Nonlinear Dynamics - Methods and Tools

—

and a Contemporary (1985 $+$) framework

to treat Linear and Nonlinear Beam Dynamics

PART 1

Reminder: Dynamical systems (types)

- **Linear dynamics,**

  Highly ordered motion, deterministic, predictable
  well understood, provides important foundation

  Typical example (ideal pendulum): $$\frac{\partial^2 \Theta}{\partial t^2} = -\frac{g}{L}\Theta$$

- **Nonlinear dynamics,**

  Disorder and irregularities, deterministic, unpredictable, chaotic motion

  Typical example (nonlinear pendulum): $$\frac{\partial^2 \Theta}{\partial t^2} = -\frac{g}{L}\sin(\Theta)$$

  Gravitational system: $$m\frac{d^2 u}{dt^2} = -\frac{\alpha u}{|u|^3}$$

Glossary: Dynamical systems (classes)

- Continuous in time domain:

  Typically described by differential equations

  In general hard to solve, hard to analyse,

- Discrete in time domain:

  Typically described by difference equations, matrices, maps, simulation programs, measurements (!)

  Easier to extract relevant information, good tools exist (Poincare sections, ...)

  Visualization allows much better understanding than formulae

If you do not believe me: how do these countours look like ?

General linear case (rather easy to see):

$$f(x,y) \;=\; x^2 + y^2 \;=\; const.$$

Simple example nonlinear case (a lot harder):

$$f(x,y) \;=\; -a\cdot x - \frac{3}{8}\cdot a\cdot b\cdot(2cx)^{3/2}\cdot\left[\frac{\sin(3y+\frac{3a}{2})}{\sin(\frac{3a}{2})} - \frac{\sin(y+\frac{a}{2})}{\sin(\frac{a}{2})}\right] \;=\; const.$$

(a, b, c are constants)

Some basic issues:

- **Linear dynamics**
  - A closed solution often exists
  - Well established methods for solving the equations
  - Combination of two solutions is another solution

- **Nonlinear dynamics**
  - A closed solution usually does <u>not</u> exist
  - Traditional methods and analysis are problem dependent
  - Combination of two solutions is <u>not</u> a solution

When can we not use a linear approach ?

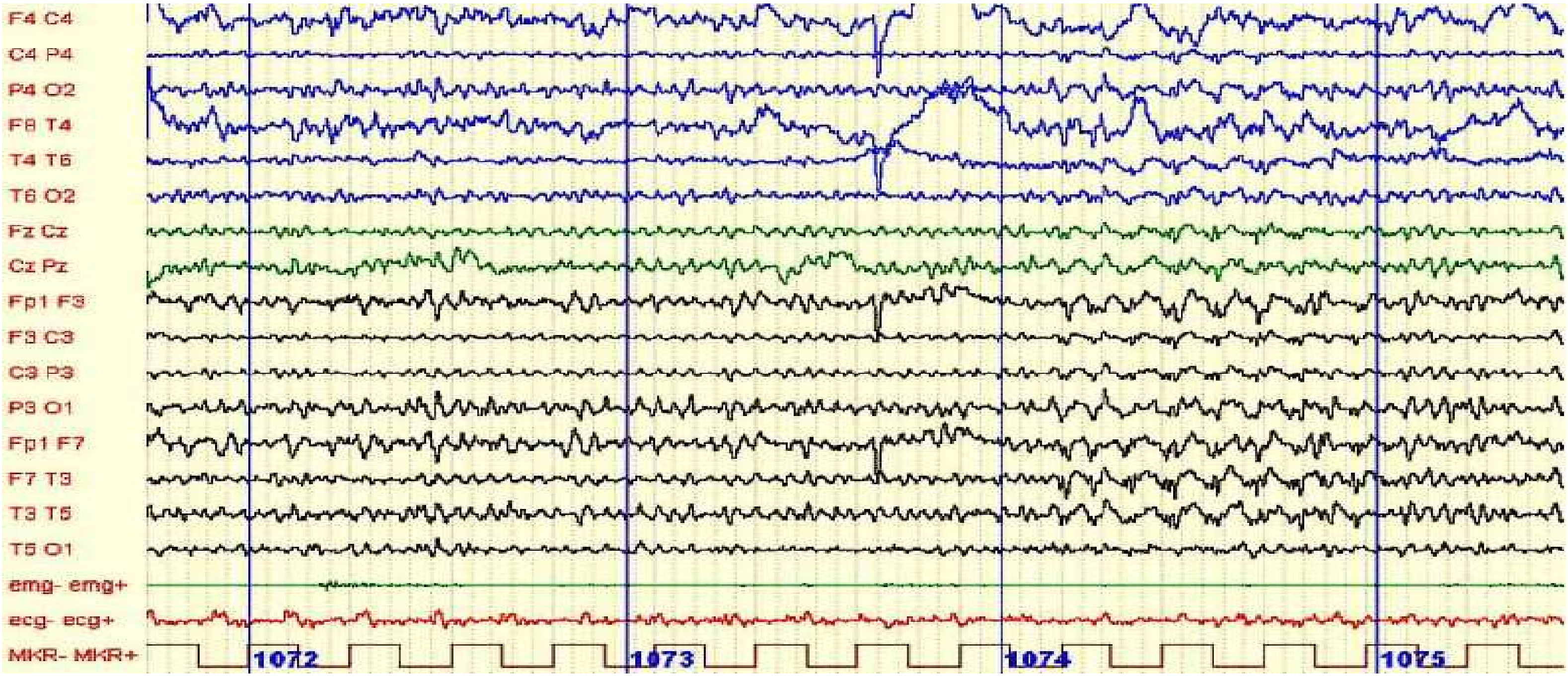
A few examples relevant for accelerators and driving forces for developments:

- **In the presence of nonlinear fields (e.g. single particle effects, non regular and chaotic motion, mostly non-integrable, ..)**
- **Problems with "small" machines, fringe fields**
- **In the presence of more than one particle (e.g. self fields, beam-beam*), ...)**

Good news: enormous progress in the last 30 years, gradual assimilation into our field: reset your beam dynamics, time warp from 1952 to 2014 summarized in 3 hours

*) **worst case scenario for large machine/storage rings**

A highly non-linear signal - can we get something out of it ?



- Modern tools (e.g. filtering, fractal analysis (in this case !), wavelets, variogram, ...) can get substantial information
- What is the situation for beam dynamics ?

Worse: Linear cases well described using heuristic (but effective) techniques, however insufficient to treat nonlinear systems

Consequence:
most introductory textbooks switch to "panic mode", nonlinear dynamics treated with some tinkered add-ons, often leading students to a dead end

This approach: show (at least some) basics of state-of-the-art:

- Introduce concepts that handle complex nonlinear cases and linear beam dynamics with the same general formalism

- Fully exploit computing capabilities

- Demonstrate features for "rings" (not a restriction for the arguments): but approximations used for linear systems most critical

Rely on many examples to illustrate the concepts and ideas, not on mathematical rigour. For more details, see [EF1]

Recommended Bibliography (and acknowledgement):

[AW] A. Wolski, Beam Dynamics in High Energy Particle Accelerators, Imperial College Press, 2014.

[AC1] A. Chao, Lecture Notes on Topics in Accelerator Physics, SLAC, 2001.
[WH] W. Herr, Mathematical and Numerical Methods for Nonlinear Dynamics in Proceedings of the CERN Accelerator School: Advanced Accelerator Physics, Trondheim, Norway, 18-29 August 2013, CERN-2014-009, pp. 157-198.

[EF1] E. Forest, Beam Dynamics - A New Attitude and Framework, Harwood Academic Publishers, 1998.
[EF2] E. Forest, From Tracking Code to Analysis, Springer, 2016.
[AD] A. Dragt, Lie Methods for Nonlinear Dynamics with Applications to Accelerator Physics
[MB] M. Berz, Modern Map Methods in Beam Physics, Academic Press, 1999.

Some general remarks:

Quite some overlap unavoidable, start slowly

Strictly follow conventions as defined in the syllabus

Demonstrated for transverse motion, but generally applicable

To save typing, slides, space and time:
where possible I use 1D for demonstration of the concepts,
but everything valid for more dimensions without change of formalism

⇨ **no need to assume uncoupled motion**

New concepts should only be introduced if they are <u>useful !</u>

→ Spend some time on the motivation !

→ Everyday applications instead of "textbook" examples ..

- Lecture 1
  - Key concepts, Linear and Nonlinear maps
  - Analysis Methods - linear Normal Forms
  - Symplectic integrators - vital tool for tracking codes
- Lecture 2
  - Lie operators and transformations
  - Analysis Methods - nonlinear Normal Forms
- Lecture 3
  - Use of Truncated Power Series Algebra, TPSA

Life time ..

There was a machine at CERN where a beam was regularly kept (1982, 1983) for $\approx$ 1 week, (max. up to 3 weeks)

→ **HERA and LHC are lousy mchines**

For what follows one should (!) always use canonical$^{*)}$ variables !

In Cartesian coordinates: $R \;=\; (X, P_X, Y, P_Y, Z, P_Z, t)$

If the energy is constant (i.e. P_Z **= const.), we use:** (X, P_X, Y, P_Y, Z, t)

This system is rather inconvenient, what we want is the description of the particle in the neighbourhood of the reference orbit/trajectory:

$$R_d \;=\; (X, P_X, Y, P_Y, Z, t)$$

which are considered now the deviations from the reference and which are zero for a particle on the reference trajectory

Very important: it is the <u>reference</u> not the <u>design</u> trajectory !

(so far it is a straight line along the Z-direction)

$^{*)}$ **see lecture by Yannis**

The independent variable is usually the time t (Newton)

Problem: particles with different initial conditions generally require different times to pass through an element. Better to measure the progress using a longitudinal coordinate Z.

We therefore replace time t by Z and eventually by s using:

$s = Z + ct \rightarrow \boxed{R = (X, P_X, Y, P_Y, s)}$ s is the distance along reference path

Non-trivial: strictly speaking requires the Hamiltonian formalism

$\rightarrow$ **using s is Hamiltonian in disguise ...**

For a "curved" trajectory, in general not circular, with a local radius of curvature $\rho(s)$ in the horizontal (X - Z plane), we transform to a new coordinate system (x, y, s) (co-moving frame) with (see e.g. [AW]):

$$X = (x+\rho)\cos\left(\frac{s}{\rho}\right) - \rho \qquad \text{(needed tomorrow)}$$

$$Y = y$$

$$Z = (x+\rho)\sin\left(\frac{s}{\rho}\right)$$

The new canonical momenta become:

$$p_x = P_X \cos\left(\frac{s}{\rho}\right) + P_Z \sin\left(\frac{s}{\rho}\right)$$

$$p_y = P_Y$$

$$p_s = P_Z\left(1 + \frac{x}{\rho}\right)\cos\left(\frac{s}{\rho}\right) - P_X\left(1 + \frac{x}{\rho}\right)\sin\left(\frac{s}{\rho}\right)$$

finally for the transverse coordinates: $\boxed{r = (x, p_x, y, p_y)}$

Some clarification (again):

F.A.Q.: Phase Space ($x, p_x, \ldots$) or Trace Space ($x, x', \ldots$)

It is <u>not</u> laziness nor stupidity to use one or the other:

- **Beam dynamics is strictly correct$^{*)}$ only with ($x, p_x, \ldots$), but in general quantities cannot be measured easily**
- **Beam dynamics with ($x, x', \ldots$) needs special precaution$^{**)}$, but quantities much easier to measure**
- **Some quantities are different (e.g. emittance)**

Be aware of that when you do the calculations ...

$^{*)}$ **Using ($x, x', \ldots$) implies that a particle can receive an infinite amount of transverse momentum (e.g. kick from a magnet) without changing its total energy !!**
$^{**)}$ **Strictly speaking, not valid in the presence of electromagnetic fields ...**

Usual starting point: Linear dynamics in synchrotrons

Each element at position s acts as a source of forces, i.e. we must write for the forces $K \Rightarrow K(s)$ (so long harmonic oscillator ..... !)

To justify the Courant-Snyder ansatz:
linear (uncoupled !) optics in rings often introduced using 1D*) Hill type equation where $K(s)$ is assumed to be a periodic function in s:

$$\frac{d^2x(s)}{ds^2} + \underbrace{\left(a_0 + 2\sum_{n=1}^{\infty} a_n \cdot \cos(2ns)\right)}_{K(s)} x(s) = 0 \qquad \text{and} \qquad \underbrace{K(s+C) \;=\; K(s)}_{ring...}$$

Solution of a Boundary Value Problem (rings !) must be periodic too !

Not applicable in the general case (e.g. Linacs, Beamlines, FFAG, Recirculators, ...), much better to treat it as an Initial Value Problem

*) **What about 2D ??**

As an Initial Value Problem - what follows immediately:

<u>First:</u> **For any linear, 1st order equation of the type**

$$\frac{dx(s)}{ds} = K(s)\, x(s) \qquad \text{(and initial values at } s_0\text{)}$$

the solution can always be written as (Floquet, Hamilton, e.g. [AD]):

$$\begin{aligned} x(s) &= a \cdot x(s_0) + b \cdot x'(s_0) \\ x'(s) &= c \cdot x(s_0) + d \cdot x'(s_0) \end{aligned} \Longrightarrow \begin{pmatrix} x \\ x' \end{pmatrix}_s = \overbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}^{A} \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$

(now $K(s)$ does <u>not</u> have to be periodic)

<u>Second:</u> The determinant of A is always 1

<u>Third:</u> No need for an "ansatz"

→ **Much better to use matrices for our <u>linear</u> systems from the start just have to know what is A between the locations s and s_0**

Real life: adding nonlinear elements (e.g. magnetic fields)

Nonlinear elements can be described by polynomials of higher order:

$$\boxed{\frac{d^2x(s)}{ds^2} + K(s)x(s) = \sum_{i,j\geq 0} p_{ij}(s)x^i y^j}$$

Electromagnetic fields can be described with the multipole expansion:

$$B_y + iB_x \;=\; \sum_{n=1} (b_n + ia_n)(x + iy)^{n-1}$$

(in LHC need up to $n \;=\; 20$ **!)**

Equations of motions become (here horizontal plane):

$$\boxed{\frac{d^2x(s)}{ds^2} + K(s)x(s) = \frac{F_x(x, y, s)}{v \times p} = -\frac{B_y(x, y, s)}{p}}$$

(Note: we have now coupling between the planes if $i \neq 0$ **and** $j \neq 0$ **!!)**

Some problems with this approach:

- **It is rather hopeless to describe a complicated system**
- **It is totally hopeless to find a closed solution**
- **Perturbation treatment required, but does not always give satisfactory results and does not fully exploit potential of computing techniques**
- **...**

Many concepts (more or less) valid in 1D become incorrect for 2D, leading to misconceptions and permanent damage ..

It's going DownHill: Different approach needed - invest in more powerful tools ...

The most reliable tools to study realistic models (i.e. description of the machine) are simulations (e.g. tracking codes)
a.k.a: No Computer, No (good) Beam

Particle Tracking:

.. a numerical solution of the (nonlinear) Initial Value Problem:
It is a "integrator" of the equation of motion
Vast amount of tracking codes available, many analysis tools available
(Examples: Lyapunov, Chirikov, chaos detection, frequency analysis, ...)

Dilemma:

Theoretical and computational tools exist side by side without an undertaking how they can be integrated

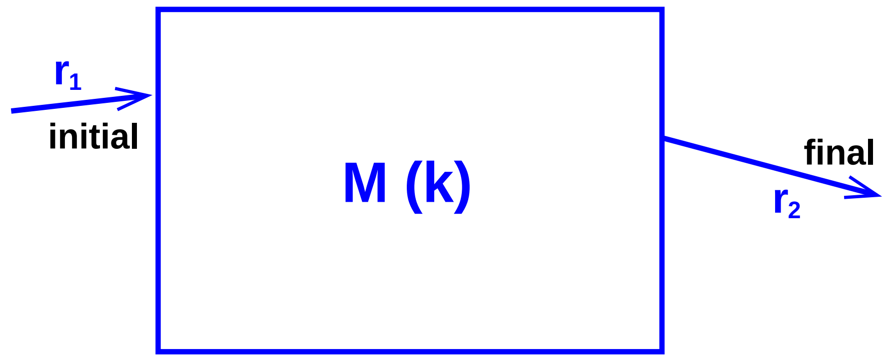
Ambition:

Find an approach to link simulations with theoretical analysis, would allow a better understanding of the physics in realistic machines

→ **Based on finite maps i.e. discrete systems**

Recap: what is a map ? (remember first week university)

A map M performs an operation on e.g. coordinates:

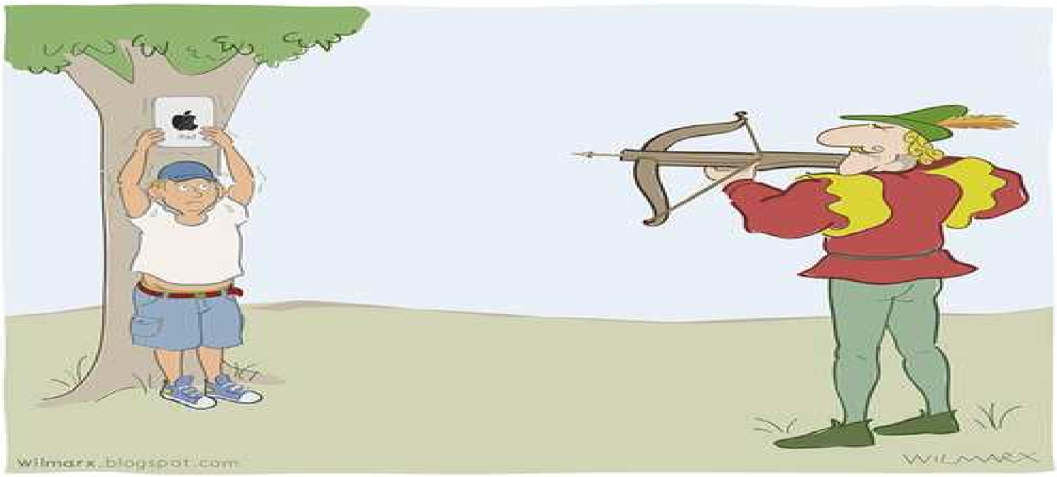


The map M depends on system parameters k (e.g. fields) and sends the initial to the final coordinates. Maps are more general than matrices which are restricted to linear systems.

What happens inside M is important only when it is constructed !!

There are different ways to do it ! (with the same result)

An every day example - a discrete system ...



- **Given initial conditions, the map would tell us where we end up**
- **Most important to know trajectory of the arrow/beam at <u>beginning</u> and <u>end of flight</u> (or any position of interest)**
- **Not important (and useless in this example) to know exact trajectory as function of <u>time</u> (speed is not the issue here ..)**
- **Overall timing not important (unless somebody shoots at you ..)**

If it cannot be measured - it is not worth the effort ...

- **We are interested in the motion at (discrete !) fixed locations where we can observe/measure it, not the full time evolution**
- **We should not invest in tools to get unnecessary and irrelevant details carried by approximate methods**
- **Relevant information for us: is the beam stable and confined and for how long, tune, beam size, α, β, γ, closed orbit, particle distribution, region of stability, a.k.a. Dynamical Aperture (DA), ...**

- **Map approach is the obvious choice**

  Maps are used by simulation codes and can be analysed

  Provides a link between simulations and analytical methods !

Look at the linear treatment first, then generalize to nonlinear theory

Linear optics was already treated in detail, I use the very basics to show the idea and demonstrate the transition

The procedure and formalism is identical

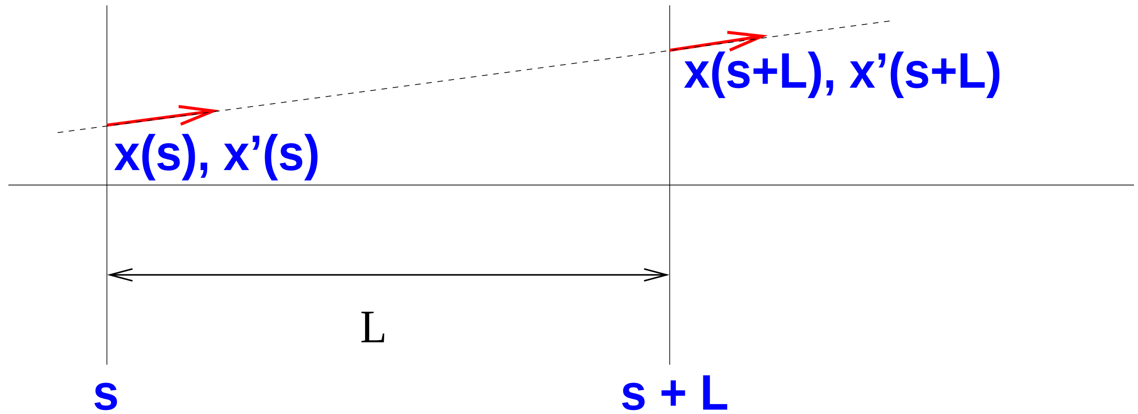
For consistency with some (classical) textbooks and other lectures I sometimes (where not critical) use x, x', y, y' **instead of** x, p_x, y, p_y

→ **Linear maps are usually written as <u>matrices</u>**

→ **Some simple (accelerator elements) examples**

First example of a map: (often derived based on intuition)

A drift space (one dimension only) of length L, starting at position s and ending at s + L



The simplest description (1D, using x, x') is (should be in 3D of course):

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s+L} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \circ \begin{pmatrix} x \\ x' \end{pmatrix}_{s} = \begin{pmatrix} x + x' \cdot L \\ x' \end{pmatrix}$$

This is only an approximation, something may go badly wrong, see later ... !

Another example (often "derived" assuming the solution):

Focusing quadrupole of length L and <u>constant</u> strength k_1 ($k_1 > 0$):

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_2} = \begin{pmatrix} \cos(L \cdot \sqrt{k_1}) & \frac{1}{\sqrt{k_1}} \cdot \sin(L \cdot \sqrt{k_1}) \\ -\sqrt{k_1} \cdot \sin(L \cdot \sqrt{k_1}) & \cos(L \cdot \sqrt{k_1}) \end{pmatrix} \circ \begin{pmatrix} x \\ x' \end{pmatrix}_{s_1}$$

similar for a defocusing quadrupole, i.e. for $k_1 < 0$

(it is the solution of $\dfrac{d^2 x(s)}{ds^2} = K(s)\, x(s)$ **when** $K(s) = k_1 = const.$**)**

However: fundamental for the map approach ➡

Can we get the maps:

1. For all elements, including nonlinear (e.g. sextupoles) where no solution exists ?

2. From first principles (i.e. fields), without reference to their use ?
(a particle does not know what the element is (supposed) to do ?)

What's the point ?

- **Primary Objects in an accelerator: e.g. magnets, drifts. They know nothing about the layout and their purpose, the formulation of the maps must not depend on concepts like closed orbit, tune etc.**

  e.g. ALS (LBNL) had no dipoles, "bending magnets" were shifted quadrupoles (non trivial change of coordinates) !

  e.g. SPS (CERN) used sextupoles, to control tunes in the collider !

- **Coordinates: magnets as such best described using Cartesian coordinates, (a rectangular magnet contains nonlinear terms in some other coordinate systems). A reasonable description of fringe fields only in Cartesian coordinates.**

→ **Traditional (in particular "advanced") dynamics treat the objects as <u>localized fluctuations</u> in an s-dependent function. Better: To study a machine (synchrotron, beam line, linac, ...) , take the magnets as they are and build a mathematical structure ("maps").**

Next: more than one element in the machine

Mechanically, a ring or beam line is a finite collection of elements
Mathematically, a ring or beam line is a finite collection of maps

- Combine the maps of all elements together
- The new map represents a bigger part of the machine
- In a ring, the elements can be lumped at one location: we obtain a "one-turn-map"

For simplicity try a combination of matrices first:
(matrices are the simplest form of a map, but limited to linear systems)

Starting from a position s_0 and combining all matrices to get the matrix to position $s_0 + L$ (shown for 1D only):

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_0 + L} = \underbrace{\mathcal{M}_N \circ \mathcal{M}_{N-1} \circ \ldots \circ \mathcal{M}_1}_{\mathcal{M}(s_0, L)} \circ \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$

For a ring with circumference C we get the One-Turn-Matrix (OTM) at s_0

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_0 + C} = \underbrace{\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}}_{\mathcal{M}_{OTM}} \circ \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$

Without proof (trust me for a few minutes), the scalar product:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_0} \cdot \mathcal{M}_{OTM} \cdot \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0} = \text{const.} = \text{J}$$

is a constant of the motion: invariant of the One-Turn-Map $\mathcal{M}_{OTM}$

Now:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_0} \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0} = J$$

always describes an ellipse (and J is its area):

$$m_{11} \cdot x^2 \;+\; (m_{12} + m_{21}) \cdot xx' \;+\; m_{22} \cdot x'^2 \;=\; J$$

true for any linear, iterative system, no assumptions or intuition needed

$\mathcal{M}_{OTM}$ **has all information what happens to the particles in one turn**
Need analysis tools to extract this information

The key: matrices can be transformed into Normal Forms

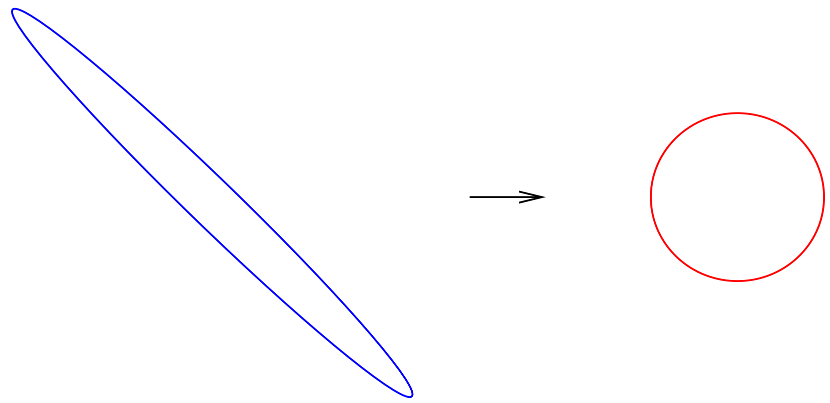
Starting with our One-Turn-Matrix $\mathcal{M}$, we try$^{*)}$ to find a (invertible) transformation $\mathcal{A}$ such that (called "similarity transformation"):

$$\boxed{\mathcal{A}\mathcal{M}\mathcal{A}^{-1} = \mathcal{R} \qquad (\text{or}: \qquad \mathcal{A}^{-1}\mathcal{R}\mathcal{A} = \mathcal{M})}$$

- **The matrix $\mathcal{R}$ is:**
  - **A "Normal Form", (or at least a very simplified form of the matrix)**
  - **Example (most important case): $\mathcal{R}$ becomes a pure rotation**
- **The matrix $\mathcal{R}$ describes the same dynamics as $\mathcal{M}$, but:**
  - **All coordinates are transformed by $\mathcal{A}$**
  - **This transformation $\mathcal{A}$ "analyses" the complexity of the motion, it contains the structure of the phase space**

$^{*)}$ Do. Or do not ! There is no try ! (it is always possible !)

Transformation to Normal Form (pictorially)



$$\mathcal{M} = \mathcal{A} \circ \mathcal{R} \circ \mathcal{A}^{-1} \quad \text{or :} \quad \mathcal{R} = \mathcal{A}^{-1} \circ \mathcal{M} \circ \mathcal{A}$$

Motion on an ellipse becomes motion on a circle (i.e. a rotation):
$\mathcal{R}$ **is the "simple" part of the map - shape is "dumped" into** $\mathcal{A}$

How to get that (i.e. $\mathcal{A}$**) ? Remember lectures on Linear Algebra (Eigenvectors, Eigenvalues ...), see also backup slides**

We find the two components of the original map:

$$\mathcal{A} = \begin{pmatrix} \sqrt{\beta(s_0)} & 0 \\ -\frac{\alpha(s_0)}{\sqrt{\beta(s_0)}} & \frac{1}{\sqrt{\beta(s_0)}} \end{pmatrix} \quad \text{and} \quad \mathcal{R} = \begin{pmatrix} \cos(\mu_x) & \sin(\mu_x) \\ -\sin(\mu_x) & \cos(\mu_x) \end{pmatrix}$$

The Normal Form transformation gives plenty of information:

- **We have stable oscillations when the eigenvalues μ_x (and μ_y etc.) are real, (forget about the $Tr(\mathcal{M}) \leq 2$ business). This concept is valid also for 2D or any complicated systems, e.g. coherent motion with 6000×6000 matrices etc: many modes !**
- μ_x **is the "tune"** $Q_x \cdot 2\pi$ **(now we can talk about <u>phase advance</u> !)**
- $\beta, \alpha, \ldots$ **are the optical parameters and describe the ellipse**
- **The closed orbit (an invariant, identical coordinates after one turn !):**

$$\mathcal{M}_{OTM} \circ (x,\ x')_{co} \equiv (x,\ x')_{co}$$

Note 1:

- The **only** assumption was that particles make more than one turn !!!
  No ansatz or any kind of other folklore needed

- Matrices $\mathcal{R}$ and $\mathcal{M}$ are called **similar** (i.e. have the same eigenvalues)
  (**to be equivalent** is not sufficient !)

Note 2:

in 2 dimensions the normal form is a 4×4 matrix:

$$\mathcal{R}^{2D} = \begin{pmatrix} \cos(\mu_x) & \sin(\mu_x) & 0 & 0 \\ -\sin(\mu_x) & \cos(\mu_x) & 0 & 0 \\ 0 & 0 & \cos(\mu_y) & \sin(\mu_y) \\ 0 & 0 & -\sin(\mu_y) & \cos(\mu_y) \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

What if the two planes (oscillators) are linearly coupled ?

Assume a one-turn-matrix in 2D (4 $\times$ 4 matrix):

$$R = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \quad \overset{\overbrace{\hphantom{xx}}^{coupling!}}{\Rightarrow} \quad T = \begin{pmatrix} M & n \\ m & N \end{pmatrix}$$

M,m,N,n are 2-by-2 block matrices.

In case of coupling: $m \neq 0, n \neq 0$ **we can try to transform as:**

$$T = \begin{pmatrix} M & n \\ m & N \end{pmatrix} = VR'V^{-1}$$

with (same procedure as before, find the simple case):

$$R' = \begin{pmatrix} A' & 0 \\ 0 & B' \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} \gamma I & C \\ -C^t & \gamma I \end{pmatrix}$$

What have we obtained ?

The matrix R' is our simple rotation, now in 2D:

- A' **and** B' **are the separate 1D rotations for the "normal modes" (what Rhodri was talking about ..)**
  Frequencies in A' **and** B' **are not the ones in** A **and** B
- **The matrix** V **transforms from the coordinates** (x, x', y, y') **into the "normal mode" coordinates** (w, w', v, v') **via the expression:**

  $(x, x', y, y') = V(w, w', v, v')$
- **The matrix** C **contains the "coupling coefficients"**

Coupling is included and handled in straightforward way with this approach !

Correction: add skew quadrupoles and you get a parametric dependence to "diagonalize"

A short comparison of the different approaches (not rigorous)

- Classical perturbation method:
  - Transform/expand <u>solution</u> in terms of distortion parameter
  - Analytical/symbolic expression for the solution
  - Solution is approximate (eigenvalues inexact, not always useful)
- Map/Normal Form approach:
  - Transform <u>Differential Equation</u> in terms of distortion parameter (Normal Form) to get an equation that can be solved
  - No symbolic expression for the solution
  - Requires some approximation of the model
  - Solutions/eigenvalues are exact

Using the map/Normal Form approach we get an exact solution at the expense of giving up a closed analytical form for the solution

What to choose (assuming we have a good computer) ??

Can we learn something from celestial mechanics (beyond Hill) ?

Not the same hardware, but both are interested in long term behaviour

→ **Numerical studies of Hamiltonian maps save the day ...**

Motion of planets is chaotic, but is it certain that it is not an artifact of the numerical integration (finite number of bits on computers) ?

For a few Myr: **numerical integration gives reliable results (i.e. GPS satellites are o.k.)**

For a few Gyr: **new approach necessary: M.Duncan et.al.,**
"The Long-Term behaviour of Orbits in the Solar System - A Mapping Approach"**, (1989)**

Even if not exact, analytical maps do not have truncation errors and can give a qualitatively correct picture (the physics) for $t \Longrightarrow \infty$!

For the very enthusiastic see also: Exact Integration using Integer Maps

TAKE

AWAY

- Exact model but approximate solution can fabricate non-existing features and conceal important underlying physics (unwanted)
- Exact solution but approximate model may give some inaccuracy, but get the physics right. Methods exist to evaluate and improve the predictions (preferred)

Impact on a key concept:

A central question in accelerator theory is to find, understand and quantify invariants:

A property of a system that is unchanged, i.e. conserved as the system evolves (typical examples can be: energy, momentum, angular momentum, charge, ..)

Given a map $\mathcal{M}$ we look for ζ with

$$\boxed{\mathcal{M}\,\zeta \; = \; \zeta}$$

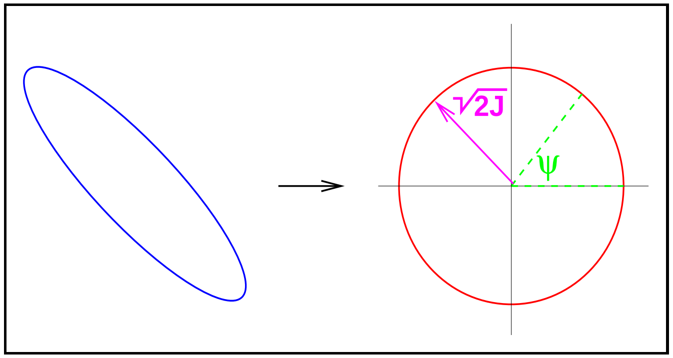
Examples (energy):

Exact solution, approximate model ⇨ energy may be slightly wrong

Exact model, approximate solution ⇨ energy may not be conserved ...

More appropriate for studies: using Action - Angle variables

Once the particles "travel" on a circle (i.e. always !), the motion is better described by the canonical variables action J_x and angle Ψ_x:



$$x = \sqrt{2J_x\beta_x}\ \cos(\Psi_x)$$

$$p_x = -\sqrt{\frac{2J_x}{\beta_x}}\ (\sin(\Psi_x) + \alpha_x\cos(\Psi_x))$$

$$J_x = \tfrac{1}{2}(\gamma_x x^2 + 2\alpha_x x p_x + \beta_x p_x^2)\ ^{*)}$$

- **Angular position along the ring Ψ becomes the independent variable !**
- **The trajectory of a particle is now <u>independent</u> of the position s !**
- **Constant Radius $\sqrt{2J}$ $\xrightarrow{defines}$ action J (invariant of motion)**

$*)$ **Never call that "emittance", this is brain clobbering !**

Interlude: If we have many particles, action is related to beam emittance (this is valid also for sources, electrons, linacs and beam lines, and non-Gaussian beams, see also recap by Hermann !):

If we can measure $< x^2 >$, $< p_x^2 >$ and $< xp_x >$ of a beam, and define a beam emittance ϵ_x (see e.g. [AW, AC2], also CERN convention):

$$\boxed{\epsilon_x \;=\; < J_x >}$$

this means:

$$\epsilon_x \;=\; \sqrt{< x^2 >< p_x^2 > \;-\; < xp_x >^2}$$

We can use action-angle variables defined before as:

$$x \;=\; \sqrt{2J_x\beta_x}\;\; \cos(\Psi_x) \qquad\qquad p_x \;=\; -\sqrt{\frac{2J_x}{\beta_x}}\;\; (\sin(\Psi_x) + \alpha_x \cos(\Psi_x))$$

and from above we get (Ψ disappears by the averaging)

$$< x^2 > \;=\; \beta_x\epsilon_x, \qquad < xp_x > \;=\; -\alpha_x\epsilon_x, \qquad < p_x^2 > \;=\; \gamma_x\epsilon_x$$

What about the amplitude function β ?

Using the above expression

$$x = \sqrt{2 J_x \beta_x(s)} \;\; \cos(\Psi_x(s))$$

and (see earlier lectures):

$$\Psi(s) = \int_{s_0}^{s} \frac{\mathrm{d}\tau}{\beta_x(\tau)}$$

plugging that into Hill's equation, we get:

$$\frac{d^2 \sqrt{\beta_x}}{ds^2} + K(s) \sqrt{\beta_x} - \frac{1}{\beta_x^{3/2}} = 0$$

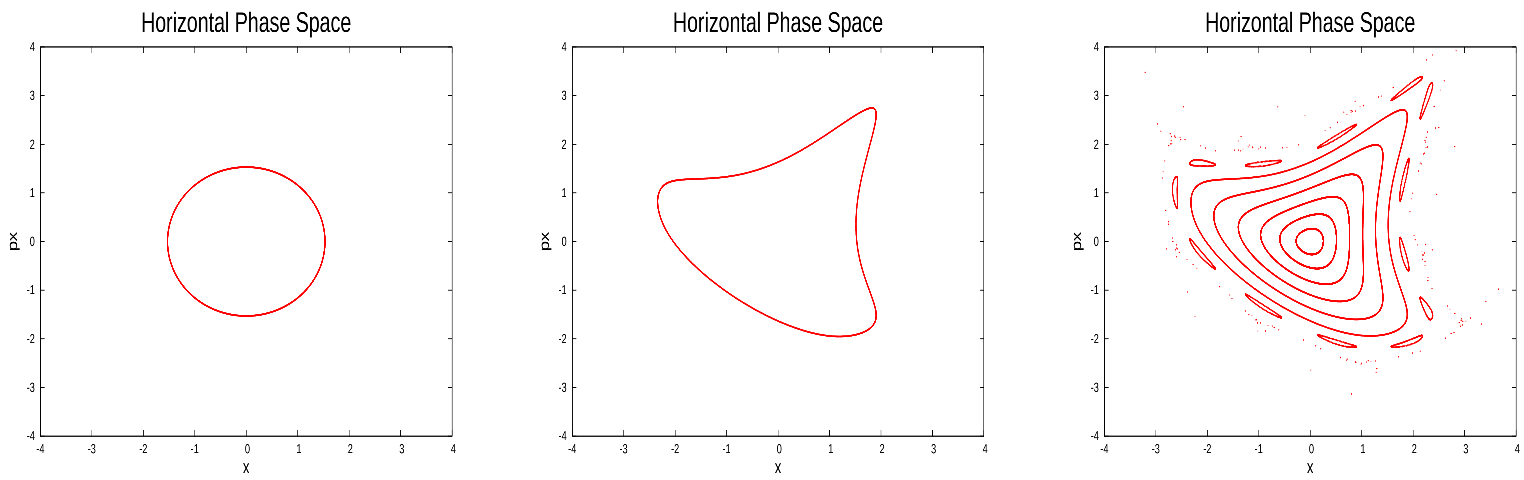
i.e. a nonlinear Differential Equation for $\beta(s)$ **!**

No solution to this equation, always need numerical integration

⇨ **requires use of maps (and computers) !**

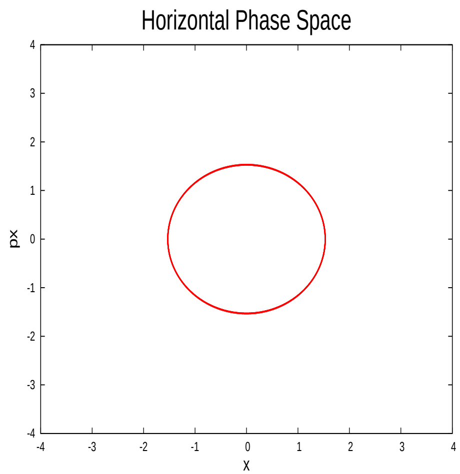
What do nonlinearities do to our particles (in phase space) ?

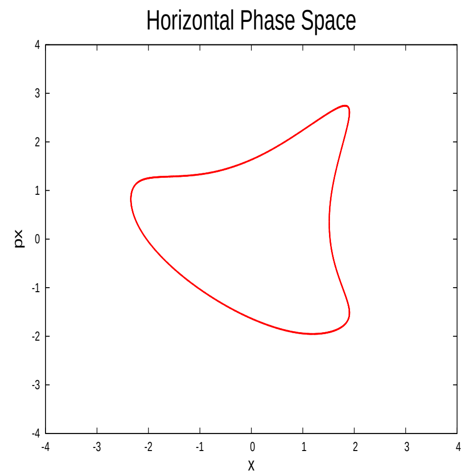
Linear motion traces only <u>circles</u> in <u>normalized</u> phase space

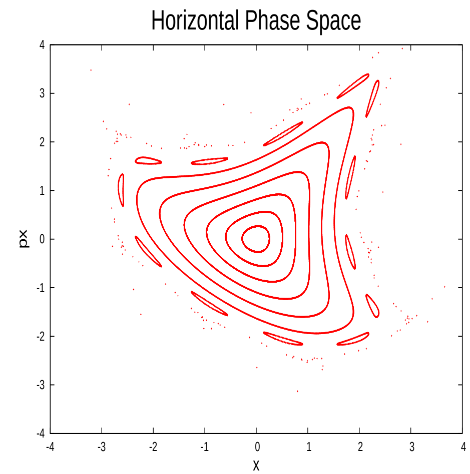


- **Here a sextupole, away and close to a (13th order) resonance**
- **Qualitatively and Quantitatively very different**
- **We find circles distorted, islands, irregular motion, chaos**
- **Transformation to a "simple" form not possible with the tools we know so far**

Side note:



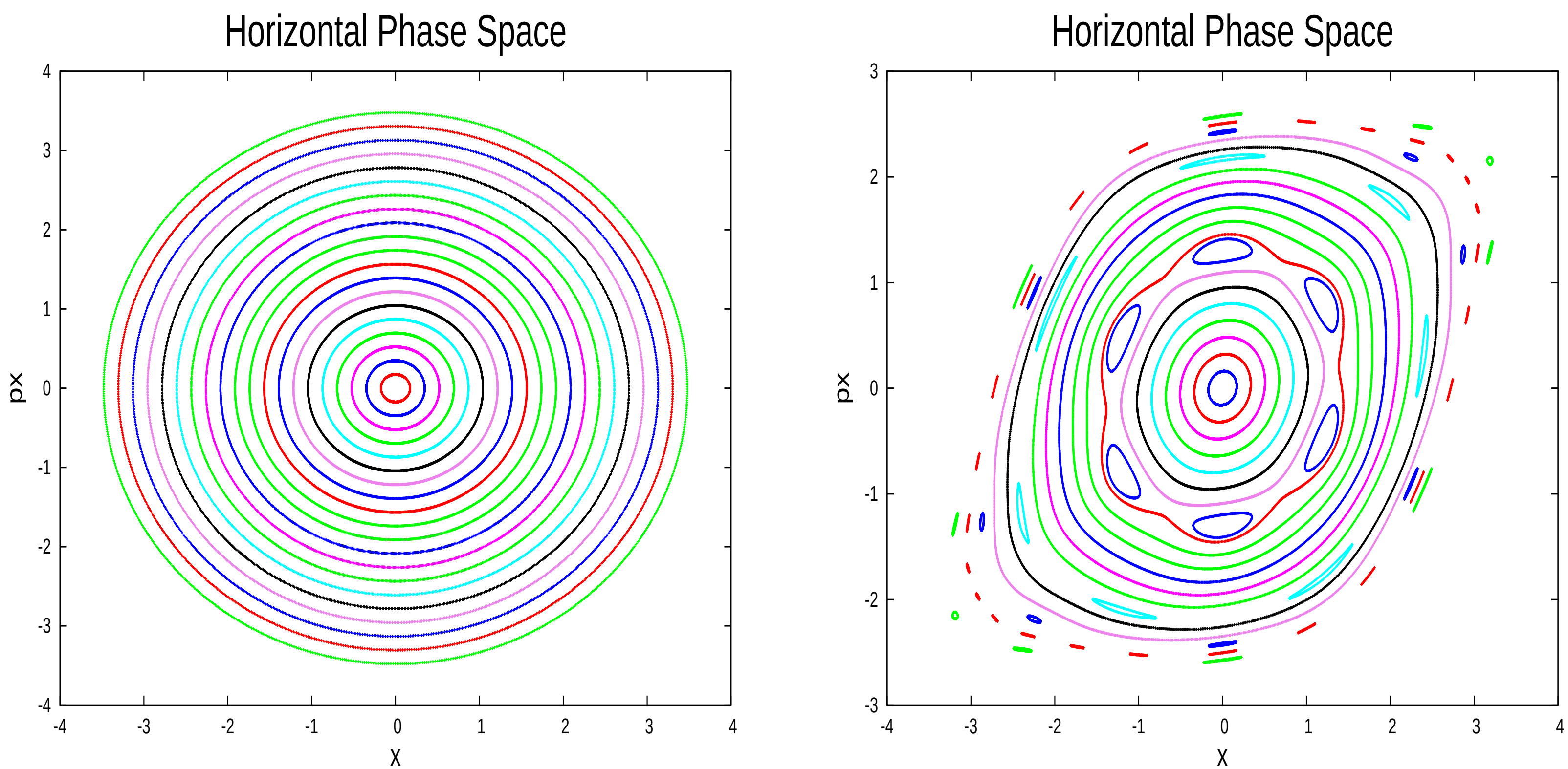




The figure in the centre is described by (1D):

$$f(J,\Psi) \;=\; -\mu\cdot J - \frac{3}{8}\cdot\mu\cdot k_2\cdot(2\beta J)^{3/2}\cdot\left[\frac{\sin(3\Psi+\frac{3\mu}{2})}{\sin(\frac{3\mu}{2})} - \frac{\sin(\Psi+\frac{\mu}{2})}{\sin(\frac{\mu}{2})}\right] \;=\; const.$$

For 2D: no space on a slide, see e.g. [AC1]



- Here a beam-beam interaction, many resonances (6th, 7th, 8th, 10th, 13th, 26th, ..) seen ...
- Some look like circles ! Can we transform them ??
- If yes: we can understand the system, let’s go ..

The general philosophy (linear to nonlinear systems):

- Describe the elements by a linear map
- Combine all maps into a ring or beam line to get the linear one turn matrix
- Normal form analysis of the linear one turn matrix will give all the information

$\Downarrow$ the hope $\Downarrow$

- Describe the elements by a nonlinear map
- Combine all maps into a ring or beam line to get the nonlinear one turn map
- Normal form analysis of the nonlinear one turn map will give all the information

all these require new techniques and methods ..

Various types of nonlinear maps

- Choice depends on the application, some examples:
  - Taylor (Power) maps
  - Lie transformations
  - Truncated Power Series Algebra (TPSA), can also generate maps from tracking
- Not all maps are allowed !
  - Key concept: Symplecticity most relevant for rings !

A symplectic matrix $\mathcal{M}$ has to fulfil the condition:

$$\mathcal{M}^T \cdot \mathcal{S} \cdot \mathcal{M} = \mathcal{S} \qquad \text{with} \qquad \mathcal{S} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$\lim_{n\to\infty} \mathcal{M}^n = \text{finite} \quad \Longrightarrow \quad \text{requires} \quad \det \mathcal{M} = 1^{*)}$$

1. $\mathcal{M}$ is <u>area preserving (x, p)</u> and J is an <u>invariant</u>:

$$\boxed{\mathcal{M}\; J = J}$$

2. <u>All</u> eigenvalues of $\mathcal{M}$ are <u>non-zero</u> and it is <u>invertible</u>

3. Products of symplectic matrices are symplectic

*) **(But note:** $\det \mathcal{M} = 1$ **alone is <u>not</u> sufficient)**

Introducing nonlinear elements (e.g. 2nd order)

Effect of a sextupole-like element with strength k_2 is (normal component):

$$\begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_{s_2} = \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_{s_1} + \begin{pmatrix} 0 \\ -\frac{1}{2} k_2 L \cdot (x_{s_1}^2 - y_{s_1}^2) \\ 0 \\ \frac{1}{2} k_2 L \cdot (x_{s_1} \cdot y_{s_1}) \end{pmatrix}$$

- ➡ **Amplitudes appear as second power**
- ➡ **(Normally) Cannot be written as a matrix**

We need something like (here for x-coordinate), i.e. Power Series:

$$x_{new} = \overbrace{R_{11} \cdot x + R_{12} \cdot x' + R_{21} \cdot y + R_{22} \cdot y'}^{\text{matrix part (power 1)}} +$$

$$\overbrace{+T_{111} \cdot x^2 + T_{112} \cdot xx' + T_{122} \cdot x'^2 + T_{113} \cdot xy + T_{114} \cdot xy' + ...}^{\text{sextupole part (power 2)}}$$

$$\overbrace{+U_{1111} \cdot x^3 + U_{1112} \cdot x^2 x' + ...}^{\text{octupole part (power 3)}}$$

$$+.......$$

and the equivalent for $x'_{new}, y_{new}, y'_{new}$ and higher orders

<u>Note:</u> for sextupoles and higher we have coupling terms $x^n y^m$, etc.

Normally, because one could write it as (1D, horizontal plane only):

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{new} = \begin{pmatrix} R_{11} & R_{12} & T_{111} & T_{112} & T_{122} \\ R_{21} & R_{22} & T_{211} & T_{212} & T_{222} \end{pmatrix} \circ \begin{pmatrix} x \\ x' \\ x^2 \\ xx' \\ x'^2 \end{pmatrix}$$

Just a fake, looks good but does not win anything ..

Easier to implement as (here up to 3rd order):

$$z_j^{new} = \sum_{k=1}^{6} R_{jk} z_k + \sum_{k=1}^{6} \sum_{l=1}^{6} T_{jkl} z_k z_l + \sum_{k=1}^{6} \sum_{l=1}^{6} \sum_{m=1}^{6} U_{jklm} z_k z_l z_m \quad \text{for} \quad j = 1..6$$

Explicit map (2D) for a sextupole with length L and strength k_2:

$$
\begin{aligned}
x_2 &= x_1 + L x'_1 &&- k_2 \left(\frac{L^2}{4}(x_1^2 - y_1^2) + \frac{L^3}{12}(x_1 x'_1 - y_1 y'_1) + \frac{L^4}{24}(x_1'^2 - y_1'^2) \right) \\
x'_2 &= x'_1 &&- k_2 \left(\frac{L}{2}(x_1^2 - y_1^2) + \frac{L^2}{4}(x_1 x'_1 - y_1 y'_1) + \frac{L^3}{6}(x_1'^2 - y_1'^2) \right) \\
y_2 &= y_1 + L y'_1 &&+ k_2 \left(\frac{L^2}{4} x_1 y_1 + \frac{L^3}{12}(x_1 y'_1 + y_1 x'_1) + \frac{L^4}{24}(x'_1 y'_1) \right) \\
y'_2 &= y'_1 &&+ k_2 \left(\frac{L}{2} x_1 y_1 + \frac{L^2}{4}(x_1 y'_1 + y_1 x'_1) + \frac{L^3}{6}(x'_1 y'_1) \right)
\end{aligned}
$$

- **Can this be used in this form ?**
- **This is not a matrix - what about the "symplectic" condition ? How to test it ?**

(if bored: find T_{234}, T_{324} ...)

➡ **It is the associated Jacobian matrix $\mathcal{J}$ (all 1st order partial derivatives) which must fulfil the symplecticity condition:**

$$\mathcal{J}_{ik} = \frac{\partial z_2^i}{\partial z_1^k} \quad \left(\text{e.g. } \mathcal{J}_{xy} = \frac{\partial x_2}{\partial y_1}\right)$$

$\mathcal{J}$ **must fulfil:** $\mathcal{J}^T \cdot \mathcal{S} \cdot \mathcal{J} = \mathcal{S}$

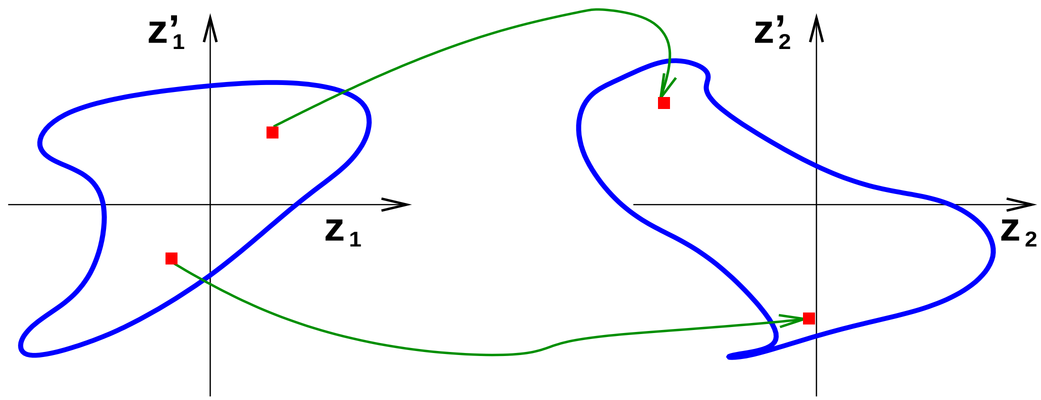
The coordinate z and the phase space dimension can be <u>very high</u> order:
<u>(number of particles) · (number of degrees of freedom)</u>

(LHC $\approx 10^{15}$, in most of my examples n = 4)

$$z_2^n = \mathcal{M}\, z_1^n$$

... an interesting consequence ⇨

Transformation of the occupied phase space



$$V_2 \;=\; \int_{V_2} d^n z_2 \;=\; \int_{V_1} \left| \frac{\partial z_2}{\partial z_1} \right| d^n z_1 \;=\; \int_{V_1} |\mathcal{M}|\, d^n z_1 \;=\; \int_{V_1} d^n z_1 \;=\; V_1$$

Under symplectic transformations ⇨ phase space volume is conserved !!

This is Liouville's theorem (see also recap by Hermann) !!

(not to be mistaken for Poincare invariant: $\int p \cdot dq \;=\; const.$ **)**

There is also a problem, I said:

$$\mathcal{J}_{ik} = \frac{\partial z_2^i}{\partial z_1^k} \quad \left(\text{e.g. } \mathcal{J}_{xy} = \frac{\partial x_2}{\partial y_1} \right)$$

$\mathcal{J}$ **must fulfil:** $\mathcal{J}^T \cdot S \cdot \mathcal{J} = \mathcal{S}$

In general: $\mathcal{J}_{ik} \neq$ **const (i.e. depend on** $x_1, x_1', ...$**)**

Confusing ?? o.k. → example sextupoles

$$
\begin{aligned}
x_2 &= x_1 + Lx_1' && - k_2\left(\tfrac{L^2}{4}(x_1^2 - y_1^2) + \tfrac{L^3}{12}(x_1x_1' - y_1y_1') + \tfrac{L^4}{24}(x_1'^2 - y_1'^2)\right) \\
x_2' &= x_1' && - k_2\left(\tfrac{L}{2}(x_1^2 - y_1^2) + \tfrac{L^2}{4}(x_1x_1' - y_1y_1') + \tfrac{L^3}{6}(x_1'^2 - y_1'^2)\right) \\
y_2 &= y_1 + Ly_1' && + k_2\left(\tfrac{L^2}{4}x_1y_1 + \tfrac{L^3}{12}(x_1y_1' + y_1x_1') + \tfrac{L^4}{24}(x_1'y_1')\right) \\
y_2' &= y_1' && + k_2\left(\tfrac{L}{2}x_1y_1 + \tfrac{L^2}{4}(x_1y_1' + y_1x_1') + \tfrac{L^3}{6}(x_1'y_1')\right)
\end{aligned}
$$

$$
\mathcal{J}_{ik} = \begin{pmatrix}
\frac{\partial x_2}{\partial x_1} & \frac{\partial x_2}{\partial x_1'} & \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_1'} \\
\frac{\partial x_2'}{\partial x_1} & \frac{\partial x_2'}{\partial x_1'} & \frac{\partial x_2'}{\partial y_1} & \frac{\partial x_2'}{\partial y_1'} \\
\frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_1'} & \frac{\partial y_2}{\partial y_1} & \frac{\partial y_2}{\partial y_1'} \\
\frac{\partial y_2'}{\partial x_1} & \frac{\partial y_2'}{\partial x_1'} & \frac{\partial y_2'}{\partial y_1} & \frac{\partial y_2'}{\partial y_1'}
\end{pmatrix}
\xrightarrow{k_2=0}
\begin{pmatrix}
1 & L & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & L \\
0 & 0 & 0 & 1
\end{pmatrix}
$$

$$
\begin{aligned}
x_2 &= x_1 + Lx_1' &&- k_2\left(\tfrac{L^2}{4}(x_1^2 - y_1^2) + \tfrac{L^3}{12}(x_1x_1' - y_1y_1') + \tfrac{L^4}{24}(x_1'^2 - y_1'^2)\right)\\
x_2' &= x_1' &&- k_2\left(\tfrac{L}{2}(x_1^2 - y_1^2) + \tfrac{L^2}{4}(x_1x_1' - y_1y_1') + \tfrac{L^3}{6}(x_1'^2 - y_1'^2)\right)\\
y_2 &= y_1 + Ly_1' &&+ k_2\left(\tfrac{L^2}{4}x_1y_1 + \tfrac{L^3}{12}(x_1y_1' + y_1x_1') + \tfrac{L^4}{24}(x_1'y_1')\right)\\
y_2' &= y_1' &&+ k_2\left(\tfrac{L}{2}x_1y_1 + \tfrac{L^2}{4}(x_1y_1' + y_1x_1') + \tfrac{L^3}{6}(x_1'y_1')\right)
\end{aligned}
$$

$$
\mathcal{J}_{ik} = \begin{pmatrix}
\frac{\partial x_2}{\partial x_1} & \frac{\partial x_2}{\partial x_1'} & \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_1'} \\
\frac{\partial x_2'}{\partial x_1} & \frac{\partial x_2'}{\partial x_1'} & \frac{\partial x_2'}{\partial y_1} & \frac{\partial x_2'}{\partial y_1'} \\
\frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_1'} & \frac{\partial y_2}{\partial y_1} & \frac{\partial y_2}{\partial y_1'} \\
\frac{\partial y_2'}{\partial x_1} & \frac{\partial y_2'}{\partial x_1'} & \frac{\partial y_2'}{\partial y_1} & \frac{\partial y_2'}{\partial y_1'}
\end{pmatrix}
\xrightarrow{k_2=0}
\begin{pmatrix}
1 & L & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & L \\
0 & 0 & 0 & 1
\end{pmatrix}
$$

For $k_2 \neq 0$ coefficients depend on initial values, e.g.:

$$
\frac{\partial y_2}{\partial y_1} = 1 + k_2\left(\frac{L^2}{4}x_1 + \frac{L^3}{12}x_1'\right) \Rightarrow
$$

Power series are not symplectic, cannot be used

Directly using finite power series maps is ruled out ...

Position and momentum change inside the magnet, i.e. the symplecticity condition does not hold for all initial conditions

Is this a dead end ? Do we have to wave the white flag ?

"Desperate disease requires dangerous remedy ..." $\rightarrow$

In previous example: $\Delta S \propto L^2, L^3$

Small error for small L**, no error for** $L = 0$ **!**

Zero length elements:

are technically difficult, but much easier to use ...

Thick "magnet":

Length and Strength specified for computation

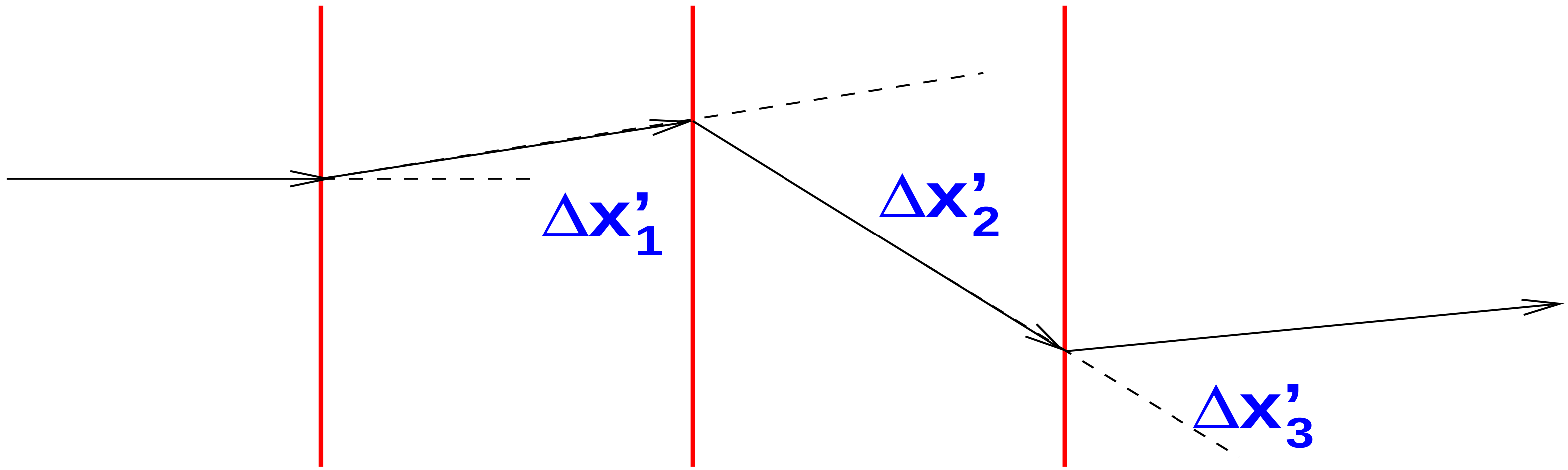
Example sextupole: L **and** k_2

Thin "magnet":

let the length go to zero, but keep Field Integral finite (L **and** k_n **are not specified separately):**

Example sextupole: $L \cdot k_2$

Moving through thin elements (shown for 1D)



The "momentum" x' receives an amplitude dependent deflection, "kick" $\quad x' \rightarrow x' + \Delta x'$

➡ $\Delta x'$ **=** $f(x)$ **(polynomials of some - possibly high - order)**

Always symplectic: no change of amplitude inside the element, no dependence on initial angle

Can we approximate a thick element by one or more thin element(s) ?

- **Yes, when the length is small or does not matter**
- **Symplecticity o.k.**

- **What about accuracy, what have we lost ??**

→ **Demonstrate with some simple examples**

(What follows is valid for all elements and provides the tools !!!)

Check out a quadrupole:

- **Start with "exact"*) map, compare with thin quadrupole**

$$\mathcal{M}_{s\to s+L} = \begin{pmatrix} \cos(L\cdot\sqrt{K}) & \frac{1}{\sqrt{K}}\cdot\sin(L\cdot\sqrt{K}) \\ -\sqrt{K}\cdot\sin(L\cdot\sqrt{K}) & \cos(L\cdot\sqrt{K}) \end{pmatrix}$$

- **Thick to thin: make L smaller and smaller, this permits:**

  Taylor/power expansion (of sin and cos) in "small" length L:

$$\mathcal{M} = \underbrace{L^0\cdot\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + L^1\cdot\begin{pmatrix} 0 & 1 \\ -K & 0 \end{pmatrix}}_{\text{thin lens in ''linear lectures''}} + L^2\cdot\begin{pmatrix} -\frac{K}{2} & 0 \\ 0 & -\frac{K}{2} \end{pmatrix} + \ldots$$

*) **.. it isn't**

- **Keep up to <u>first order</u> term in L (contribution with L^2 is small)**

$$\mathcal{M} = L^0 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + L^1 \cdot \begin{pmatrix} 0 & 1 \\ -K & 0 \end{pmatrix}$$

$$\mathcal{M} = \begin{pmatrix} 1 & L \\ -K \cdot L & 1 \end{pmatrix} + \mathcal{O}(L^2)$$

- **Precise to first order $\mathcal{O}(L^1)$**
- **det $\mathcal{M} = (1 + KL^2) \neq$ 1, non-symplectic !**

A possible (dangerous ?) Remedy:

If we add a term $-KL^2$ the matrix becomes symplectic:

$$\Rightarrow \quad \mathcal{M} = \begin{pmatrix} 1 & L \\ -K \cdot L & 1-KL^2 \end{pmatrix}$$

$$\Rightarrow \quad \textbf{det}\ \mathcal{M} = (1 - KL^2 + KL^2) = \mathbf{1}$$

(we have not damaged the accuracy too much, the original truncated matrix is inaccurate to order $\mathcal{O}(L^2)$ anyway ...)

Carry on:

Keep up to <u>second order</u> term in L

$$\mathcal{M} = \begin{pmatrix} 1 - \frac{1}{2}KL^2 & L \\ -K \cdot L & 1 - \frac{1}{2}KL^2 \end{pmatrix} + \mathcal{O}(L^3)$$

- **Precise to second order** $\mathcal{O}(L^2)$
- **More accurate than before, but again not symplectic**

Make it symplectic by adding $-\frac{1}{4}KL^3$

$$\mathcal{M} = \begin{pmatrix} 1 - \frac{1}{2}KL^2 & L - \frac{1}{4}KL^3 \\ -K \cdot L & 1 - \frac{1}{2}KL^2 \end{pmatrix} + \mathcal{O}(L^3)$$

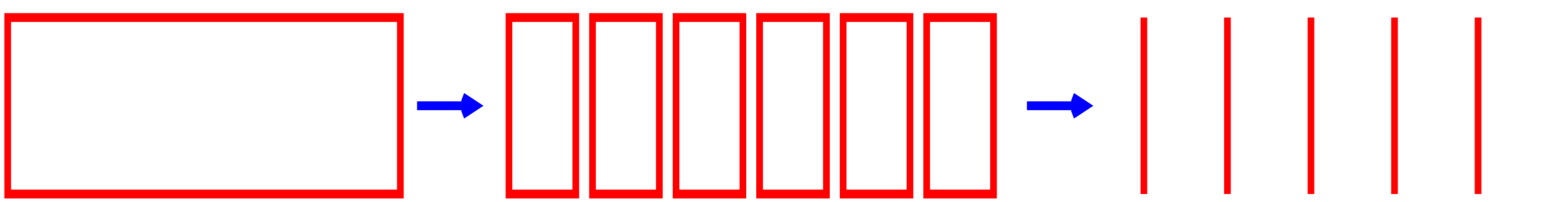
A symplectic model closer to ideal model ...

Looks like we made some arbitrary changes
(just trust me for a few minutes ..)

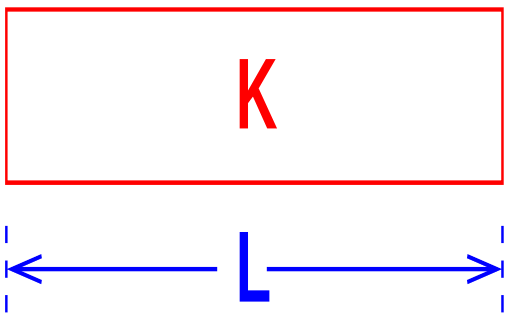
Are we silly or is there a physical picture behind the approximations ?

No / Yes → geometry of thin lens kicks ...

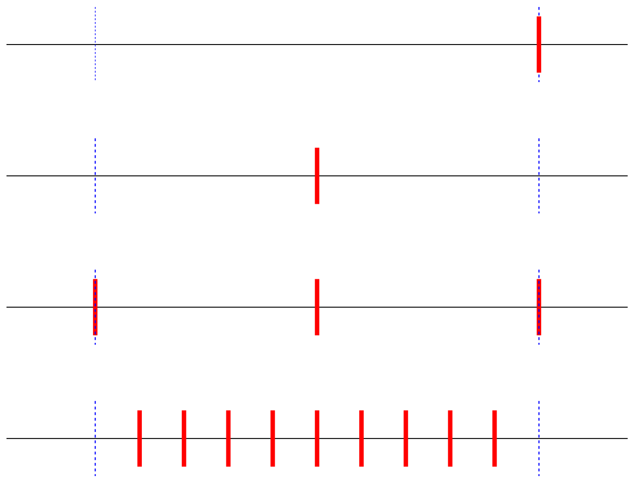
A thick element we should split into one or more thin elements with drifts between them, e.g.:



Thick quadrupole ⇨ thin quadrupoles ⇨ "kicks"



some options:

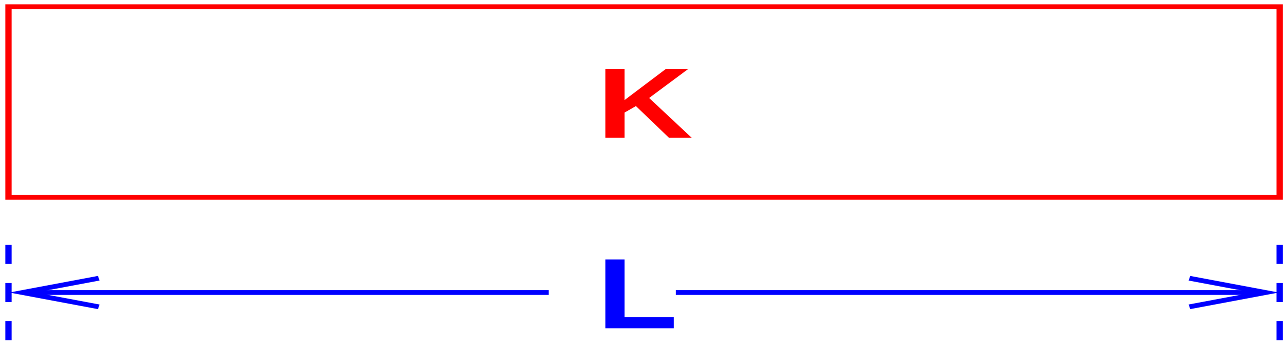


Represented by one or more "thin" lenses (kicks)

How many and where ?

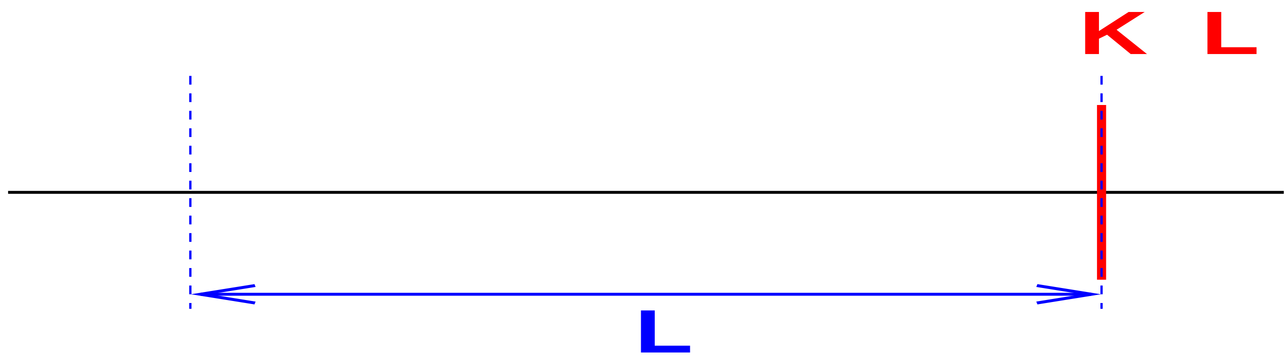
Which is a good strategy ? → accuracy and simplicity

Thick quadrupole ..



Go to thin quadrupoles → various options

Option 1



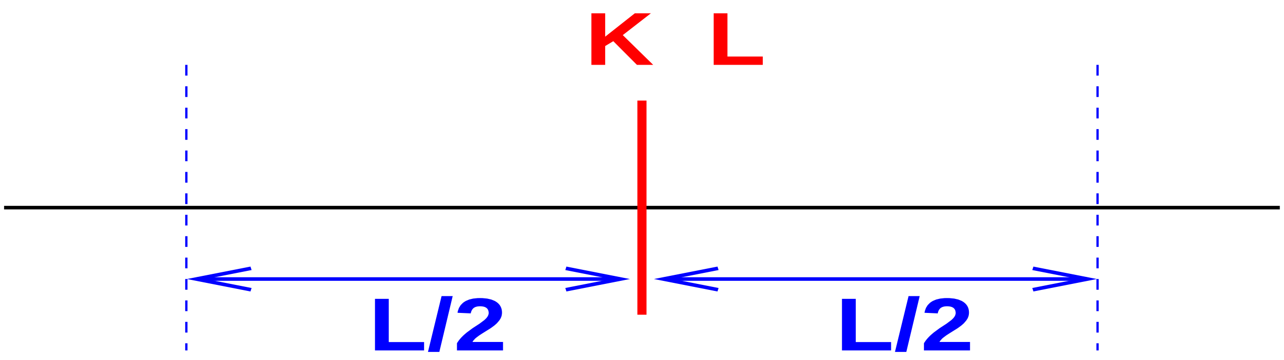
One thin quadrupole "kick" and one drift combined

$$\mathcal{M}_{drift\,+\,kick} = \overbrace{\begin{pmatrix} 1 & 0 \\ -K\cdot L & 1 \end{pmatrix}}^{M_{kick}} \overbrace{\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}}^{M_{drift}} = \begin{pmatrix} 1 & L \\ -K\cdot L & 1-KL^2 \end{pmatrix}$$

Reminder: product of symplectic matrices is symplectic

Resembles our "symplectification" of order O(1)

Option 2



One thin quadrupole "kick" between two drifts of half length

$$\mathcal{M} = \begin{pmatrix} 1 & \frac{1}{2}L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -K \cdot L & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{2}L \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 - \frac{1}{2}KL^2 & L - \frac{1}{4}KL^3 \\ -K \cdot L & 1 - \frac{1}{2}KL^2 \end{pmatrix}$$

Resembles more accurate "symplectification" of order O(2)

Accuracy of thin lenses

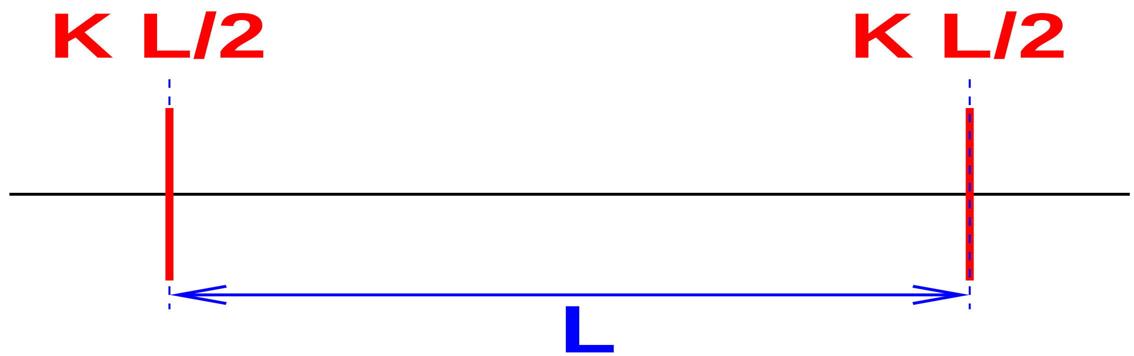
- **One kick at the end (or beginning):**
  - → **Error (inaccuracy) of second order** $\mathcal{O}(L^2)$
- **One kick in the centre:**
  - → **Error (inaccuracy) of third order** $\mathcal{O}(L^3)$

It is very relevant how to apply thin lenses !

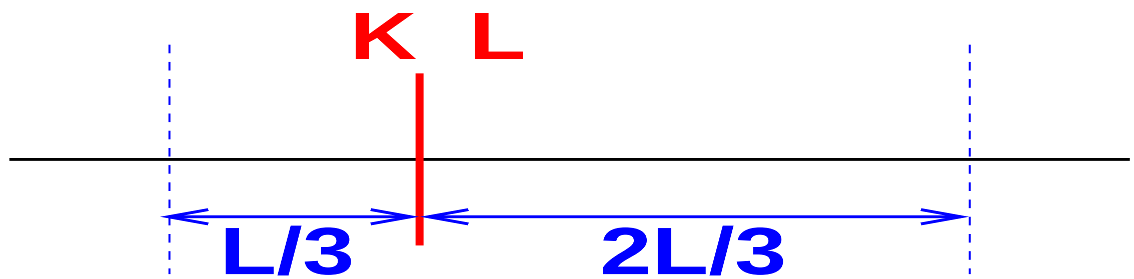
If you describe a quadrupole like : $\begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}$

The aim should be to be precise and fast (and simple to implement)

What about these options ?

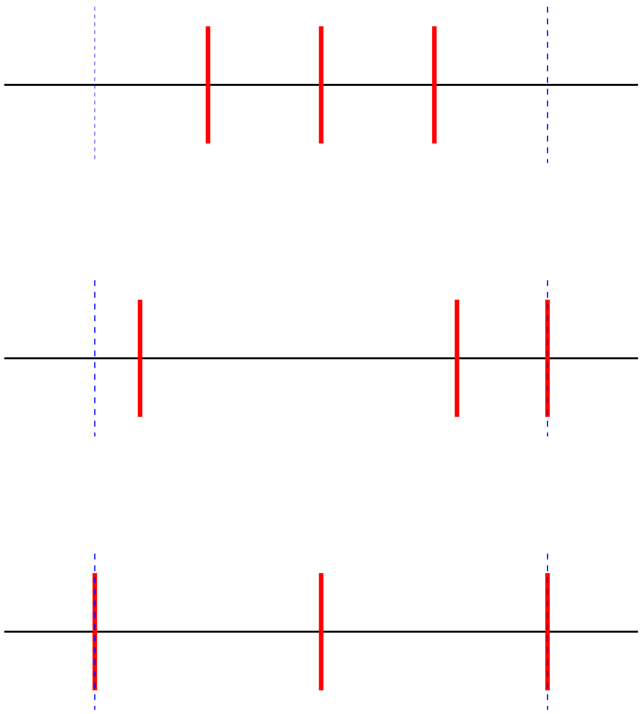


or:



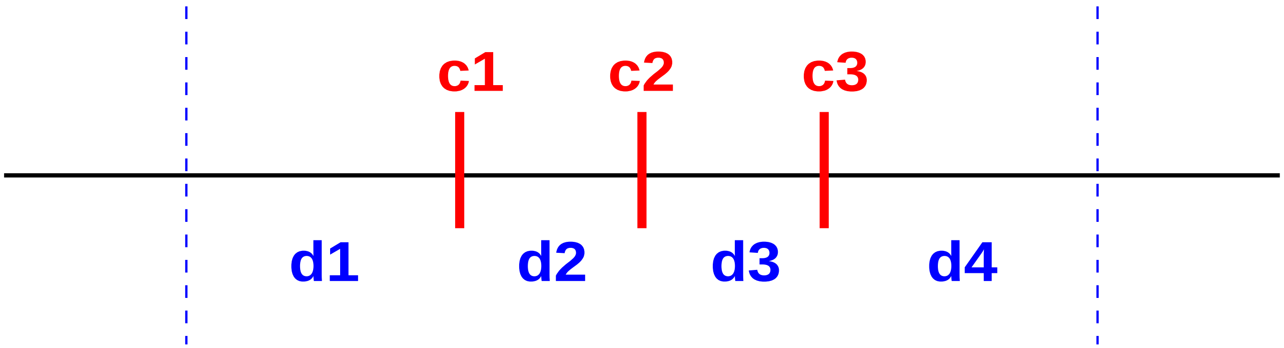
- Home exercises (optional, you need about 5 minutes)
- Are they symplectic ? (you have about 5 seconds)

How to go on - can we do better ?



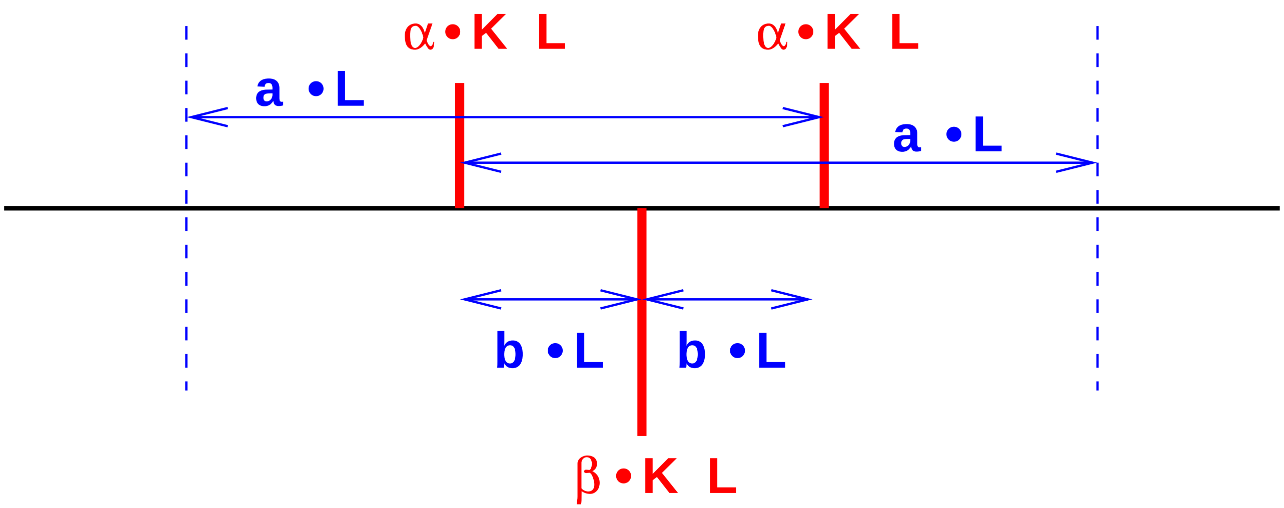
- **Try more slices, e.g. 3 kicks:**
- **How to put them ?**
- **Allow that they are at any positions and have different strengths**
- **Find positions and strengths which minimize the inaccuracy**

- **Try a model with 3 kicks:**



- **To get best accuracy (i.e. deviation from exact solution):**
- **You have 7 free parameters to minimize deviation:**
  - **Kicks c1, c2, c3 (allow different strength)**
  - **Drifts d1, d2, d3, d4 (allow any position of the kicks)**

The optimization gives us: (for the derivation, e.g. [AC1])



with:

$$a = \frac{1}{2} \cdot \frac{1}{2 - 2^{1/3}}, \quad b = \frac{1 - 2^{1/3}}{2} \cdot \frac{1}{2 - 2^{1/3}}$$

$$\alpha = \frac{1}{2 - 2^{1/3}}, \quad \beta = -2^{1/3} \cdot \frac{1}{2 - 2^{1/3}}$$

We have a $\mathcal{O}(4)$ integrator ... (without proof)

Resulting matrix $\mathcal{M}$ (from 7 matrices: 4 drifts, 3 kicks) becomes:

$$\mathcal{M}(\mathcal{O}4) = \begin{pmatrix} 1 - \frac{1}{2}k^2L^2 + \frac{1}{24}k^4L^4 & L - \frac{1}{6}k^2L^3 + \frac{1-2^{1/3}}{24(2-2^{1/3})^2}k^4L^5 \\ +\frac{2^{1/3}}{48(2-2^{1/3})^3}k^6L^6 & +\frac{2^{1/3}}{96(2-2^{1/3})^4}k^6L^7 \\ & \\ -k^2L + \frac{1}{6}k^4L^3 & 1 - \frac{1}{2}k^2L^2 + \frac{1}{24}k^4L^4 \\ +\frac{2^{1/3}}{24(2-2^{1/3})^2}k^6L^5 & +\frac{2^{1/3}}{48(2-2^{1/3})^3}k^6L^6 \end{pmatrix}$$

For the ambitious - Prove that it is symplectic

(MATHEMATICA® is really a good friend ...)

Why all that ? (answer in a few minutes)

Symplectic integration

- **What we do is a Symplectic Integration**
- **From a lower order integration scheme (1 kick), construct higher order scheme:**

$$\overbrace{\mathcal{O}(2)}^{1\ kick} \Rightarrow \overbrace{\mathcal{O}(4)}^{3\ kicks} \Rightarrow \overbrace{\mathcal{O}(6)}^{?\ kicks} \Rightarrow \ldots$$

Formally (for the formulation of S_k see later):

From a 2nd order scheme (1 kick) S_2 we construct a 4th order scheme (3 kicks = 3 x 1 kick) like:

$S_4 = S_2(x_1) \circ S_2(x_0) \circ S_2(x_1)$ **with scaling coefficients:**

$$x_0 = \frac{-2^{1/3}}{2 - 2^{1/3}} \qquad x_1 = \frac{1}{2 - 2^{1/3}}$$

- **Can be considered an iterative scheme (see e.g. H. Yoshida, 1990, E. Forest, 1998):**

If S_{2k} is a symmetric integrator of order $2k$, then:

$$S_{2k+2} = S_{2k}(x_1) \circ S_{2k}(x_0) \circ S_{2k}(x_1)$$

with : $$x_0 = \frac{-\sqrt[2k+1]{2}}{2 - \sqrt[2k+1]{2}} \qquad x_1 = \frac{1}{2 - \sqrt[2k+1]{2}}$$

- **Higher order integrators can be obtained in a similar way:**

$$S_{2k} \Rightarrow S_{2k+2} \Rightarrow S_{2k+4} \Rightarrow S_{2k+6} \Rightarrow \ldots$$

- **Stop at the desired order, rather simple to implement on a computer (with paper and pencil makes you a lunatic)**

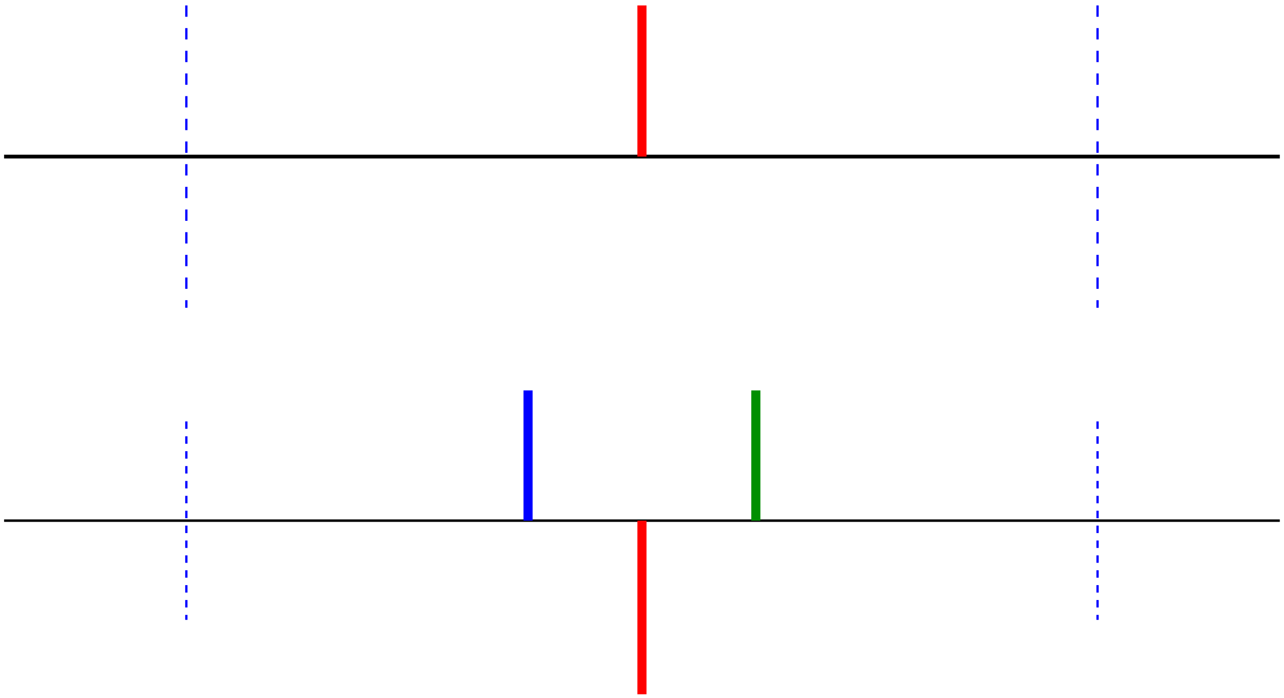
Example: From a 4th order to 6th order

$$S_6 = S_4(x_1) \circ S_4(x_0) \circ S_4(x_1)$$

Replace <u>each kick</u> of a 4th order integrator by a 4th order integrator, using the same scaling factors

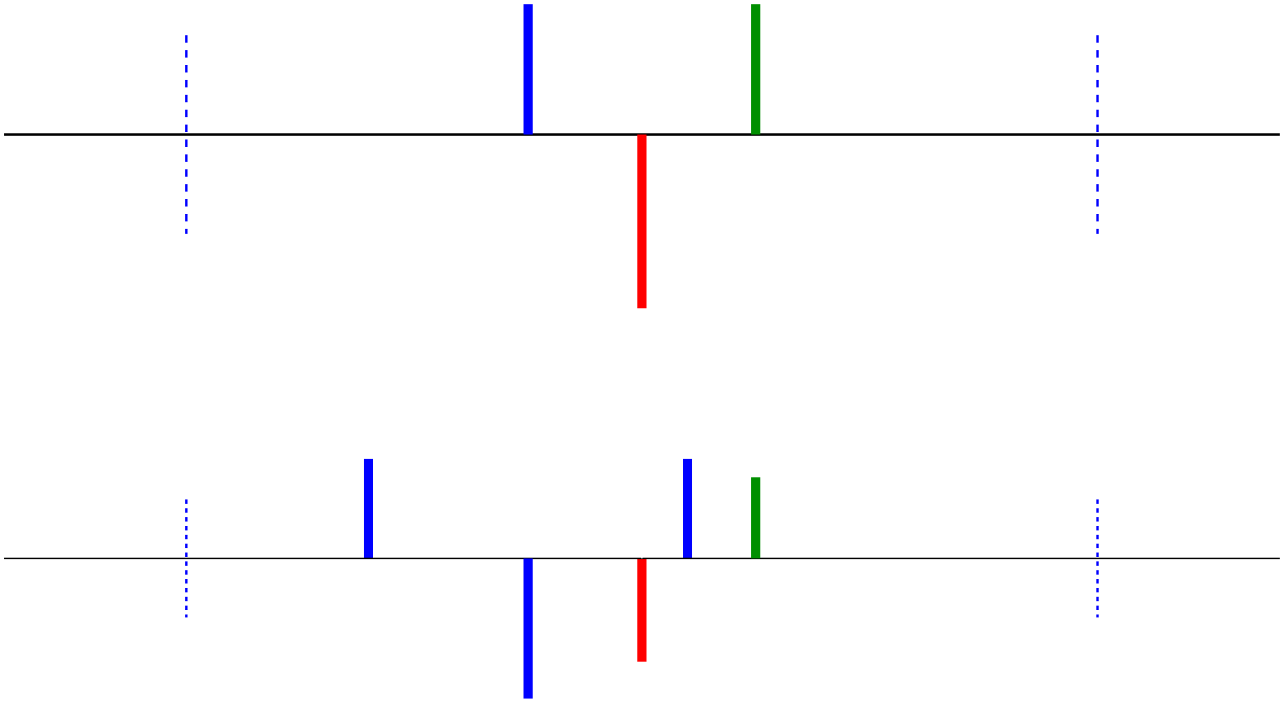
We get 3 times 4th order with 3 kicks each, we have the 9 kick, 6th order integrator mentioned earlier

Integrator of order 2 $\Longrightarrow$ 4



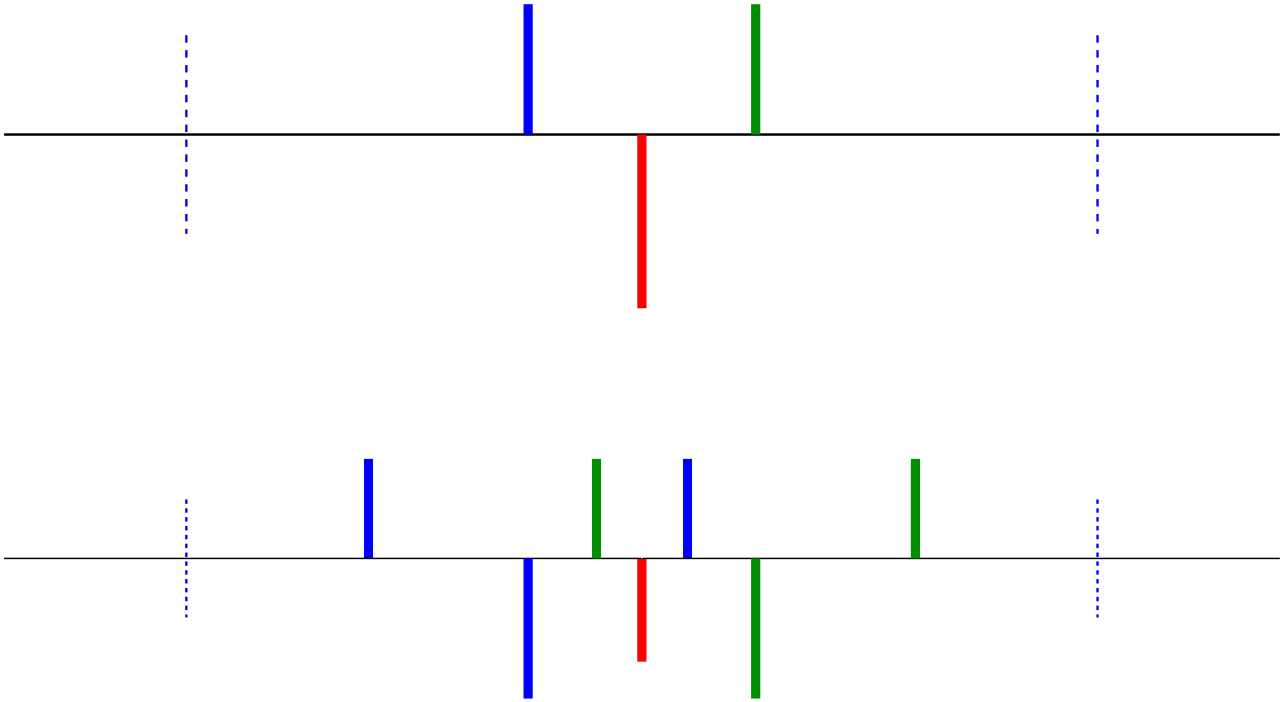
- Replace kick by 4th order integrator

Integrator of order 4 $\Longrightarrow$ 6 - step by step



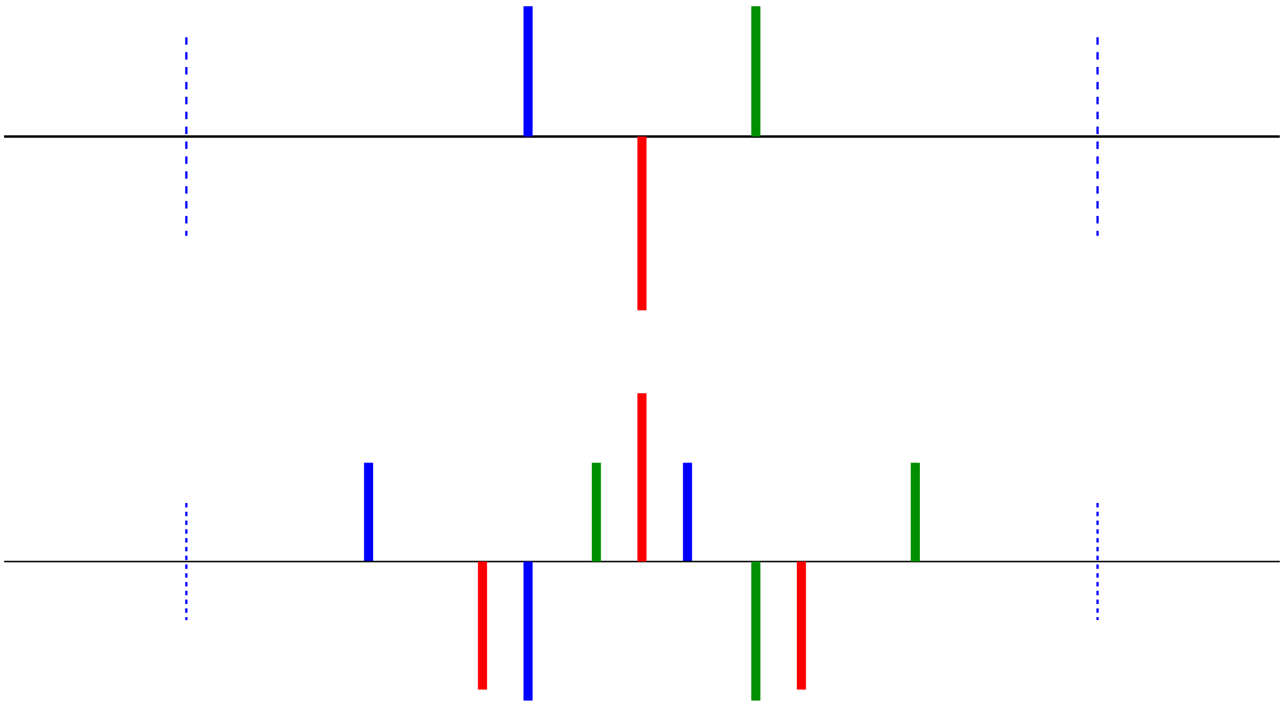
- Replace each <u>kick</u> by 4th order integrator

Integrator of order 4 $\Longrightarrow$ 6 - step by step



- **Replace each <u>kick</u> by 4th order integrator**

Integrator of order 4 $\Longrightarrow$ 6 - step by step



- Replace each <u>kick</u> by 4th order integrator, requires 9 kicks
- We have 3 interleaved 4th order integrators (compute $\mathcal{M}(\mathcal{O}6)$), repeat the procedure to go to higher orders

Some remarks:

- **We have used a linear map (quadrupole) to demonstrate the integration**
- **Can that be applied for other maps (solenoids, higher order, nonlinear maps) ?**
  - **Yes !!**
  - **We get the same integrators ! (i.e. same constants)**
  - **Proof and systematic (and easy) extension in the form of Lie operators (see later)**

→ **Without proof: best possible accuracy for thin lenses (be smart: a scheme with more thin lenses may be less precise !)**

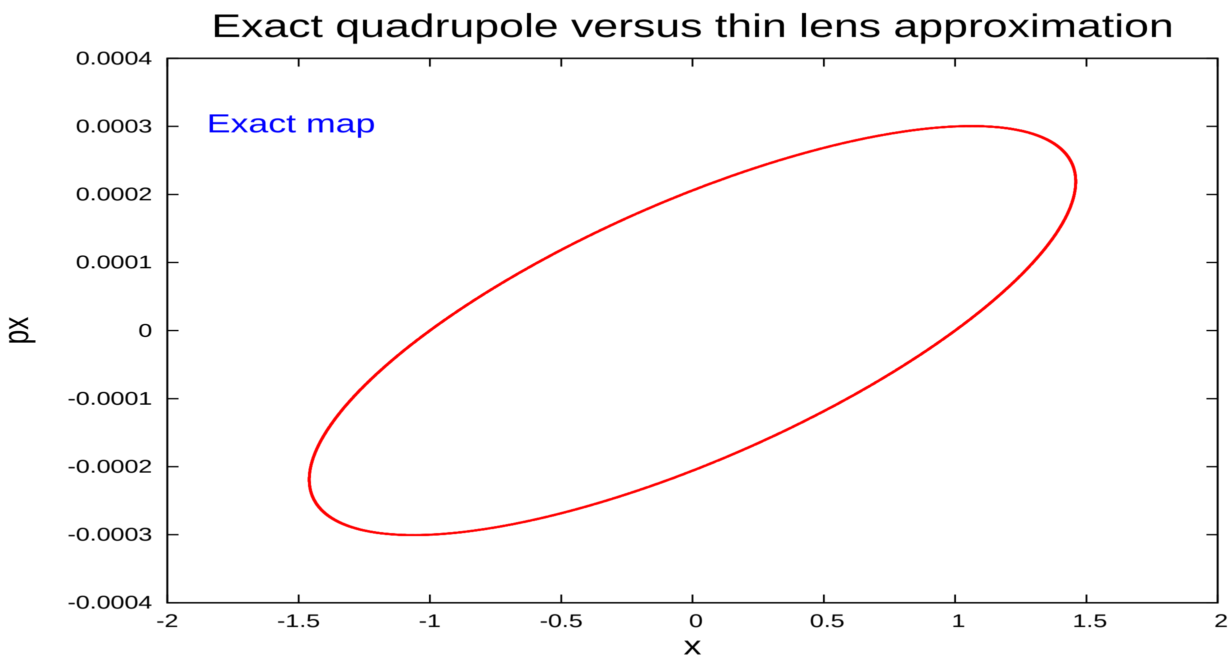
To remember:

Given a truncated Power map we construct a symplectic map whose lower order terms agree with the exact non-symplectic Power expansion and whose higher order (neglected) terms are small.

Key question:

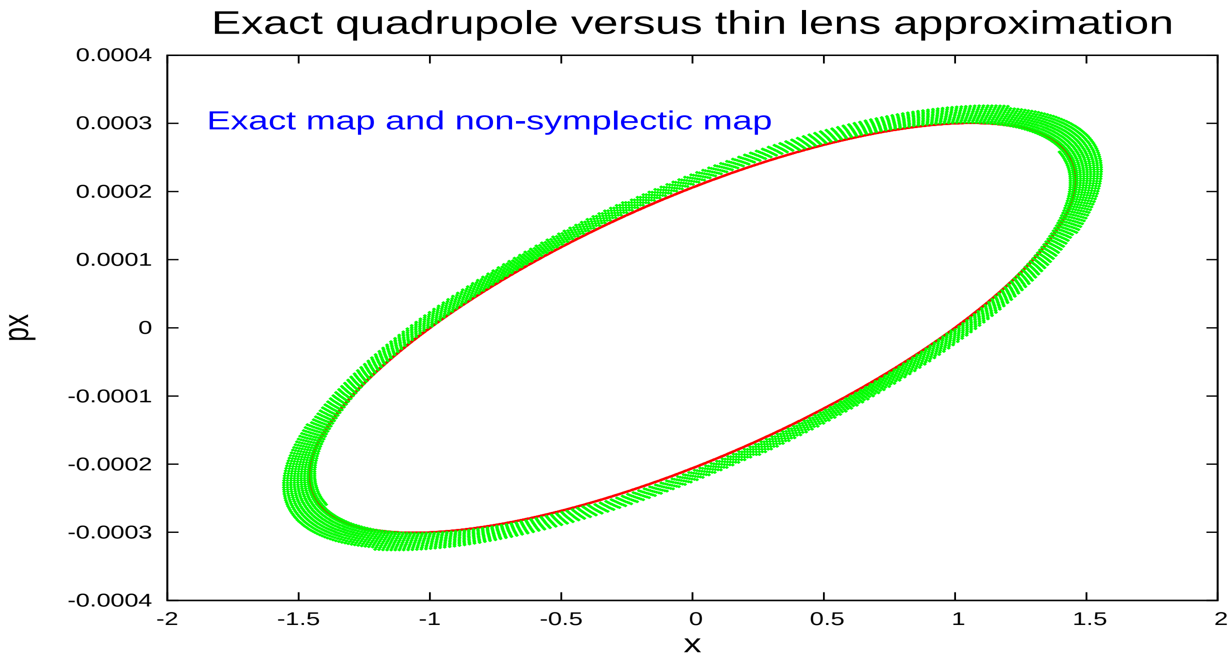
How can we say that the neglected terms do not exceed a tolerable limit ?

What is the point ???



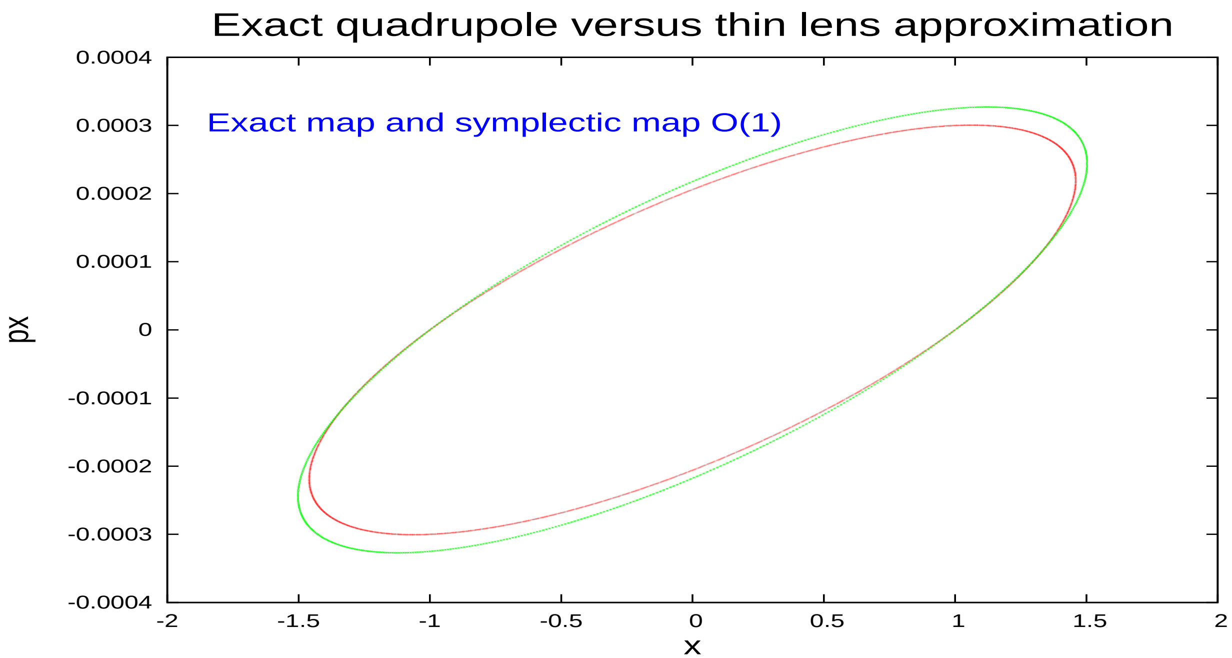
Phase space ellipse - quadrupole exact solution

Quadrupole non-symplectic solution L^1



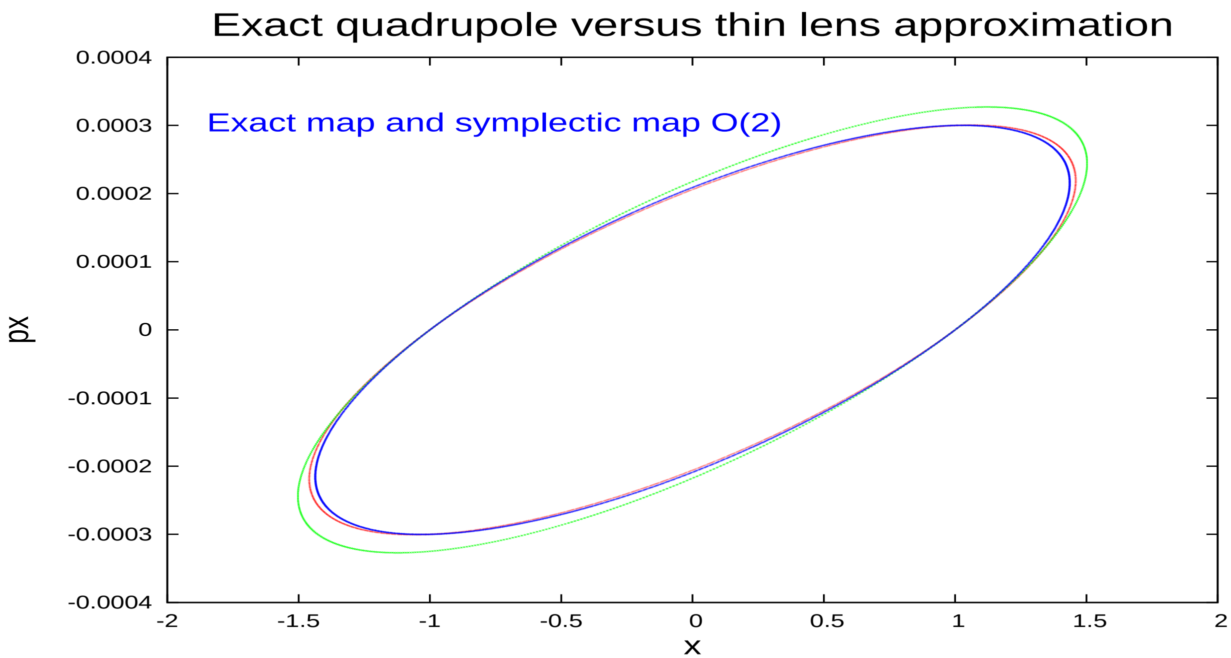
Non-symplecticity: particles spiral towards outside, artifact of approximation

Quadrupole symplectic $\mathcal{O}(L^1)$ solution



symplectic, solution order $\mathcal{O}(L^1)$, but visible inaccuracy

Quadrupole symplectic $\mathcal{O}(L^2)$ solution



symplectic, solution order $\mathcal{O}(L^2)$, but good accuracy

Quantitatively: Accuracy of (nonlinear) thin lenses

- **Nonlinear elements are usually thin (thinner than dipoles, quadrupoles ...)**
  - **Dipoles: $\approx$ 14.3 m**
  - **Quadrupole: $\approx$ 2 - 5 m**
  - **Sextupoles, Octupoles: $\approx$ 0.30 m**
  - **Decapole: $\approx$ 0.07 m**

Assume a kick from a general function of x:

deflection : $\Delta x' = f(x)$

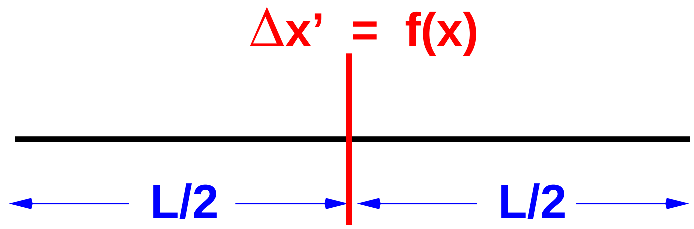
e.g. quadrupole $f(x) = k \cdot x^1$

e.g. sextupole $f(x) = k \cdot x^2$

e.g. octupole $f(x) = k \cdot x^3$

→ **Can try our simplest thin lens approximation O(2) first ...**

Drift - Kick - Drift



$$1.Step \quad \begin{pmatrix} x \\ x' \end{pmatrix}_{s_1+L/2} = \begin{pmatrix} 1 & \frac{L}{2} \\ 0 & 1 \end{pmatrix} \circ \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}_{s_1}$$

$$2.Step \quad \begin{pmatrix} x \\ x' \end{pmatrix}_{s_1+L/2} = \begin{pmatrix} x \\ x' + \Delta x' \end{pmatrix}_{s_1+L/2} = \begin{pmatrix} x \\ x' + f(x) \end{pmatrix}_{s_1+L/2}$$

$$3.Step \quad \begin{pmatrix} x \\ x' \end{pmatrix}_{s_1+L} = \begin{pmatrix} 1 & \frac{L}{2} \\ 0 & 1 \end{pmatrix} \circ \begin{pmatrix} x \\ x' \end{pmatrix}_{s_1+L/2}$$

Putting it together and written in explicit form:

$$\begin{pmatrix} x(L) \\ x'(L) \end{pmatrix} = \begin{pmatrix} x_0 + \frac{L}{2} \cdot (x'_0 + x'(L)) \\ x'_0 + L \cdot f(x_0 + \frac{L}{2} x'_0) \end{pmatrix}$$

$$\Rightarrow \quad x(L) \approx x_0 + L \cdot x'_0 \quad + \frac{L^2}{2} \cdot \underbrace{f(x_0 + \frac{L}{2} x'_0)}$$

(using: $f(z + \Delta z) \approx f(z) + f'(z) \cdot \Delta z$ **for small** Δz**)**

$$\Rightarrow \quad \boxed{x(L) \approx x_0 + L \cdot x'_0 \quad + \frac{L^2}{2} \cdot f(x_0) + \frac{L^3}{4} \cdot f'(x_0) x'_0}$$

It is symplectic (... and time reversible) !!

Comparison:

the (exact, but non-symplectic) Taylor expansion of $f(x)$ gives:

$$x(L) \quad = \quad x_0 + x_0' L + \frac{L^2}{2} f(x_0) + \frac{L^3}{6} f'(x_0) x_0' + ...$$

the (approximate, but symplectic) algorithm gives:

$$x(L) \quad = \quad x_0 + x_0' L + \frac{L^2}{2} f(x_0) + \frac{L^3}{4} f'(x_0) x_0' + ...$$

- **Errors are $\mathcal{O}(L^3)$ (is correct to $\mathcal{O}(L^2)$ by construction)**

  Errors are $\mathcal{O}(L^5)$ for the $\mathcal{O}(L^4)$ (3 kicks) scheme

- **For small L acceptable, and symplectic**

just for illustration : $\frac{L^3_{decapole}}{L^3_{dipole}} \approx 10^{-7}$

An application, a (1D) sextupole with:

$$f(x) \quad = \quad k \cdot x^2$$

using the thin lens approximation gives:

$$\begin{aligned}
x(L) \quad &= \quad x_0 + x_0' L + \frac{1}{2} k x_0^2 L^2 + \overbrace{\color{red}{\frac{1}{2} k x_0 x_0' L^3}}^{\frac{1}{3} k x_0 x_0' L^3} + \ldots \\
x'(L) \quad &= \quad x_0' + k x_0^2 L + k x_0 x_0' L^2 + \color{red}{\frac{1}{4} k x_0'^2 L^3 + \ldots}
\end{aligned}$$

Map for thick sextupole of length L in thin lens approximation, accurate to $\mathcal{O}(L^2)$

Short summary: thin lens computations

- **Are exactly symplectic**
- **Simulations based on thin lenses fast and efficient**

  successfully applied to large (storage) rings (e.g. SPS, Tevatron, LHC, LEP, ...)

But do not represent an <u>exact</u> model of the accelerator

If used blindly: .. an exact solution to a wrong problem

For (large) accelerators the thin lenses are usually a good approximation and tool (because we do not have to go to very high-order integrators to get proper results).

Rap sheet - part 1

- **Problems: Linear Beam Dynamics:**
  - **Many effects can be understood using linear approximations**
  - **Methods not applicable to complex (nonlinear) machines**
- **Problems: Nonlinear Beam Dynamics:**
  - **Strong impact on beam dynamics**
  - **Traditional methods (may) give unsatisfactory/unreliable results**
- **Initial Probation measures:**
  - **Concept of linear and nonlinear maps allow proper integration and extension, Analysis with e.g. "Normal Forms"**
  - **Symplectic integration techniques, particle tracking**
- **Progressive Probation measures → next lectures**