CERN

Instabilities Part III: Transverse wake fields – impact on beam dynamics

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Outline

We will close in into the description and the impact of **transverse wake fields**. We will discuss the **different types** of transverse wake fields, outline how they can be implemented numerically and then investigate **their impact on beam dynamics**. We will see some **examples of transverse instabilities** such as the transverse mode coupling instability (TMCI) or headtail instabilities.

Part 3: Transverse wakefields –

their different types and impact on beam dynamics

- Transverse wake function and impedance
- Numerical implementation, transverse „potential well distortion“ ad headtail instabilities
- Two particle models, transverse mode coupling instability

Signpost

- We have **discussed longitudinal wake fields** and impedances and their impact on both the machine as well as the beam.
- We have learned about **beam induced heating** and how it is related to the beam power spectrum and the machine impedance.
- We have discussed the effects of **potential well distortion** (stable phase and synchrotron tune shifts, bunch lengthening and shortening).
- We have seen some examples of **longitudinal instabilities** (Microwave, Robinson).

Part 3: Transverse wakefields –

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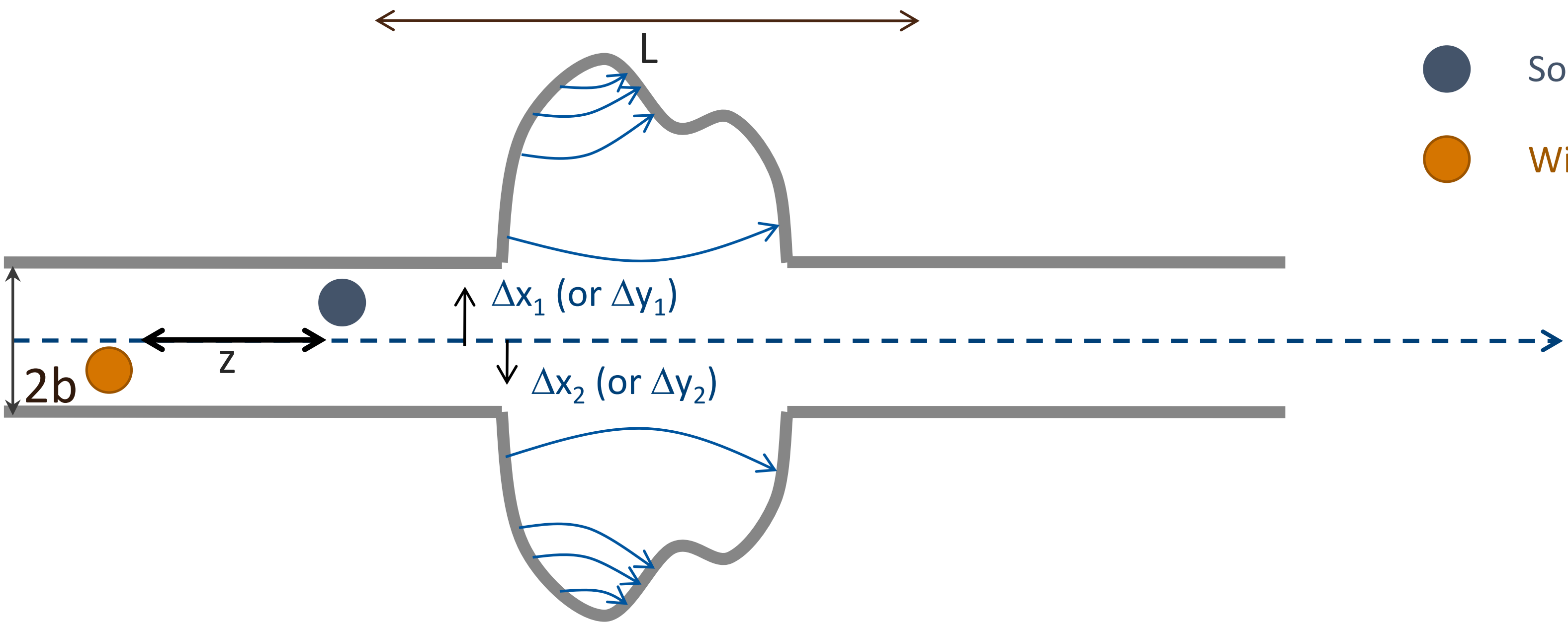
Recap: wake functions in general



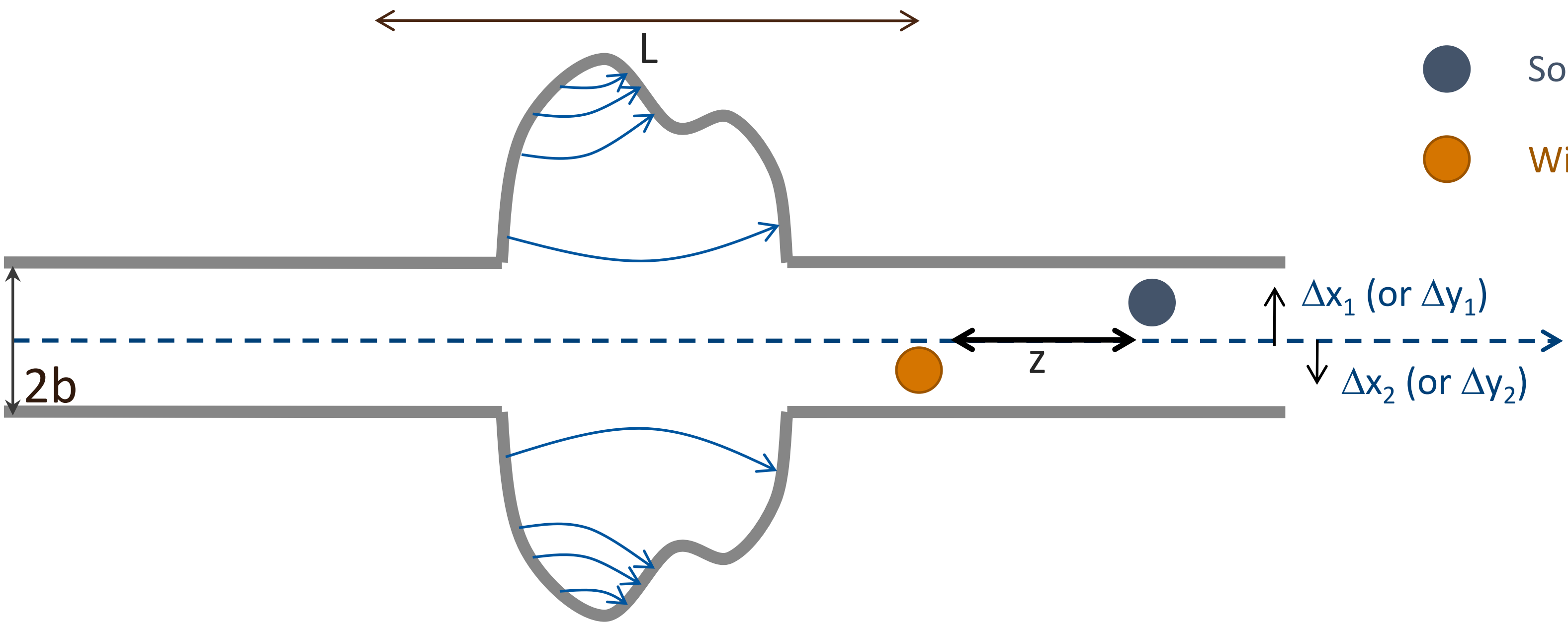
Definition as the **integrated force** associated to a change in energy:

- In general, for two point-like particles, we have

$$\Delta E_2 = \int F(x_1, x_2, z, s)\, ds = -q_1 q_2\, \boldsymbol{w(x_1, x_2, z)}$$

w is typically expanded in the transverse offsets of source and witness particles. This yields the different types of wake fields (dipole, quadrupole, coupling wakes)

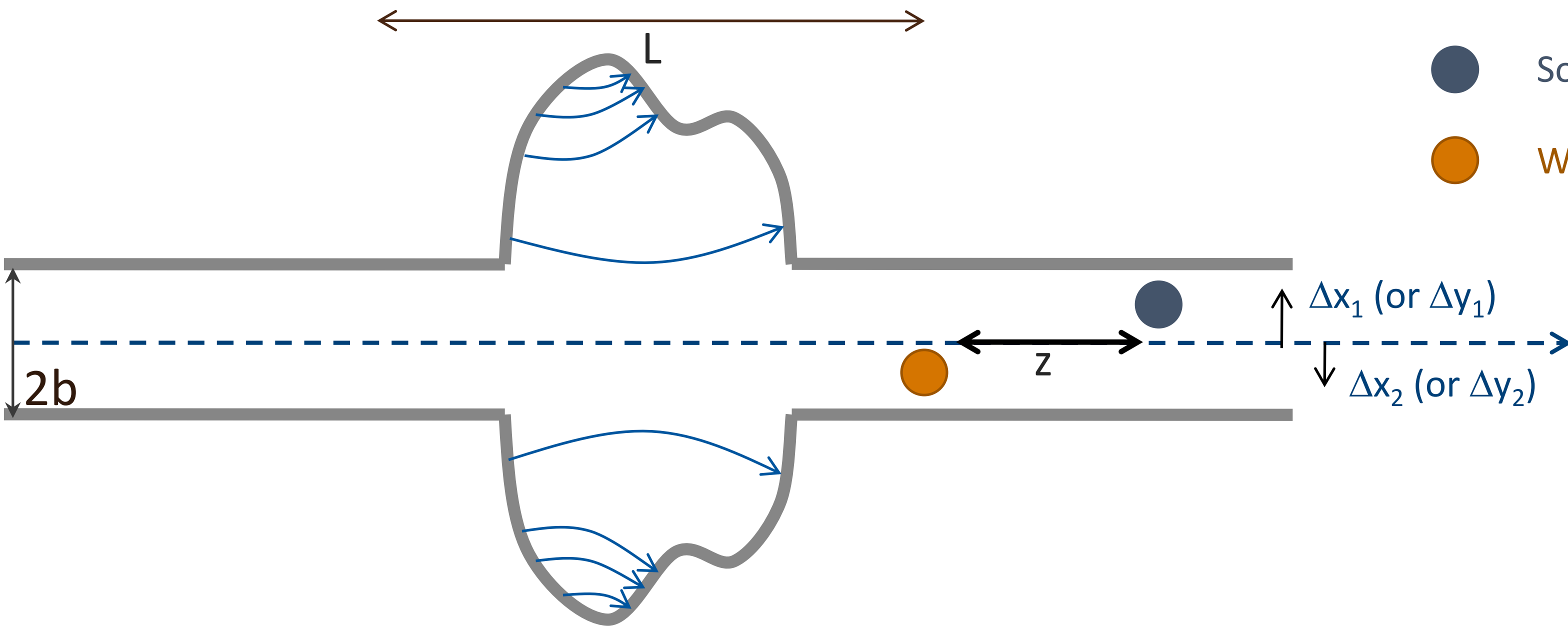
Transverse wake functions



- Transverse wake fields

$$\Delta E_{x2} = \int F_x(x_1, x_2, z, s)\, ds$$

Transverse wake functions



- Transverse wake fields

$$\Delta E_{x2} = \int F_x(x_1, x_2, z, s)\, ds = -q_1 q_2 \left(W_{C_x}(z) + W_{Dx}(z)\, \Delta x_1 + W_{Q_x}(z)\, \Delta x_2 \right)$$

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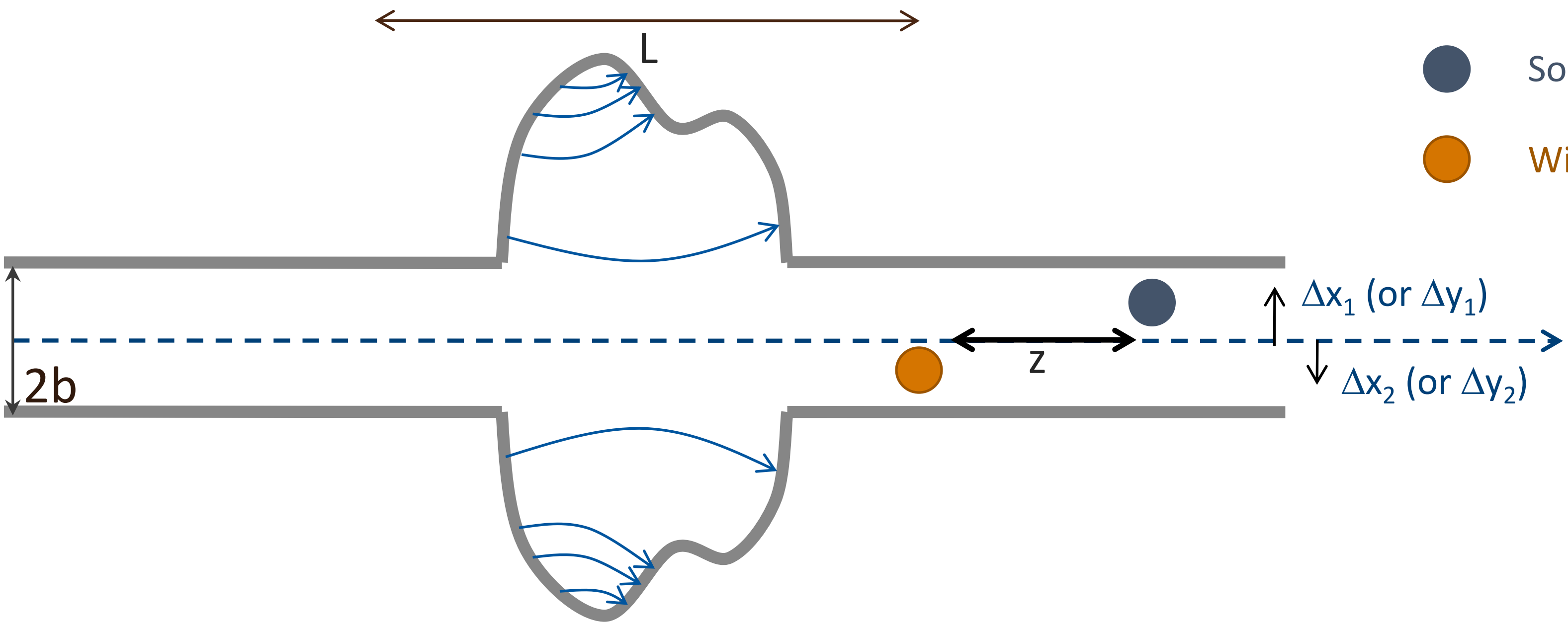

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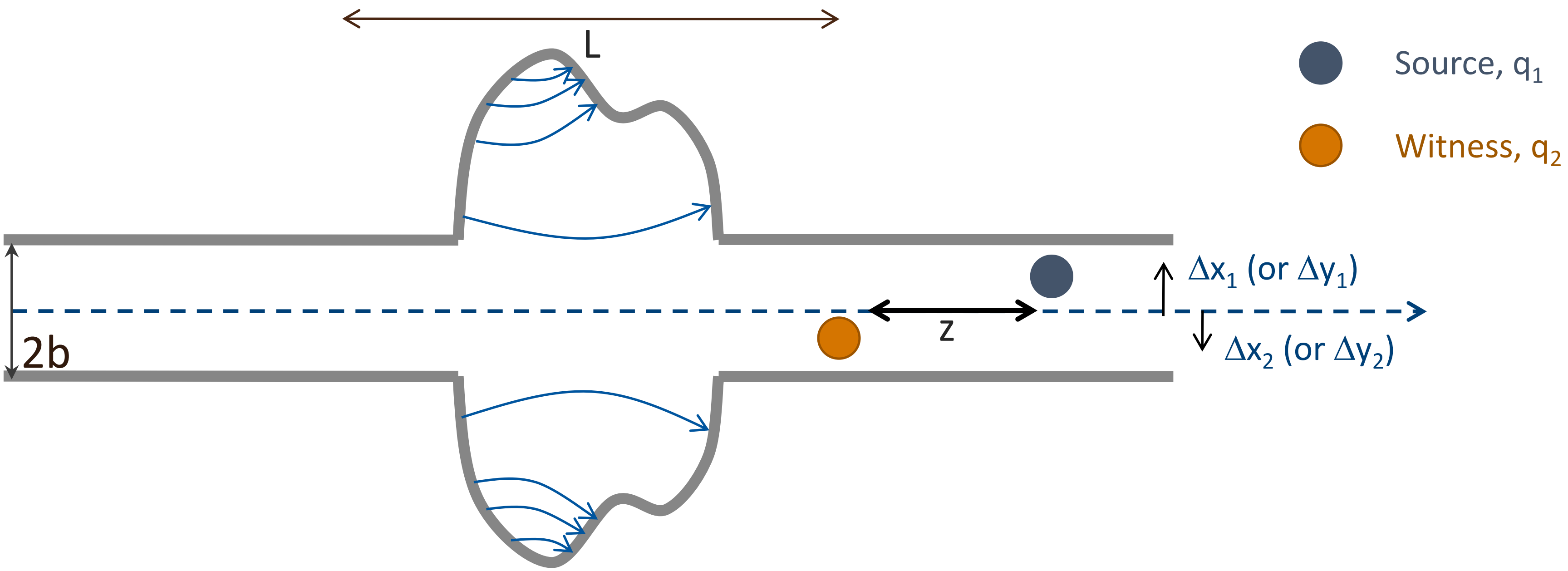
$$\longrightarrow \frac{\Delta E_{x2}}{E_0} = x_2'$$

Transverse deflecting kick of the witness particle from transverse wakes

Transverse wake functions



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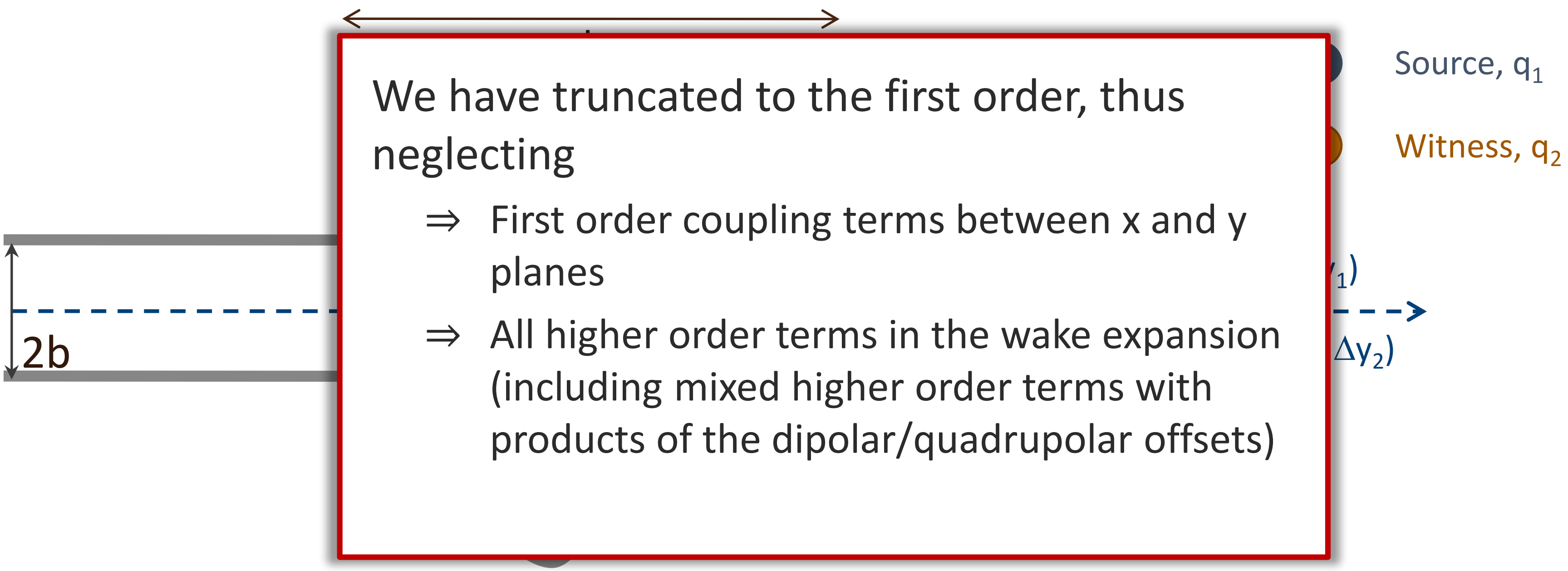
Zeroth order for asymmetric structures → Orbit offset

Dipole wakes – depends on **source particle** → Orbit offset

Quadrupole wakes – depends on **witness particle** → Detuning

Transverse wake functions



We have truncated to the first order, thus neglecting

- ⇒ First order coupling terms between x and y planes
- ⇒ All higher order terms in the wake expansion (including mixed higher order terms with products of the dipolar/quadrupolar offsets)

- Transverse wake fields

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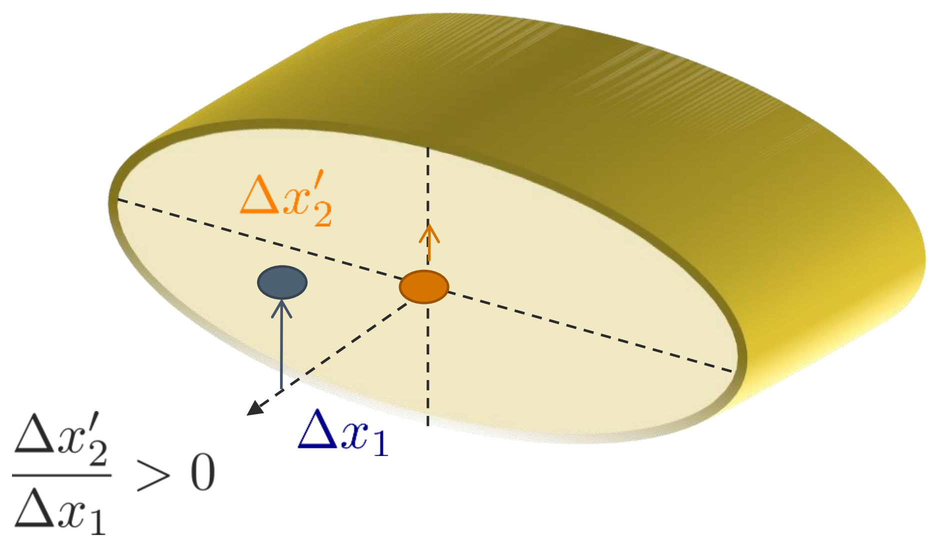
Quadrupole wakes – depends on **witness particle** → Detuning

Transverse dipole wake function

$$W_{D_x}(z) = -\frac{E_0}{q_1 q_2}\frac{\Delta x_2'}{\Delta x_1} \xrightarrow{z\to 0} W_{D_x=0}(0) = 0$$

- The value of the transverse dipolar wake function **in z=0 vanishes** because source and witness particles are traveling parallel and they can only – mutually – interact through space charge, which is not included in this framework
- ***W_{Dx}(0–)<0*** since trailing particles are **deflected toward the source particle** (Δx1 and Δx'2 have the same sign)



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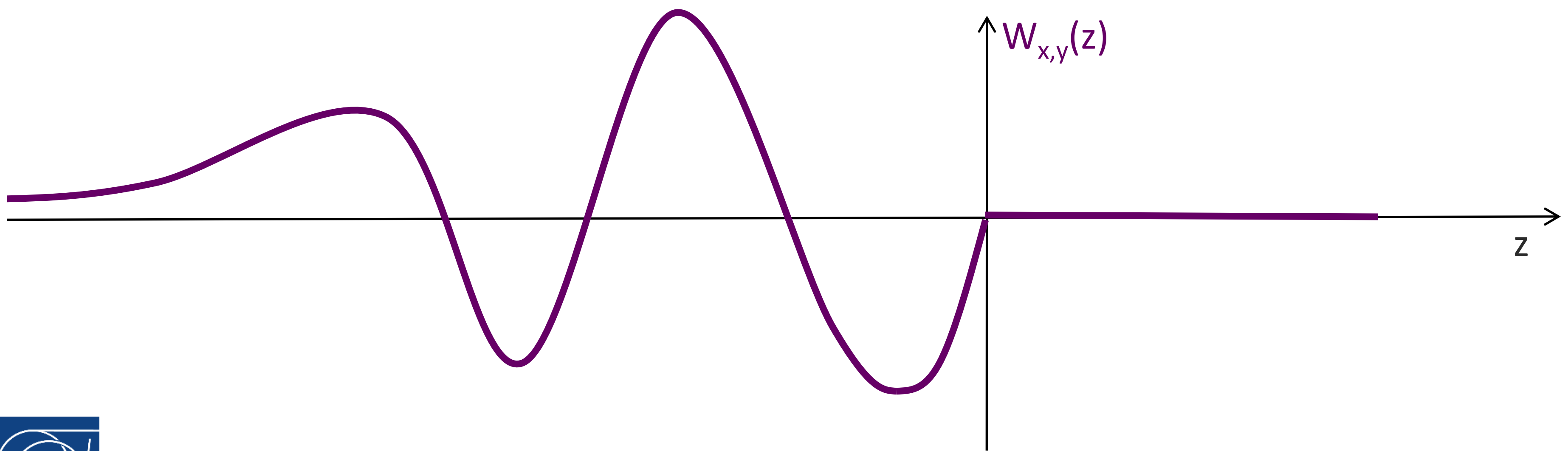
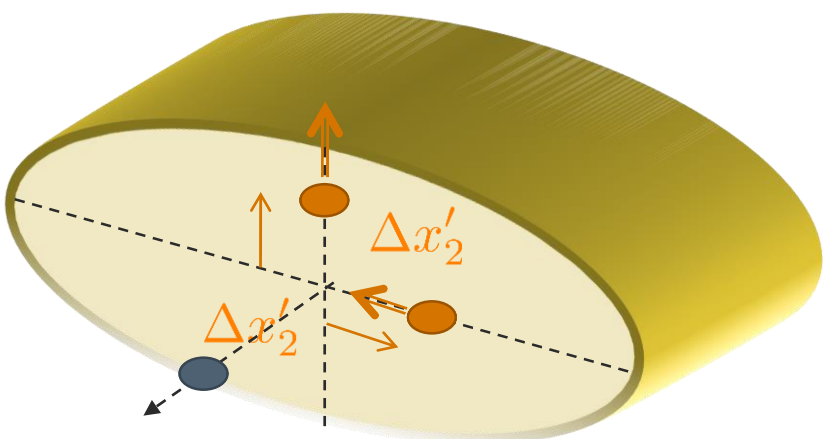
- ***W_{Dx}(z)*** has a discontinuous derivative in z=0 and it vanishes for all z>0 because of the ultra-relativistic approximation



Transverse quadrupole wake function

$$W_{Q_x}(z) = -\frac{E_0}{q_1 q_2}\frac{\Delta x_2'}{\Delta x_2} \quad \xrightarrow{z \to 0} \quad W_{Q_x=0}(0) = 0$$

- The value of the transverse quadrupolar wake function **in z=0 vanishes** because source and witness particles are traveling parallel and they can only – mutually – interact through space charge, which is not included in this framework
- ***W_{Qx}(0–)<0*** can be of either sign since trailing particles can be **either attracted or deflected yet further off axis** (depending on geometry and boundary conditions)



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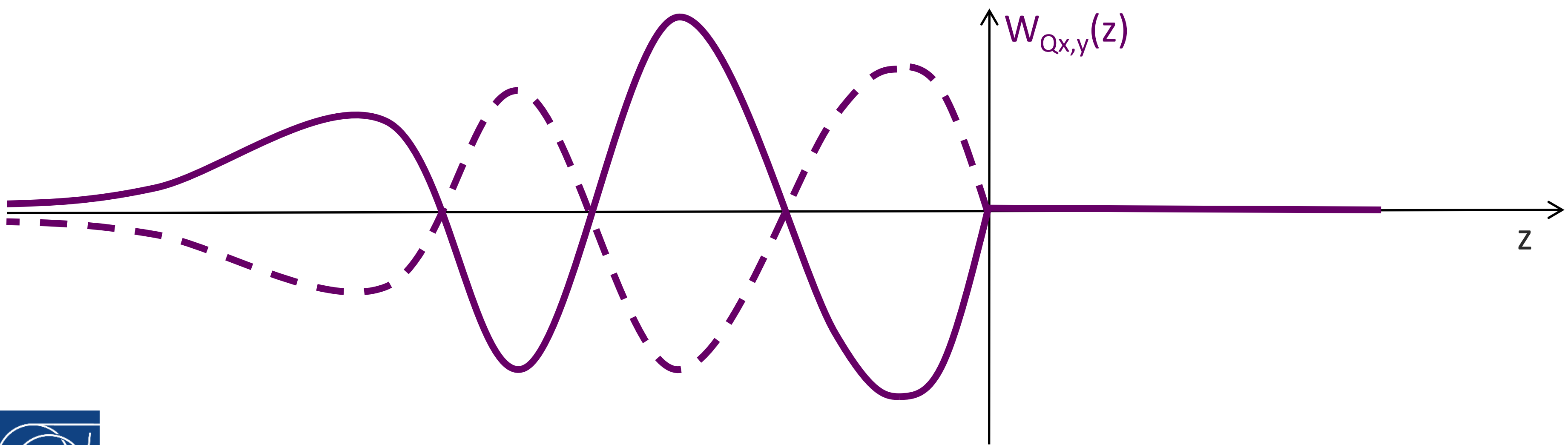


Transverse impedance

$$W_{D_x}(z) = -\frac{E_0}{q_1 q_2}\frac{\Delta x_2'}{\Delta x_1} \qquad W_{Q_x}(z) = -\frac{E_0}{q_1 q_2}\frac{\Delta x_2'}{\Delta x_2}$$

- The **wake function** of an accelerator component is basically its **Green function in time domain** (i.e., its response to a pulse excitation)
  - → Very useful for macroparticle models and simulations, because it can be used to describe the driving terms in the single particle equations of motion!
- We can also describe it as a **transfer function in frequency domain**
  - → This is the definition of **transverse beam coupling impedance** of the element under study

Dipolar

$$Z_{D_x}(\omega) = i\int_{-\infty}^{\infty} W_{D_x}(z)\exp\left(-\frac{i\omega z}{c}\right)\frac{dz}{c}$$

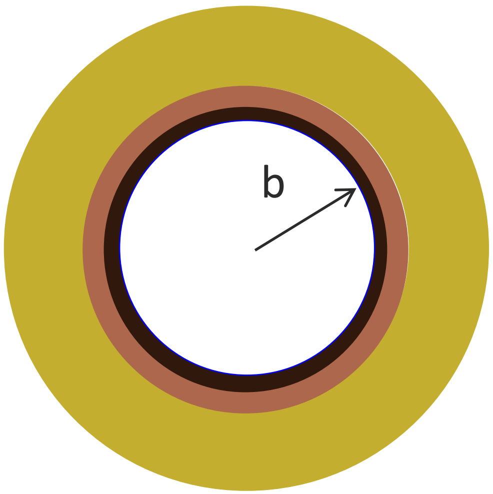
Quadrupolar

$$Z_{Q_x}(\omega) = i\int_{-\infty}^{\infty} W_{Q_x}(z)\exp\left(-\frac{i\omega z}{c}\right)\frac{dz}{c}$$

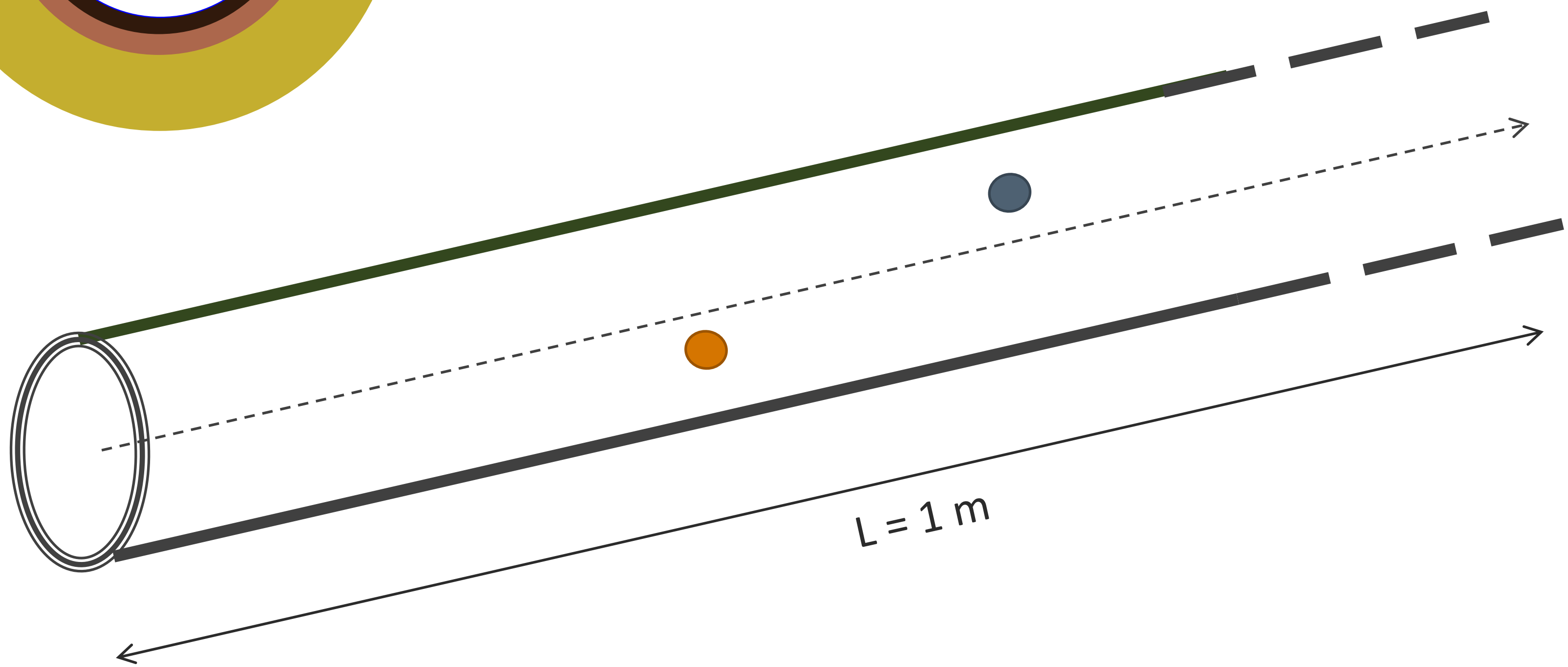
$[\Omega/\mathrm{m}]$

Examples of wakes/impedances

- Resistive wall of beam chamber



- The case of a conductive pipe with an **arbitrary number of layers** with specified EM properties can be solved **semi-analytically**
- Layers sometimes required for impedance, but also for other reasons (e.g. coating against electron cloud or for good vacuum)

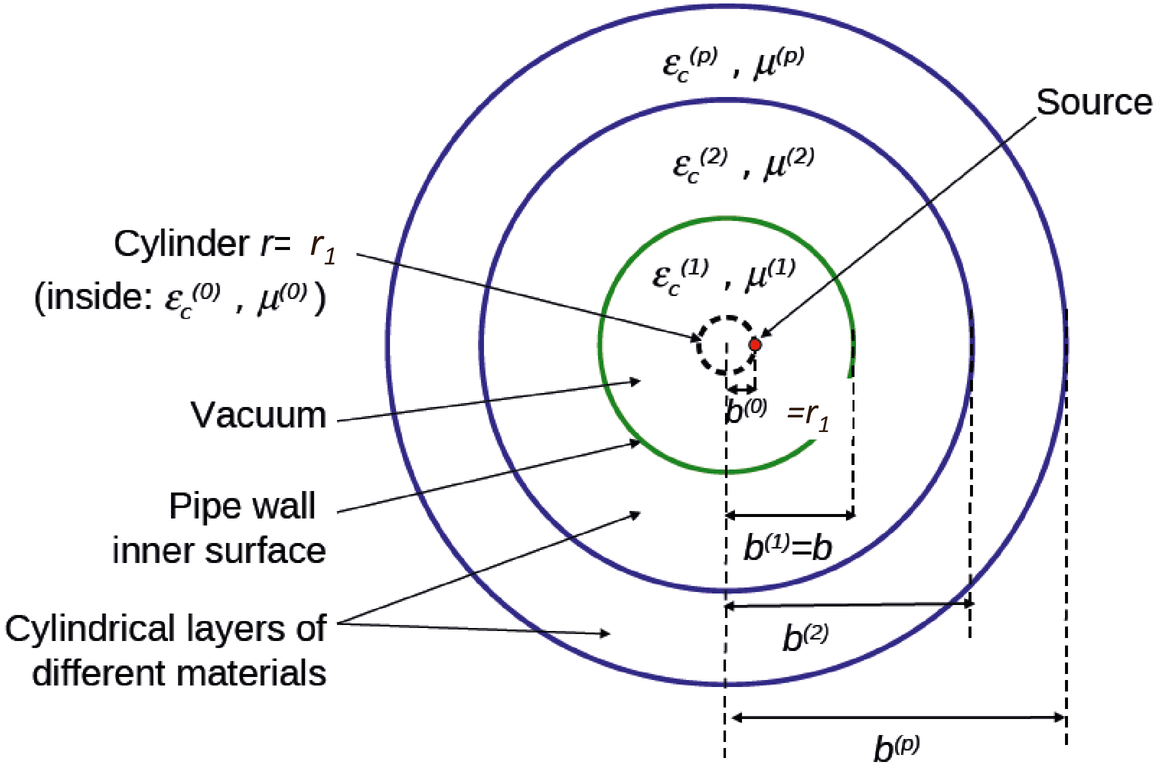


Examples of wakes/impedances

- Resistive wall of beam chamber

- The **equations for the coefficients of the azimuthal modes of E_s** must be solved in all the media and the conservation of the tangential components of the fields is applied at the boundaries between different layers

- ➔ E.g. ***ImpedanceWake2D*** code calculates impedances and then wakes. It can also deal with flat structures



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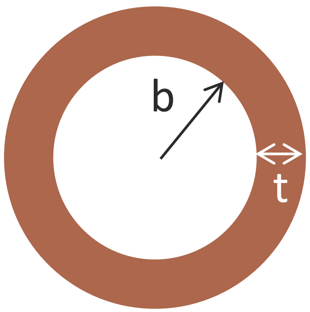
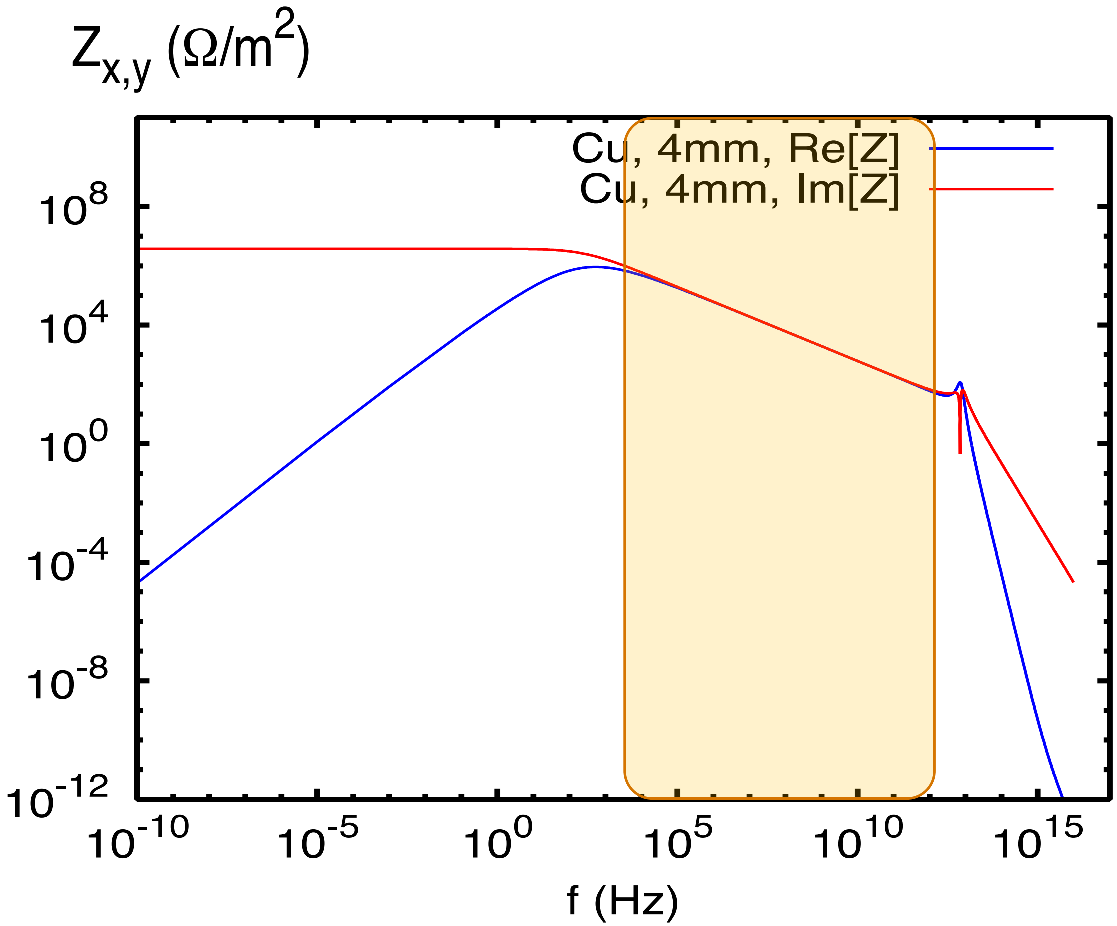


- An example: a 1 m long Cu pipe with radius b=2 cm and thickness t = 4 mm in vacuum



- Highlighted region shows the typical $\omega^{-1/2}$ scaling
- Another scaling is with respect to b where:
  - Longitudinal impedance ~ b^{-1}
  - Transverse imepdance ~b^{-3}

Signpost

- We have seen the **definition of transverse wake fields** and how they can be classified into constant, dipolar and quadrupolar wake fields.
- We have discussed the **basic features** of each of the different types of transverse wake fields.
- We will now look into how the impact of wake fields onto charged particle beams can be **modeled numerically** to prepare for investigating the different types of coherent instabilities further along.

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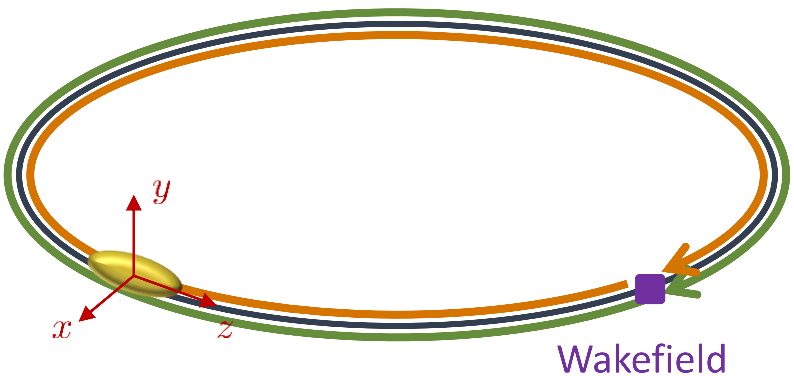
Quick summary of steps for solving numerically

- Tracking one full turn including the interaction with wake fields:



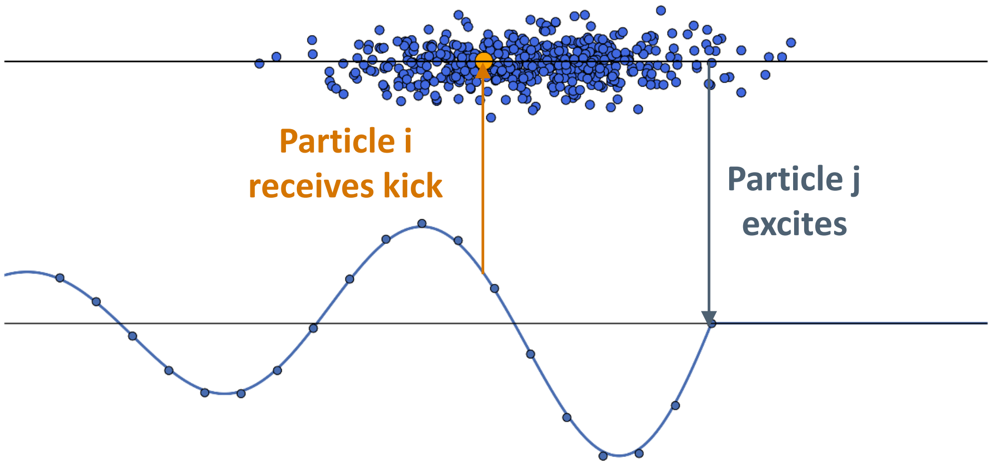
$$\begin{pmatrix} x_i \\ x_i' \end{pmatrix}\Bigg|_{k+1} = \mathcal{M}_i \begin{pmatrix} x_i \\ x_i' \end{pmatrix}\Bigg|_{k}$$

$$\begin{pmatrix} z_i \\ \delta_i \end{pmatrix}\Bigg|_{k+1} = \mathcal{I}\left[\begin{pmatrix} z_i \\ \delta_i \end{pmatrix}\Bigg|_{k}\right]$$

$$(x_i')\big|_{k+1} = (x_i')\big|_{k} + \mathcal{WK}$$

1. Initialise a macroparticle distribution with a given emittance
2. Update transverse coordinates and momenta according to the linear periodic transfer map – adjust the individual phase advance according to chromaticity and detuning with amplitude
3. Update the longitudinal coordinates and momenta according to the leap-frog integration scheme
4. Update momenta only (apply kicks) according to wake field generated kicks

Numerical implementation of wakefields



- The **wake functions** are obtained **externally** from electromagnetic codes such as ACE3P, CST, GdfidL, HFSS…
- In the tracking code, the **wake fields** at a given point need to **update the particle/macroparticle momenta** (i.e. they provide a kick)
- The **kick** on to a particle/**macroparticle 'i'** generated by **all particles/macroparticles 'j'** via the wake fields is:

$$\Delta x_i' = -\frac{e^2}{m\gamma\beta^2 c^2} \times \sum_{j=0}^{\text{n\_macroparticles}} \begin{cases} W_{Cx}(z_i - z_j) \\ \Delta x_j \cdot W_{Dx}(z_i - z_j) \\ W_{Qx}(z_i - z_j)\,\Delta x_i \end{cases}$$

Numerical implementation of wakefields

- To be **numerically more efficient**, the beam is **longitudinally sliced** into a set of slices
- Provided the slices are thin enough to sample the wake fields, the wakes can be **assumed constant within a single slice**
- The **kick** on to the set of **macroparticles in slice 'i'** generated by the set of **macroparticles in slice 'j'** via the wake fields now becomes:

$$\Delta x'[i] = -\frac{e^2}{m\gamma\beta^2c^2}$$

$$\times \sum_{j=0}^{\text{n\_slices}} \begin{cases} N[j] \cdot W_{Cx}[i-j] \\ N[j]\langle x\rangle[j] \cdot W_{Dx}[i-j] \\ N[j] \cdot W_{Qx}[i-j]\,\Delta x[i] \end{cases}$$

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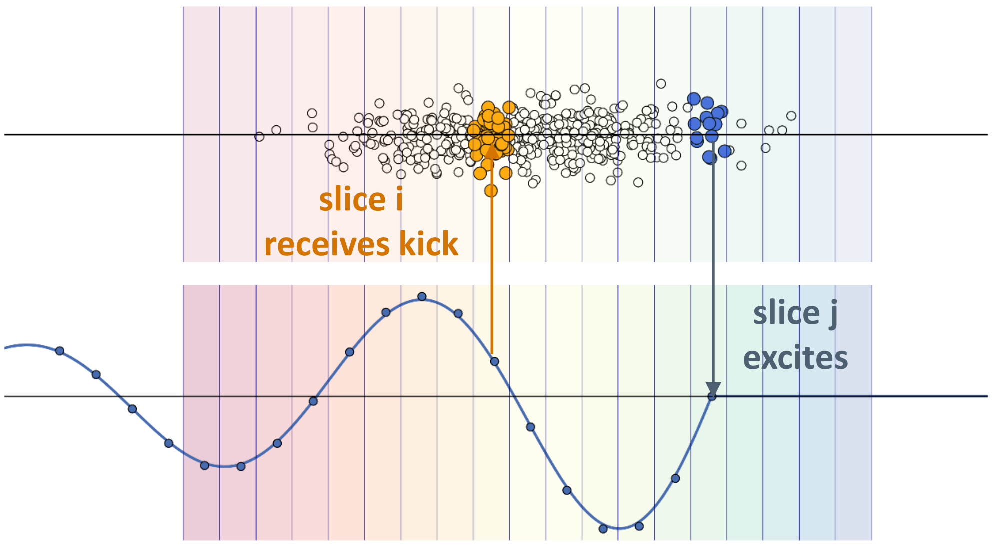

$$\Delta x_i - \frac{\quad}{m\gamma\beta^2c^2}$$

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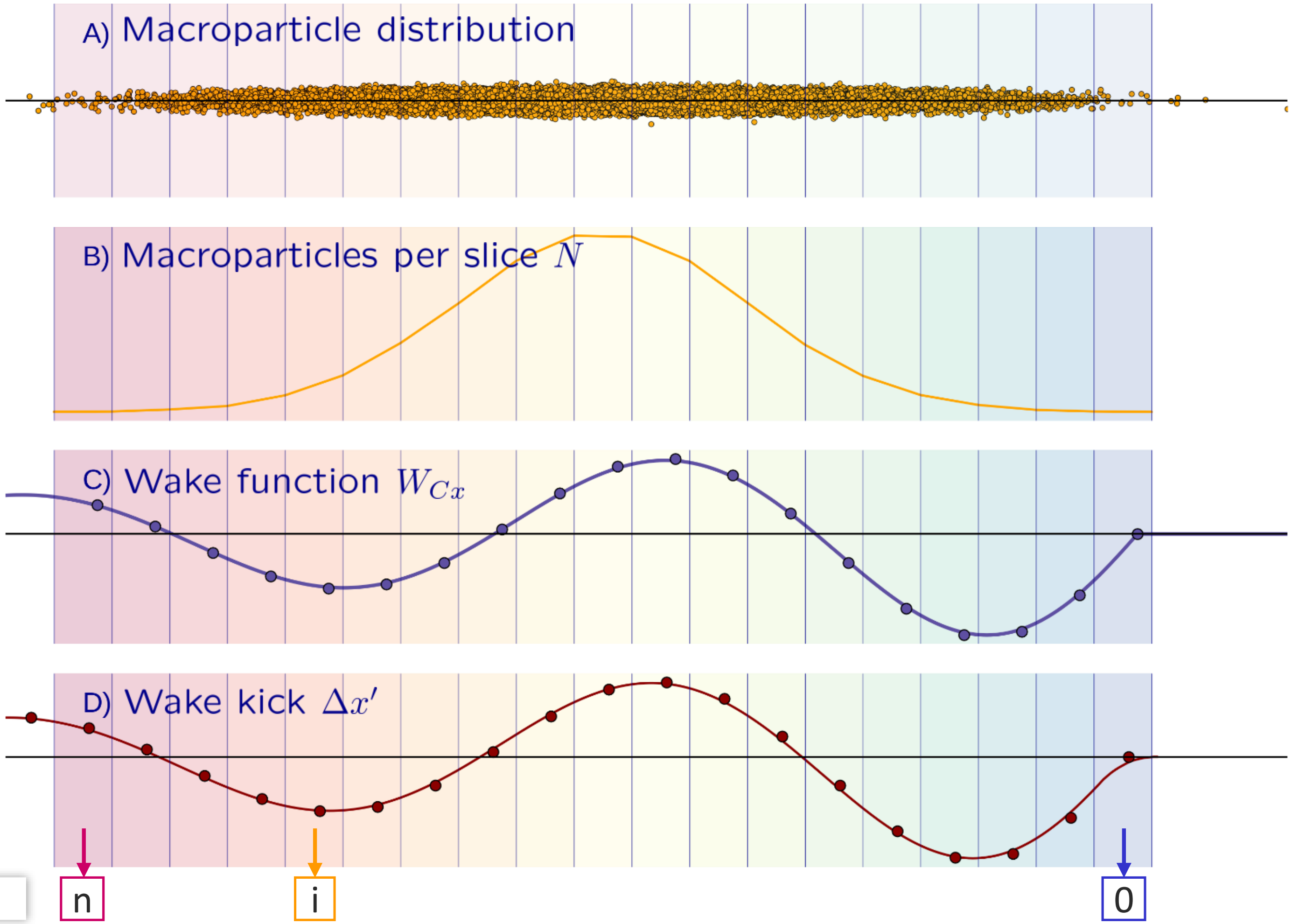
- N[i]: **number of macroparticles in slice 'i'** → can be pre-computed and stored in memory
- W[i]: **wake function** pre-computed and stored in memory **for all differences i-j**

Count	0	1	2	3	4	5	6
N[i]	…	…	…	…	…	…	…
W[i]	…	…	…	…	…	…	…

Numerical implementation of wakefields

A. Bin macroparticles into discrete set of slices – binning needs to be fine enough as to sample the wake function
B. Compute number of macroparticles per slice
C. Perform convolution with wake function to obtain wake kicks
D. Apply wake kicks (momentum update)



$$\Delta x'[i] = -\frac{e^2}{m\gamma\beta^2c^2}\sum_{j=0}^{i} N[j]\cdot W_{Cx}[i-j]\,, \qquad x'[i] \to x'[i] + \Delta x'[i]\,, \qquad i = 1, \ldots, \text{n\_slices}$$

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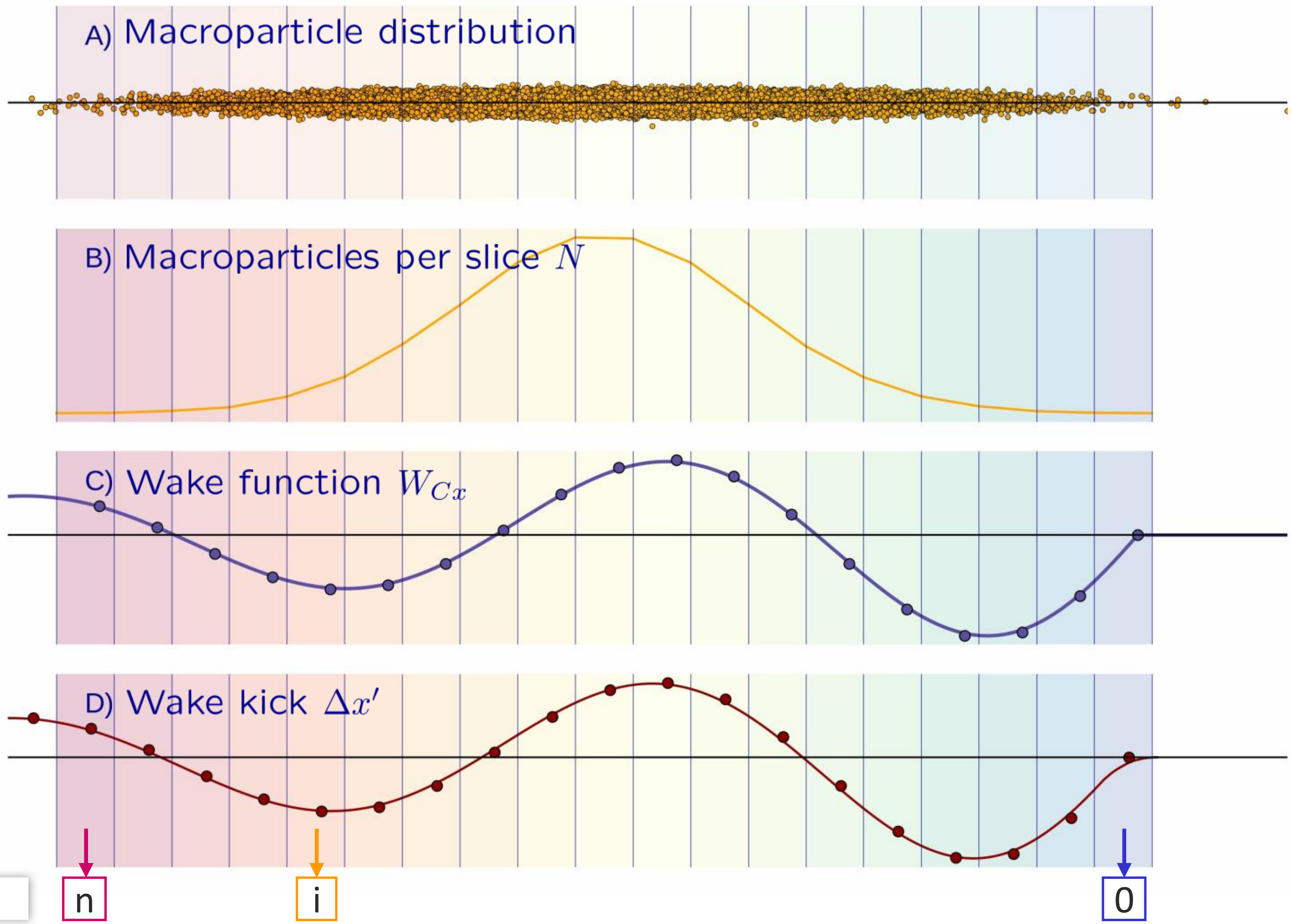

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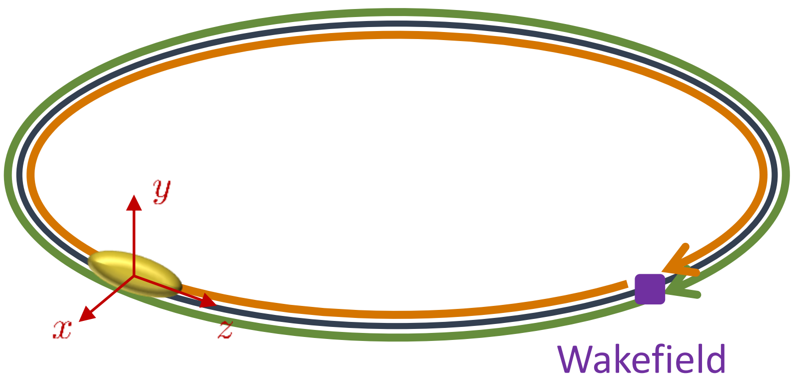

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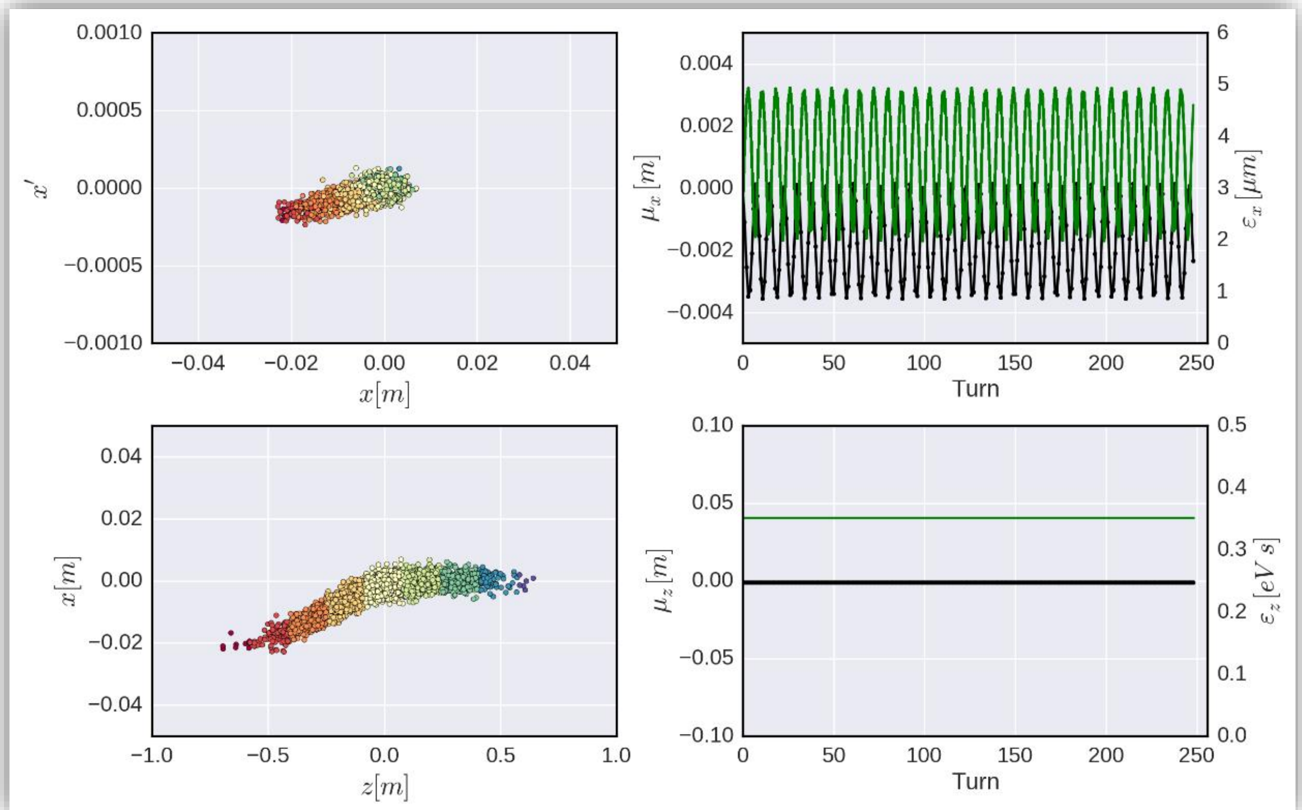
3. Update the longitudinal coordinates and momenta according to the leap-frog integration scheme
4. Update momenta only (apply kicks) according to wake field generated kicks
5. Repeat turn-by-turn…

Examples – constant wakes

$$\Delta x'[i] = -\frac{e^2}{m\gamma\beta^2 c^2} \sum_{j=0}^{i} N[j] \cdot W_{Cx}[i-j]$$

Dipolar term → orbit kick

Slice dependent change of closed orbit (if line density does not change)



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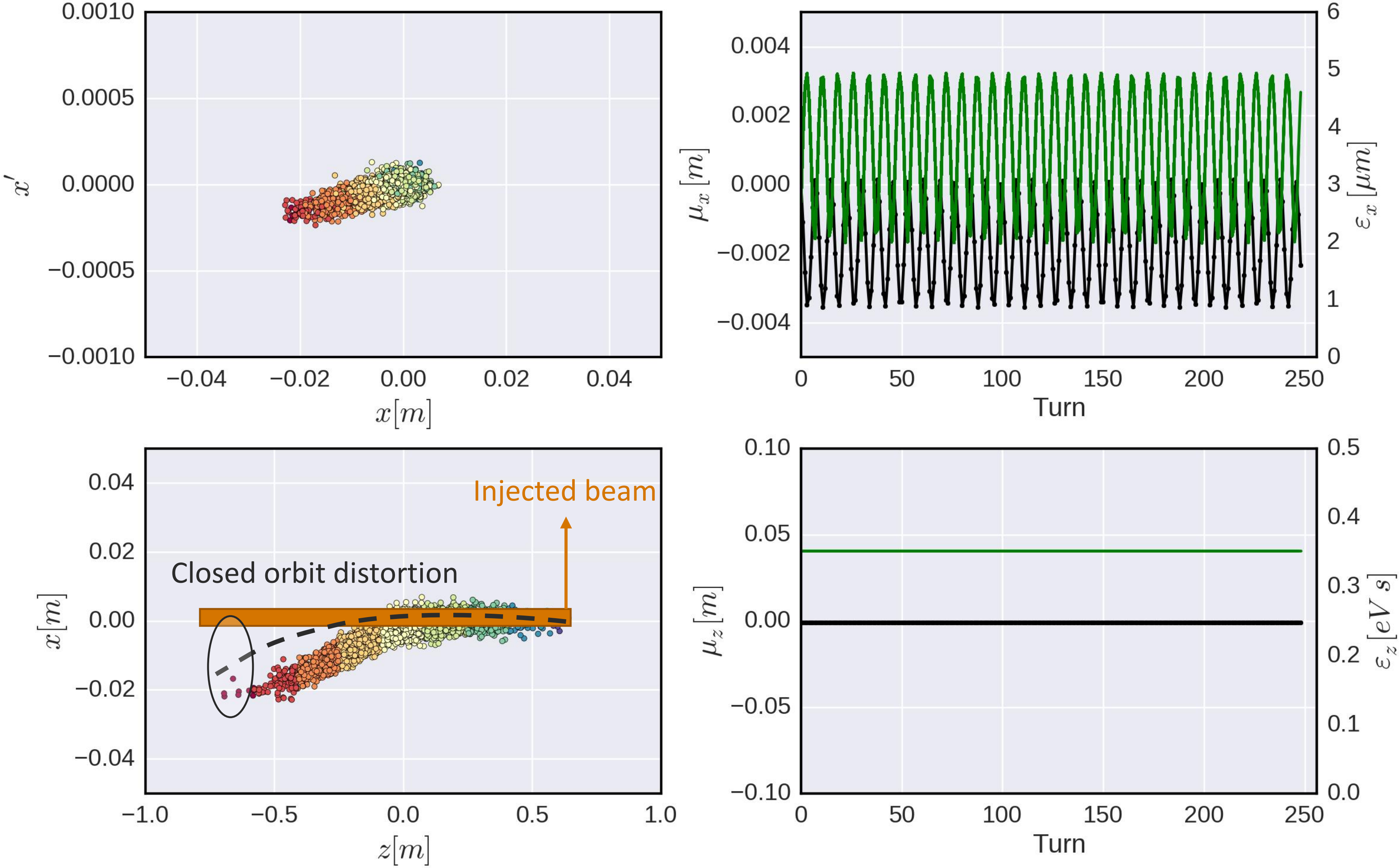

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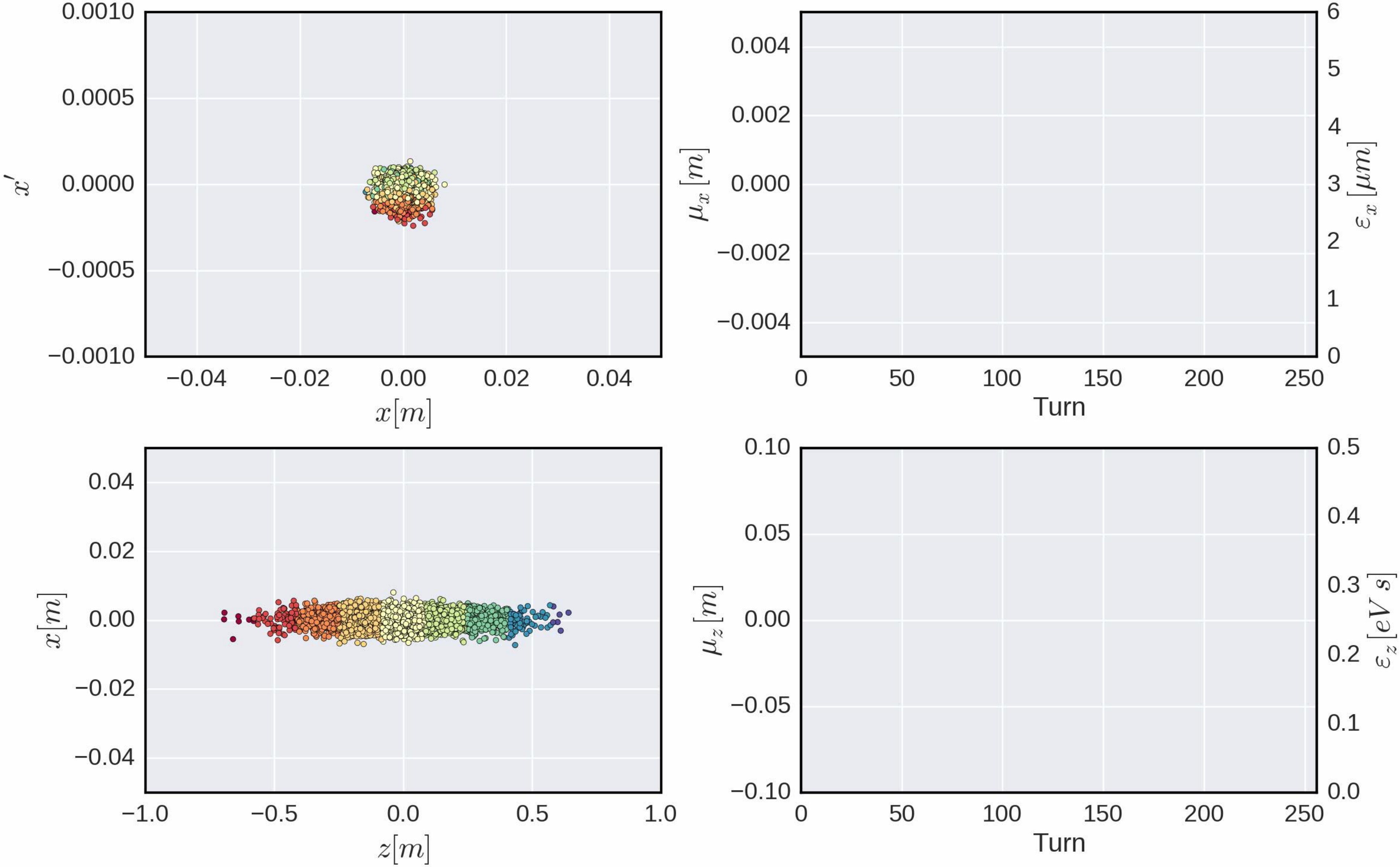

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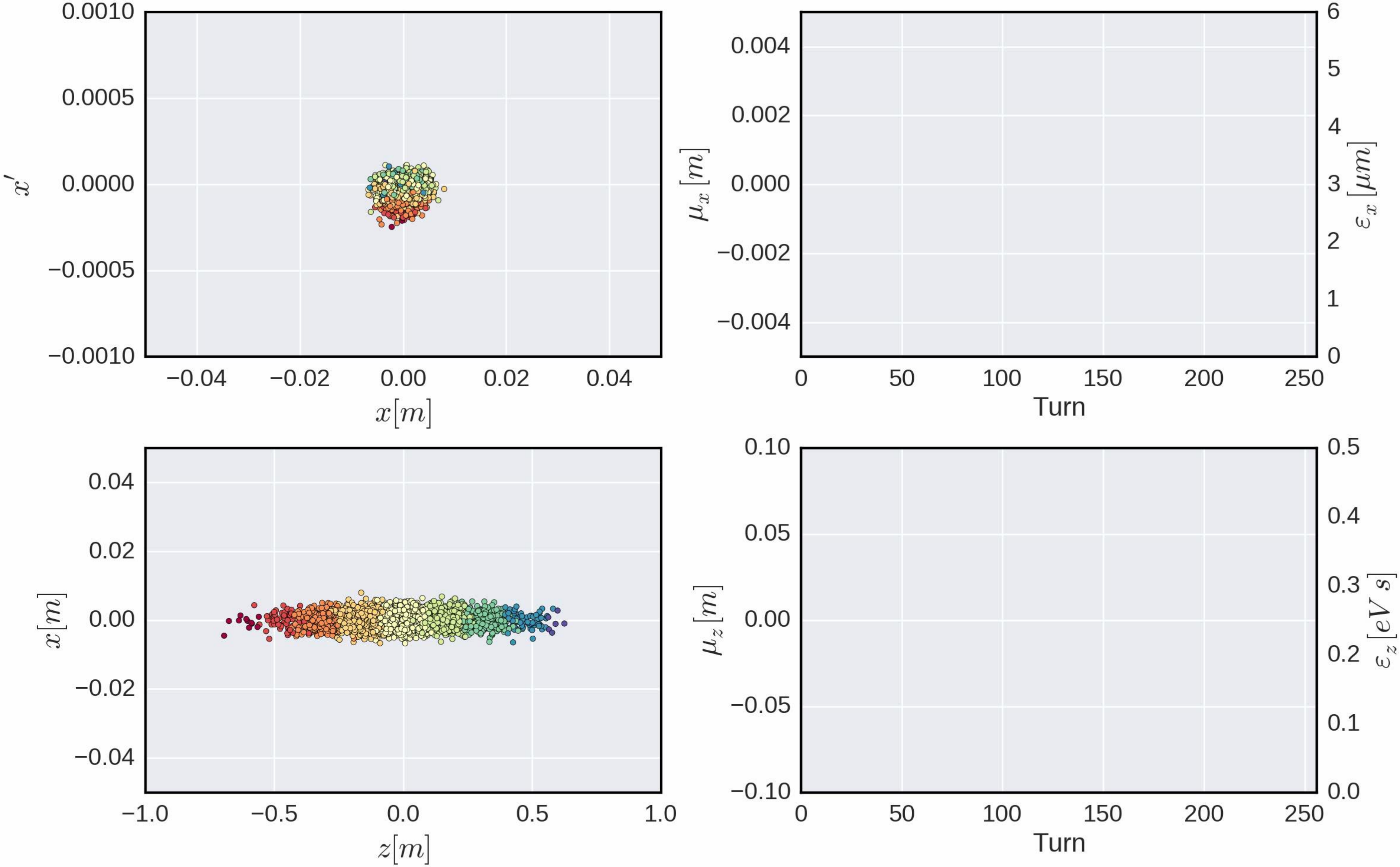

Examples – dipole wakes

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Dipolar term → orbit kick

Offset dependent orbit kick
→ kicks can accumulate

- Without synchrotron motion:
  kicks accumulate turn after turn – the **beam is unstable** → beam break-up in linacs

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With synchrotron motion we can get into a feedback loop

- Without synchrotron motion:
  kicks accumulate turn after turn – the **beam is unstable** → beam break-up in linacs
- With synchrotron motion:
  - Chromaticity ≠ 0
    - **Headtail modes** → beam is unstable (can be very weak and often damped by non-linearities)
  - Chromaticity = 0
    - Synchrotron sidebands are well separated → **beam is stable**
    - Synchrotron sidebands couple → **(transverse) mode coupling instability**

Examples – dipole wakes

The CERN Accelerator School

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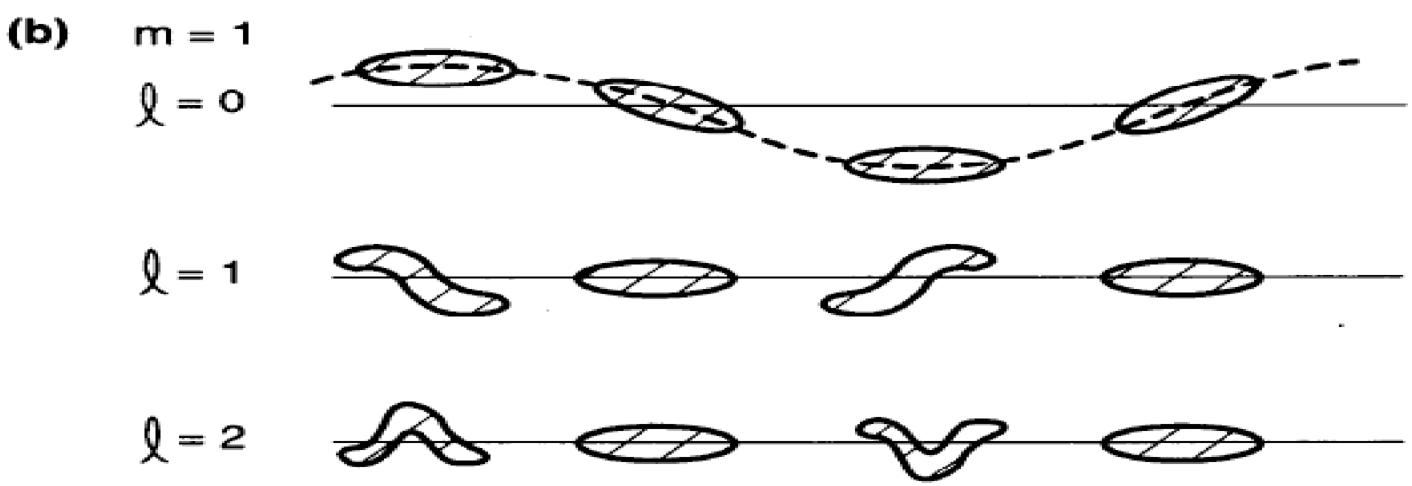
Dipole wakes – headtail modes

- As soon as **chromaticity is non-zero**, a ‘resonant’ condition can be met as particles now can ‘synchronize’ their synchrotron amplitude dependent betatron motion with the action of the wake fields.
- **Headtail modes arise** – the order of the respective mode depends on the chromaticity together with the impedance and bunch spectrum
- Different transverse head-tail modes **correspond to different parts of the bunch** oscillating with relative phase differences, for example:
  - Mode 0 is a rigid bunch mode
  - Mode 1 has head and tail oscillating in counter-phase
  - Mode 2 has head and tail oscillating in phase and the bunch center in opposition



Dipole wakes – headtail modes

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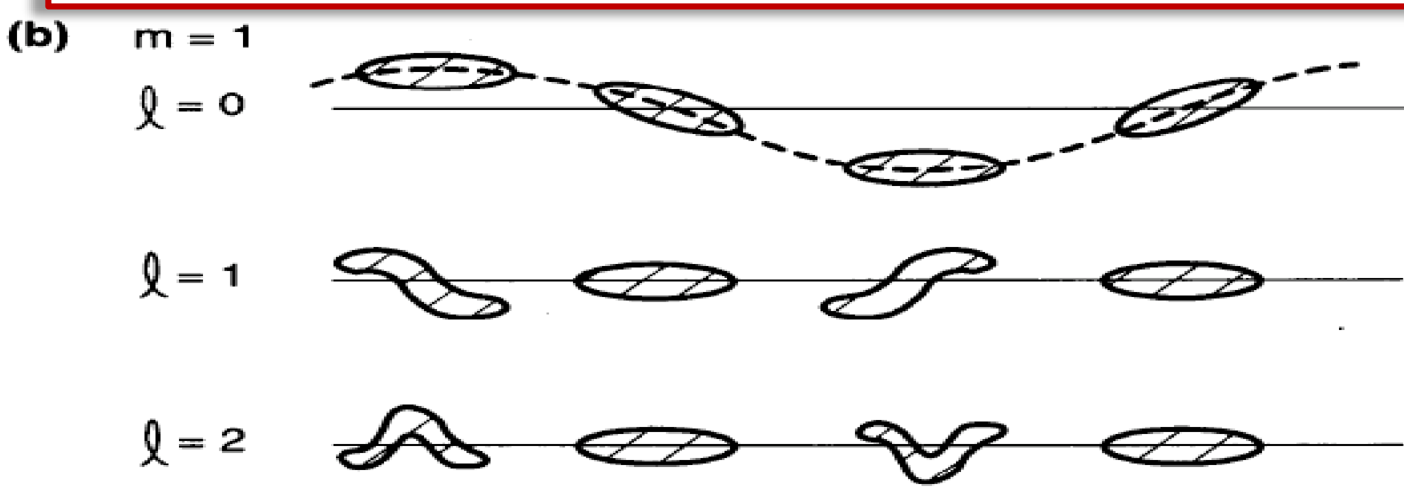
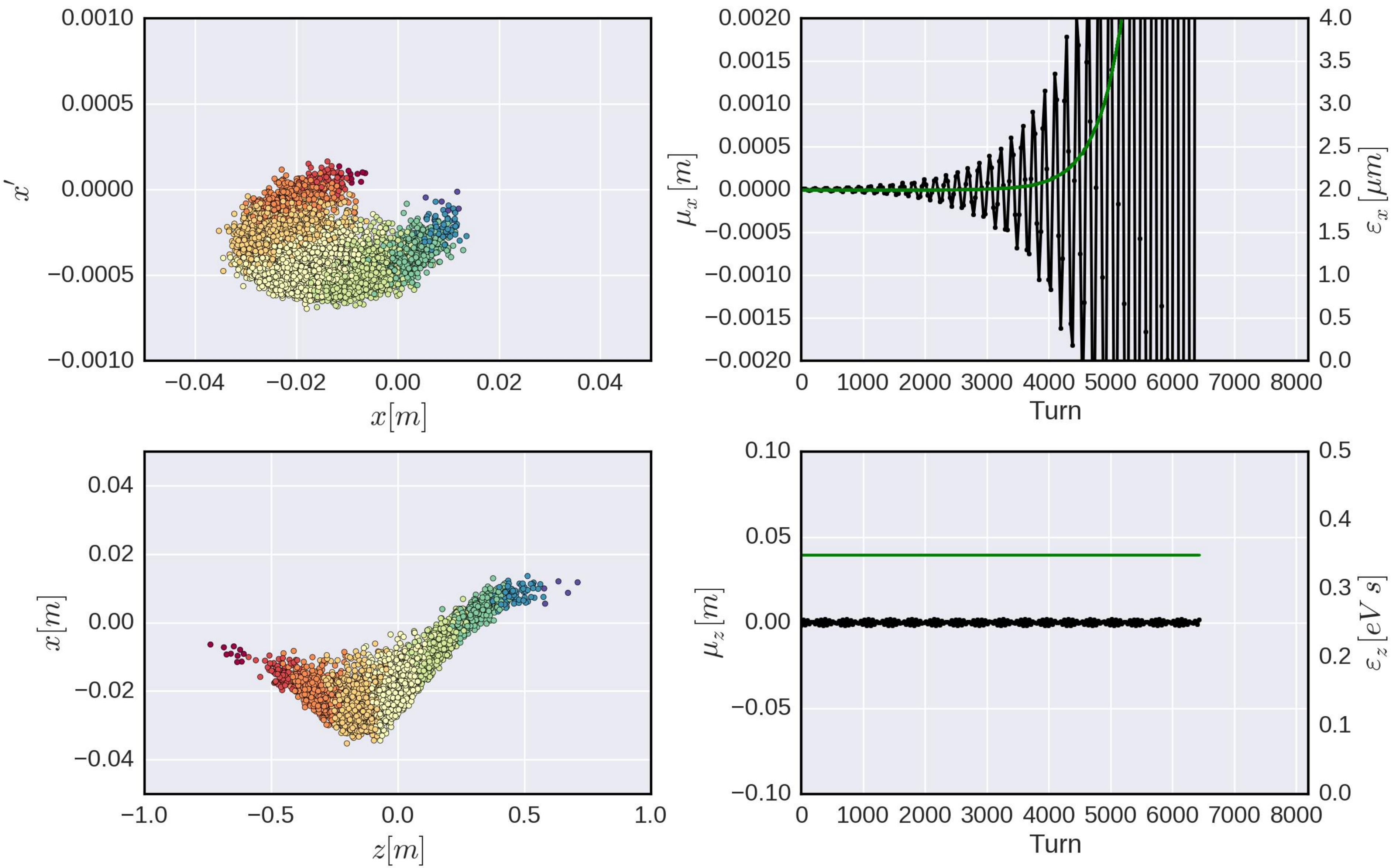
- **Headt…** … s on the ch… m
- Differ… **s of the bu…**
  - Mo…
  - Mo…
  - Mo… osition

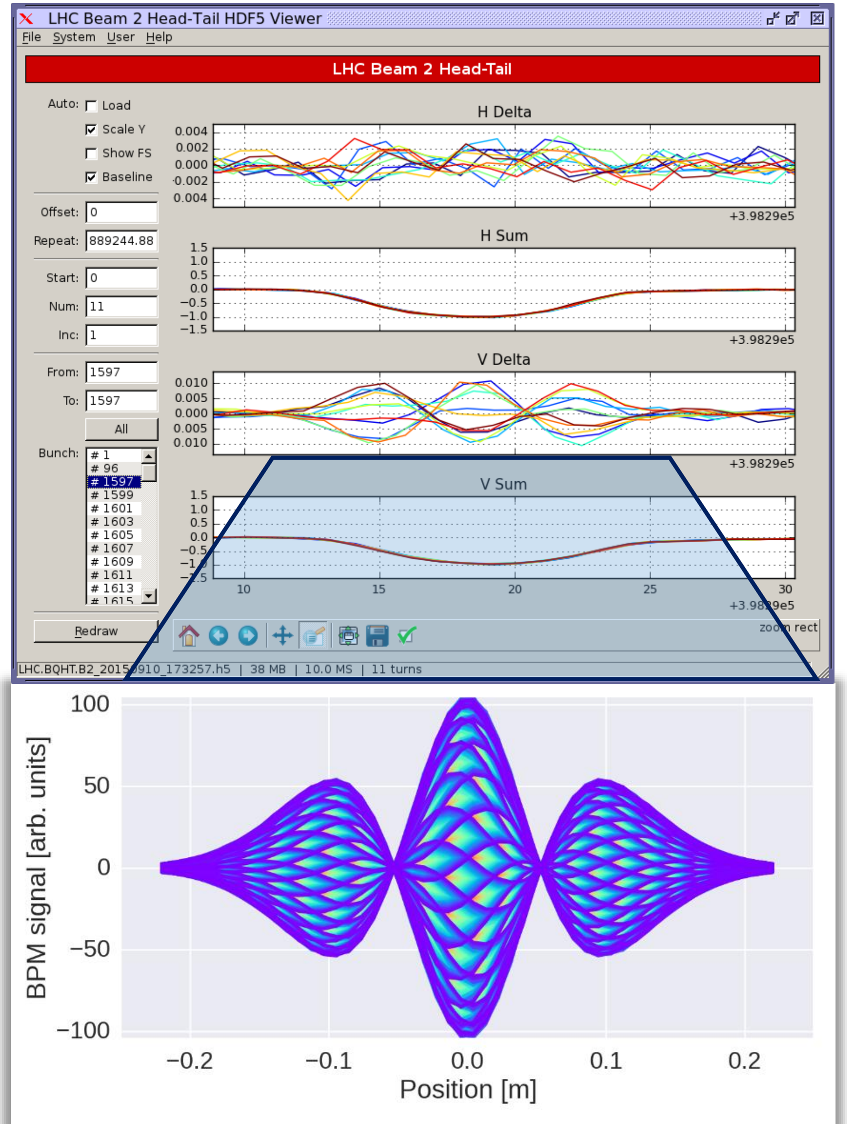
> Remark:
>
> Due to this ‚synchronicity', **below transition** (η<0):
>
> - Mode 0 is unstable if Q'>0.
> - Higher order modes tend to be unstable if Q'<0 (though at lower growth rates).
>
> The **situation is reversed** when a machine is operated above transition.

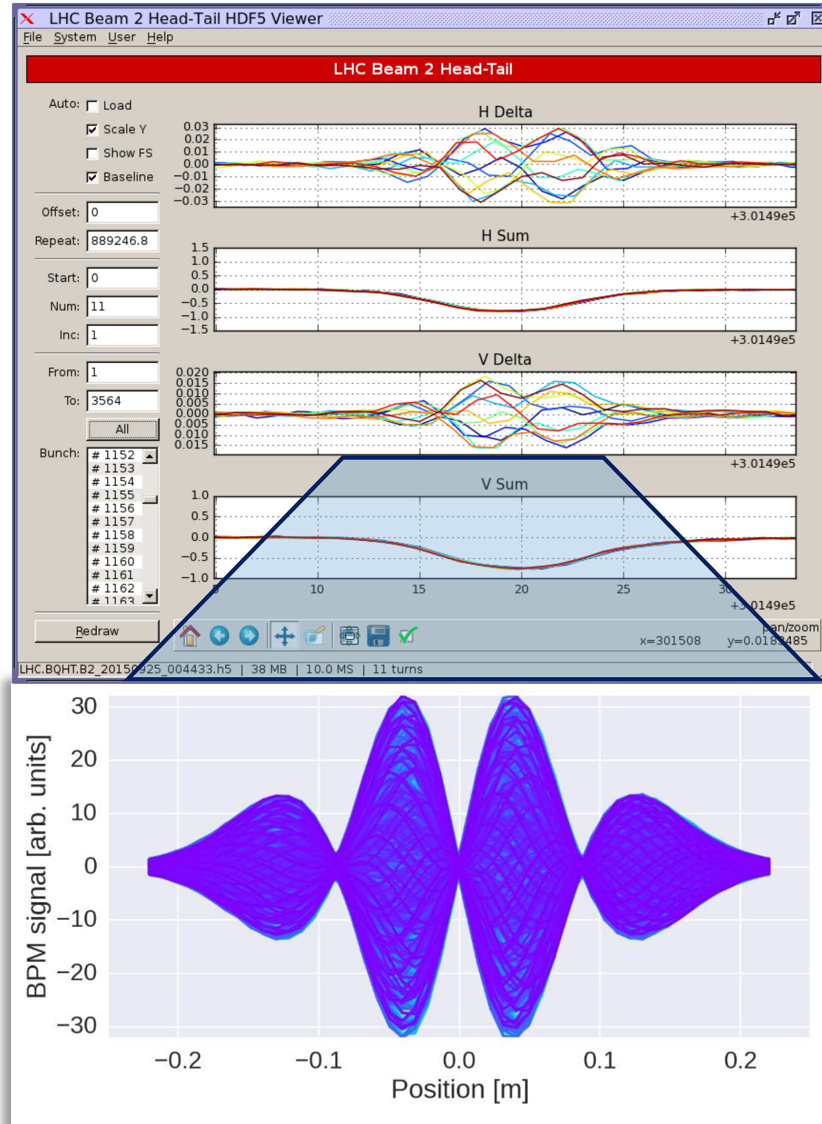


Dipole wakes – headtail modes



Example: Headtail modes in the LHC





Signpost

- We have seen how the impact of **wake fields on charged particle beams** can be **implemented numerically** in an efficient manner via **the longitudinal discretization** of bunches.
- We have used the simulation models to show **orbit effects** and **headtail instabilities** from transverse wake fields.
- We will now **derive another fundamentally limiting effect** using **analytical models**. One very simple but already quite powerful tool are **two-particle models**.

Part 3: Transverse wakefields – their different types and impact on beam dynamics

- Transverse wake function and impedance
- Numerical implementation, transverse „potential well distortion“ ad headtail instabilities
- Two particle models, transverse mode coupling instability

The Strong Head Tail Instability

- Aka the Transverse Mode Coupling Instability:
  - To illustrate TMCI we will need to make use of **some simplifications**:
    - The bunch **is represented through two particles** carrying half the total bunch charge and placed in opposite phase in the longitudinal phase space
    - They both feel external linear focusing in all three directions (i.e. linear betatron focusing + linear synchrotron focusing).
    - Zero chromaticity (Q'x,y=0)
    - Constant transverse wake left behind by the leading particle
    - Smooth approximation → constant focusing + distributed wake

↓

We will:

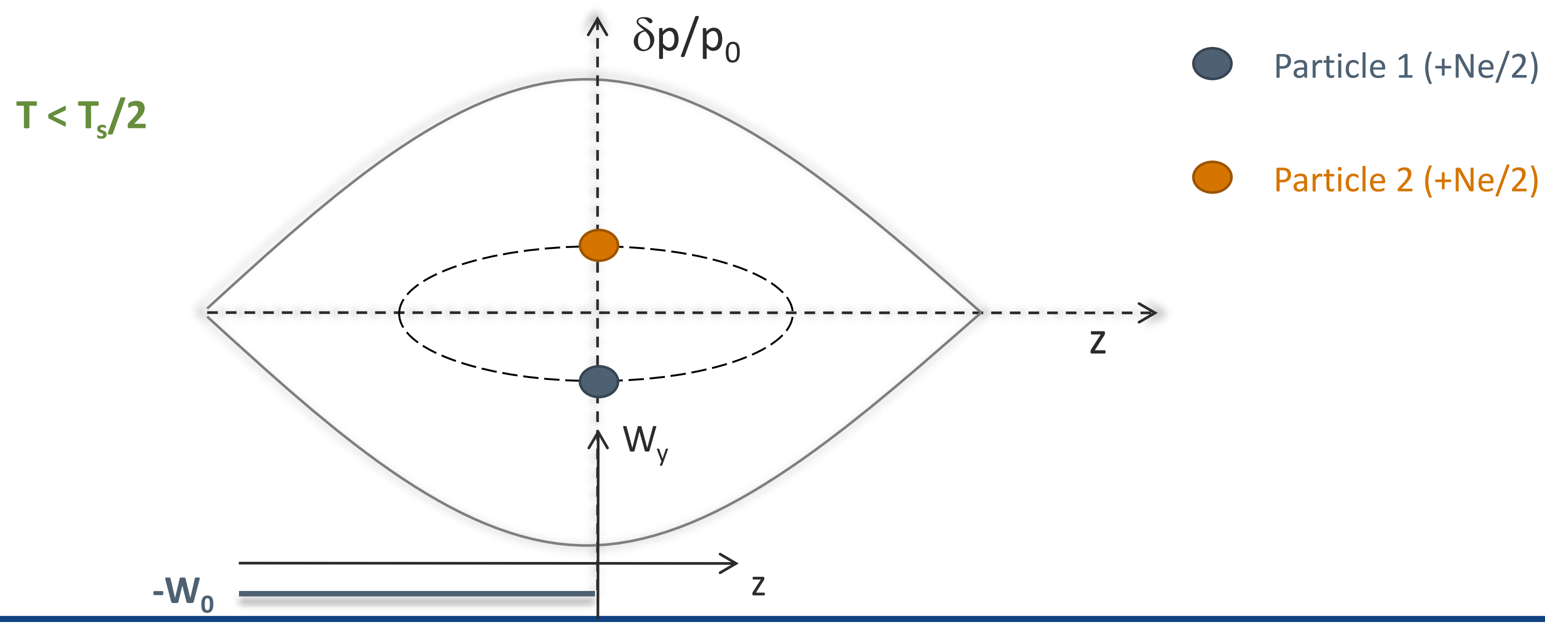
- Calculate a stability condition (threshold) for the transverse motion
- Have a look at the excited oscillation modes of the centroid

The Strong Head Tail Instability

- During the first half of the synchrotron motion, particle 1 is leading and executes free betatron oscillations, while particle 2 is trailing and feels the defocusing wake of particle 1

$$\begin{cases} \dfrac{d^2 y_1}{ds^2} + \left(\dfrac{\omega_\beta}{c}\right)^2 y_1 = 0 \\ \dfrac{d^2 y_2}{ds^2} + \left(\dfrac{\omega_\beta}{c}\right)^2 y_2 = \left(\dfrac{e^2}{m_0 c^2}\right) \dfrac{N\,W_0}{2\gamma C}\, y_1(s) \end{cases}$$

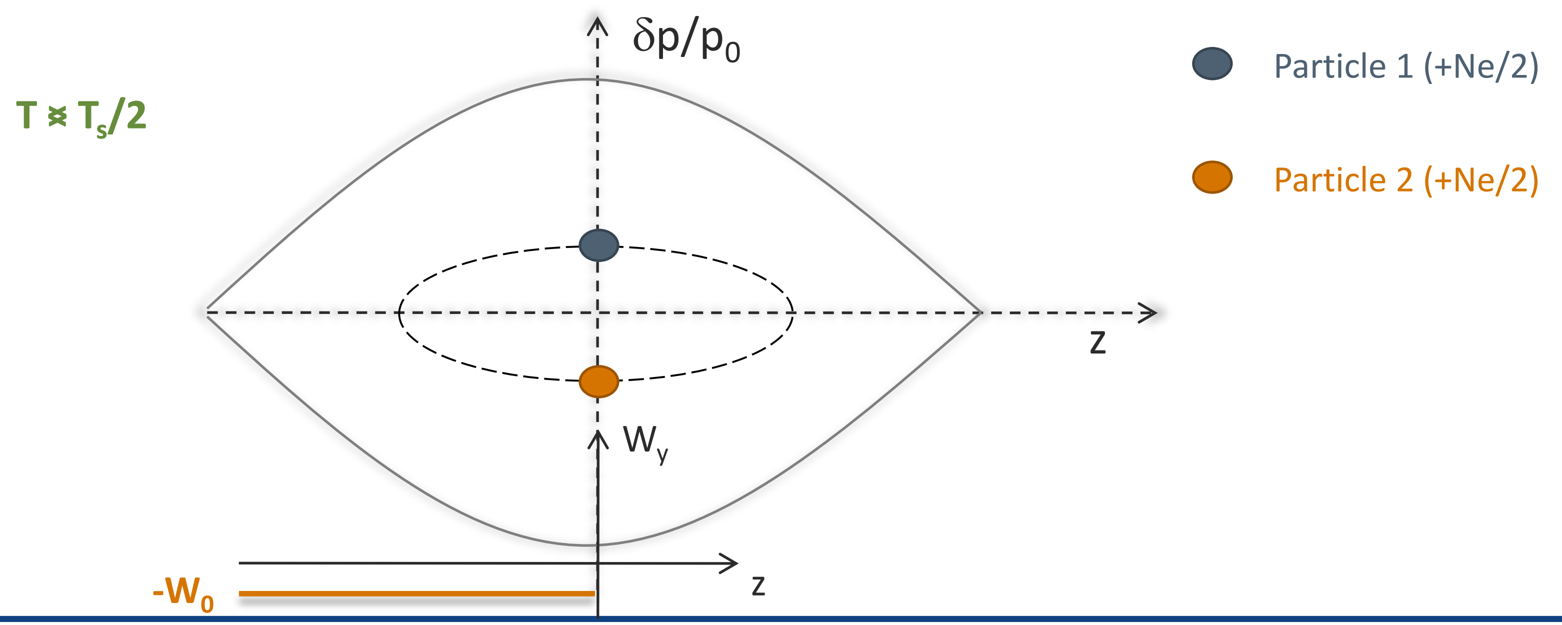
$$0 < s < \frac{\pi c}{\omega_s}$$



The Strong Head Tail Instability

- During the first half of the synchrotron motion, particle 1 is leading and executes free betatron oscillations, while particle 2 is trailing and feels the defocusing wake of particle 1
- During the second half of the synchrotron period, the situation is reversed:

$$
\begin{cases}
\dfrac{d^2 y_1}{ds^2} + \left(\dfrac{\omega_\beta}{c}\right)^2 y_1 = \left(\dfrac{e^2}{m_0 c^2}\right) \dfrac{N W_0}{2\gamma C} y_2(s) \\
\dfrac{d^2 y_2}{ds^2} + \left(\dfrac{\omega_\beta}{c}\right)^2 y_2 = 0
\end{cases}
\qquad \frac{\pi c}{\omega_s} < s < \frac{2\pi c}{\omega_s}
$$



The Strong Head Tail Instability

- We solve with respect to the complex variables defined below during the first half of synchrotron period
- y1(s) is a free betatron oscillation
- y2(s) is the sum of a free betatron oscillation plus a driven oscillation with y1(s) being its driving term

$$\tilde{y}_{1,2}(s) = y_{1,2}(s) + i\frac{c}{\omega_\beta}\, y'_{1,2}(s)$$

$$\tilde{y}_1(s) = \tilde{y}_1(0)\, \exp\left(-\frac{i\omega_\beta s}{c}\right)$$

$$\tilde{y}_2(s) = \underbrace{\tilde{y}_2(0)\, \exp\left(-\frac{i\omega_\beta s}{c}\right)}_{\text{Free oscillation term}}$$

since we consider $s = \frac{\pi c}{\omega_s}$

$$+\underbrace{i\frac{Ne^2 W_0}{4\, m_0 \gamma c\, C\omega_\beta}\left(\frac{c}{\omega_\beta}\tilde{y}_1^*(0)\sin\left(\frac{\omega_\beta s}{c}\right) + \tilde{y}_1(0)\, s\, \exp\left(-\frac{i\omega_\beta s}{c}\right)\right)}_{\text{Driven oscillation term}}$$

- Second term in RHS equation for y2(s) negligible if $\omega_s << \omega_\beta$
- We can now transform these equations into linear mapping across half synchrotron period

The Strong Head Tail Instability

- We can now transform these equations into **linear mapping** across half synchrotron period

$$\begin{pmatrix}\tilde{y}_1\\ \tilde{y}_2\end{pmatrix}_{s=\pi c/\omega_s} = \left[\exp\left(-\frac{i\pi\omega_\beta}{\omega_s}\right)\cdot\begin{pmatrix}1 & 0\\ i\Upsilon & 1\end{pmatrix}\right]\cdot\begin{pmatrix}\tilde{y}_1\\ \tilde{y}_2\end{pmatrix}_{s=0}, \qquad \Upsilon = \frac{\pi N e^2 W_0}{4\, m_0 \gamma\, C \omega_\beta \omega_s}$$

- In the second half of synchrotron period, **particles 1 and 2 exchange their roles** – we can therefore find the transfer matrix over the full synchrotron period for both particles. We can **analyze the eigenvalues** of the two particle system

$$\begin{aligned}\begin{pmatrix}\tilde{y}_1\\ \tilde{y}_2\end{pmatrix}_{s=2\pi c/\omega_s} &= \left[\exp\left(-\frac{i\,2\pi\omega_\beta}{\omega_s}\right)\cdot\begin{pmatrix}1 & i\Upsilon\\ 0 & 1\end{pmatrix}\cdot\begin{pmatrix}1 & 0\\ i\Upsilon & 1\end{pmatrix}\right]\cdot\begin{pmatrix}\tilde{y}_1\\ \tilde{y}_2\end{pmatrix}_{s=0}\\ &= \left[\exp\left(-\frac{i\,2\pi\omega_\beta}{\omega_s}\right)\cdot\begin{pmatrix}1-\Upsilon^2 & i\Upsilon\\ i\Upsilon & 1\end{pmatrix}\right]\cdot\begin{pmatrix}\tilde{y}_1\\ \tilde{y}_2\end{pmatrix}_{s=0}\end{aligned}$$

Strong Head Tail Instability – stability condition

$$\lambda_1 \cdot \lambda_2 = 1 \Rightarrow \lambda_{1,2} = \exp\left(\pm i\varphi\right)$$

$$\lambda_1 + \lambda_2 = 2 - \Upsilon^2 \Rightarrow \sin\left(\frac{\varphi}{2}\right) = \frac{\Upsilon}{2}$$

$$\Rightarrow \Upsilon = \frac{\pi N e^2 W_0}{4\, m_0 \gamma\, C \omega_\beta \omega_s} \leq 2$$

- Since the product of the eigenvalues is 1, the only condition for stability is that they both be purely imaginary exponentials
- From the second equation for the eigenvalues, it is clear that this is true only when $\sin(\varphi/2)<1$
- This translates into a **stability condition** on the beam/wake parameters

Strong Head Tail Instability – stability condition

$$N \leq N_{\text{threshold}} = \frac{8}{\pi e^2}\frac{p_0 \omega_s}{\beta_y}\left(\frac{C}{W_0}\right)$$

- Proportional to p_0 → bunches with higher energy tend to be more stable
- Proportional to ω_s → the quicker is the longitudinal motion within the bunch, the more stable is the bunch
- Inversely proportional to β_y → the effect of the impedance is enhanced if the kick is given at a location with large beta function

- Inversely proportional to the wake per unit length along the ring, W_0/C → a large integrated wake (impedance) lowers the instability threshold

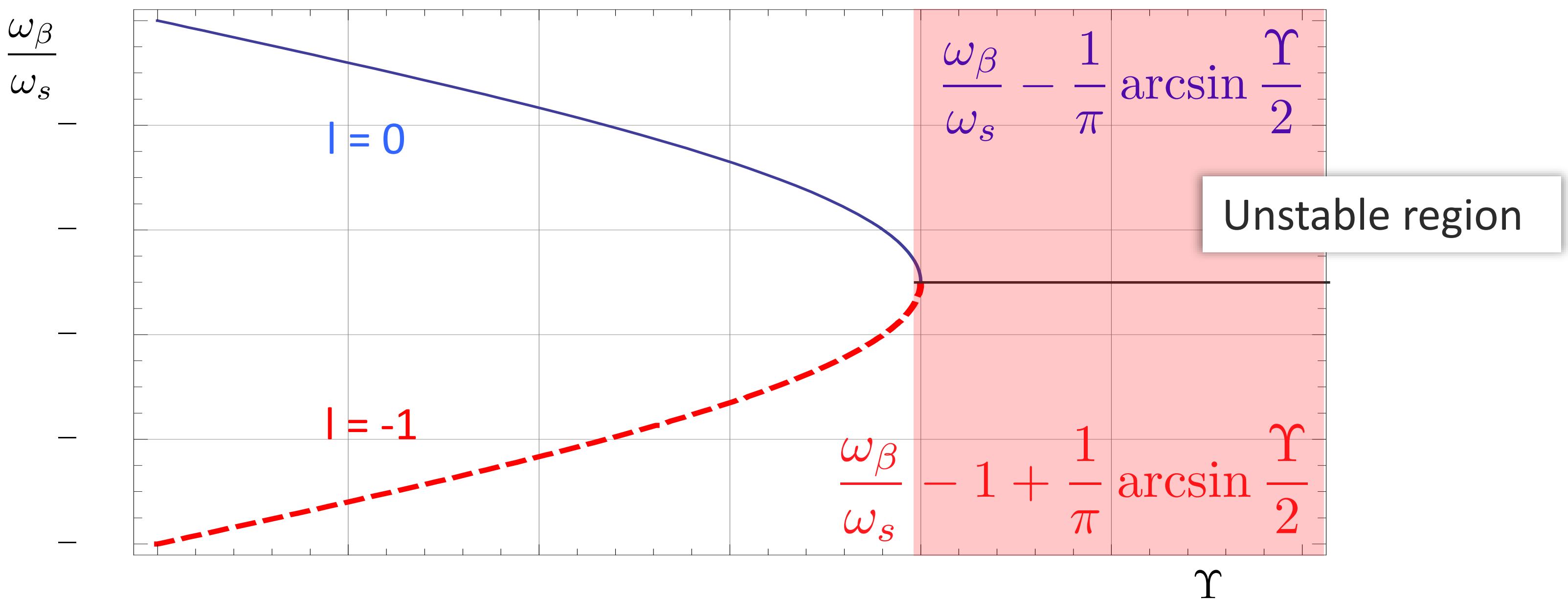
Strong Head Tail Instability – mode frequencies

- The evolution of the eigenstates follows:

$$\begin{pmatrix} \tilde{V}_{+n} \\ \tilde{V}_{-n} \end{pmatrix} = \exp\left(-i\frac{2\pi\omega_\beta}{\omega_s}n\right)\cdot\begin{pmatrix} \exp\left[-2i\arcsin\left(\frac{\Upsilon}{2}\right)\cdot n\right] & 0 \\ 0 & \exp\left[2i\arcsin\left(\frac{\Upsilon}{2}\right)\cdot n\right] \end{pmatrix}\begin{pmatrix} \tilde{V}_{+0} \\ \tilde{V}_{-0} \end{pmatrix}$$

Eigenfrequencies: $$\omega_\beta + l\omega_s \pm \frac{\omega_s}{\pi}\arcsin\frac{\Upsilon}{2}$$

They shift with increasing intensity



Strong Head Tail Instability – mode frequencies

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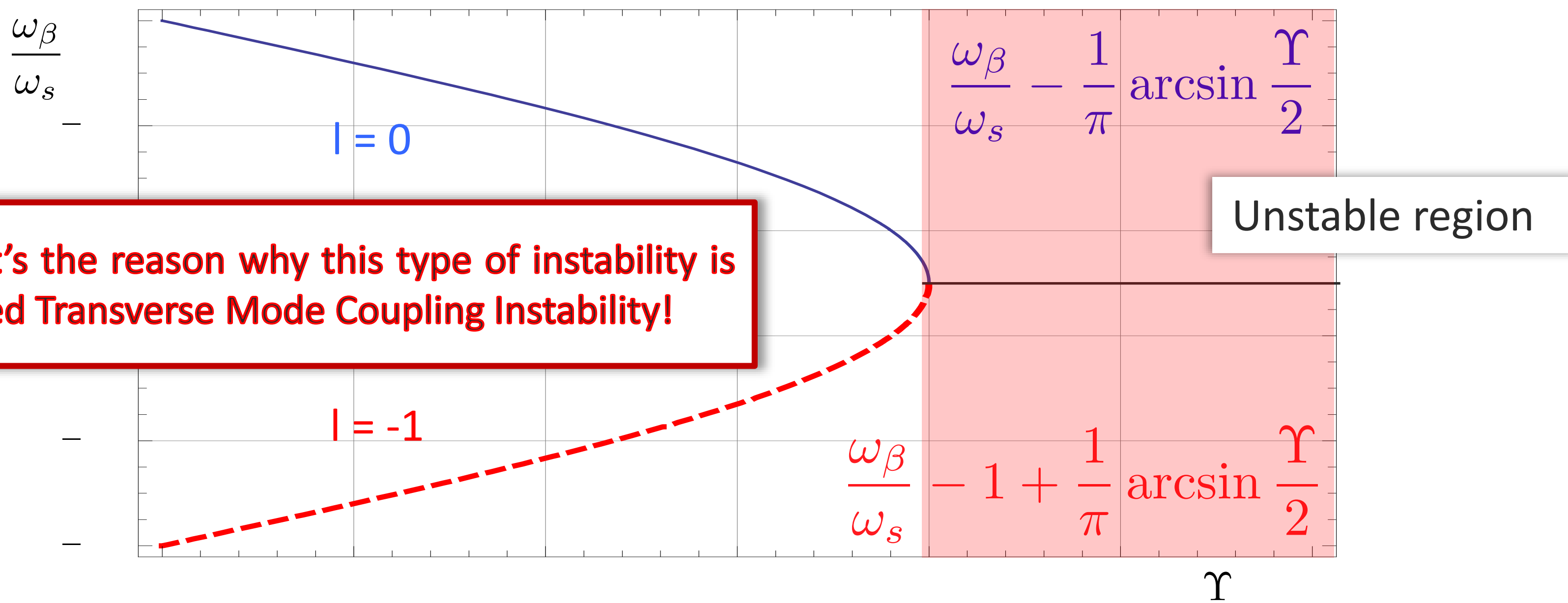
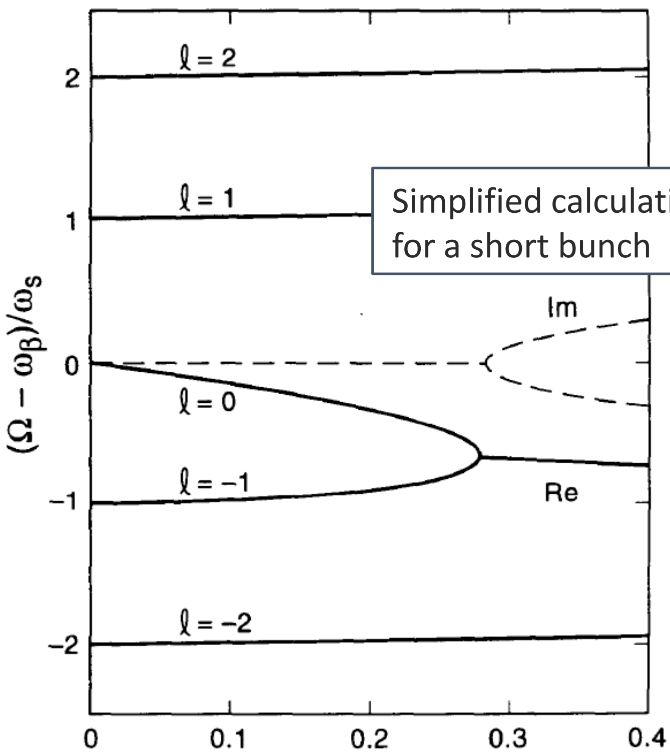


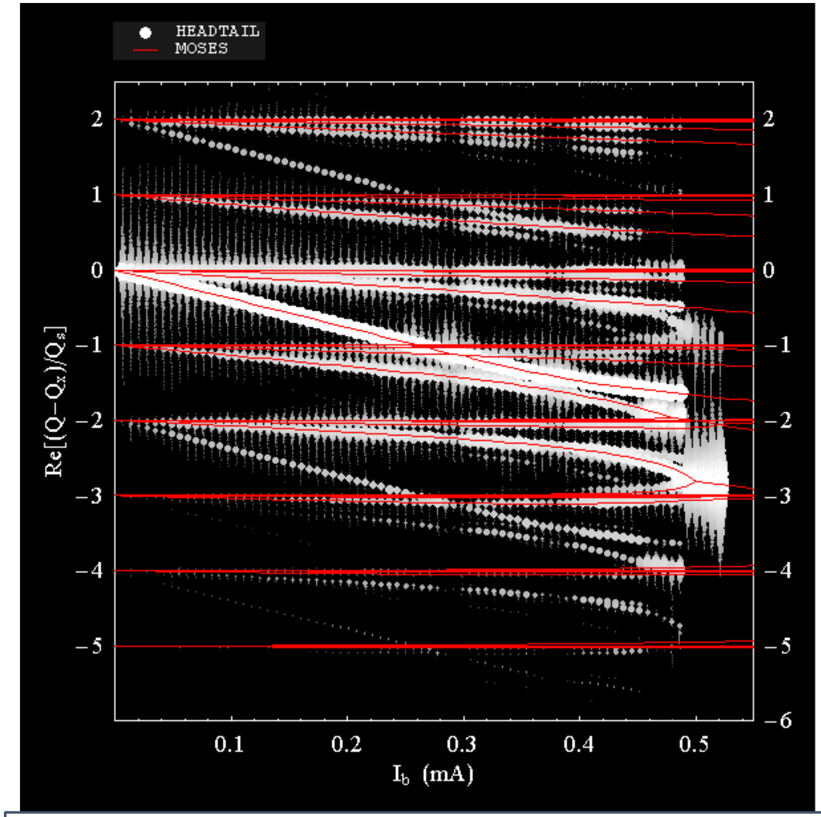
That's the reason why this type of instability is called Transverse Mode Coupling Instability!

Strong Head Tail Instability – why TMCI?

- For a real bunch, modes exhibit a more complicated shift pattern
- The shift of the modes can be calculated via Vlasov equation or can be found through macroparticle simulations



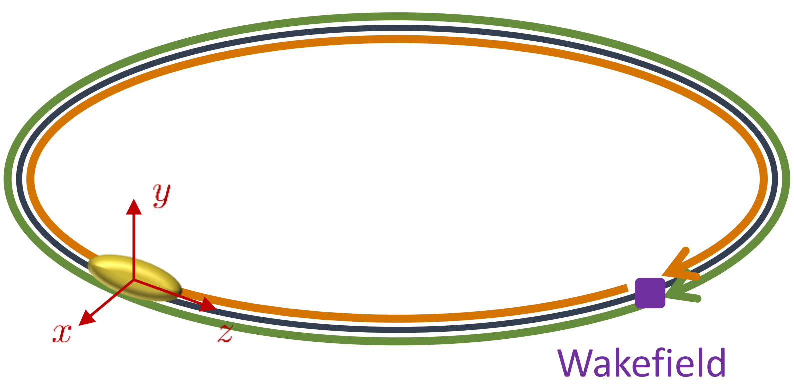
Simplified calculation for a short bunch



Full calculation for a relatively long SPS bunch (red lines) + macroparticle simulation (white traces)

Quick summary of steps for solving numerically

- Tracking one full turn including the interaction with wake fields:



$$\begin{pmatrix} x_i \\ x_i' \end{pmatrix}\Bigg|_{k+1} = \mathcal{M}_i \begin{pmatrix} x_i \\ x_i' \end{pmatrix}\Bigg|_{k}$$

$$\begin{pmatrix} z_i \\ \delta_i \end{pmatrix}\Bigg|_{k+1} = \mathcal{I}\left[\begin{pmatrix} z_i \\ \delta_i \end{pmatrix}\Bigg|_{k}\right]$$

$$(x_i')\big|_{k+1} = (x_i')\big|_{k} + \mathcal{WK}$$

1. Initialise a macroparticle distribution with a given emittance
2. Update transverse coordinates and momenta according to the linear periodic transfer map – adjust the individual phase advance according to chromaticity and detuning with amplitude
3. Update the longitudinal coordinates and momenta according to the leap-frog integration scheme
4. Update momenta only (apply kicks) according to wake field generated kicks
5. Repeat turn-by-turn…

Examples – dipole wakes

The CERN Accelerator School

$$\Delta x'[i] = -\frac{e^2}{m\gamma\beta^2 c^2\, C} \sum_{j=0}^{i} N[j]\,\langle x\rangle[j] \cdot W_{Dx}[i-j]$$

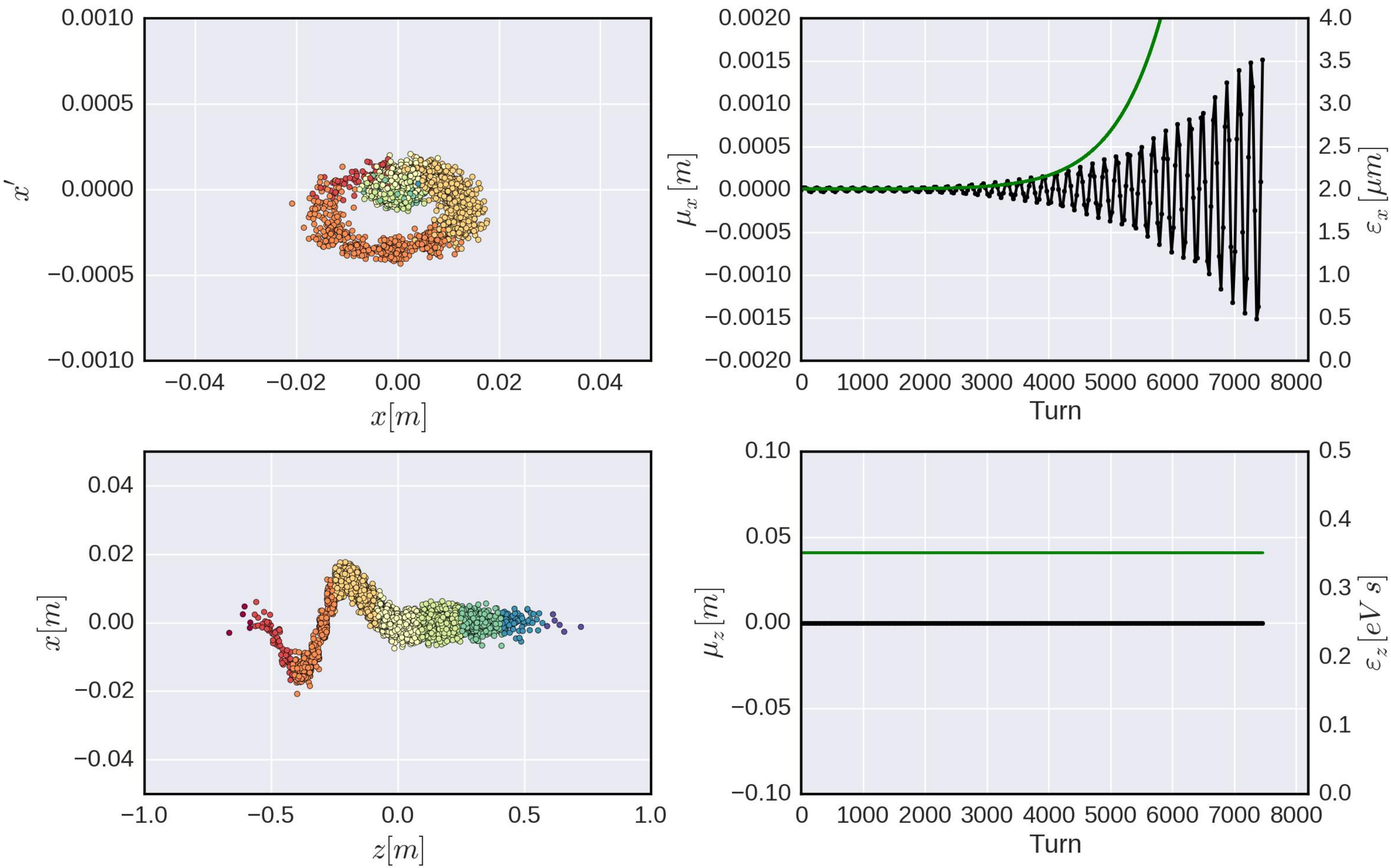
Dipolar term → orbit kick

Offset dependent orbit kick
→ kicks can accumulate

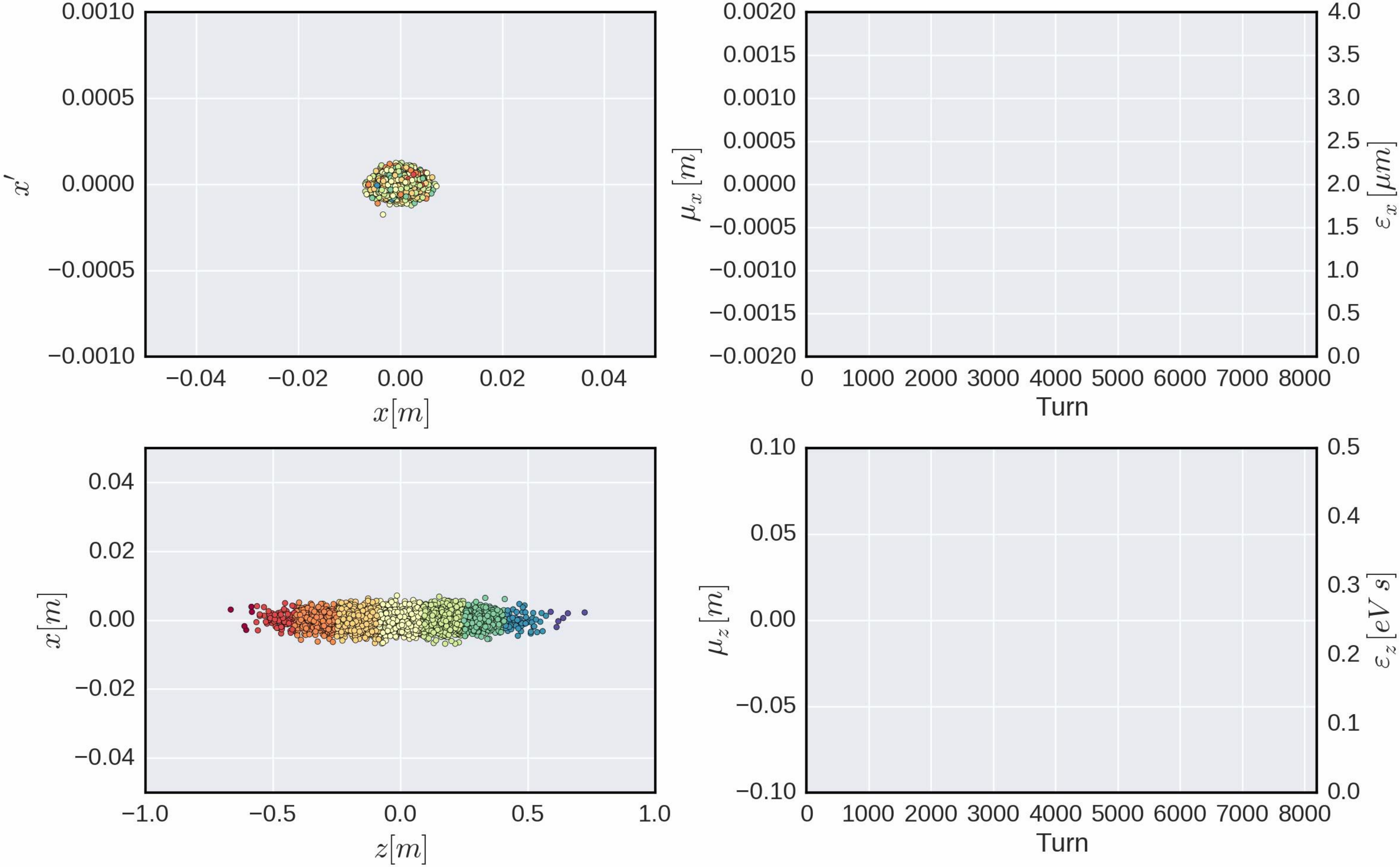
With synchrotron motion we can get into a feedback loop

- Without synchrotron motion:
  kicks accumulate turn after turn – the **beam is unstable** → beam break-up in linacs
- With synchrotron motion:
  - Chromaticity ≠ 0
    - **Headtail modes** → beam is unstable (can be very weak and often damped by non-linearities)
  - Chromaticity = 0
    - Synchrotron sidebands are well separated → **beam is stable**
    - Synchrotron sidebands couple → **(transverse) mode coupling instability**

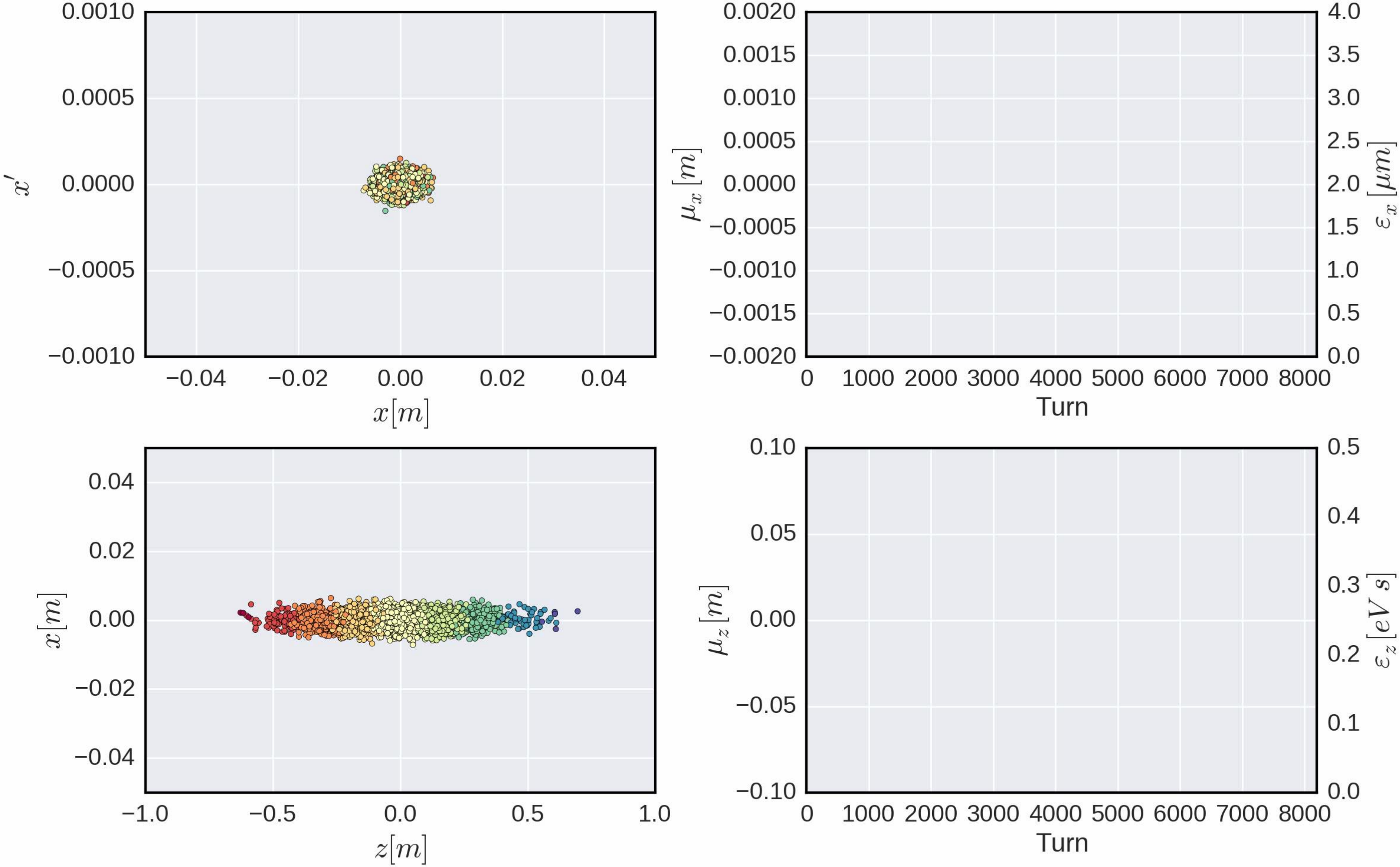
Dipole wakes – beam break-up



Dipole wakes – beam break-up

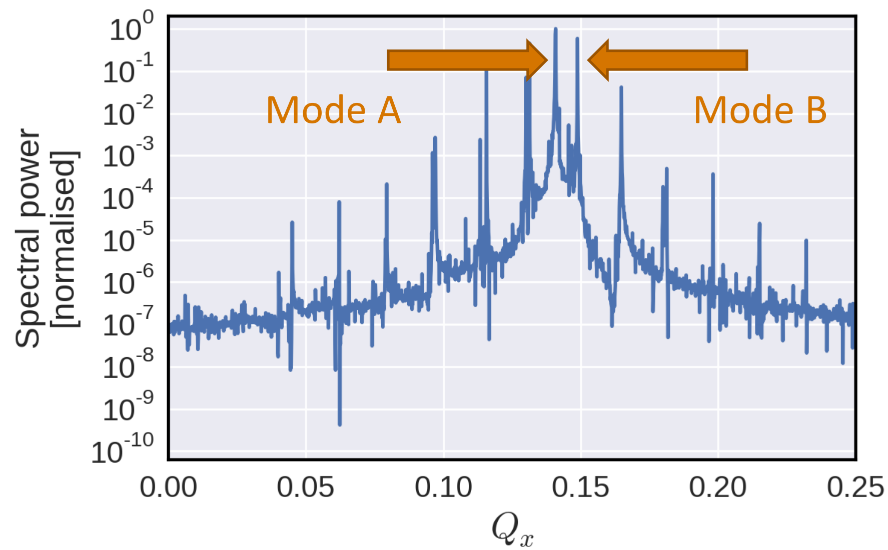


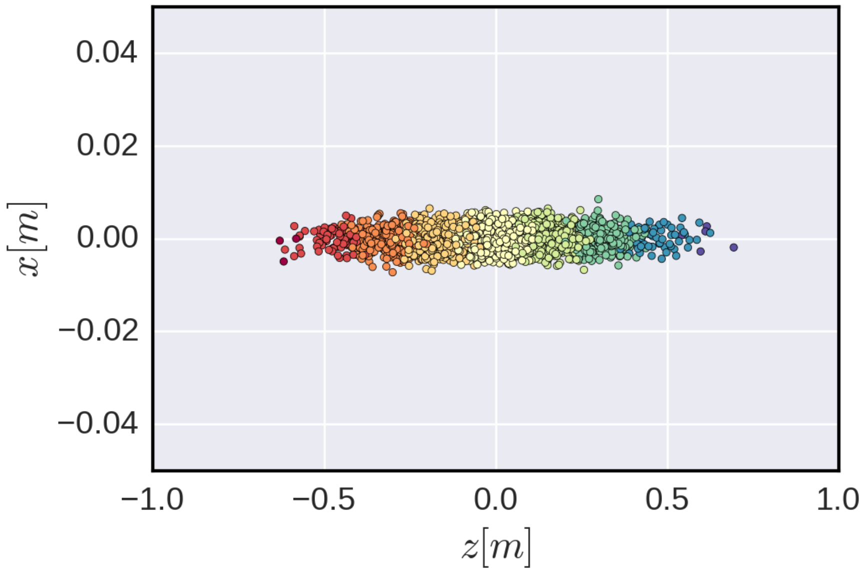
Dipole wakes – TMCI below threshold

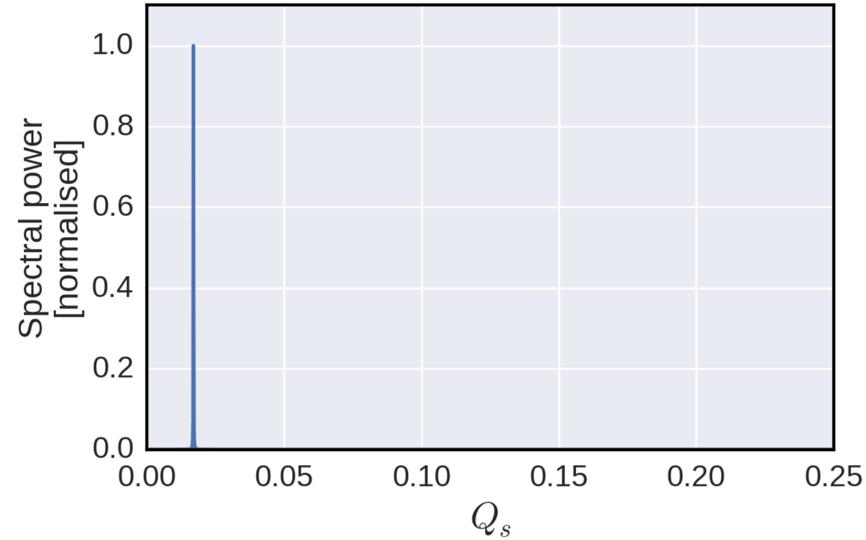


Dipole wakes – TMCI below threshold

As the intensity increases the coherent modes shift – here, modes A and B are approaching each other



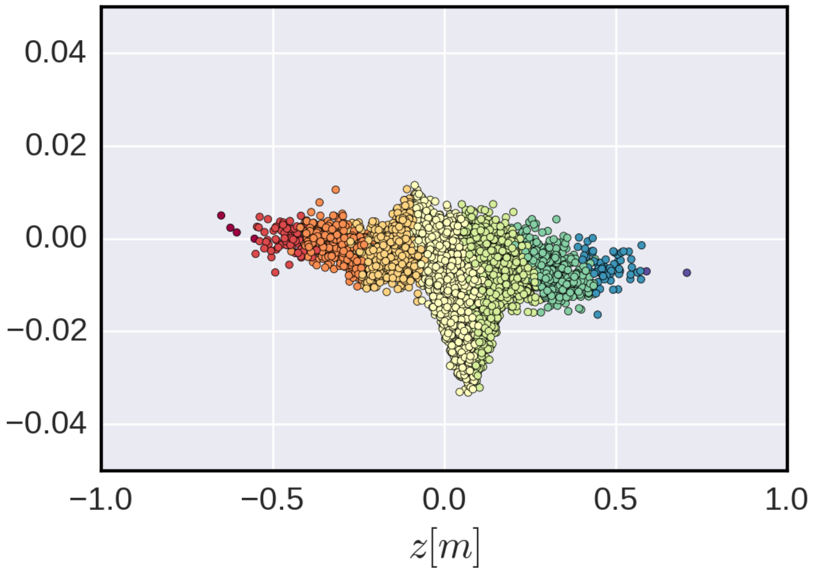


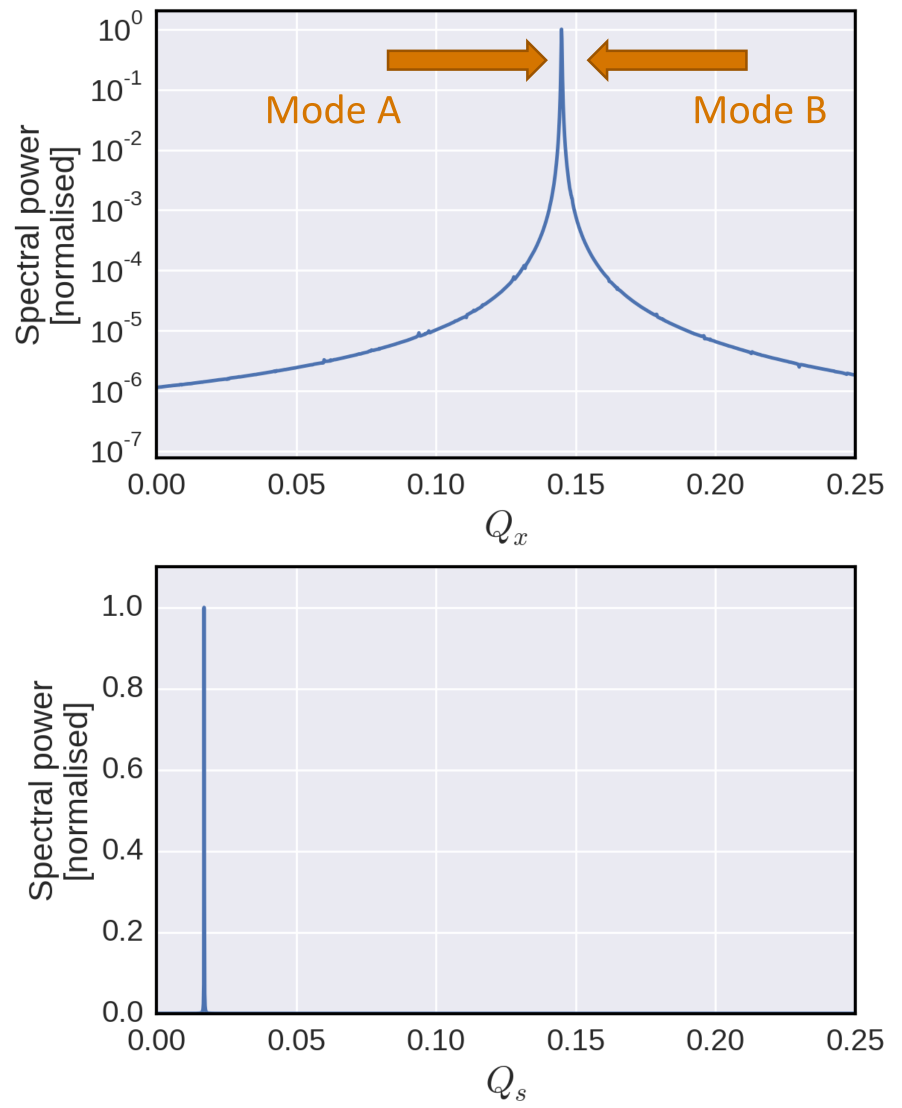


Turn

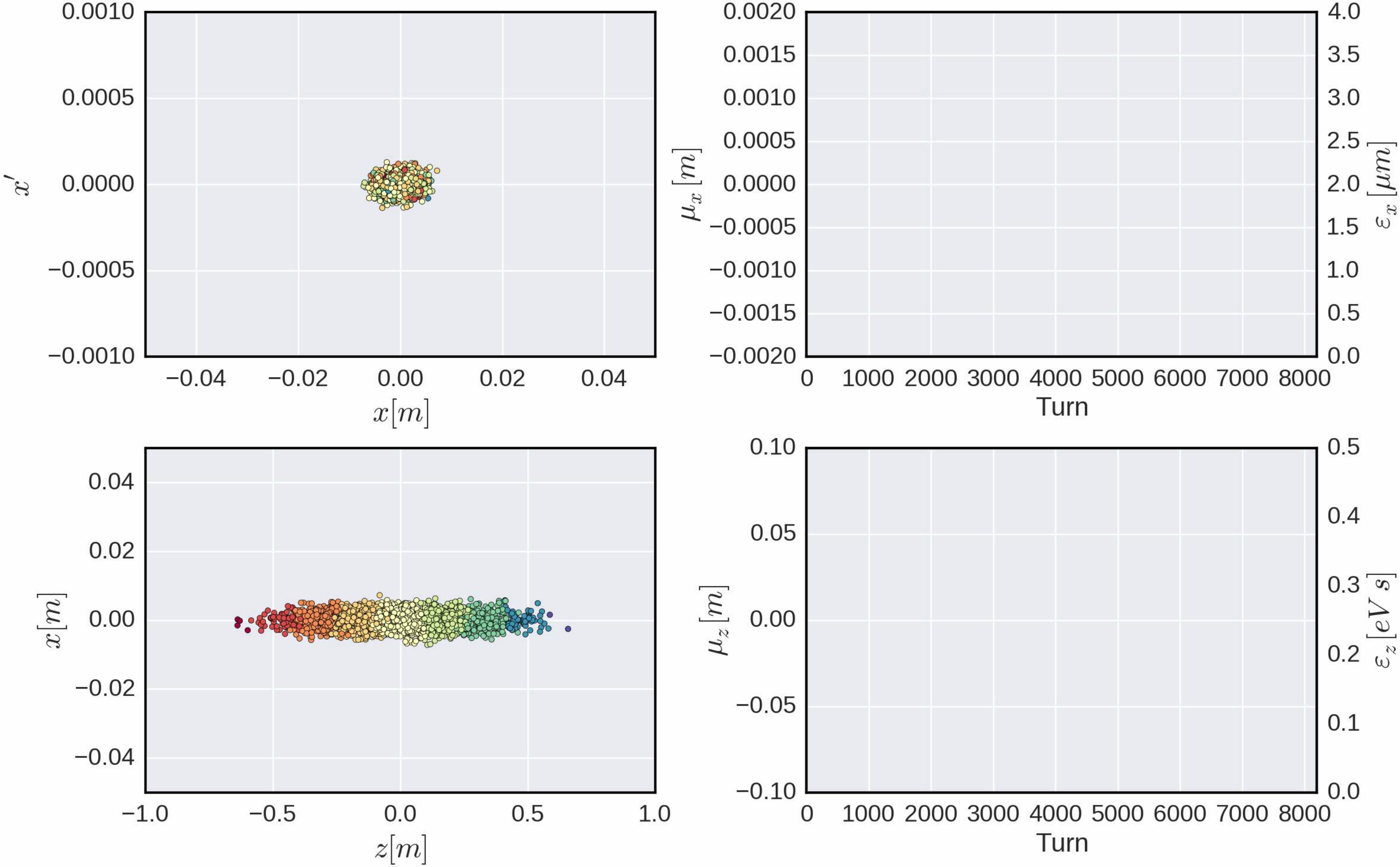
Dipole wakes – TMCI above threshold

When the two modes merge a fast coherent instability arises – the transverse mode coupling instability (TMCI) which often is a hard intensity limit in many machines





Dipole wakes – TMCI above threshold



Signpost

- We have **discussed transverse wake fields** and impedances, their classification into different types along with their impact on the beam dynamics.
- We have seen how the wake field interaction with a charged particle beam can be carried out numerically in an efficient manner.
- We have seen some examples of the effects of transverse wake fields on the beam such as **orbit distortion or headtail instabilities**.
- We have discussed the two-particle model and an analytically solvable problem which led to the description of the **transverse mode coupling instability (TMCI)**.

Part 3: Transverse wakefields –

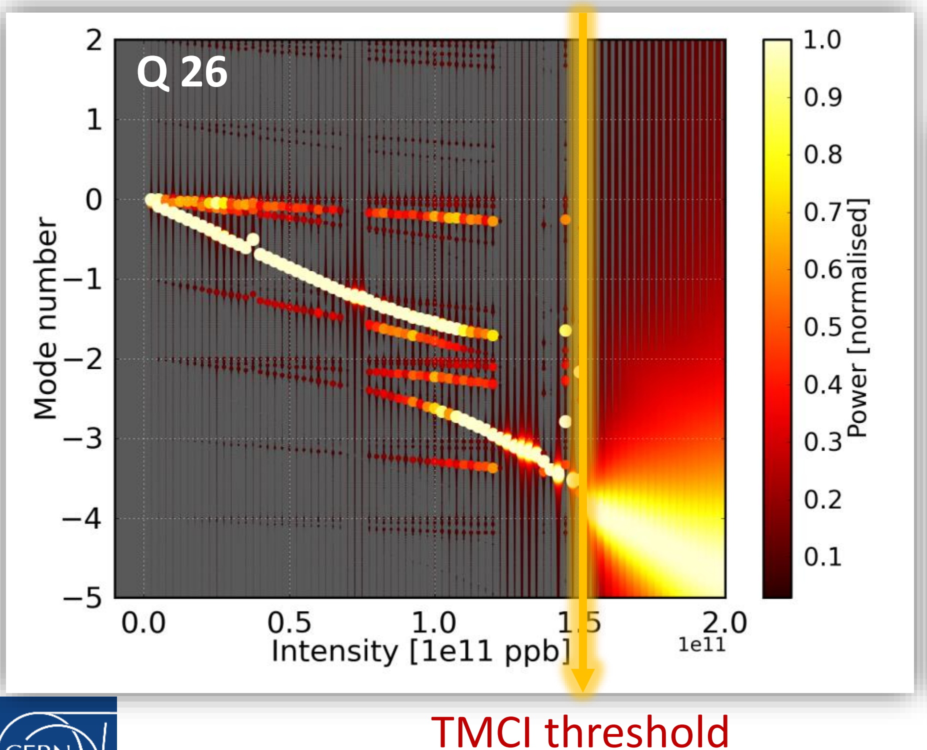
their different types and impact on beam dynamics

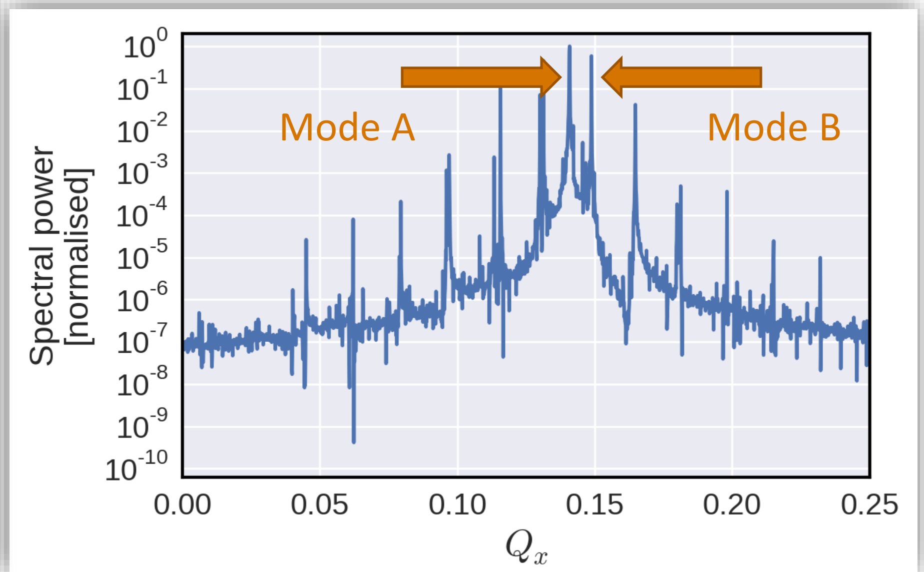
- Transverse wake function and impedance
- Numerical implementation, transverse „potential well distortion“ ad headtail instabilities
- Two particle models, transverse mode coupling instability

End part 3

Raising the TMCI threshold – SPS Q20 optics

- In **simulations** we have the possibility to perform **scans of variables**, e.g. we can run **100 simulations in parallel** changing the beam intensity
- We can then perform a **spectral analysis** of **each simulation**…
- … and stack all obtained plot behind one another to obtain…
- … the typical **visualization plots of TMCI**





Raising the TMCI threshold – SPS Q20 optics

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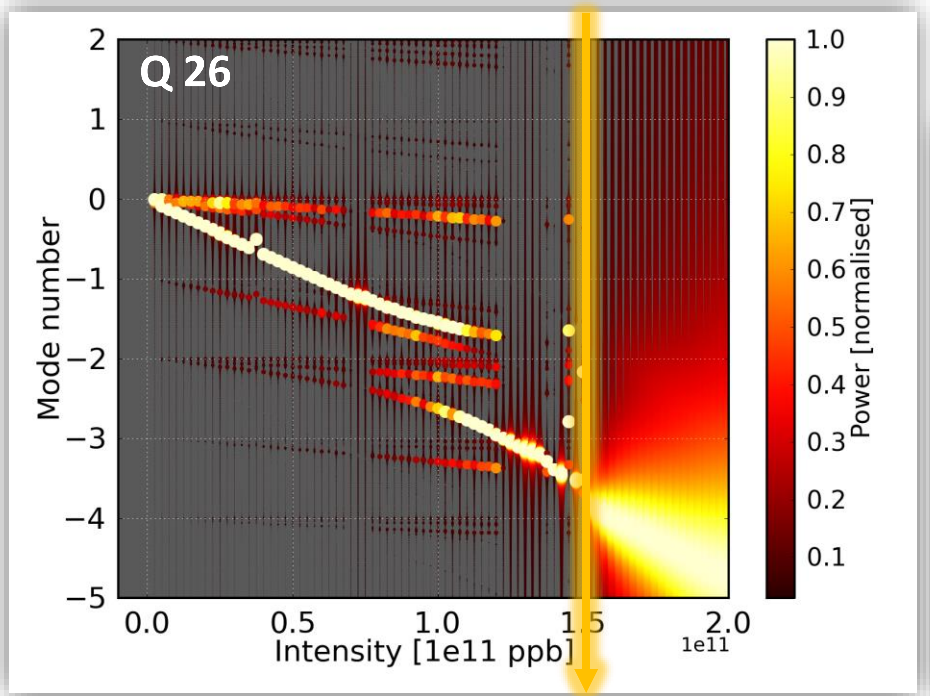

TMCI threshold

The mode number is given as

$$m = \frac{Q_x - Q_{x0}}{Q_s}$$

The modes are separated by the synchrotron tune.

Raising the TMCI threshold – SPS Q20 optics

- In **simulations** we have the possibility to perform **scans of variables**, e.g. we can run **100 simulations in parallel** changing the beam intensity
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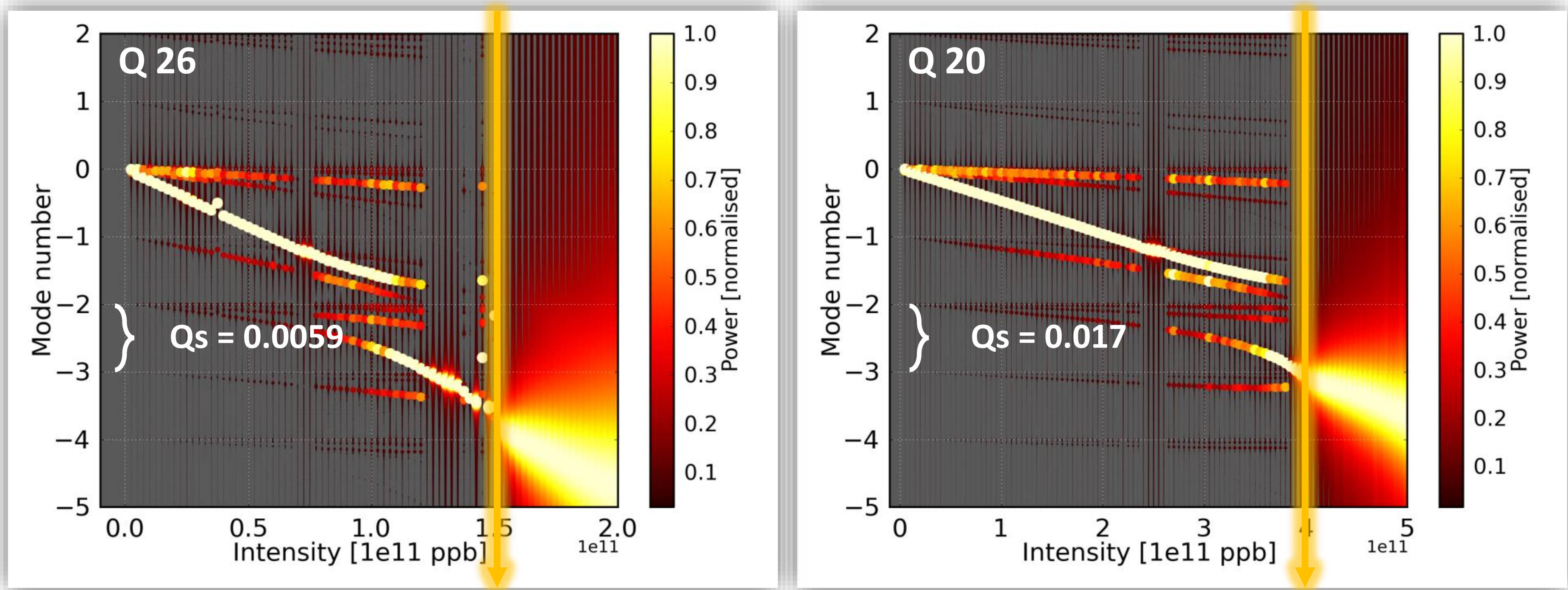


TMCI threshold   TMCI threshold

Backup

Wakefields – rough formalism

$$H = \frac{1}{2}p_x^2 + \frac{1}{2}K(s)x^2 + \sum_k \frac{e^2}{m\gamma\beta^2c^2\,C} \iiint \rho(x_s, z_s)\, w(x, x_s, z - z_s - kC)\, dx_s\, dz_s\, dx$$

Wakefields – rough formalism

$$H = \frac{1}{2}p_x^2 + \frac{1}{2}K(s)x^2 + \sum_k \frac{e^2}{m\gamma\beta^2c^2\,C} \iiint \rho(x_s, z_s)\, w(x, x_s, z - z_s - kC)\, dx_s\, dz_s\, dx$$

$$= \ldots + \sum_k \frac{e^2}{m\gamma\beta^2c^2\,C} \iiint \rho(x_s, z_s) \sum_{mn} x^n x_s^m\, W_{mn}(z - z_s - kC)\, dx_s\, dz_s\, dx$$

$$= \ldots + \sum_k \frac{e^2}{m\gamma\beta^2c^2\,C} \sum_{mn} \int x^n \int \lambda_m(z_s)\, W_{mn}(z - z_s - kC)\, dz_s\, dx$$

$$\lambda_m(z_s) = \int \rho(x_s, z_s) x_s^m\, dx_s$$

- Expansion

Wakefields – rough formalism

$$H = \frac{1}{2}p_x^2 + \frac{1}{2}K(s)x^2 + \sum_k \frac{e^2}{m\gamma\beta^2c^2\,C}\sum_{mn}\int x^n \int \lambda_m(z_s)\,W_{mn}(z - z_s - kC)\,dz_s\,dx$$

$$\lambda_m(z_s) = \int \rho(x_s, z_s)x_s^m\,dx_s$$

$$H = \frac{1}{2}p_x^2 + C + \boxed{A\,x} + \boxed{\frac{1}{2}B\,x^2} + \ldots, \qquad \text{with}\ \frac{dq}{ds} = \frac{\partial H(p,q)}{\partial p}, \qquad \frac{dp}{ds} = -\frac{\partial H(p,q)}{\partial q}$$

Dipole term (n=1) → change of orbit

Quadrupole term (n=2) → change of tune

- Expansion – up to second order:

n	m	type
0	0, 1	
1	0	

Constant transverse wake (n=0, m=0)

Dipole transverse wake (n=0, m=1)

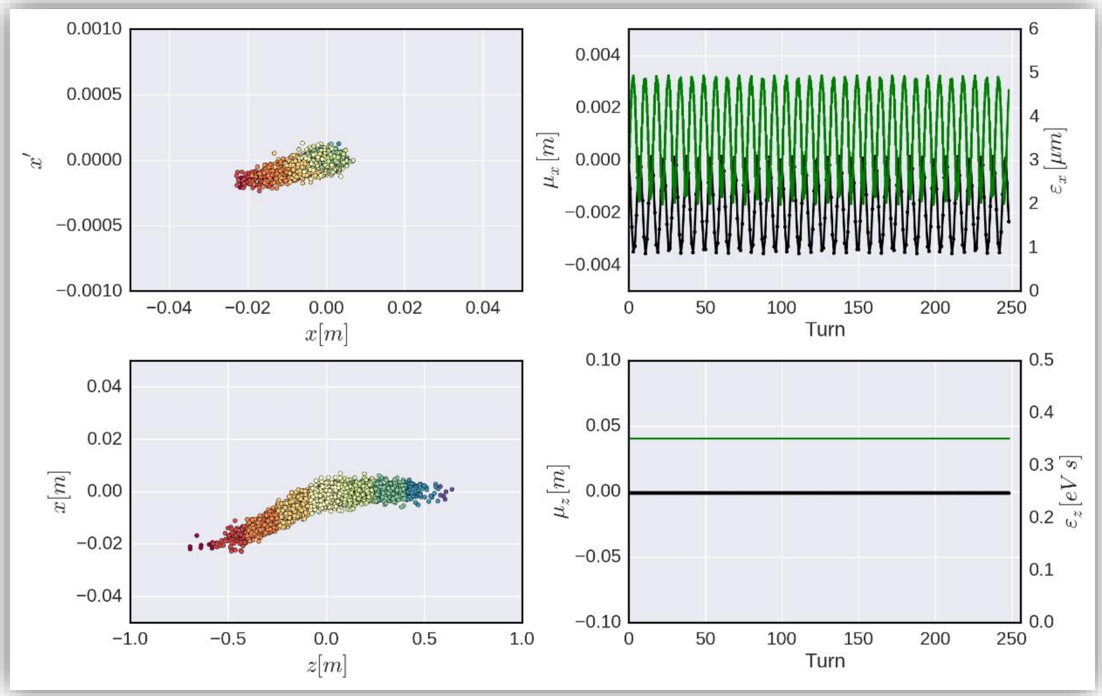
Quadrupole transverse wake (n=1, m=0)

Examples – constant wakes

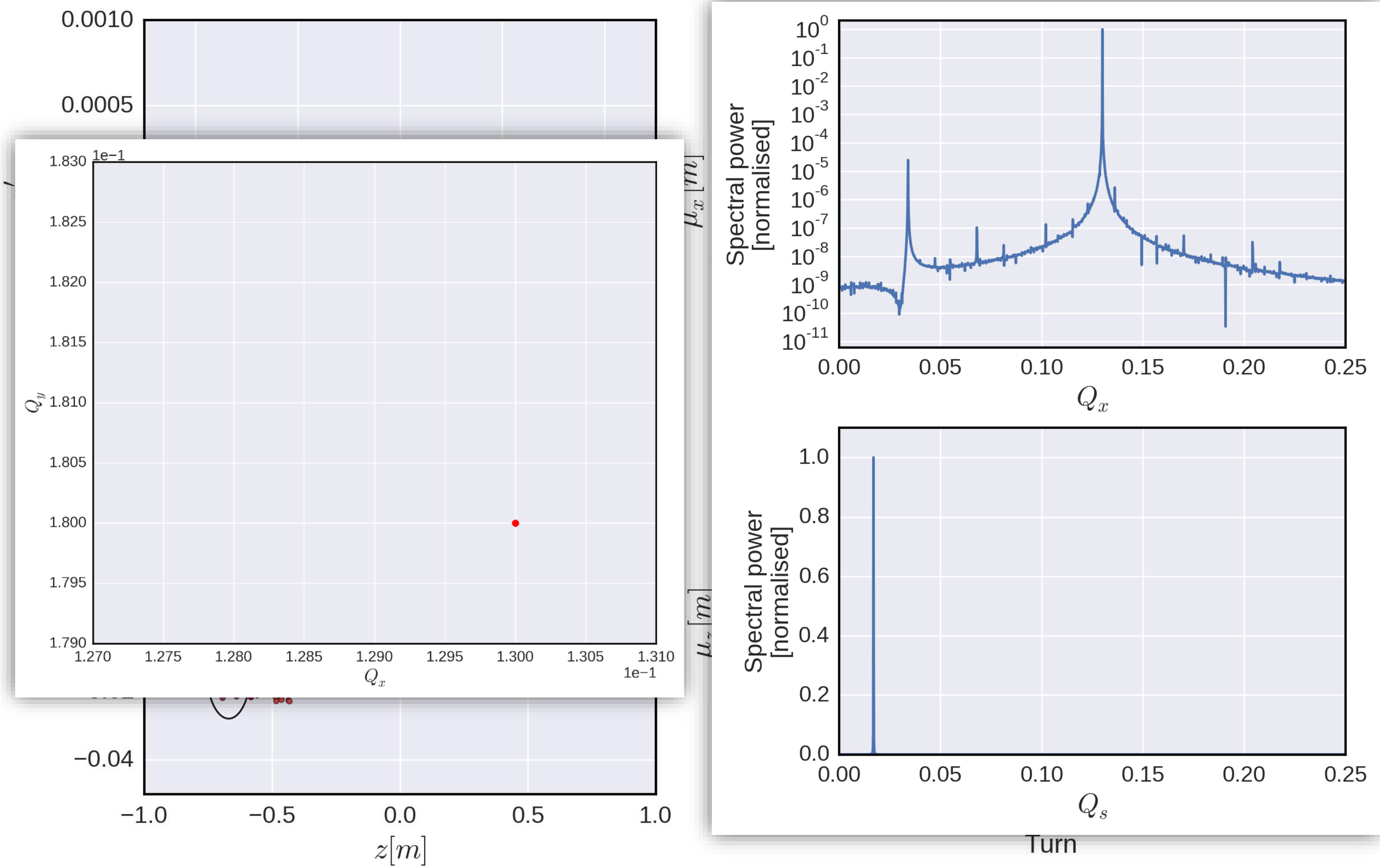
$$H = \frac{1}{2}p_x^2 + \frac{1}{2}K(s)x^2 + \frac{e^2}{m\gamma\beta^2 c^2 C} x \sum_{j=0}^{n\_slices-1} \lambda(z_j)\, W_{01}(z - z_j)\, \Delta z_j$$

Dipolar term → orbit kick

Slice dependent change of closed orbit (if line density does not change)



Examples – constant wakes

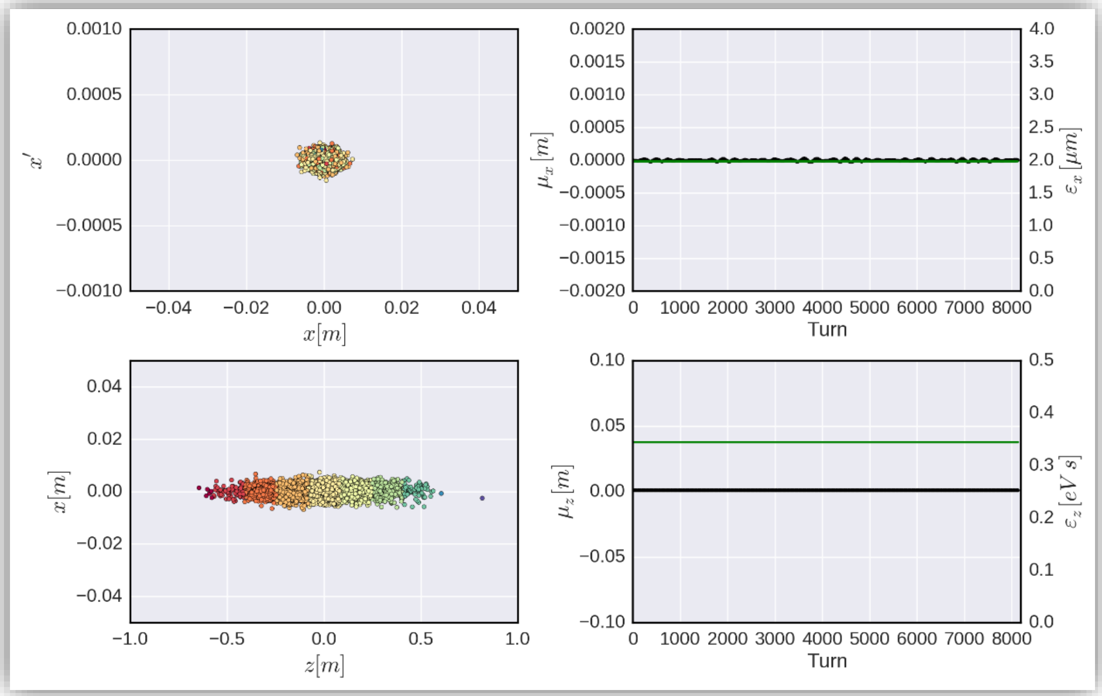


Examples – quadrupole wakes

$$H = \frac{1}{2}p_x^2 + \frac{1}{2}K(s)x^2 + \frac{e^2}{m\gamma\beta^2c^2\,C}\,x^2 \sum_{j=0}^{\mathrm{n\_slices}-1} \lambda(z_j)\,W_{02}(z-z_j)\,\Delta z_j$$

Quadrupole term → tune kick

Slice dependent change of tune
(if line density does not change)



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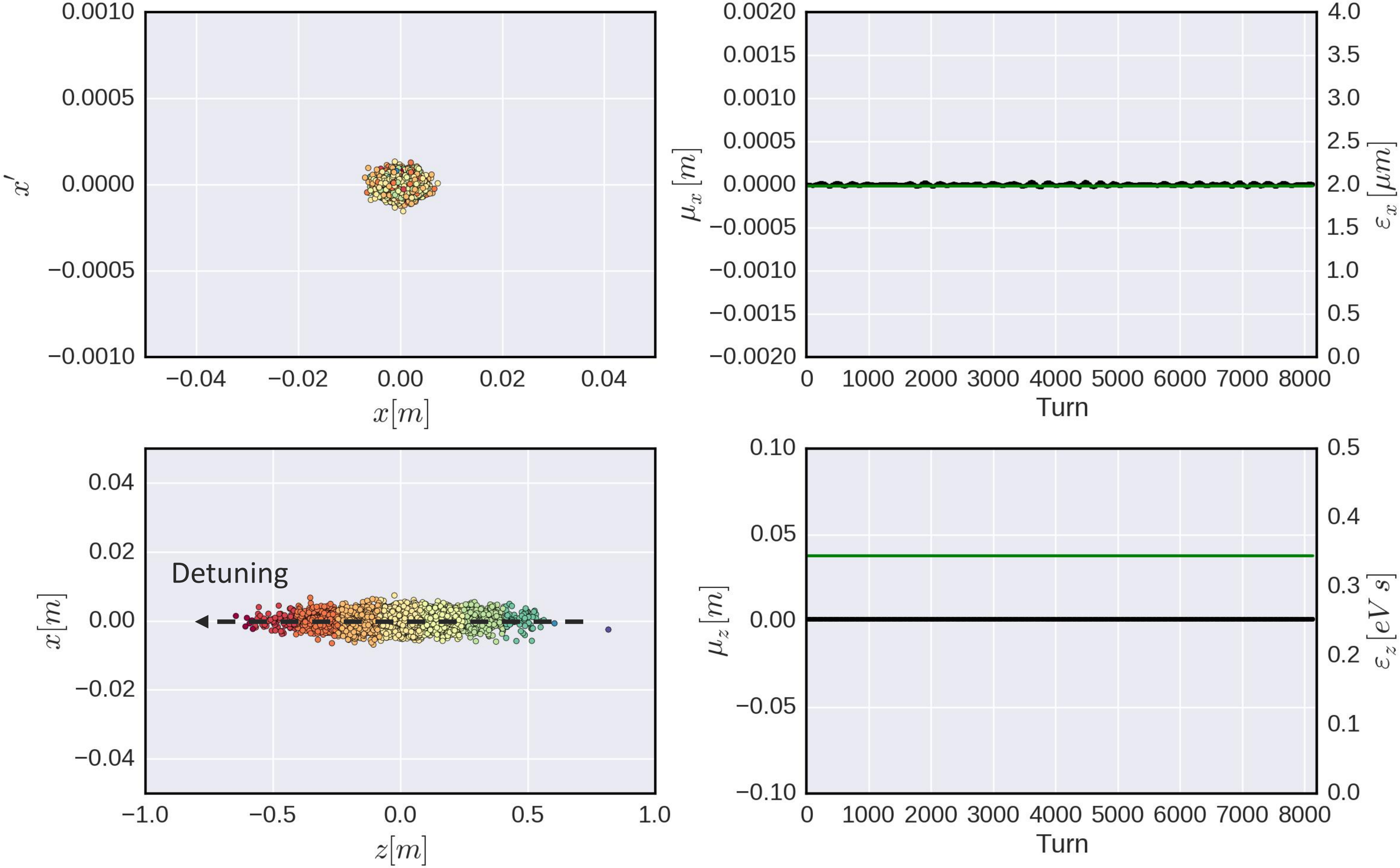
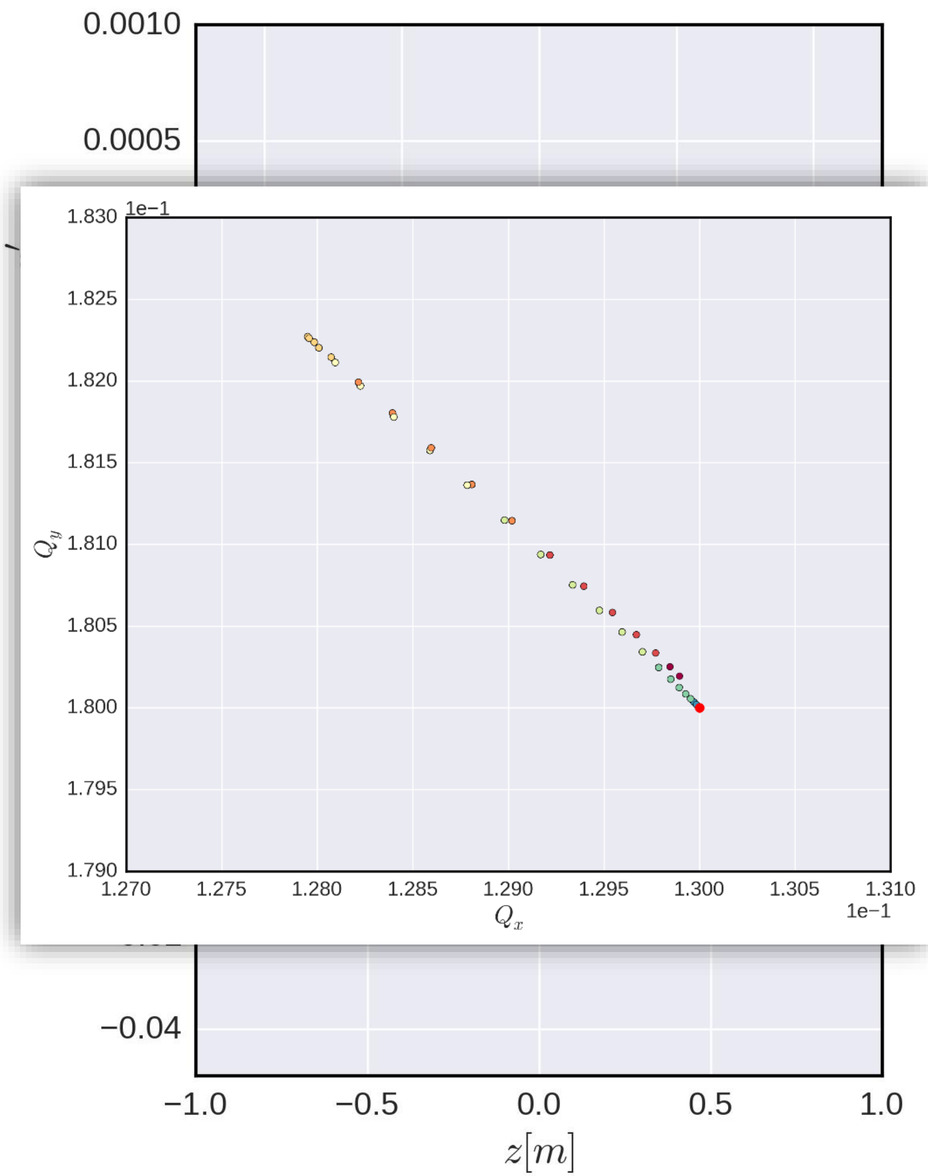
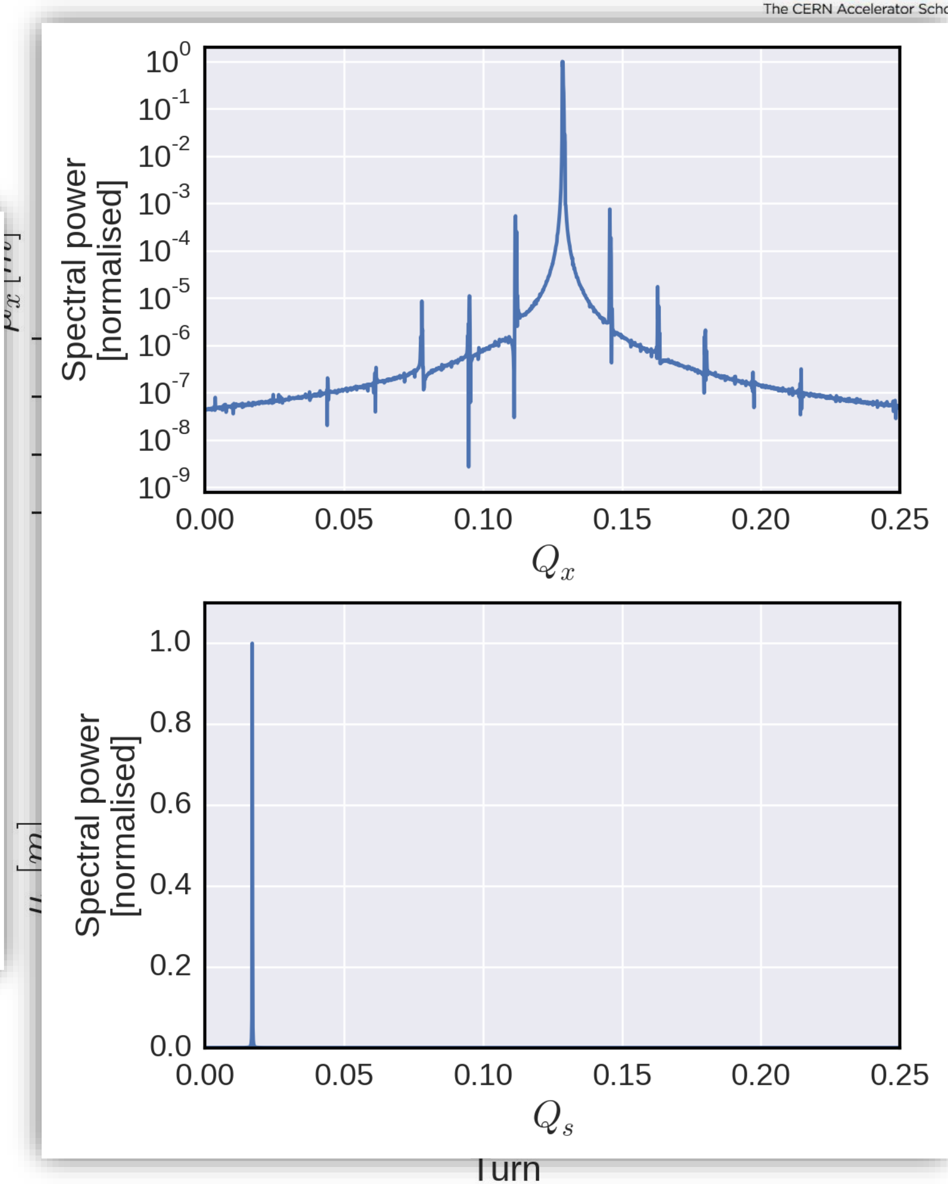


Examples – quadrupole wakes





Examples – dipole wakes

$$H = \frac{1}{2}p_x^2 + \frac{1}{2}K(s)x^2 + \frac{e^2}{m\gamma\beta^2c^2\,C}\,x \sum_{j=0}^{n\_\text{slices}-1} \lambda(z_j)\,\langle x\rangle|_j\; W_{11}(z-z_j)\,\Delta z_j$$

Dipolar term → orbit kick

Offset dependent orbit kick
→ kicks can accumulate

- Without synchrotron motion:
  kicks accumulate turn after turn – the beam is unstable → beam break-up in linacs

Examples – dipole wakes

$$H = \frac{1}{2}p_x^2 + \frac{1}{2}K(s)x^2 + \frac{e^2}{m\gamma\beta^2c^2\,C}\,x \sum_{j=0}^{n_\text{slices}-1} \lambda(z_j)\,\langle x\rangle\big|_j\, W_{11}(z-z_j)\,\Delta z_j$$

Dipolar term → orbit kick

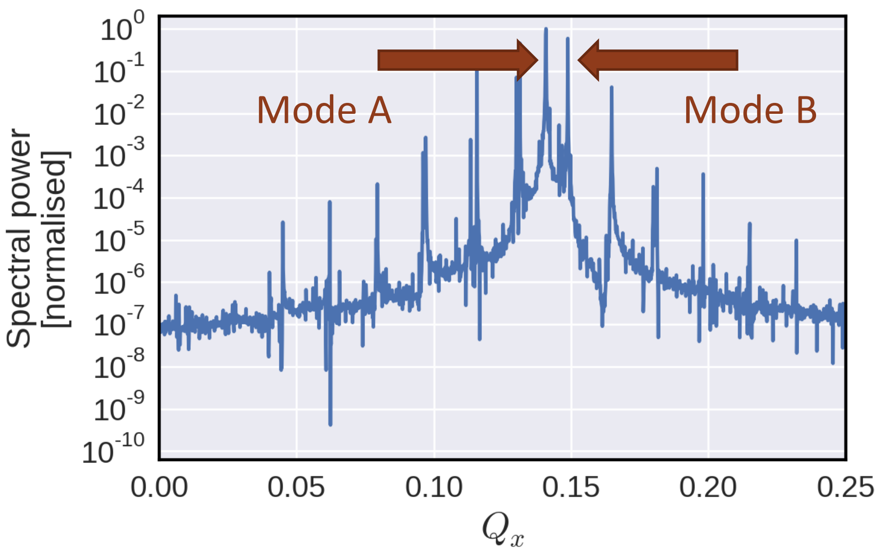
With synchrotron motion we can get into a feedback loop

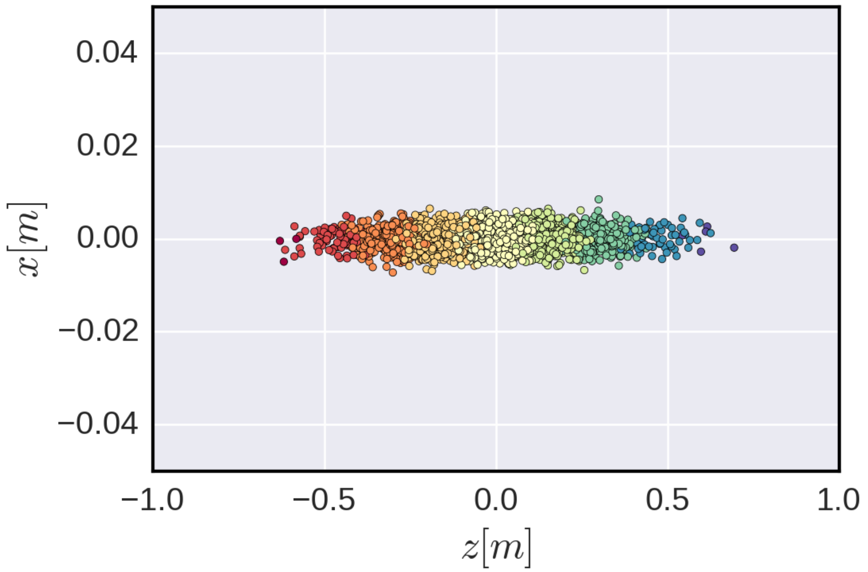
Offset dependent orbit kick
→ kicks can accumulate

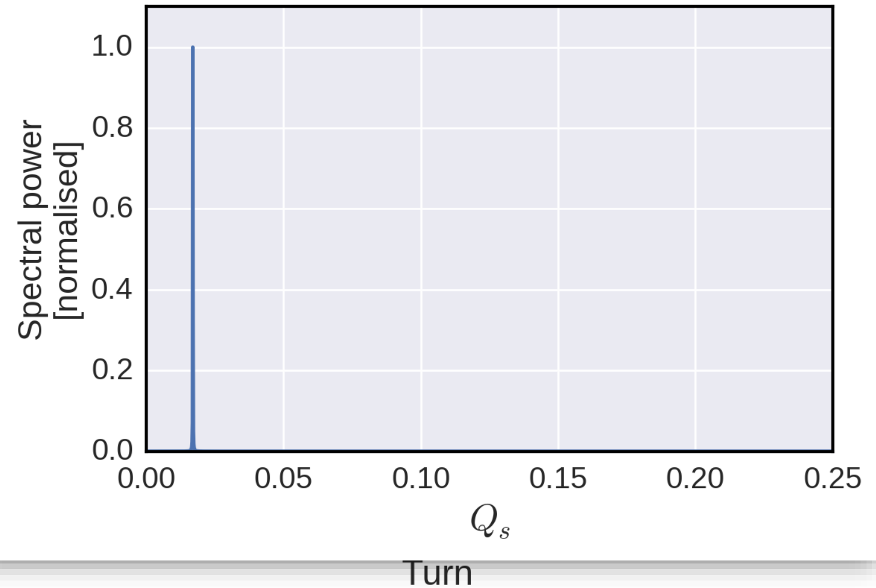
- Without synchrotron motion:
  kicks accumulate turn after turn – the beam is unstable → beam break-up in linacs
- With synchrotron motion:
  - Chromaticity = 0
    - Synchrotron sidebands are well separated → beam is stable
    - Synchrotron sidebands couple → (transverse) mode coupling instability
  - Chromaticity ≠ 0
    - Headtail modes → beam is unstable (can be very weak and often damped by non-linearities)

Dipole wakes – TMCI below threshold

As the intensity increases the coherent modes shift – here, modes A and B are approaching each other

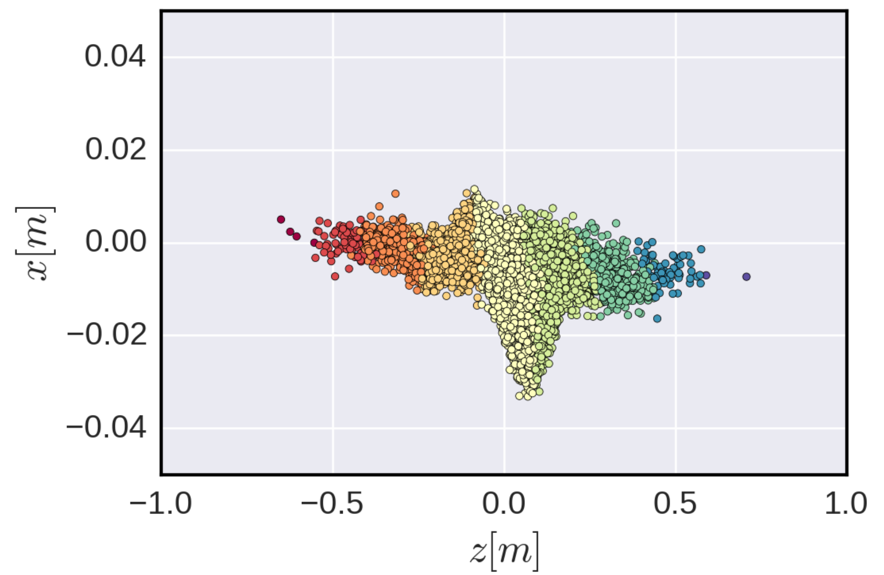


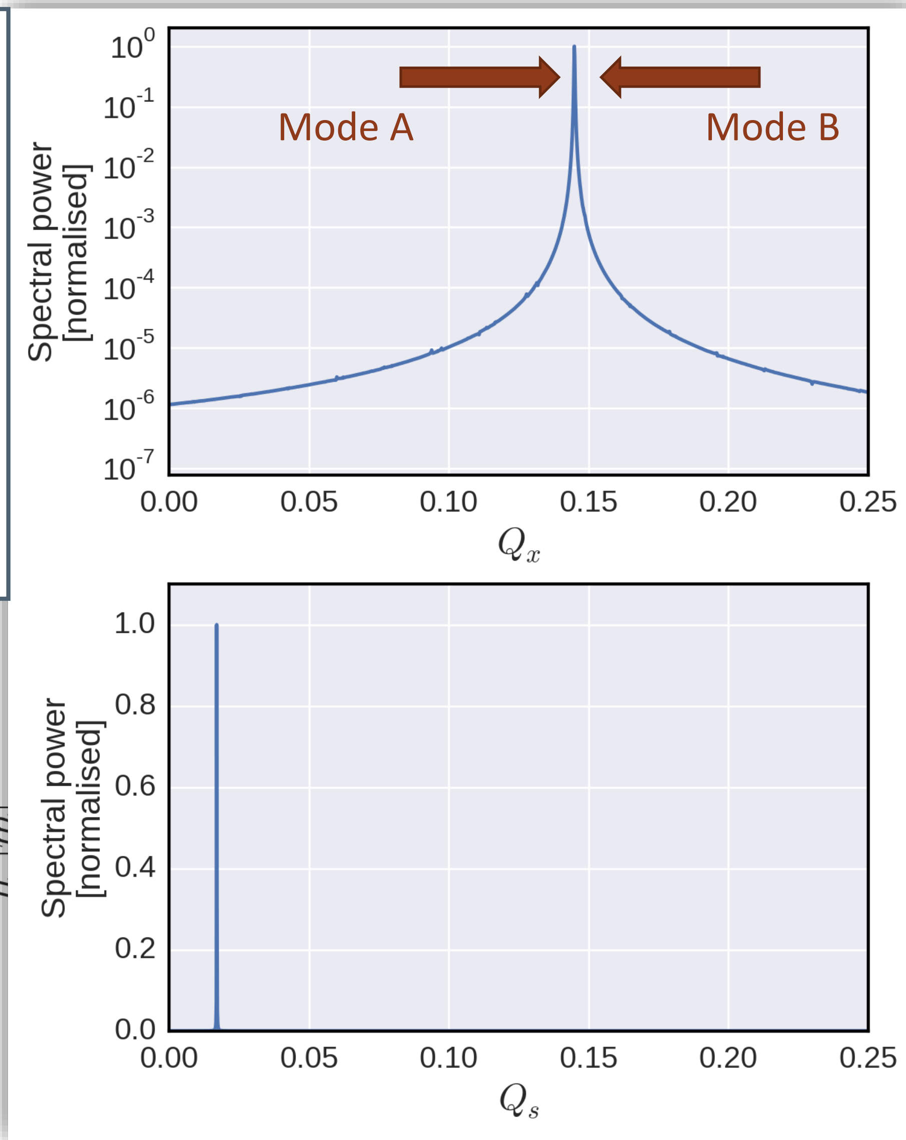




Dipole wakes – TMCI above threshold

When the two modes merge a fast coherent instability arises – the transverse mode coupling instability (TMCI) which often is a hard intensity limit in many machines



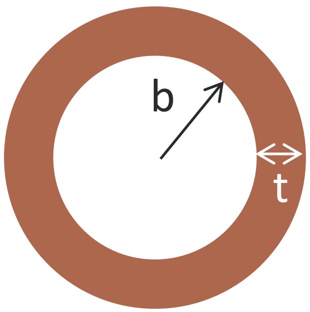


Backup - wakefields

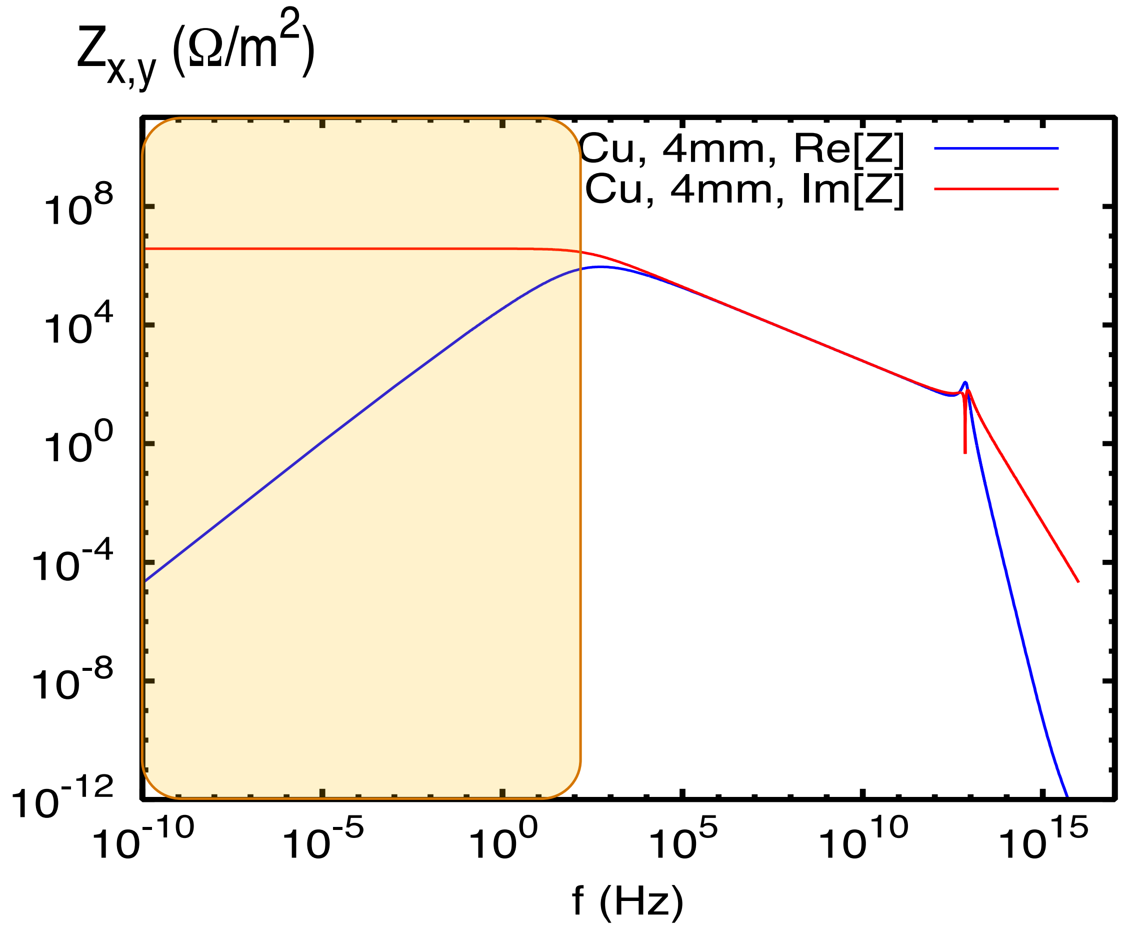
Break

Examples of wakes/impedances

- Resistive wall of beam chamber



  - An interesting example: a 1 m long Cu pipe with radius b=2 cm and thickness t = 4 mm in vacuum

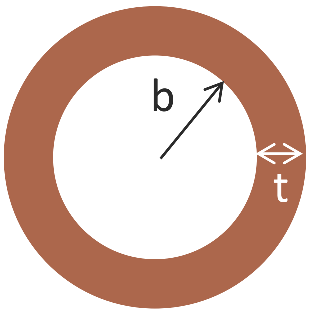


3 frequency regions of interest:

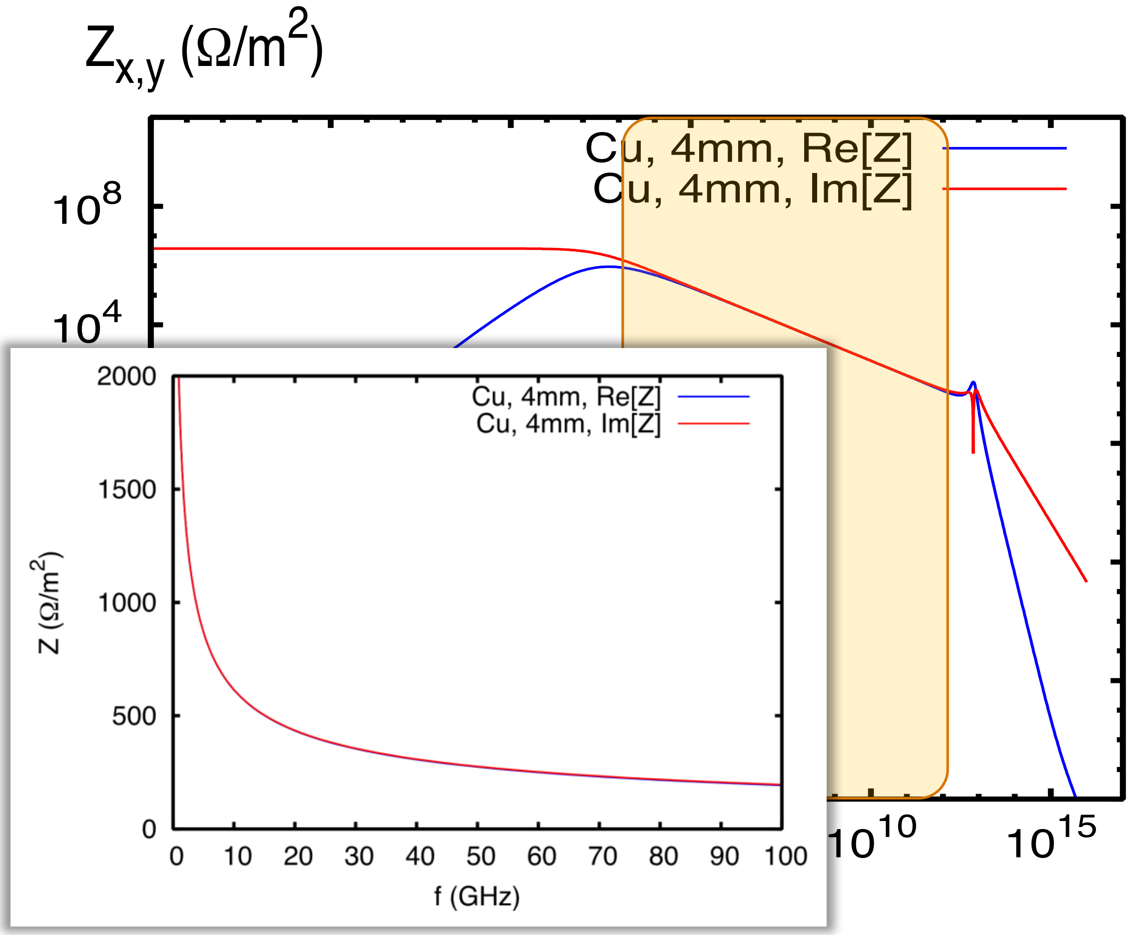
1. Below 0.1 kHz, impedance is basically purely imaginary, EM field is shielded by image charges → Indirect space charge

Examples of wakes/impedances

- Resistive wall of beam chamber



○ An interesting example: a 1 m long Cu pipe with radius b=2 cm and thickness t = 4 mm in vacuum

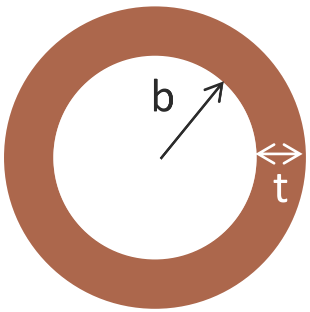


3 frequency regions of interest:

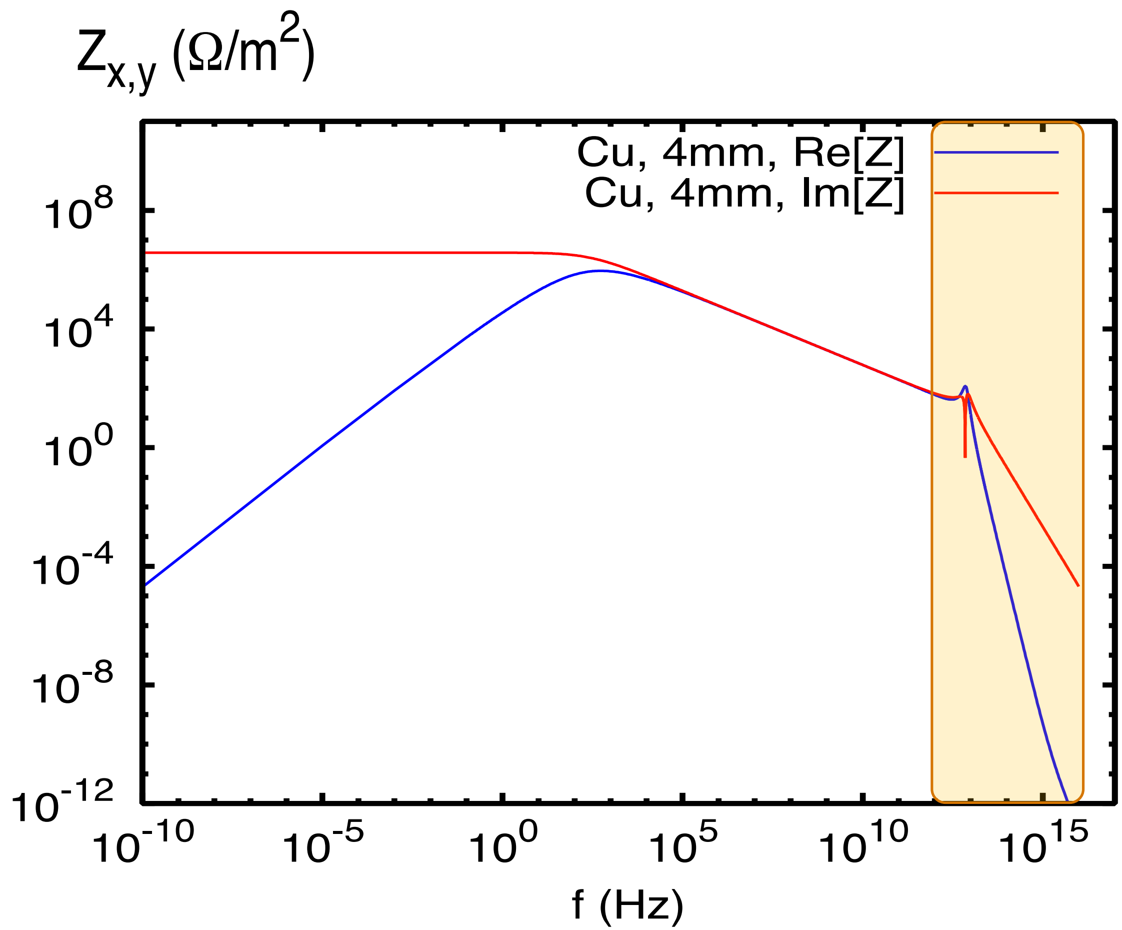
2. Between 10 kHz and 1 THz, the EM field is fully attenuated in the Cu layer and the impedance is like the one calculated assuming infinitely thick wall

Examples of wakes/impedances

- Resistive wall of beam chamber



○ An interesting example: a 1 m long Cu pipe with radius b=2 cm and thickness t = 4 mm in vacuum

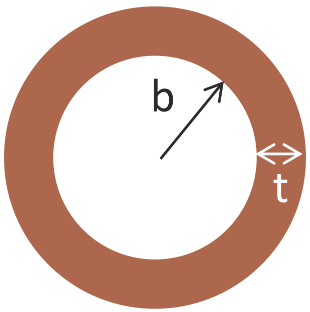


3 frequency regions of interest:

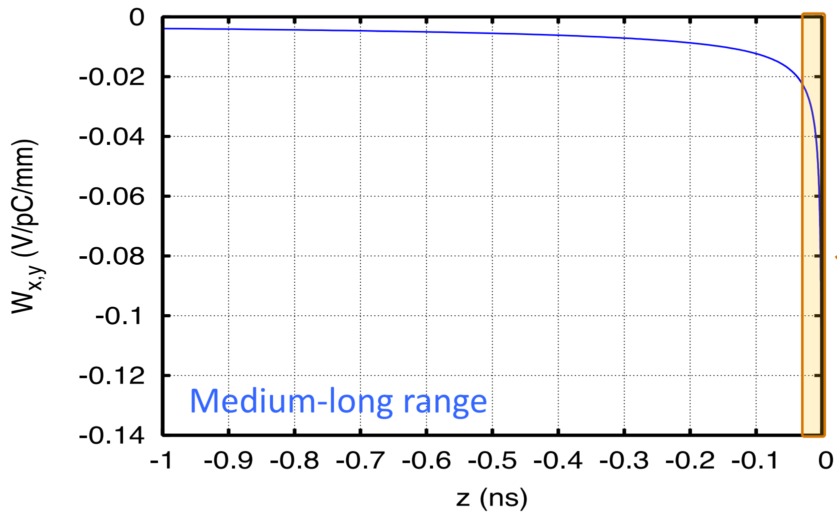
3. Above 1 THz, there is a resonance (100 THz region). In this region also ac conductivity should be taken into account

Examples of wakes/impedances

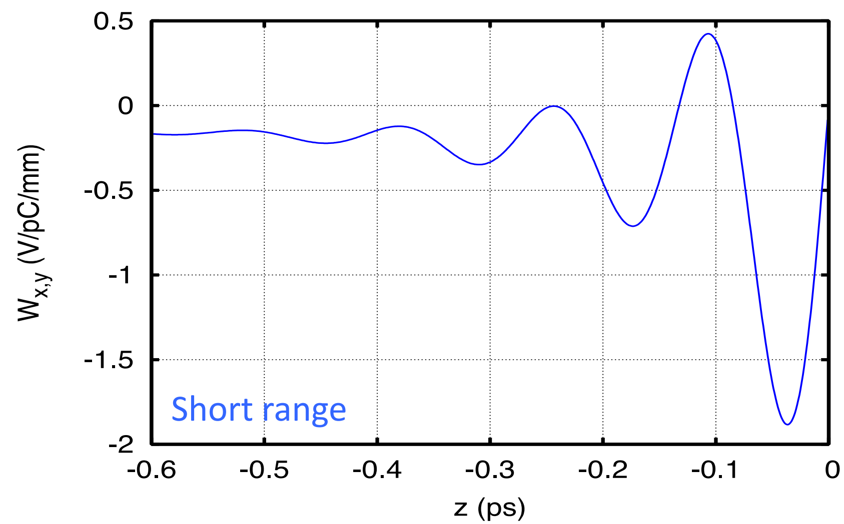
- Resistive wall of beam chamber



- Correspondingly, in time domain, the wake exhibits different behaviours in short and long range



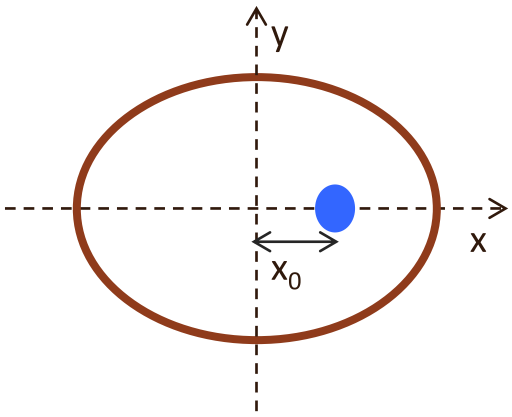
In the range of **tenths of ns up to fractions of ms** (e.g. bunch length to several turns for the SPS) **monotonically decaying wake**



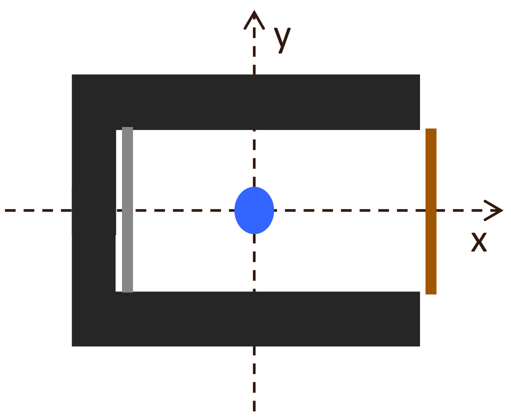
From **behind the source to ~1ps** the wake has an **oscillatory behaviour**, associated to the high frequency resonance

Beam deflection kick

Off-axis traversal of symmetric chamber



Traversal of asymmetric chamber

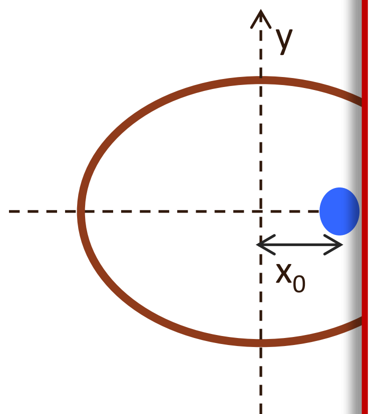


$$\Delta x'(z) = -\frac{e^2 x_0}{E_0} \int_{-\hat{z}}^{\hat{z}} \lambda(z') \Big(W_{D_x}(z-z') + W_{Q_x}(z-z') \Big)\, dz'$$

$$\Downarrow$$

Beam deflection kick

Off-axis traversal of
symmetric chamb



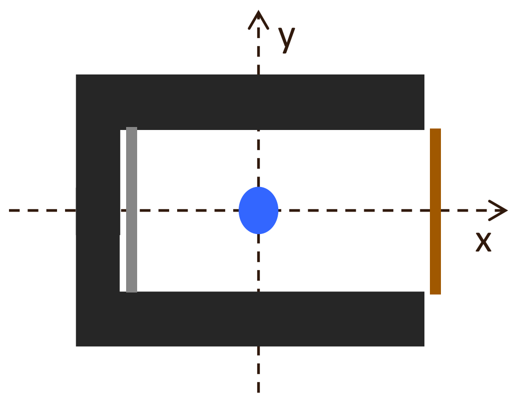
$$) + W_{Q_x}(z-z')\Big)\,dz'$$

$$_x(\omega) + Z_{Q_x}(\omega)\Big)\,d\omega$$

The beam deflection kicks

- ⇒ Are responsible for intensity dependent orbit variations
- ⇒ Cause z-dependent orbits and can determine tilted equilibrium bunch distributions for long bunches

Traversal of asymmetric
chamber



$$\langle \Delta x' \rangle = -\frac{e^2 x_0 \omega_0}{E_0} \sum_{p=0}^{\infty} \left| \tilde{\lambda}(p\omega_0) \right|^2 \mathrm{Im}\Big(Z_{D_x}(p\omega_0) + Z_{Q_x}(p\omega_0) \Big)$$

Asymmetric chambers with constant wake:

$$\langle \Delta x' \rangle = -\frac{e^2 x_0 \omega_0}{E_0} \sum_{p=0}^{\infty} \left| \tilde{\lambda}(p\omega_0) \right|^2 \mathrm{Im}\Big(Z_{C_x}(p\omega_0) \Big)$$

Some hints for energy loss estimations

The CERN Accelerator School

$$\lambda(z) = \frac{N}{\sqrt{2\pi}\sigma_z} \exp\left(-\frac{z^2}{2\sigma_z^2}\right) \quad \overset{\mathcal{F}}{\Longleftrightarrow} \quad \tilde{\lambda}(\omega) = N \exp\left(-\frac{\omega^2\sigma_z^2}{2c^2}\right)$$

$$\int_{-\infty}^{\infty} |\tilde{\lambda}(\omega)|^2 \mathrm{Re}\left[Z_{||}(\omega)\right] d\omega$$ can be calculated

1) With $Z_{||}(\omega) = Z_{||}^{Res}(\omega)$ from slide 77 in the two limiting cases

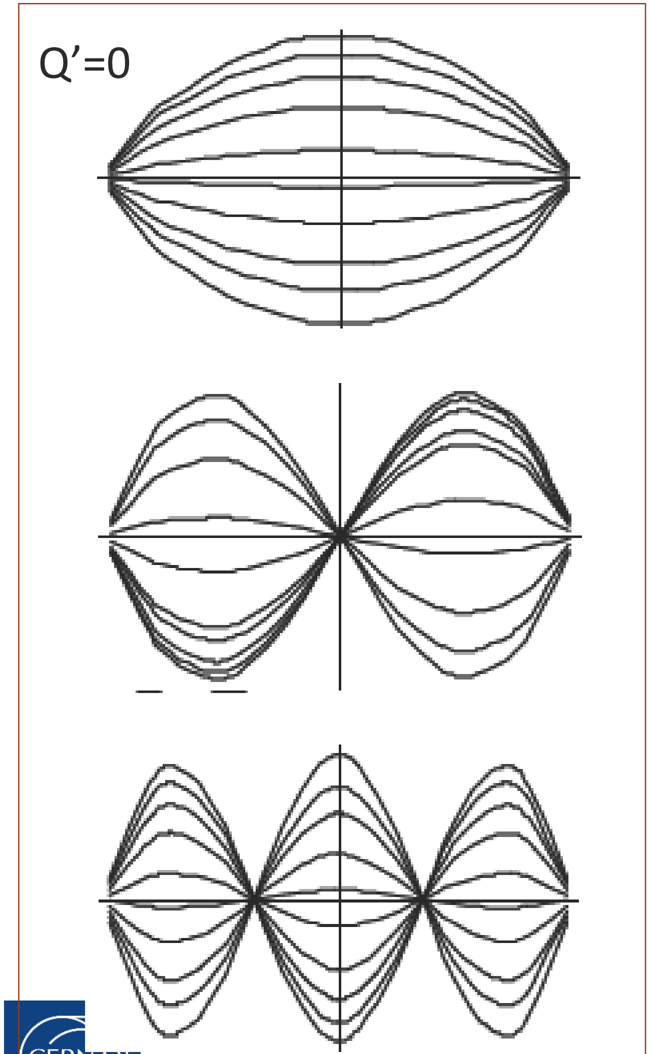
$$\sigma_z \gg \frac{c}{\omega_r}$$ Need to expand $\mathrm{Re}[Z_{||}(\omega)]$ for small ω

$$\sigma_z \ll \frac{c}{\omega_r}$$ Need to assume $|\lambda(\omega)|$ constant over $\mathrm{Re}[Z_{||}(\omega)]$

2) With $Z_{||}(\omega) = Z_{||RW}(\omega)$ from slide 64

CERN

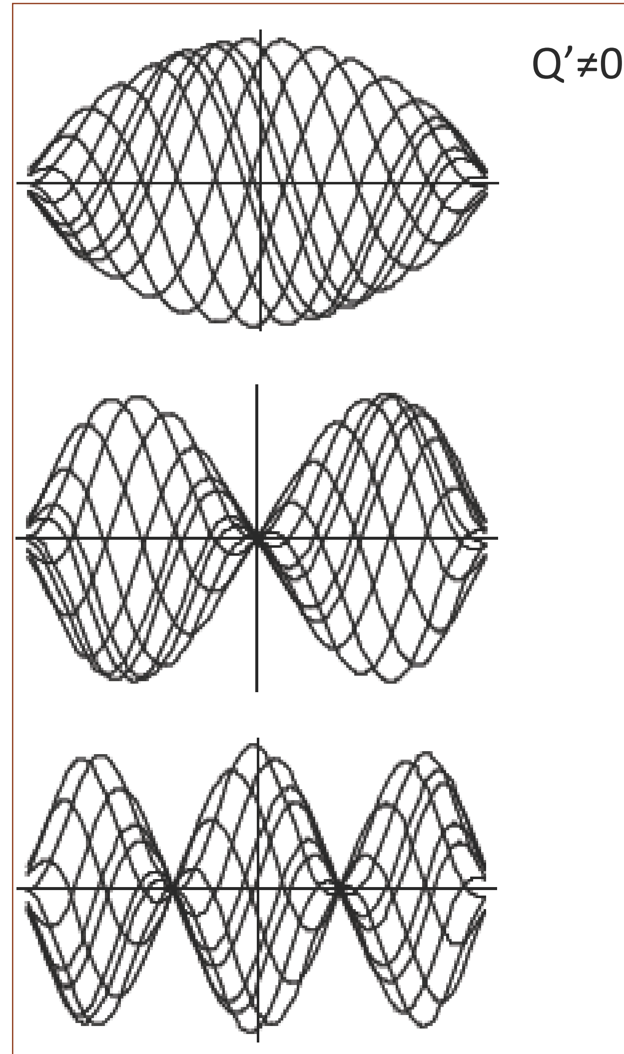
glance into the head-tail modes (as seen at a wide-band BPM)



l=0

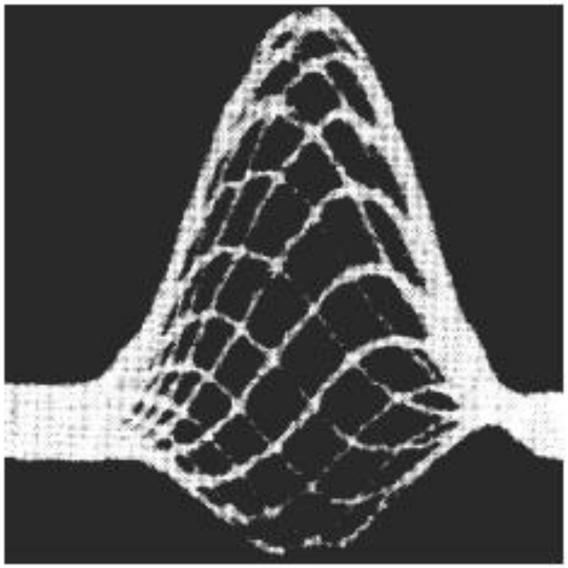
l=1

l=2

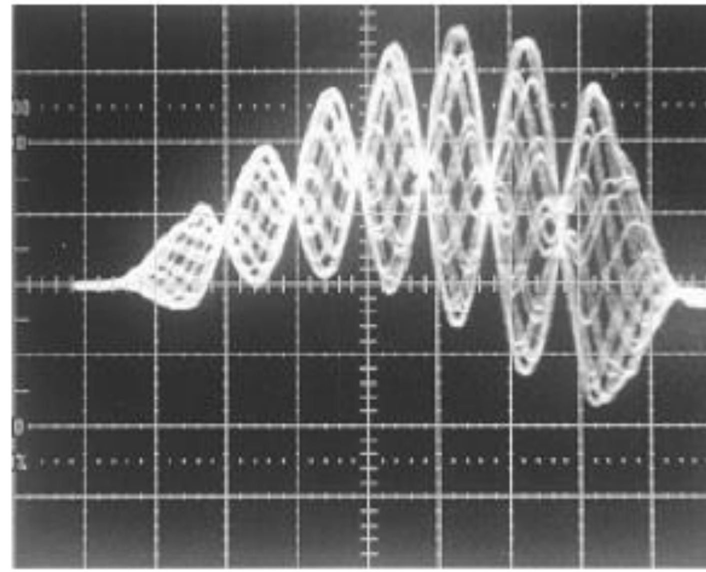


glance into the head-tail modes (experimental observations)

Observation in the CERN PSB in ~1974
(J. Gareyte and F. Sacherer)



Observation in the CERN PS in 1999

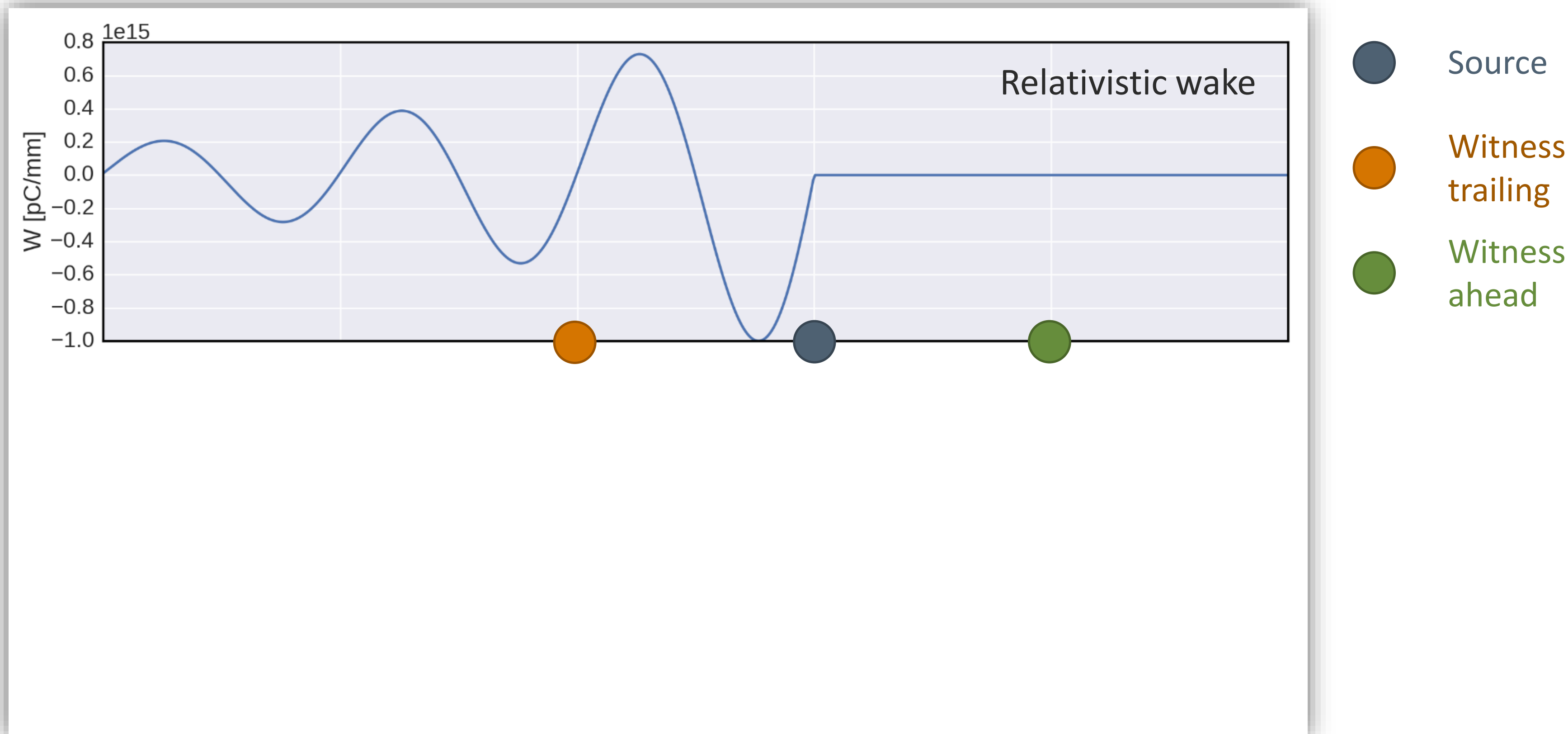


- The mode that gets first excited in the machine depends on
  - The spectrum of the exciting impedance
  - The chromaticity setting
- Head-tail instabilities are a good diagnostics tool to identify and quantify the main impedance sources in a machine

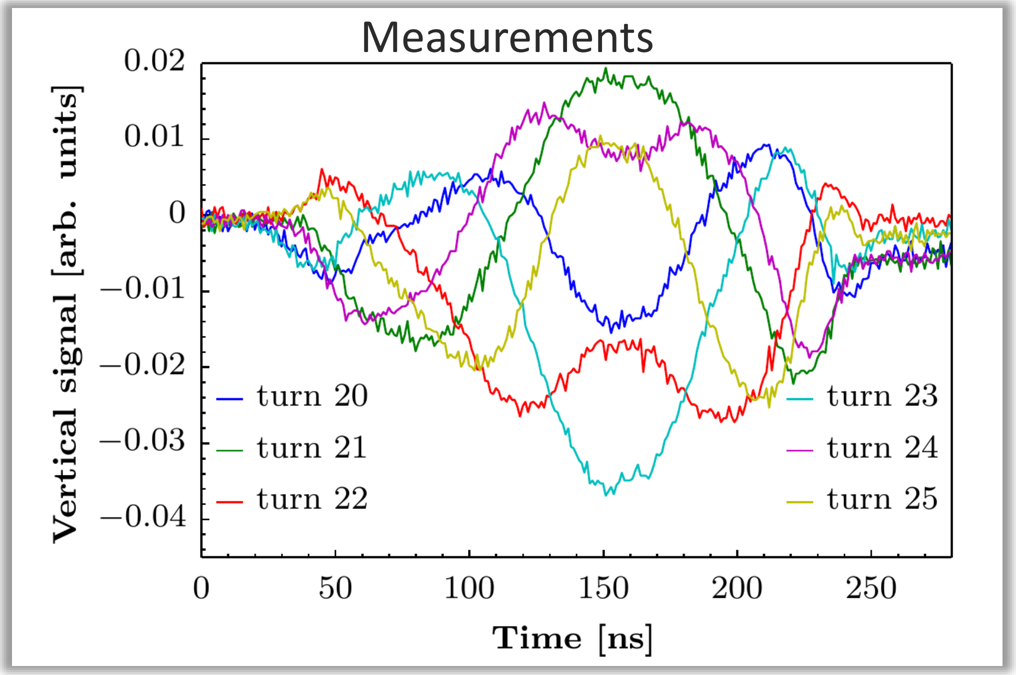
Relativistic vs. non-relativistic wakes

- Relativistic wakes only affect trailing particles following the source particle
- Finite values range for negative distances, i.e. (-L, 0) or “tail – head”
  - L: bunch length

- Nonrelativistic wakes can also affect particles ahead of the source particle
- Finite values extend from (-L, L) or “tail –head” & “head – tail”
  - L: bunch length

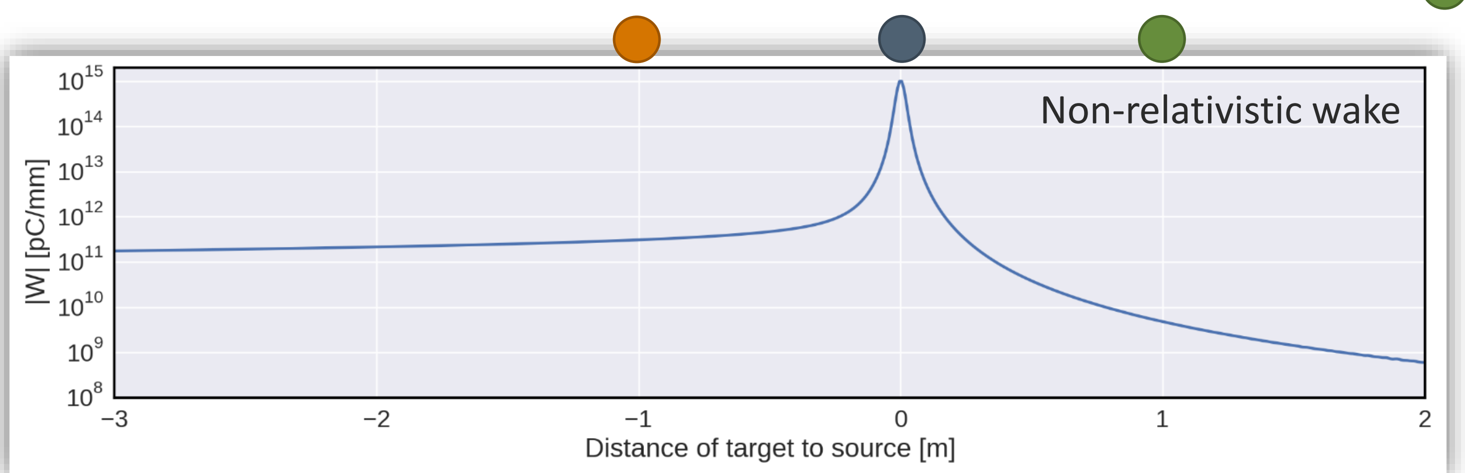


Example non-relativistic wakes

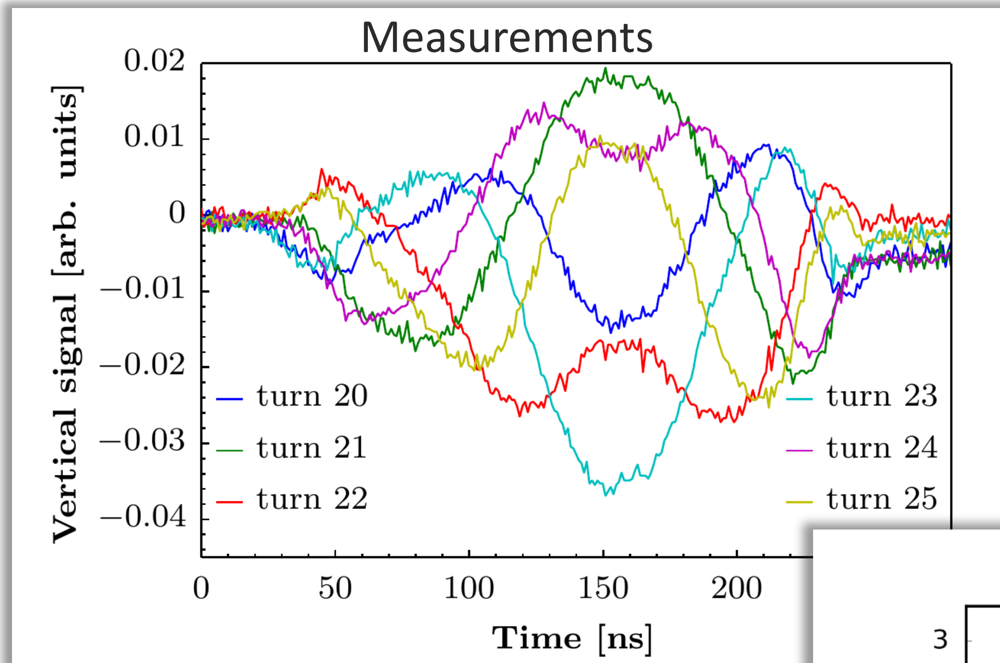


- PS injection oscillations show intra-bunch modulations
- These can only be reproduced when adding non-relativistic wakes caused by indirect space charge fields

A. Huschauer et al



Example non-relativistic wakes



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A. Huschauer et al