Acceleration & Longitudinal Beam Dynamics

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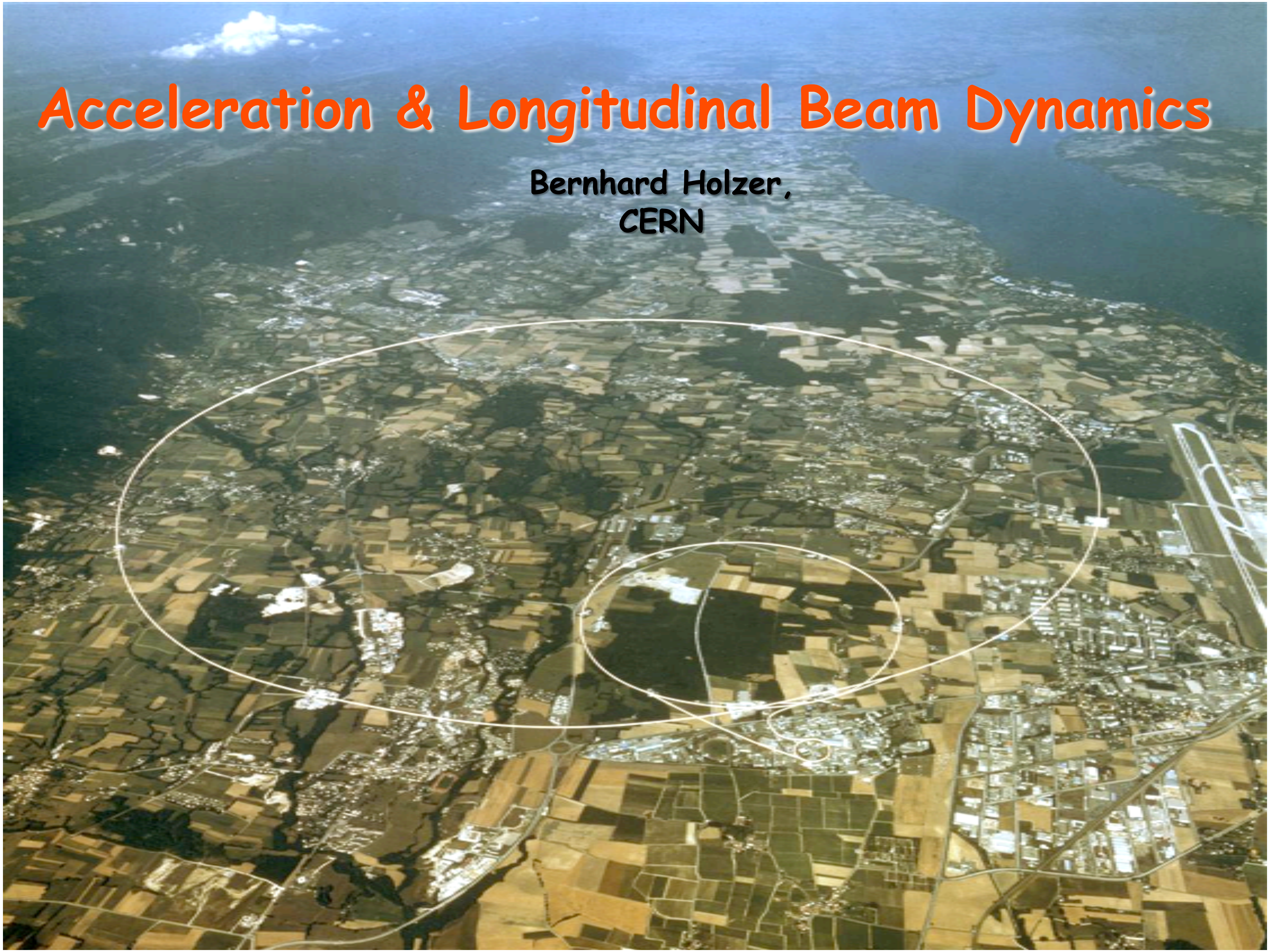

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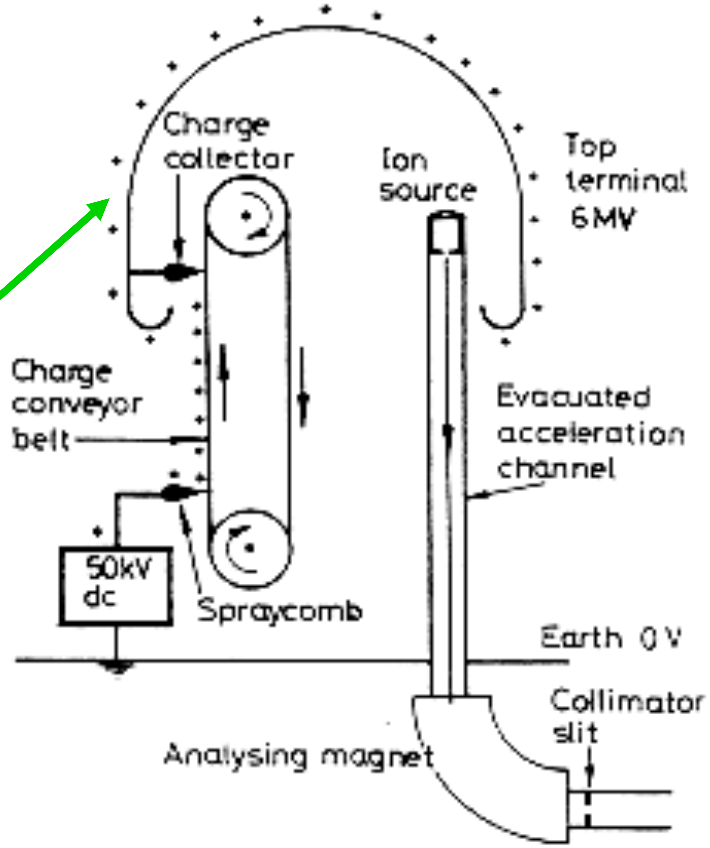
1.) Electrostatic Machines:
(Tandem –) van de Graaff Accelerator (1930 ...)

creating high voltages by mechanical transport of charges



* Terminal Potential: $U \approx 12 \ldots 28$ MV
  using high pressure gas to suppress discharge (SF_6)

Problems: * Particle energy limited by high voltage discharges
* high voltage can only be applied once per particle ...
  ... or twice ?

Energy Gain

... we have to start again from the basics

Lorentz force $\vec{F} = q * (\vec{E} + \cancel{\vec{v} \times \vec{B}})$

in long. direction the B-field creates no force

$\boldsymbol{v} \parallel \boldsymbol{B}$

$$\boxed{\vec{F} = \frac{d\vec{p}}{dt} = e\vec{E}}$$

acc. force is given by the electr. Field

In relativistic dynamics, energy and momentum satisfy the relation:

$$E^2 = E_0^2 + p^2c^2 \qquad (E = E_0 + W)$$

Hence:

$$dE = \int F ds = v dp$$

and the kinetic energy gained from the field along the z path is:

$$dW = dE = eE_z ds \quad \Rightarrow \quad W = e\int E_z ds = eV$$

The „Tandem principle": Apply the accelerating voltage twice ...
... by working with negative ions (e.g. H^-) and stripping the electrons in the centre of the structure

$$dW = dE = eE_z ds \quad \Rightarrow \quad W = e\int E_z ds = eV$$

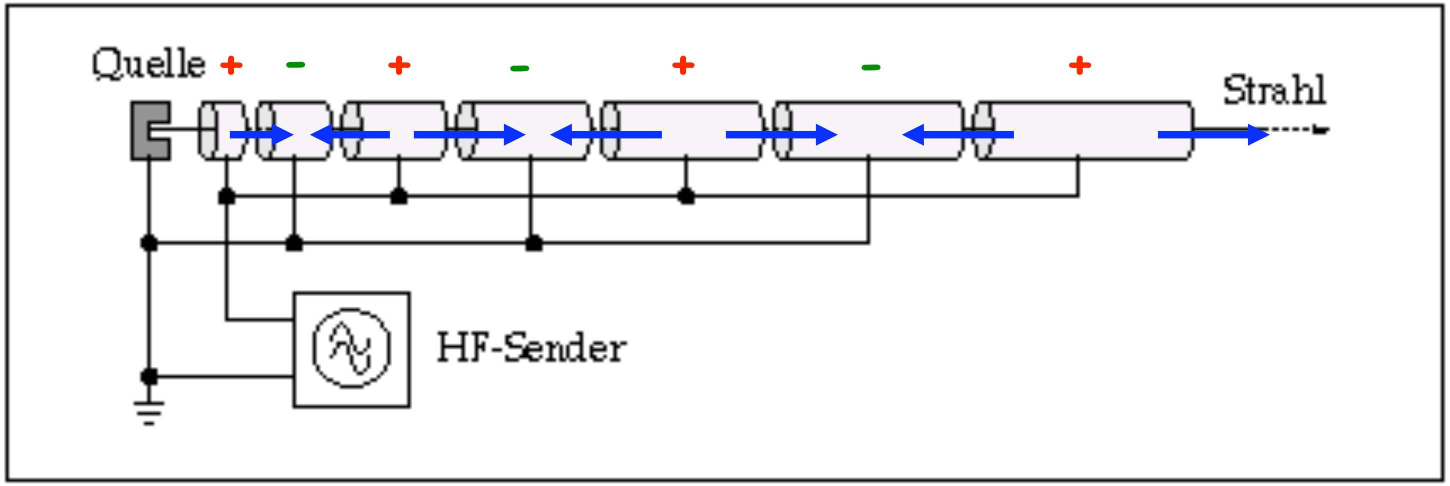
nota bene: all particles are "synchron" with the acceleration potential

Electro Static Accelerator: 12 MV-Tandem van de Graaff Accelerator at MPI Heidelberg

2.) The first RF-Accelerator: „Linac"

1928, Wideroe: how can the acceleration voltage be applied several times to the particle beam

schematic Layout:



Energy gained after n acceleration gaps

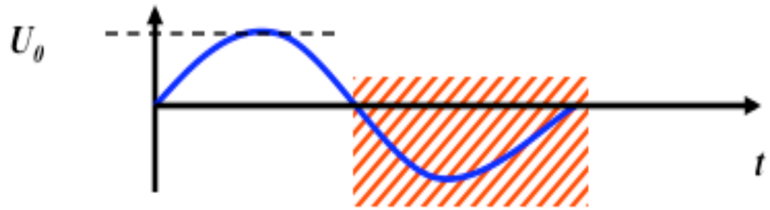
$$E_n = n * q * U_0 * \sin\psi_s$$

n number of gaps between the drift tubes
q charge of the particle
$\mathbf{U_0}$ Peak voltage of the RF System
Ψ_S synchronous phase of the particle

*** the problem of synchronisation ... between the particles and the rf voltage**
*** „voltage has to be flipped" to get the right sign in the second gap**
→ shield the particle in drift tubes during the negative half wave of the RF voltage

Wideroe-Structure: the drift tubes

shielding of the particles during the negative half wave of the RF



Time span of the negative half wave: $\tau_{RF}/2$

Length of the Drift Tube: $l_i = v_i * \frac{\tau_{rf}}{2}$

Kinetic Energy of the Particles $E_i = \frac{1}{2} m v^2$

$$\rightarrow v_i = \sqrt{2E_i/m}$$

$$l_i = \frac{1}{\nu_{rf}} * \sqrt{\frac{i * q * U_{0*\sin\psi_s}}{2m}}$$

valid for non relativistic particles ...

Alvarez-Structure: 1946, surround the whole structure by a rf vessel

Energy: ≈ 20 MeV per Nucleon β ≈ 0.04 … 0.6, Particles: Protons/Ions

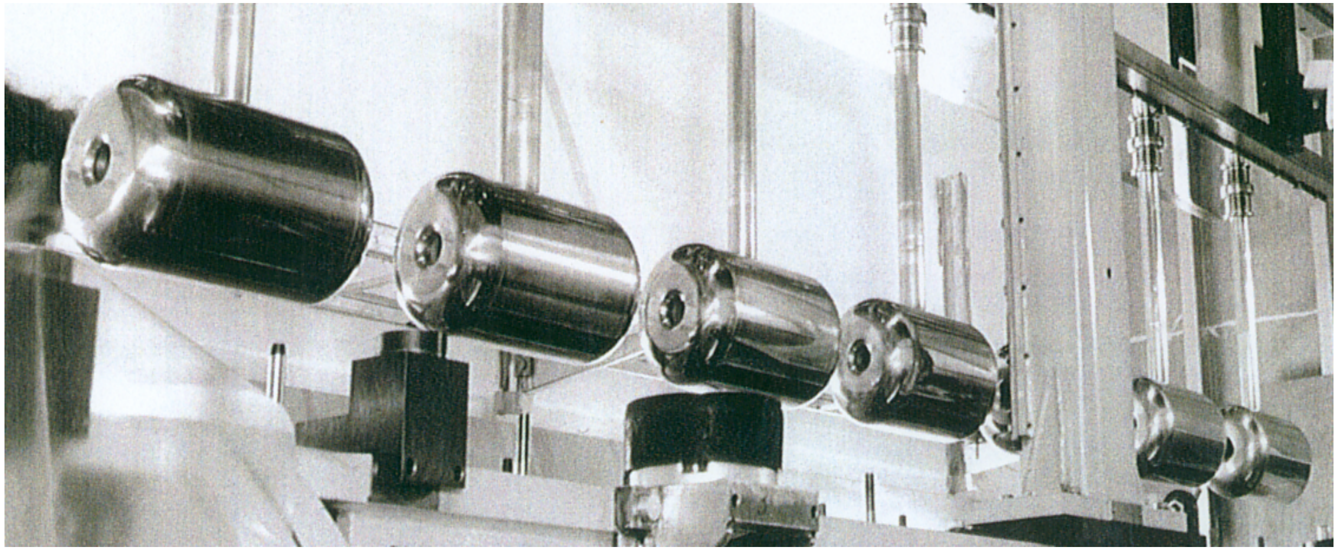
Example: DESY Accelerating structure of the Proton Linac

$$E_{total} = 988\, MeV$$

$$m_0 c^2 = 938\, MeV$$

$$p = 310\, MeV / c$$

$$E_{kin} = 50\, MeV$$



Beam energies

1.) reminder of some relativistic formula

rest energy $E_0 = m_0 c^2$

total energy $E = \gamma * E_0 = \gamma * m_0 c^2$

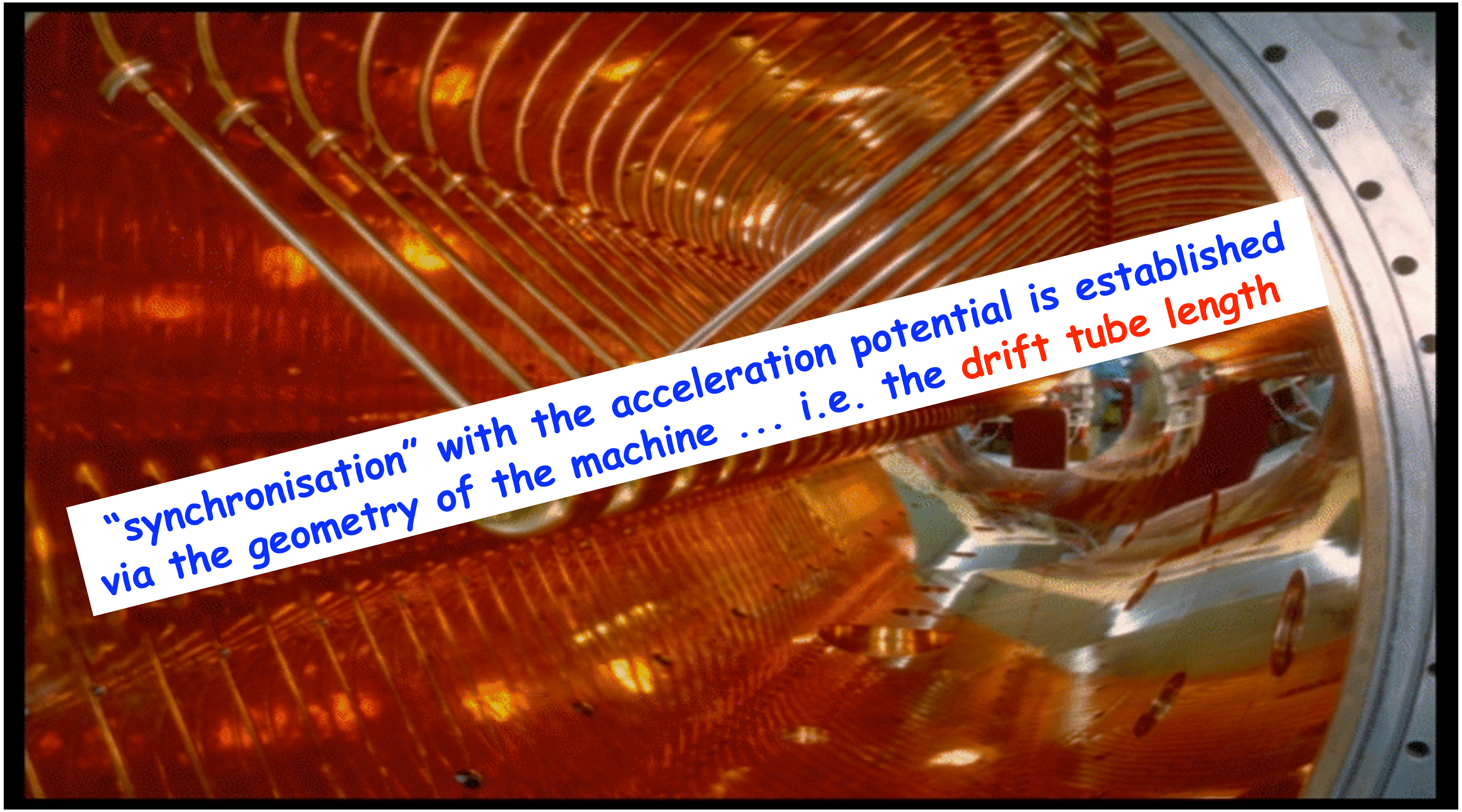
momentum $E^2 = c^2 p^2 + m_0^2 c^4$

kinetic energy $E_{kin} = E_{total} - m_0 c^2$

GSI: **Unilac, typical Energie ≈ 20 MeV per Nukleon, $\beta \approx 0.04 \dots 0.6$, Protons/Ions, $\nu = 110$ MHz**

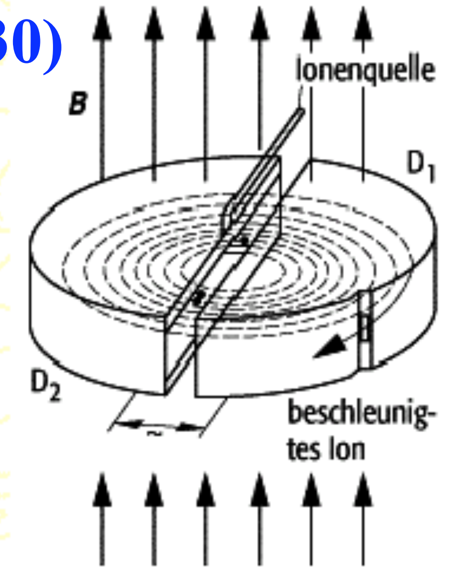
Energy Gain per „Gap“:

$$W = q\,U_0 \sin \omega_{RF} t$$



Application: until today THE standard proton / ion pre-accelerator
CERN Linac 4 is being built at the moment

3.) The Cyclotron: (Livingston / Lawrence ~1930)



Idea: B = const, RF = const
Synchronisation particle / RF via orbit

Lorentzforce

$$\vec{F} = q * (\vec{v} \times \vec{B}) = q * v * B$$

circular orbit

$$q * v * B = \frac{m * v^2}{R} \quad \longrightarrow \quad B * R = p/q$$

increasing radius for
increasing momentum
➔ Spiral Trajectory

revolution frequency

$$\omega_z = \frac{v}{R} = \frac{q}{m} * B_z$$

the cyclotron (rf-) frequency
is independent of the momentum

rf-frequency = h* revolution frequency, h = “harmonic number”

Cyclotron:

exact equation for revolution frequency:

$$\omega_z = \frac{v}{R} = \frac{q}{\gamma * m} * B_z$$

Cyclotron SPIRAL at GANIL

1.) if $v << c \Rightarrow \gamma \cong 1$

2.) γ increases with the energy
$\Rightarrow$ no exact synchron

"synchronisation" with the acceleration potential is established
via the spiraling orbit length

Sy... ron

B = constant
$\gamma\, \omega_{RF}$ = constant
ω_{RF} decreases with time

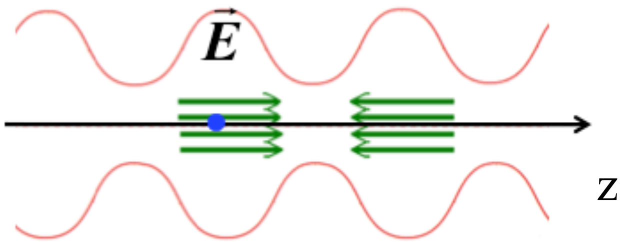
$$\omega_s(t) = \omega_{rf}(t) = \frac{q}{\gamma(t) * m_0} * B$$

keep the synchronisation condition by varying the rf frequency

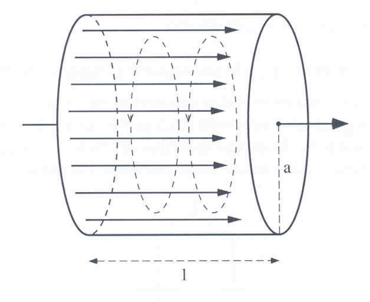
4.) RF Cavities, Acceleration and Energy Gain

$$dW = dE = eE_z ds \quad \Rightarrow \quad W = e\int E_z ds = eV$$

RF acceleration: $V \neq const$



In this case the electric field is oscillating. So it is for the potential. The energy gain will depend on the RF phase experienced by the particle.

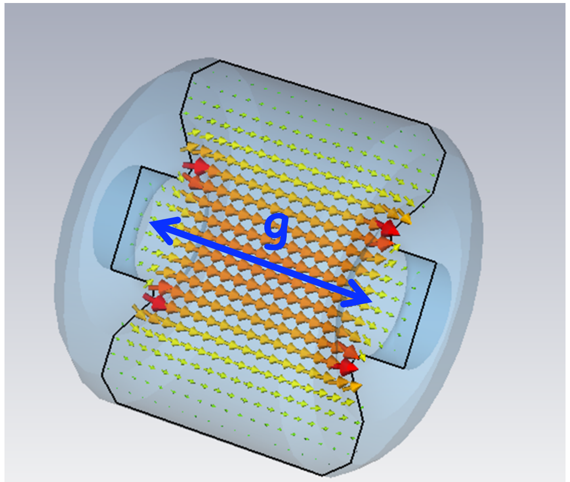


$$\int \hat{E}_z dz = \hat{V} \qquad E_z = \hat{E}_z \cos \omega_{RF} t \quad = \quad \hat{E}_z \cos \Phi(t)$$

$$W = e\hat{V} \cos \Phi$$

Neglecting the transit time in the gap.

Energy Gain in RF structures: Transit Time Factor



Oscillating field at frequency **ω** (amplitude is assumed to be constant all along the gap)

$$E_z = E_0 \cos\omega t = \frac{V}{g}\cos\omega t$$

Consider a particle passing through the middle of the gap at time t=0 : $z = vt$

The total energy gain is: $$\Delta W = \frac{eV}{g}\int_{-g/2}^{g/2}\cos\omega\frac{z}{v}dz$$

$$\Delta W = eV\frac{\sin\theta/2}{\theta/2} = eVT$$

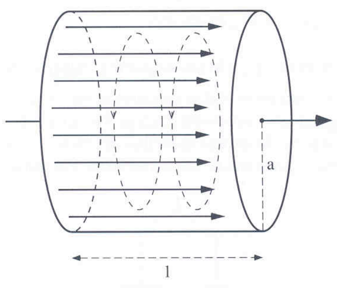
$$T = \frac{\sin\theta/2}{\theta/2}$$ transit time factor ($0 < T < 1$)

$$\theta = \frac{\omega g}{v}$$ *transit angle*

ideal case: $T = \dfrac{\sin\theta/2}{\theta/2} \to 1 \quad \Leftrightarrow \quad \theta/2 \to 0$

el. static accelertors $\omega \to 0$

minimise acc. gap $g \to 0$



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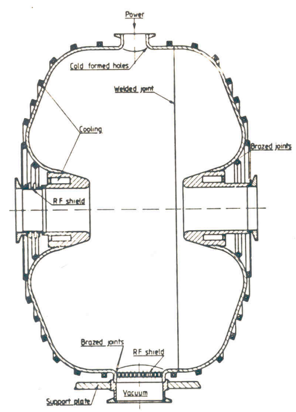
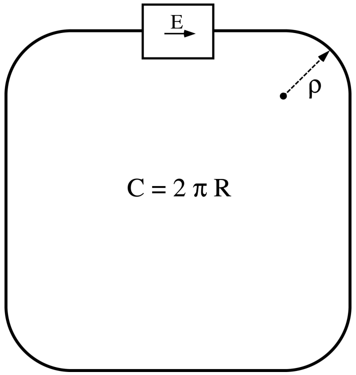


5.) The Synchrotron (Mac Millan, Veksler, 1945)



The synchrotron: Ring Accelerator of const. R where the increase in momentum (i.e. B-field) is automatically synchronised with the correct synchronous phase of the particle in the rf cavities

"synchronisation" as basic principle of the machine

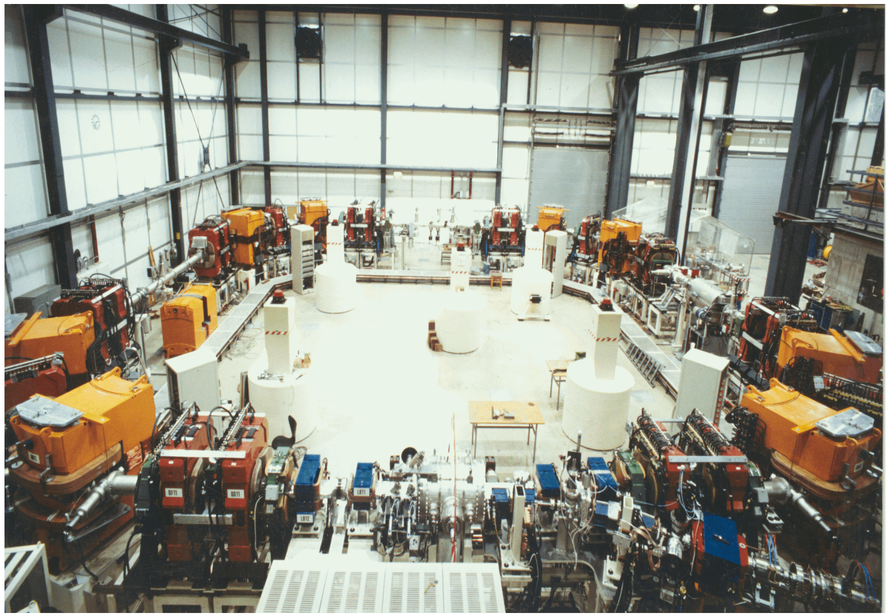
$e\hat{V}$: → Energy gain per turn

$\Phi = \Phi_s = cte$ → Synchronous particle

$\omega_{RF} = h\omega_r$ → RF synchronism

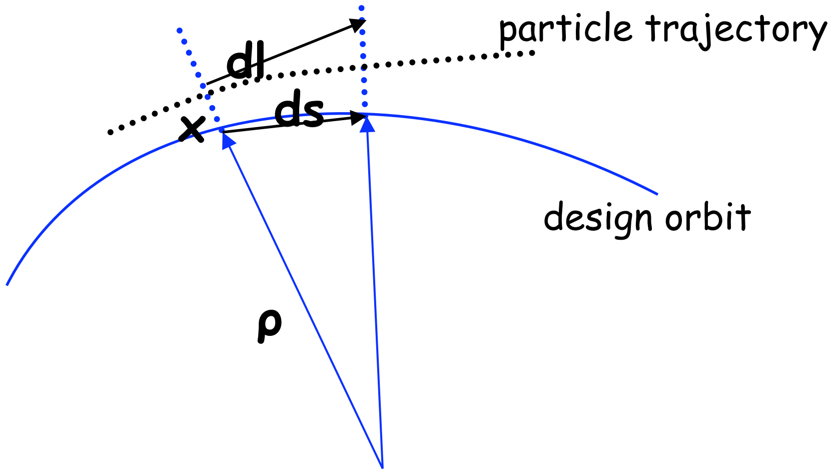
$\rho = cte \quad R = cte$ → Constant orbit

$B\rho = P/e \Rightarrow B$ → Variable magnetic field



6.) Momentum Compaction Factor: α_p

particle with a displacement x to the design orbit
→ path length dl ...



$$\frac{dl}{ds} = \frac{\rho + x}{\rho}$$

$$\rightarrow \; dl = \left(1 + \frac{x}{\rho(s)}\right) ds$$

circumference of an off-energy closed orbit

$$l_{\Delta E} = \oint dl = \oint \left(1 + \frac{x_{\Delta E}}{\rho(s)}\right) ds$$

remember: $x_{\Delta E}(s) = D(s) \frac{\Delta p}{p}$

$$\delta l_{\Delta E} = \frac{\Delta p}{p} \oint \left(\frac{D(s)}{\rho(s)}\right) ds$$

* The lengthening of the orbit for off-momentum particles is given by the dispersion function and the bending radius.

Definition: $$\frac{\delta l_{\varepsilon}}{L} = \alpha_p \frac{\Delta p}{p}$$

$$\rightarrow \alpha_p = \frac{1}{L}\oint\left(\frac{D(s)}{\rho(s)}\right) ds$$

For first estimates assume: $$\frac{1}{\rho} = const.$$

$$\int_{dipoles} D(s)\, ds \approx l_{\Sigma(dipoles)} \cdot \langle D \rangle_{dipole}$$

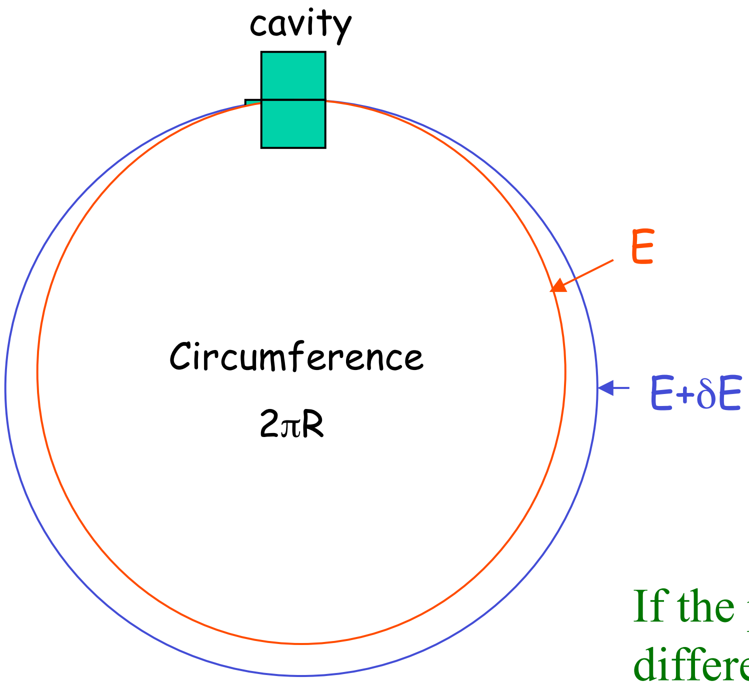
$$\alpha_p = \frac{1}{L}\, l_{\Sigma(dipoles)} \cdot \langle D \rangle \frac{1}{\rho} = \frac{1}{L}\, 2\pi\rho \cdot \langle D \rangle \frac{1}{\rho} \quad \rightarrow \quad \alpha_p \approx \frac{2\pi}{L} \langle D \rangle \approx \frac{\langle D \rangle}{R}$$

Assume: $v \approx c$

$$\rightarrow \quad \frac{\delta T}{T} = \frac{\delta l_{\varepsilon}}{L} = \alpha_p \frac{\Delta p}{p}$$

α_p combines via the dispersion function the momentum spread with the longitudinal motion of the particle.

7.) Dispersion Effects in a Synchrotron



If a particle is slightly shifted in momentum it will have a different orbit:

$$\alpha = \frac{p}{R}\frac{dR}{dp}$$

This is the "momentum compaction" generated by the bending field.

If the particle is shifted in momentum it will have also a different velocity. As a result of both effects **the revolution frequency changes:**

$$\eta = \frac{p}{f_r}\frac{df_r}{dp}$$

p=particle momentum

R=synchrotron physical radius

f_r=revolution frequency

Dispersion Effects in a Synchrotron

$$\boxed{\eta = \frac{p}{f_r}\frac{df_r}{dp}} \qquad f_r = \frac{\beta c}{2\pi R} \Rightarrow \frac{df_r}{f_r} = \frac{d\beta}{\beta} - \frac{dR}{R}$$

$$\rightarrow \frac{dR}{R} = \alpha \frac{dp}{p}$$

$$p = mv = \beta\gamma \frac{E_0}{c} \Rightarrow \frac{dp}{p} = \frac{d\beta}{\beta} + \frac{d\left(1-\beta^2\right)^{-\frac{1}{2}}}{\left(1-\beta^2\right)^{-\frac{1}{2}}} = \left(1-\beta^2\right)^{-1}\frac{d\beta}{\beta}$$

$$\rightarrow \frac{d\beta}{\beta} = \frac{1}{\gamma^2}\frac{dp}{p}$$

$$\frac{df_r}{f_r} = \left(\frac{1}{\gamma^2} - \alpha\right)\frac{dp}{p} \longrightarrow \boxed{\eta = \frac{1}{\gamma^2} - \alpha}$$

The change of revolution frequency depends on the particle energy γ and changes sign during acceleration.

Particles get faster in the beginning – and arrive earlier at the cavity: classic regime

Particles travel at v =c and get more massive – and arrive later at the cavity: relativistic regime

boundary between the two regimes: no frequency dependence on dp/p,
η =0 "transition energy"

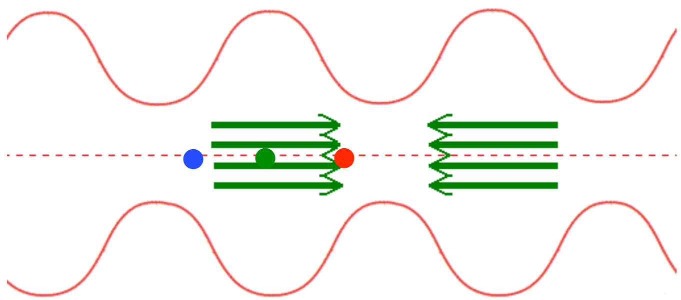
$$\boxed{\gamma_{tr} = \frac{1}{\sqrt{\alpha}}}$$

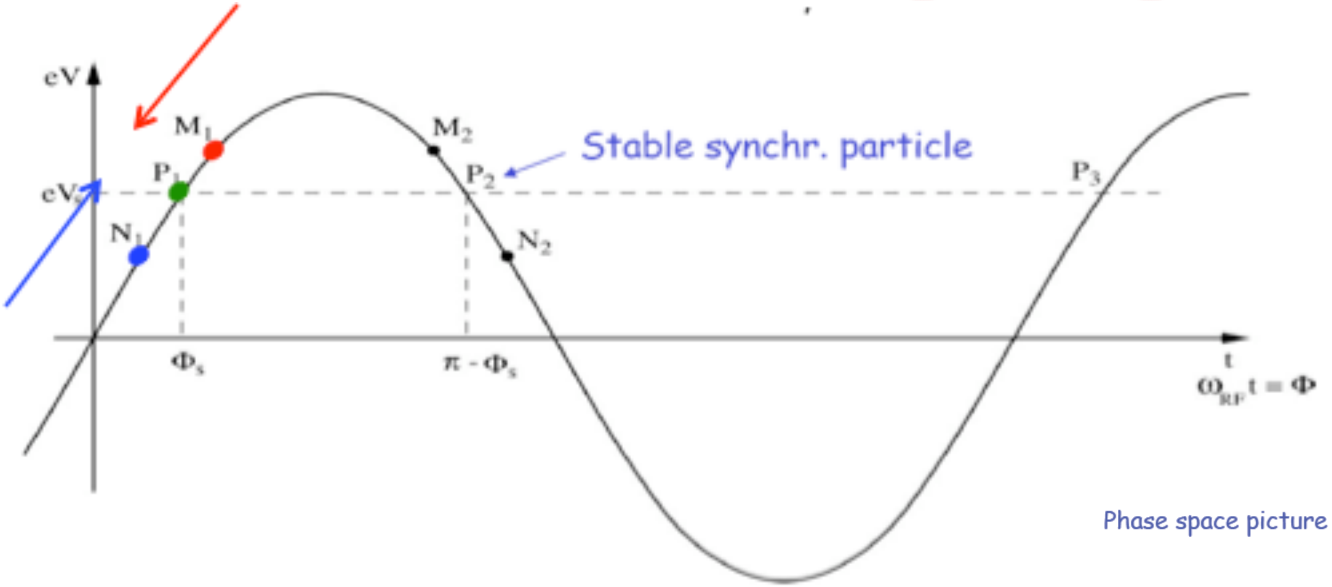
8.) The Acceleration for $\Delta p/p \neq 0$
"Phase Focusing" below transition

ideal particle ●

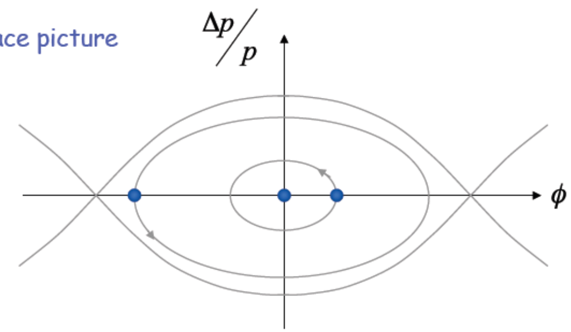
particle with $\Delta p/p > 0$ ● faster

particle with $\Delta p/p < 0$ ● slower



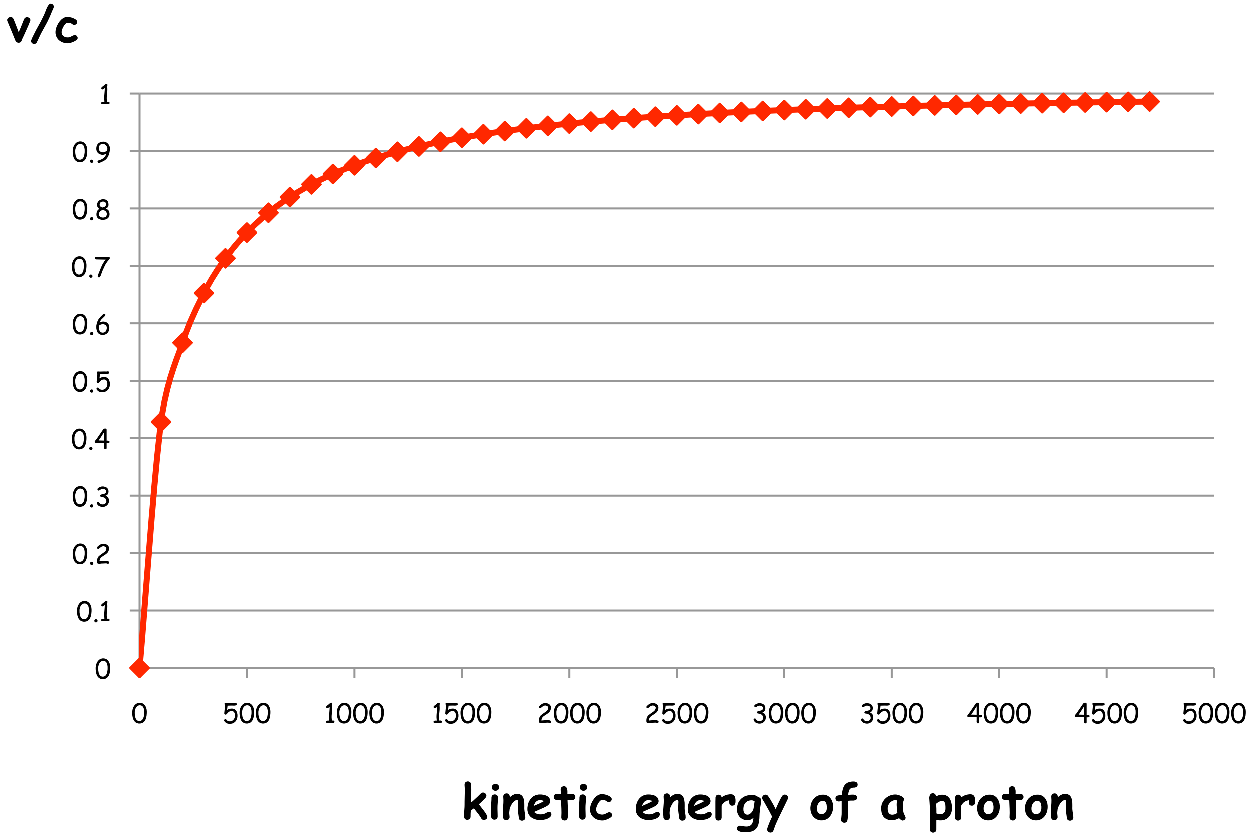


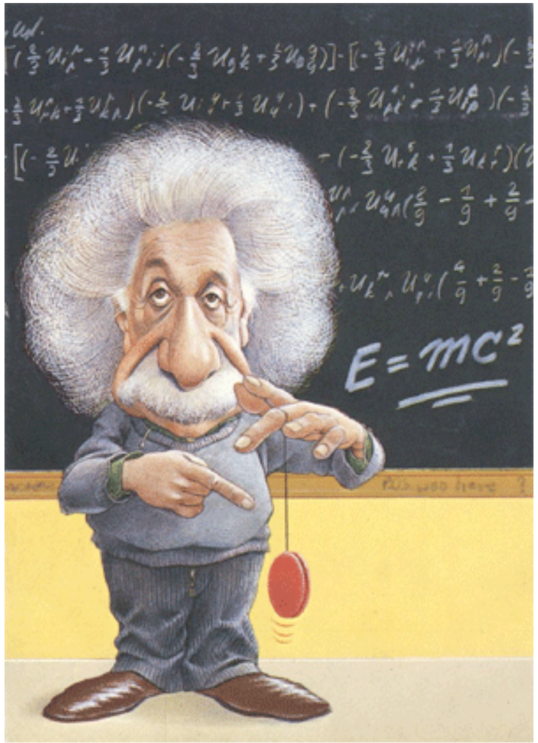
Focussing effect in the longitudinal direction keeping the particles close together ... forming a "bunch"



... so sorry, here we need help from Albert:

$$\gamma = \frac{E_{total}}{mc^2} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \longrightarrow \frac{v}{c} = \sqrt{1 - \frac{mc^2}{E^2}}$$





... some when the particles do not get faster anymore

.... but heavier !

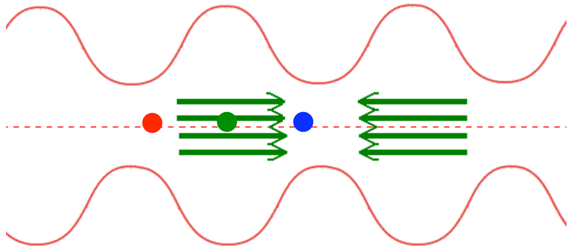
9.) The Acceleration for Δp/p≠0

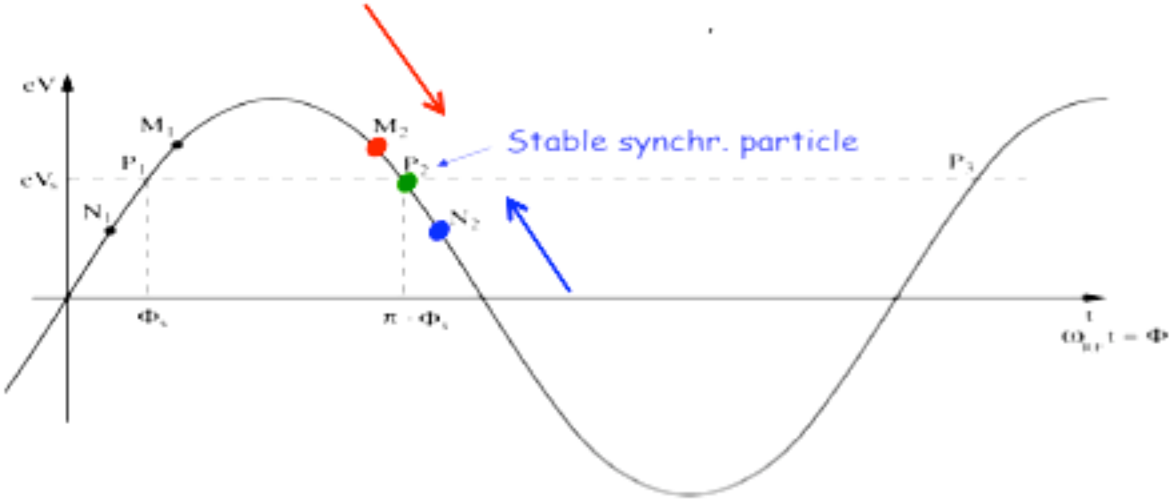
"Phase Focusing" above transition

ideal particle ●

particle with Δp/p > 0 ● heavier

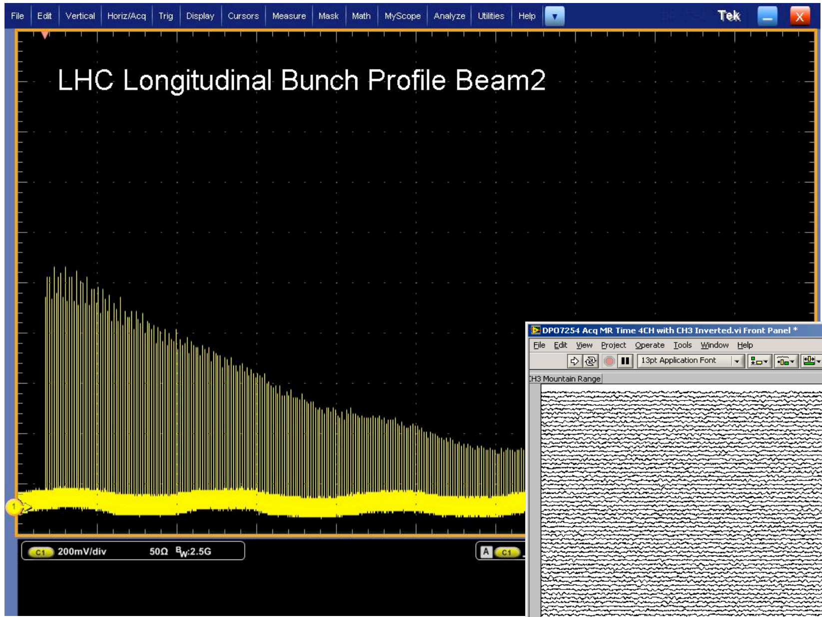
particle with Δp/p < 0 ● lighter

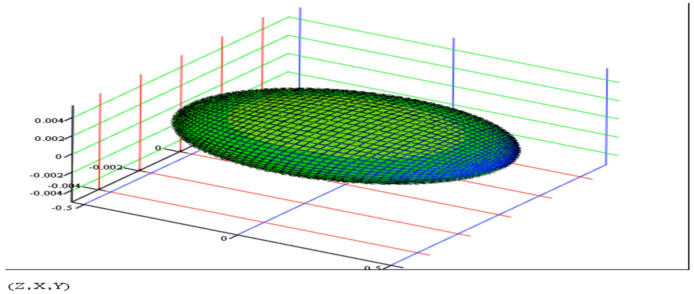




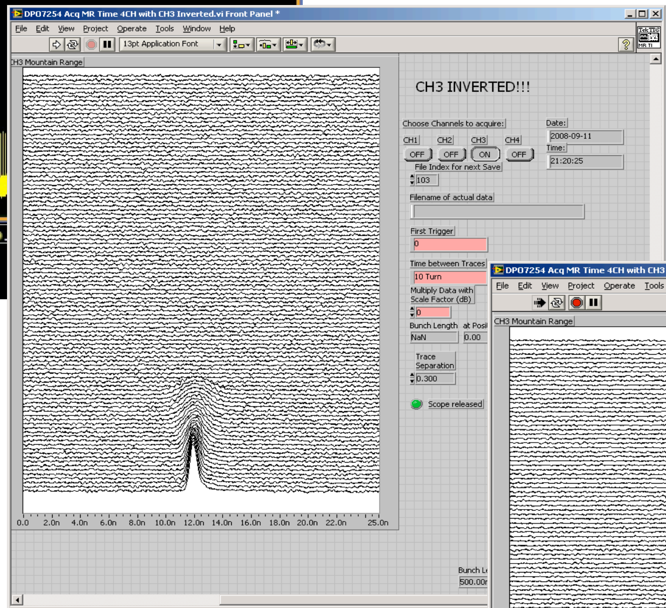
oscillation frequency ≈ some Hz

LHC Commissioning: RF

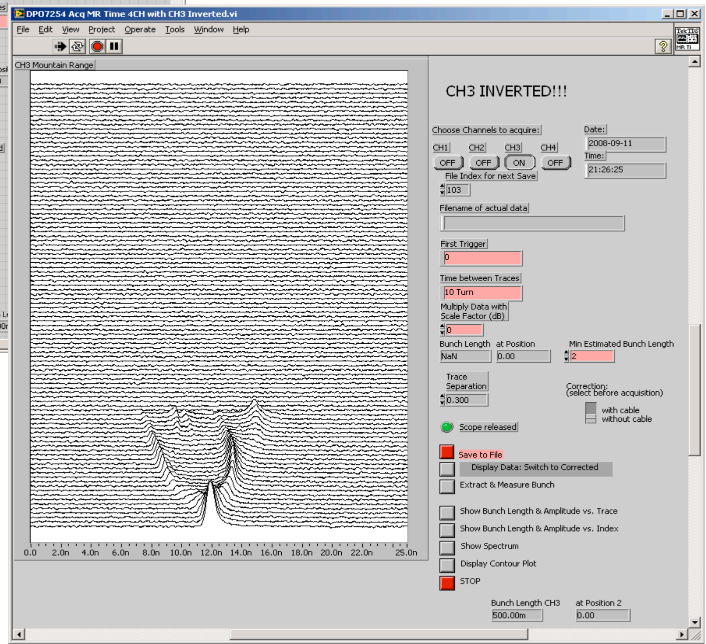




a proton bunch: focused longitudinal by the RF field



RF off



RF on,
wrong phase condition

... and how do we accelerate now ???

with the dipole magnets !

Energy ramping is simply obtained by varying the B field:

$$p = eB\rho \quad \Rightarrow \quad \frac{dp}{dt} = e\rho\,\dot{B} \quad \Rightarrow (\Delta p)_{turn} = e\rho\,\dot{B}T_r = \frac{2\pi\ e\rho\ R\dot{B}}{v}$$

Energy Gain per turn: $$E^2 = E_0^2 + p^2c^2 \quad \Rightarrow \quad \Delta E = v\Delta p$$

$$\boxed{\Delta E_{turn} = \Delta W_{turn} = 2\pi e\rho R\dot{B} = e\hat{V}\sin\phi_s}$$

* The energy gain depends on the rate of change of the dipole field
* The number of stable synchronous particles is equal to the harmonic number h. They are equally spaced along the circumference.
* Each synchronous particle satifies the relation p = eBρ. They have the nominal energy and follow the nominal trajectory.

10.) Longitudinal Dynamics: synchrotron motion

We have to follow two coupled variables:

* the energy gained by the particle
* and the RF phase experienced by the same particle.

Since there is a well defined synchronous particle which has always the same phase ϕ_s, and the nominal energy E_s, it is sufficient and elegant to follow other particles with respect to that particle.

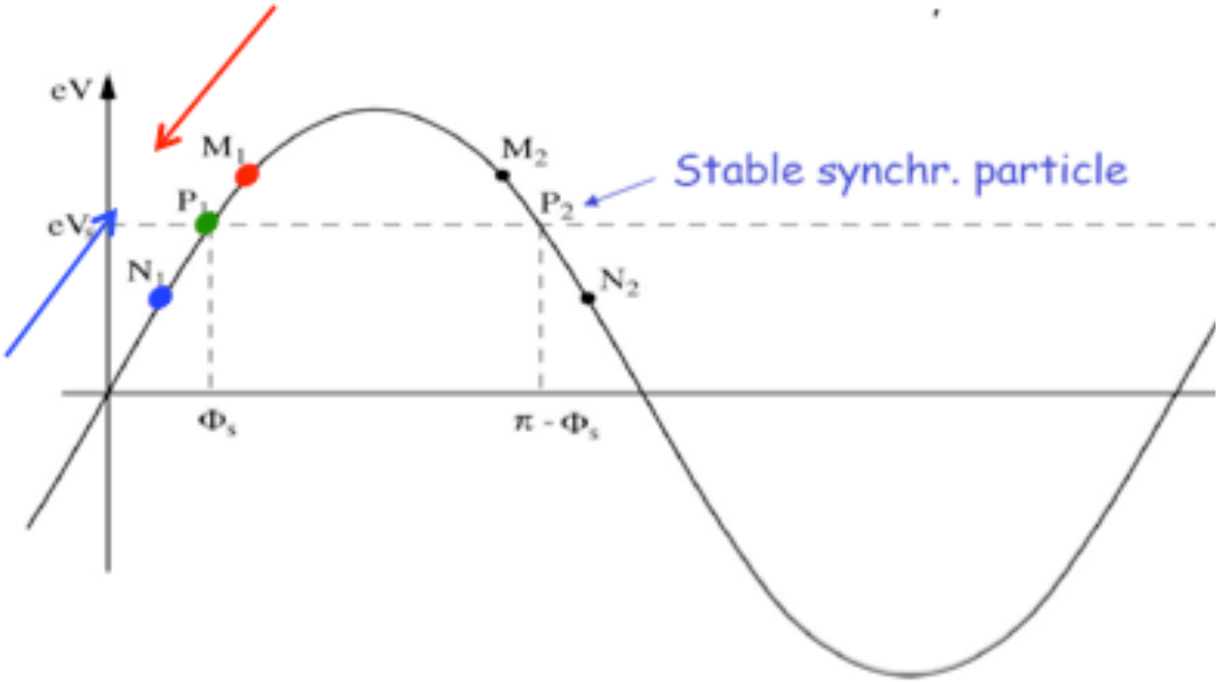
We will introduce the following relative variables:

revolution frequency :	$\Delta f_r = f_r - f_{rs}$
particle RF phase :	$\Delta\phi = \phi - \phi_s$
particle momentum :	$\Delta p = p - p_s$
particle energy :	$\Delta E = E - E_s$
azimuth angle :	$\Delta\theta = \theta - \theta_s$

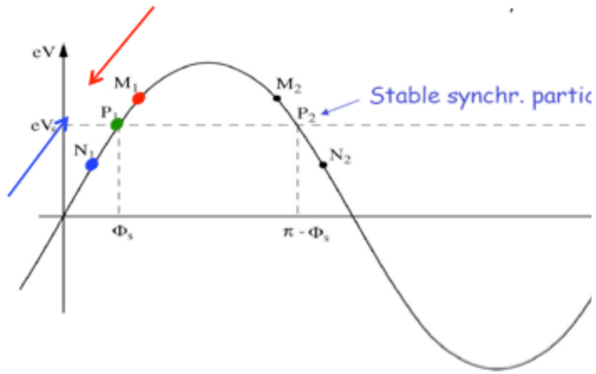
The Equation of Motion:

Energy-Phase Relations in a Synchrotron

energy offset ←→ phase change



Equation of Motion:



Relation between momentum difference and difference in revolution frequency:

$$\frac{df_r}{f_r} = \left(\frac{1}{\gamma^2} - \alpha\right)\frac{dp}{p}$$

which translates into difference in revolution time:

$$\frac{dT}{T_0} = \left(\alpha - \frac{1}{\gamma^2}\right)\frac{dp}{p}$$

A particle with higher momentum travels faster (in the classical regime)

The result is a difference in phase at the cavity

$$\Delta\psi = 2\pi\frac{\Delta T}{T_{rf}} = \omega_{rf} * \Delta T$$

$$= h * \omega_0 * \Delta T = h * 2\pi\frac{\Delta T}{T_0}$$

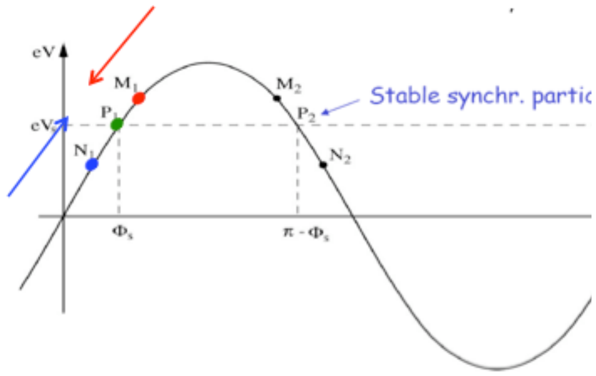
$$= h * 2\pi\left(\alpha - \frac{1}{\gamma^2}\right)\frac{dp}{p}$$

$$= \frac{h * 2\pi}{\beta^2}\left(\alpha - \frac{1}{\gamma^2}\right)\frac{dE}{E}$$

The RF frequency has to be a integer multiple of the revolution frequency, “h” called harmonic number

difference in energy and difference in phase are related via the momentum compaction

Equation of Motion:



$$\Delta\psi = \frac{h*2\pi}{\beta^2}\left(\alpha - \frac{1}{\gamma^2}\right)\frac{dE}{E}$$

differentiate to time

1 $$\Delta\dot{\psi} = \frac{\Delta\psi}{T_0} = \frac{h*2\pi}{\beta^2 T_0}\left(\alpha - \frac{1}{\gamma^2}\right)\frac{dE}{E}$$

rate of change of the phase difference
wrt to the ideal particle

the energy change is given by the RF system:

$$\Delta E = e*U_0(\sin(\psi_s + \Delta\psi) - \sin\psi_s)$$

$$\sin(\psi_s + \Delta\psi) - \sin\psi_s =$$
$$\sin\psi_s \underbrace{\cos\Delta\psi}_{\approx 1} - \cos\psi_s \underbrace{\sin\Delta\psi}_{\Delta\psi} - \sin\psi_s$$

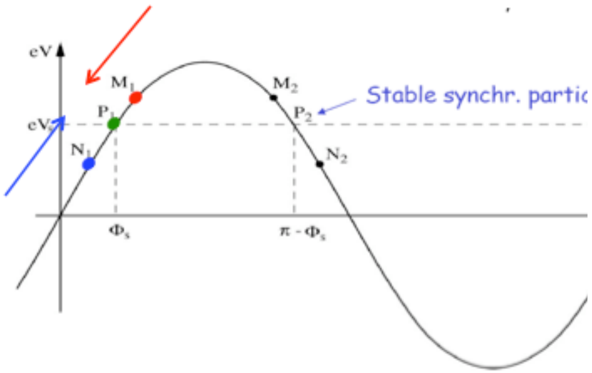
and the phase difference determines the rate of energy change per turn

$$\Delta\dot{E} = e*\frac{U_0}{T_0}\Delta\psi\cos\psi_s$$

differentiate a second time

2 $$\Delta\ddot{E} = e*\frac{U_0}{T_0}\Delta\dot{\psi}\cos\psi_s$$

Equation of Motion:



1 $$\Delta\dot{\psi} = \frac{\Delta\psi}{T_0} = \frac{h*2\pi}{\beta^2 T_0}\left(\alpha - \frac{1}{\gamma^2}\right)\frac{dE}{E}$$

2 $$\Delta\ddot{E} = e*\frac{U_0}{T_0}\Delta\dot{\psi}\cos\psi_s$$

***put** 1 **into** 2 **et c'est ca** Equation of Motion in Phase Space E-ψ:*

$$\Delta\ddot{E} = e*\frac{U_0}{T_0}\frac{2\pi h}{\beta^2 T_0}\left(\alpha - \frac{1}{\gamma^2}\right)\frac{dE}{E}\cos\psi_s$$

Definition:

$$\Omega = \omega_0 * \sqrt{\frac{-eU_0 h\cos\psi_s}{2\pi\beta^2 E}\left(\alpha - \frac{1}{\gamma^2}\right)}$$

$$\Delta\ddot{E} + \Omega^2\Delta E = 0$$

We get a differential equation that describes the difference in energy of a particle to the ideal (i.e. synchronous) particle under the influence of the phaes focusing effect of our sinusoidal RF function.

And it is a harmonic oscillation !!!

The oscillation frequency Ω is called synchrotron frequency and usually in the range of some Hz ... kHz.

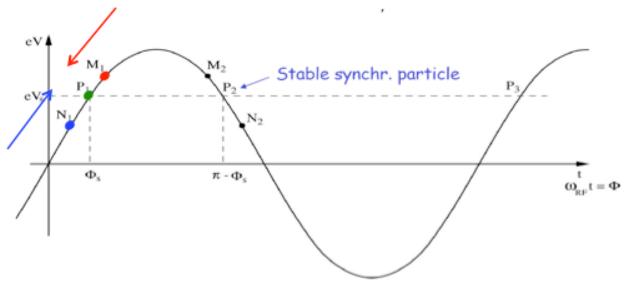
Small Amplitude Oscillations: phase stability

We get - in equivalent way - the harmonic oscillation of the particle phase with the oscillation frequency

$$\Delta\ddot{\psi} + \Omega_s^2 \Delta\psi = 0 \qquad \Omega_s = \omega_0 * \sqrt{\frac{eU_0 h \cos\psi_s}{2\pi\beta^2 E}\eta} \qquad \textit{remember} \quad \eta = \frac{1}{\gamma^2} - \alpha$$

Stability condition: $\mathbf{\Omega_s}$ real $\quad \Omega_s^2 > 0$

$\gamma < \gamma_{tr}$	$\eta > 0$	$0 < \phi_s < \pi/2$
$\gamma > \gamma_{tr}$	$\eta < 0$	$\pi/2 < \phi_s < \pi$

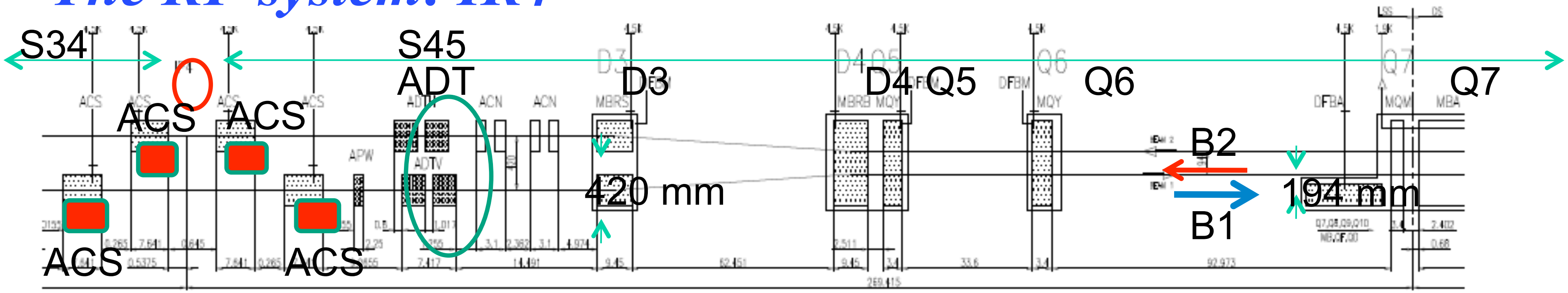


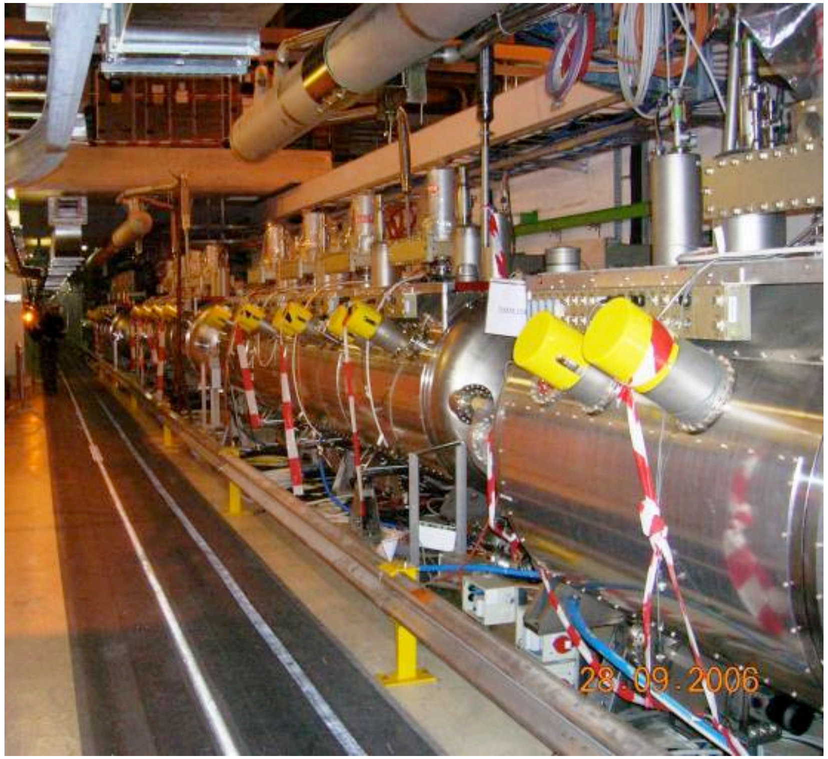
And we will find this situation "h"-times in the machine

LHC:
35640 Possible Bunch Positions ("buckets")
2808 Bunches

oscillation frequency depends on
- * ***the square root***
- * ***of an electrical potential***
- ***-> weak force <-> small frequncy***

The RF system: IR4



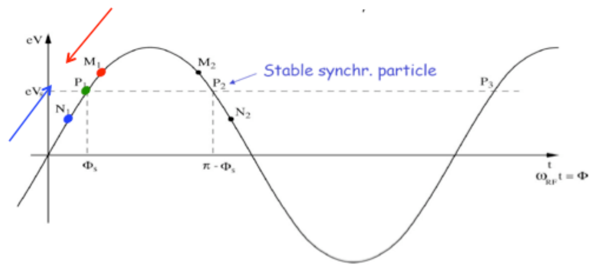


4xFour-cavity cryo module 400 MHz, 16 MV/beam
Nb on Cu cavities @4.5 K (=LEP2)
Beam pipe diam.=300mm

Bunch length (4σ)	*ns*	*1.06*
Energy spread (2σ)	*10^{-3}*	*0.22*
Synchr. rad. loss/turn	*keV*	*7*
Synchr. rad. power	*kW*	*3.6*
RF frequency	*MHz*	*400*
Harmonic number		*35640*
RF voltage/beam	*MV*	*16*
Energy gain/turn	*keV*	*485*
Synchrotron frequency	*Hz*	*23.0*

(small) ... Synchrotron Oscillations in Energy and Phase

$$\Delta\ddot{\psi} + \Omega_s^2 \Delta\psi = 0$$



Ansatz: $\Delta\psi = \Delta\psi_{\max} * \cos(\Omega_s t)$

take the first derivative and put it into the first energy-phase relation $\Delta\dot{\psi} = \frac{h * 2\pi}{\beta^2 T_0}\left(\alpha - \frac{1}{\gamma^2}\right)\frac{dE}{E}$

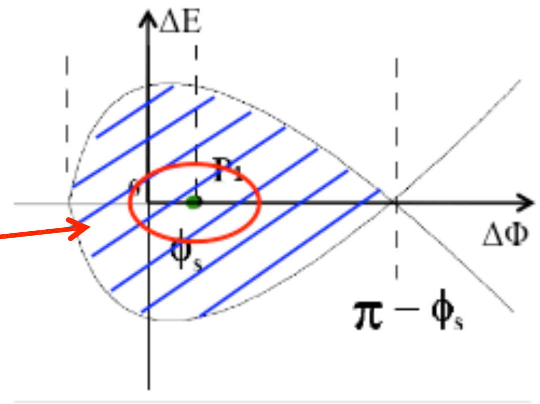
$$\frac{d(\Delta\psi)}{dt} = -\Delta\psi_{\max} * \sin(\Omega_s t) * \Omega_s$$

to get the energy oscillations $\Delta E = \underbrace{\frac{\beta^2 T_0 \Omega_s \Delta\psi_{\max} E_s}{2\pi h \eta}}_{\Delta E_{\max}} \sin(\Omega_s t)$

$$\Delta E = \Delta E_{\max} * \sin(\Omega_s t)$$

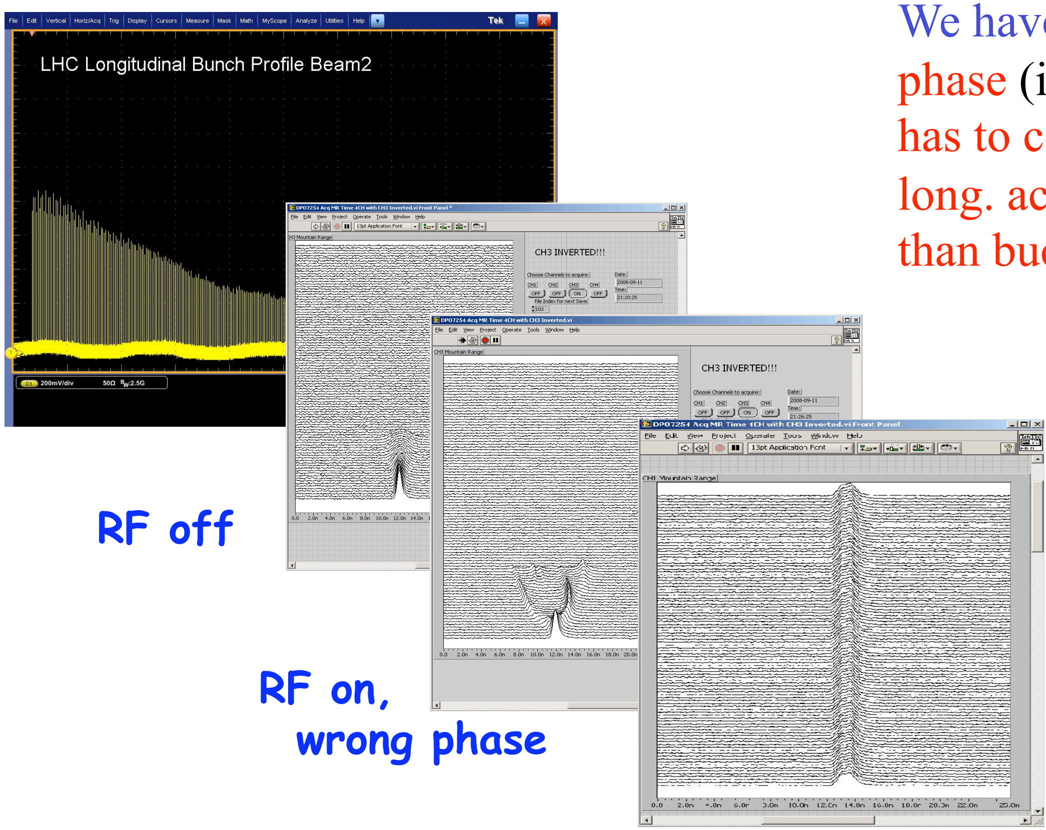
which defines an ellipse in phase space Δψ, ΔE:

$$\left(\frac{\Delta\psi}{\Delta\psi_{\max}}\right)^2 + \left(\frac{\Delta E}{\Delta E_{\max}}\right)^2 = 1$$



LHC Commissioning: RF

We have to match these conditions:
phase (i.e. timing between rf and injected bunch) has to correspond to ϕ_s
long. acceptance of injected beam has to be smaller than bucket area of the synchrotron.



RF off

RF on,
wrong phase

RF on, phase adjusted,
beam captured

max stable energy: set $\phi = \phi_s$ and calculate ΔE

$$(\Delta E_{\max})_{sep} = \sqrt{\frac{p_s v_s e U_0}{2\pi h \eta_s}} * \sqrt{\left|4\cos\phi_s - (2\pi - 4\phi_s)\sin\phi_s\right|}$$

LHC injection:
acceptance: 1.4eVs
long emittance: 1.0 eVs

Than'x

Appendix: Relativistic Relations

court. Chris Prior, Trinity College / CAS

	$\frac{\Delta\beta}{\beta}$	$\frac{\Delta p}{p}$	$\frac{\Delta T}{T}$	$\frac{\Delta E}{E} = \frac{\Delta\gamma}{\gamma}$
$\frac{\Delta\beta}{\beta} =$	$\frac{\Delta\beta}{\beta}$	$\frac{1}{\gamma^2}\frac{\Delta p}{p}$ $\frac{\Delta p}{p} - \frac{\Delta\gamma}{\gamma}$	$\frac{1}{\gamma(\gamma+1)}\frac{\Delta T}{T}$	$\frac{1}{\beta^2\gamma^2}\frac{\Delta\gamma}{\gamma}$ $\frac{1}{\gamma^2-1}\frac{\Delta\gamma}{\gamma}$
$\frac{\Delta p}{p} =$	$\gamma^2\frac{\Delta\beta}{\beta}$	$\frac{\Delta p}{p}$	$\frac{\gamma}{\gamma+1}\frac{\Delta T}{T}$	$\frac{1}{\beta^2}\frac{\Delta\gamma}{\gamma}$
$\frac{\Delta T}{T} =$	$\gamma(\gamma+1)\frac{\Delta\beta}{\beta}$	$\left(1+\frac{1}{\gamma}\right)\frac{\Delta p}{p}$	$\frac{\Delta T}{T}$	$\frac{\gamma}{\gamma-1}\frac{\Delta\gamma}{\gamma}$
$\frac{\Delta E}{E} =$	$(\beta\gamma)^2\frac{\Delta\beta}{\beta}$	$\beta^2\frac{\Delta p}{p}$	$\left(1-\frac{1}{\gamma}\right)\frac{\Delta T}{T}$	$\frac{\Delta\gamma}{\gamma}$
$\frac{\Delta\gamma}{\gamma} =$	$(\gamma^2-1)\frac{\Delta\beta}{\beta}$	$\frac{\Delta p}{p} - \frac{\Delta\beta}{\beta}$		