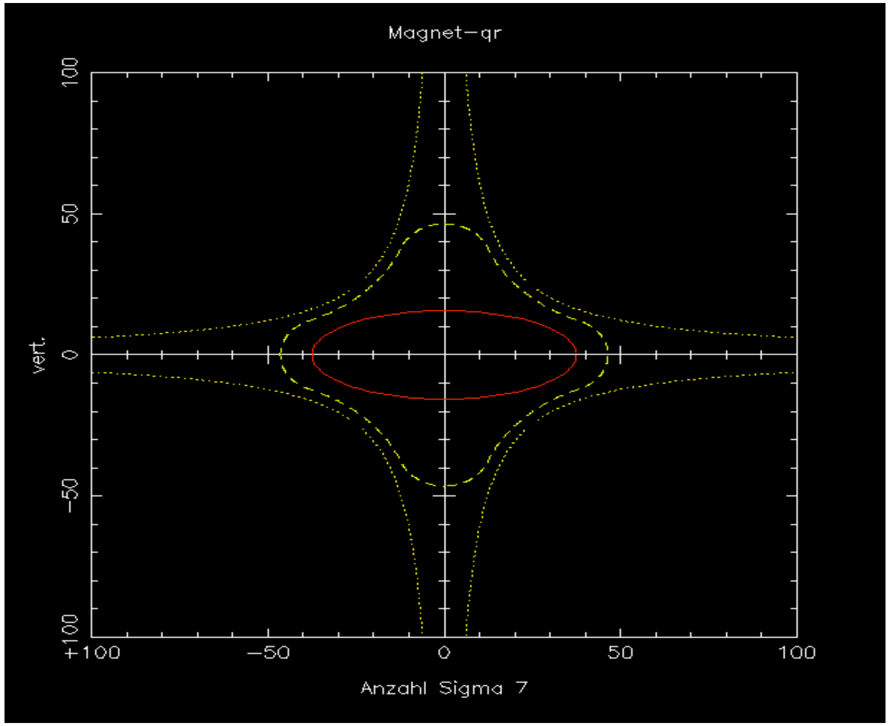


LHC injection optics at 450 GeV

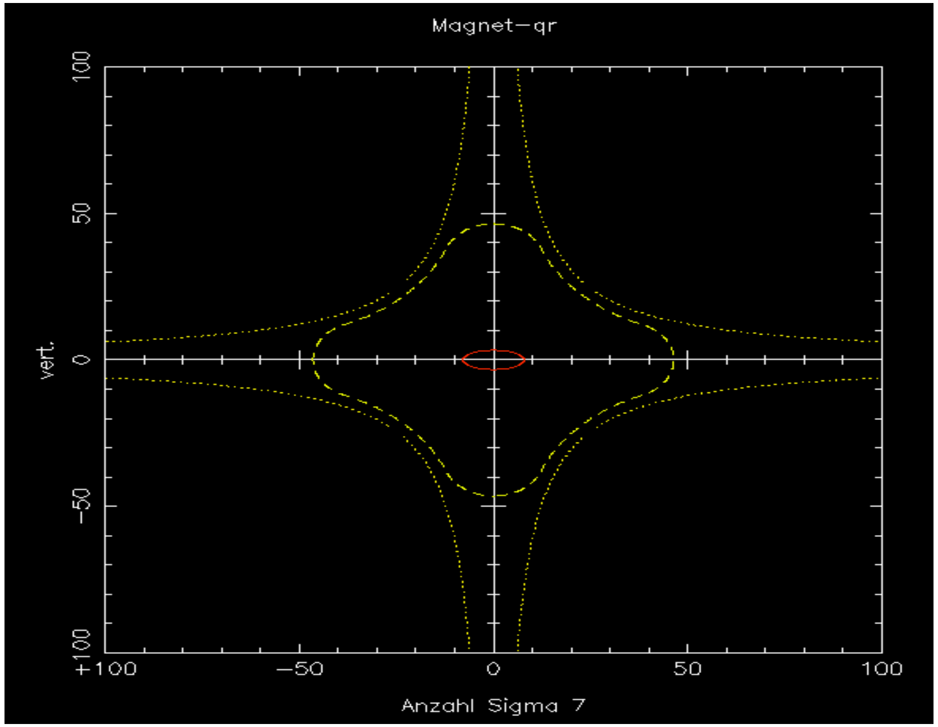
Example: HERA proton ring

injection energy: 40 GeV $\gamma = 43$
flat top energy: 920 GeV $\gamma = 980$

emittance $\varepsilon\,(40GeV) = 1.2 * 10^{-7}$
$\varepsilon\,(920GeV) = 5.1 * 10^{-9}$



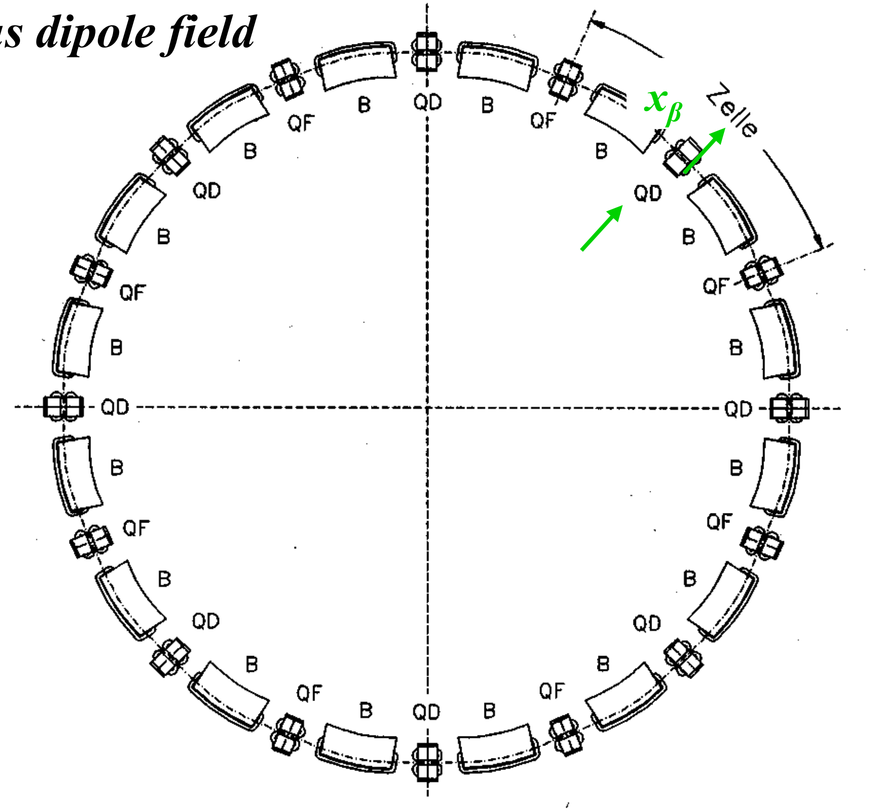
7 σ beam envelope at E = 40 GeV



… and at E = 920 GeV

Dispersion

Example: homogeneous dipole field



it for $\Delta p/p > 0$

$: D(s) \cdot \frac{\Delta p}{p}$

Matrix formalism:

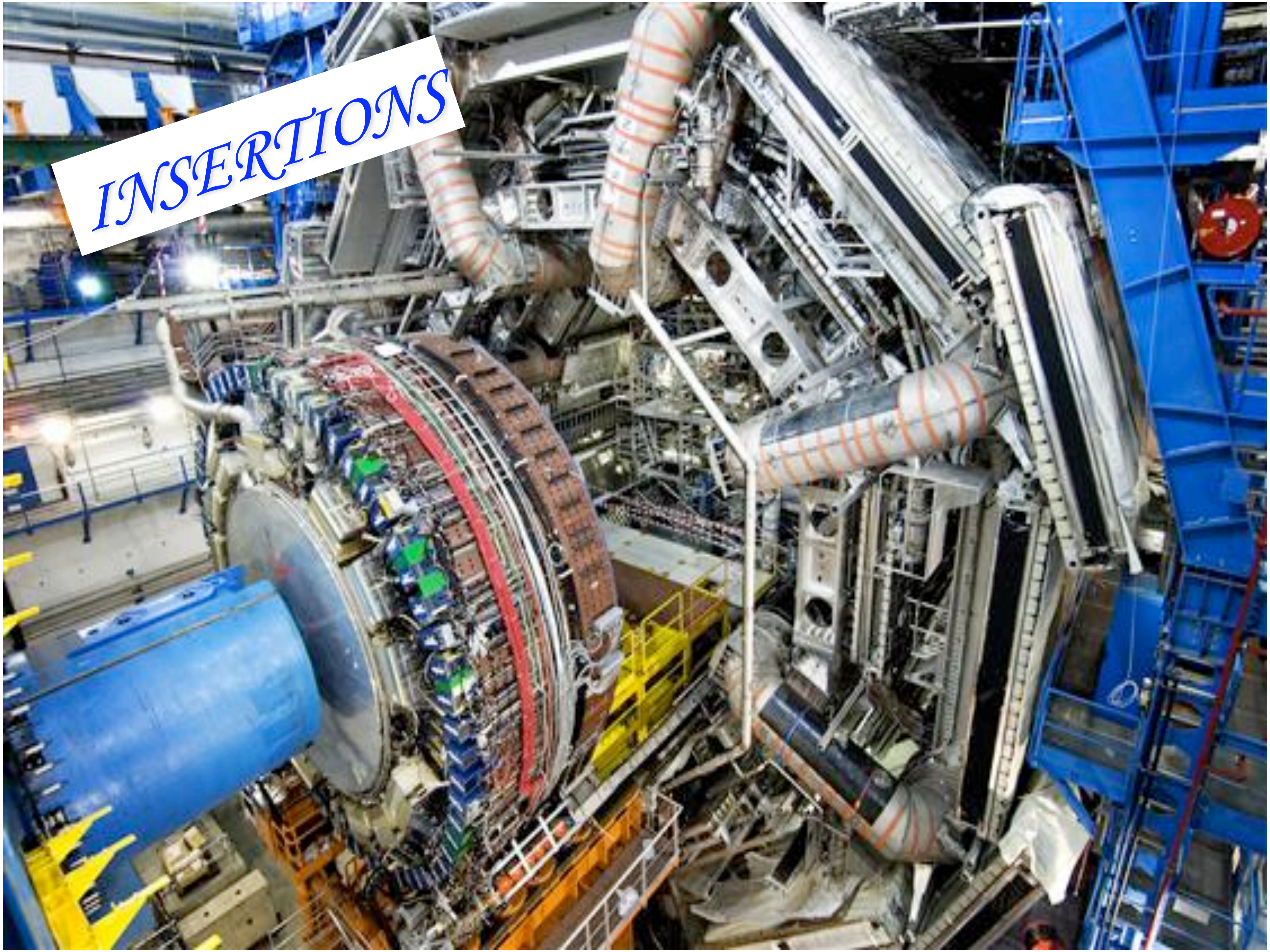
$$x(s) = x_\beta(s) + D(s) \cdot \frac{\Delta p}{p}$$

$$x(s) = C(s) \cdot x_0 + S(s) \cdot x_0' + D(s) \cdot \frac{\Delta p}{p}$$

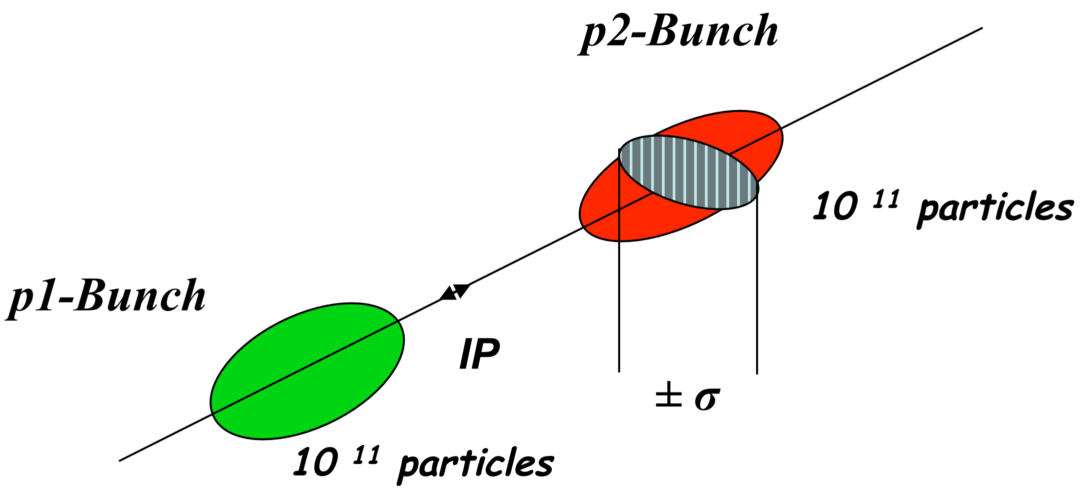
$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_0 + \frac{\Delta p}{p} \begin{pmatrix} D \\ D' \end{pmatrix}$$

$$C = \cos(\sqrt{|k|}s) \qquad S = \frac{1}{\sqrt{|k|}} \sin(\sqrt{|k|}s)$$

$$C' = \frac{dC}{ds} \qquad S' = \frac{dS}{ds}$$


INSERTIONS

Luminosity



Example: Luminosity run at LHC

$$\beta_{x,y} = 0.55\ \boldsymbol{m} \qquad f_0 = 11.245\ \boldsymbol{kHz}$$

$$\varepsilon_{x,y} = 5 * 10^{-10}\ \boldsymbol{rad\ m} \qquad n_b = 2808$$

$$\sigma_{x,y} = 17\ \mu\boldsymbol{m}$$

$$\boldsymbol{L} = \frac{1}{4\pi e^2 f_0 n_b} * \frac{I_{p1} I_{p2}}{\sigma_x \sigma_y}$$

$$I_p = 584\ \boldsymbol{mA}$$

$$\boldsymbol{L} = 1.0 * 10^{34}\ {}^{1}\!/_{cm^2 s}$$

Mini-β Insertions: Betafunctions

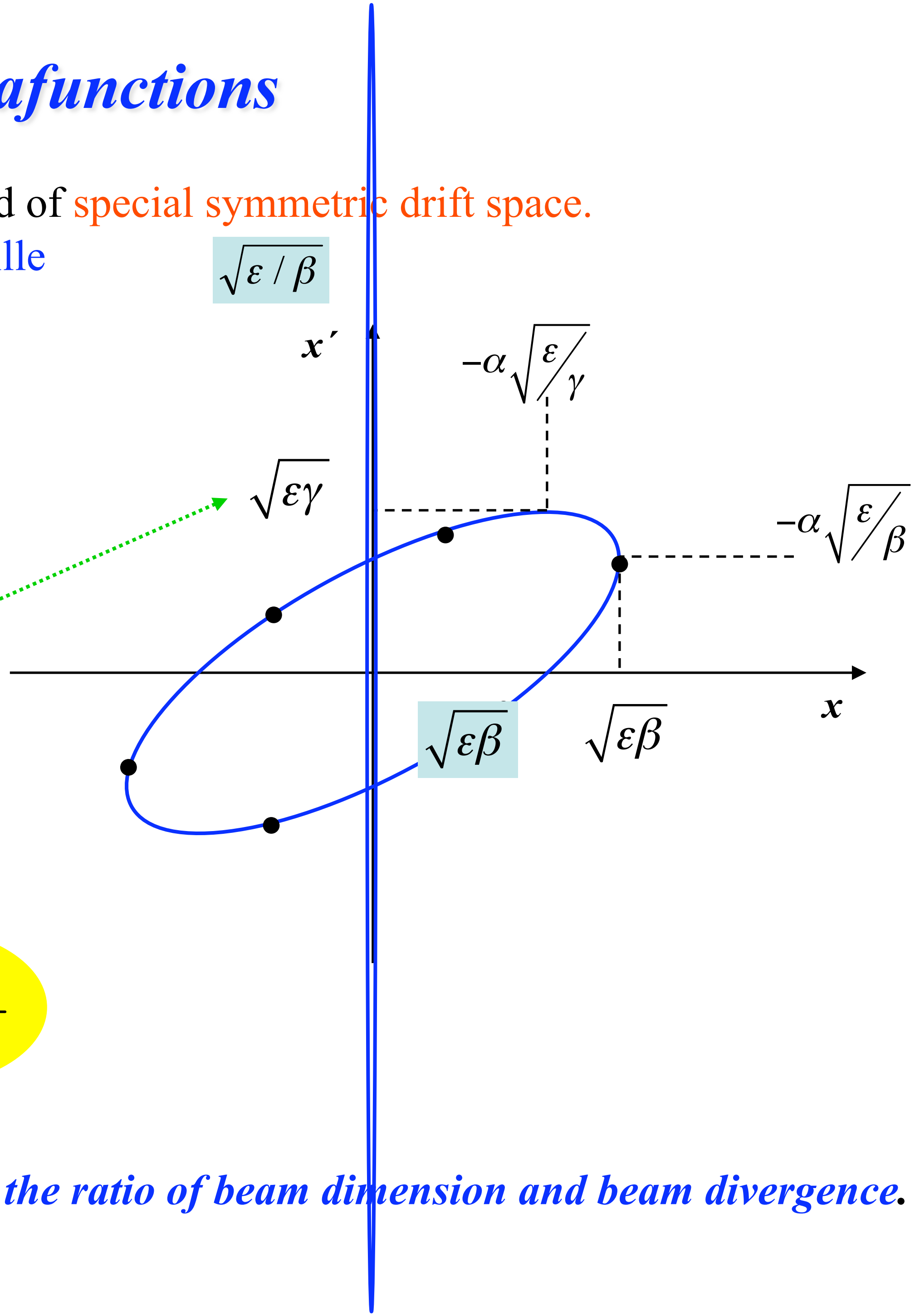
A mini-β insertion is always a kind of special symmetric drift space.

→greetings from Liouville

$$\alpha^* = 0$$

$$\gamma^* = \frac{1+\alpha^2}{\beta} = \frac{1}{\beta^*}$$

$$\sigma'^* = \sqrt{\frac{\varepsilon}{\beta^*}}$$



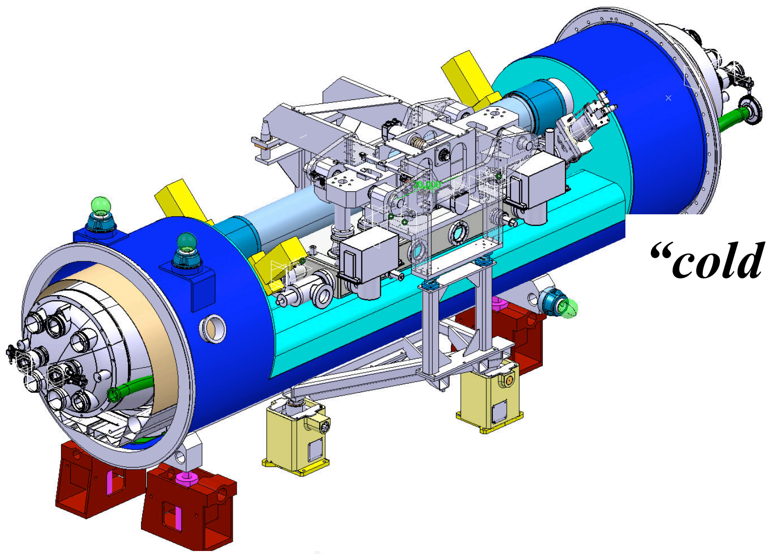
$$\beta^* = \frac{\sigma^*}{\sigma'^*}$$

at a symmetry point β is just the ratio of beam dimension and beam divergence.

Transverse Dynamics and Magnet Field Quality

$$\frac{B(x)}{p/e} = \frac{1}{\rho} + k\,x + \cancel{\frac{1}{2!} m\,x^2} + \cancel{\frac{1}{3!} n\,x^3} + \ldots$$

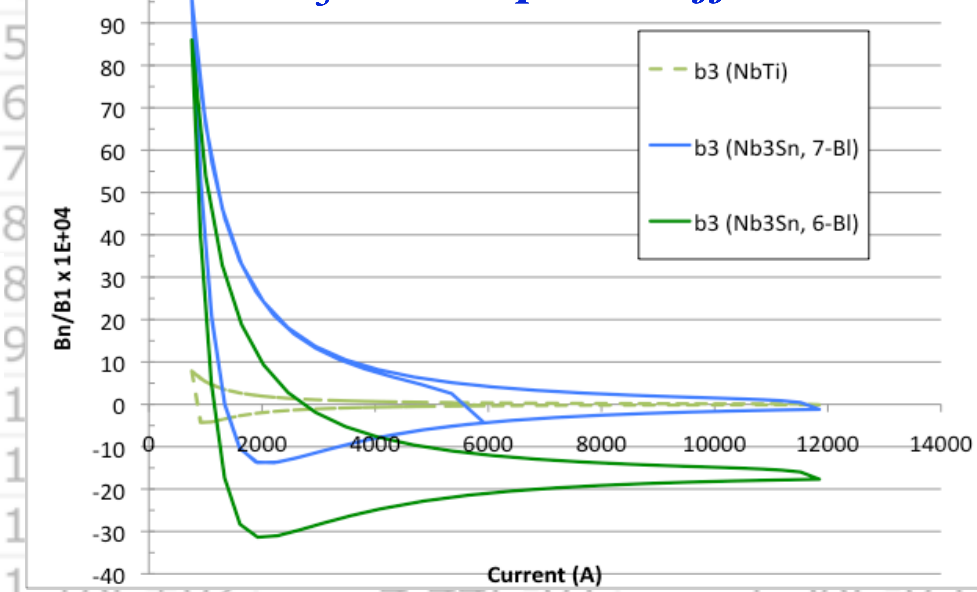
LHC Collimator Upgrade Project: installation of new high field dipoles



"cold collimator"

tracking studies & prototype measurements needed

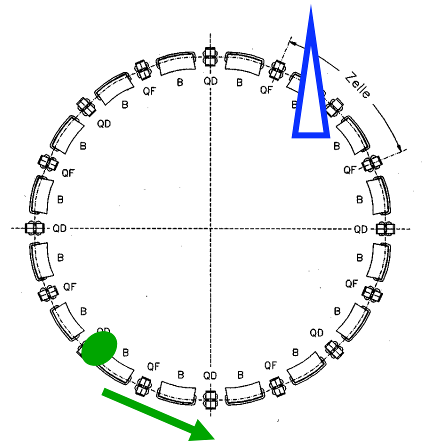
Nb_3Sn sextupole coefficient

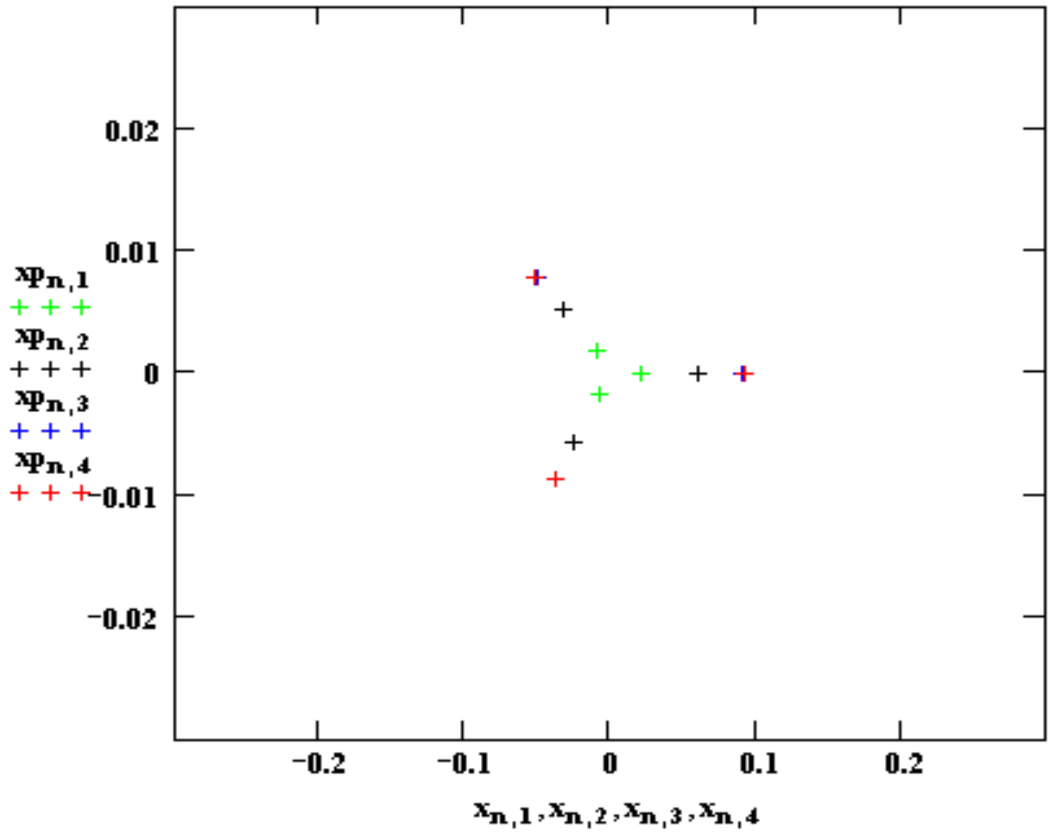


Effect of a strong (!!!) Sextupole …

Again: the phase space ellipse

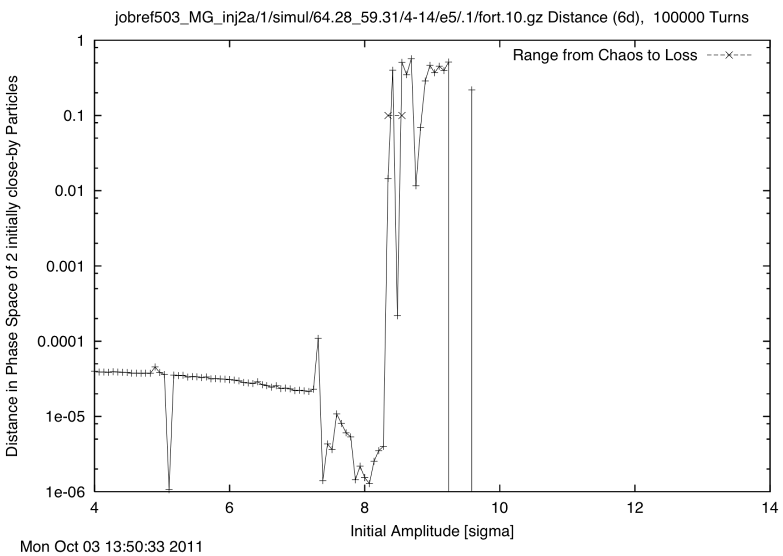
for each turn write down – at a given position „s" in the ring – the single partilce amplitude x and the angle x´... and plot it.





→ *Catastrophy !*

„dynamic aperture"



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