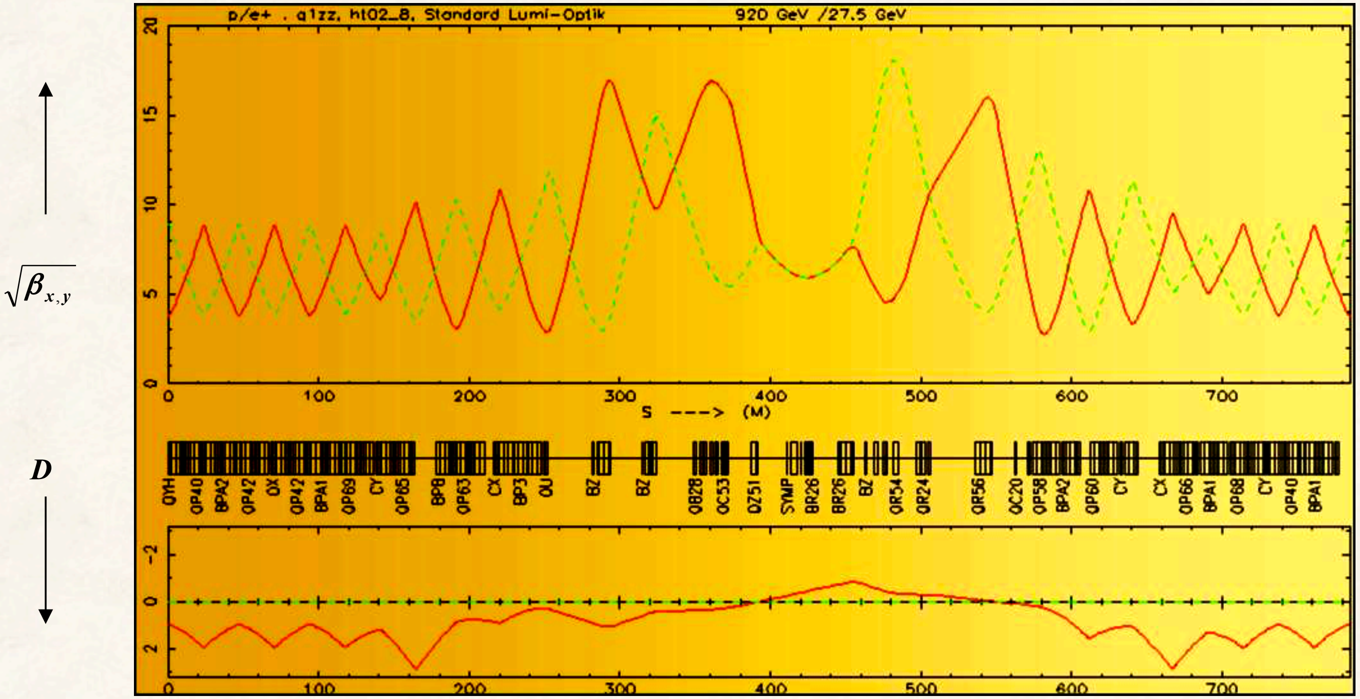
Lattice Design in Particle Accelerators

Bernhard Holzer, CERN



1952: Courant, Livingston, Snyder:

Theory of strong focusing in particle beams

Lattice Design: ***„… how to build a storage ring"***

High energy accelerators → circular machines

somewhere in the lattice we need a number of dipole magnets,
that are bending the design orbit to a closed ring

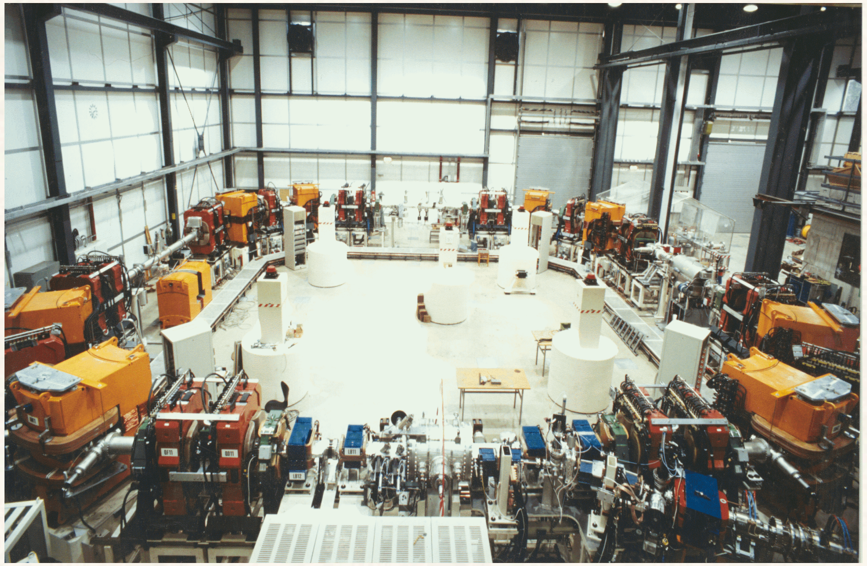
Geometry of the ring:

centrifugal force = Lorentz force

$$e * v * B = \frac{mv^2}{\rho}$$

$$\rightarrow e * B = \frac{mv}{\rho} = p / \rho$$

$$\rightarrow B * \rho = p / e$$



Example: heavy ion storage ring TSR
8 dipole magnets of equal bending strength

p = momentum of the particle,
ρ = curvature radius

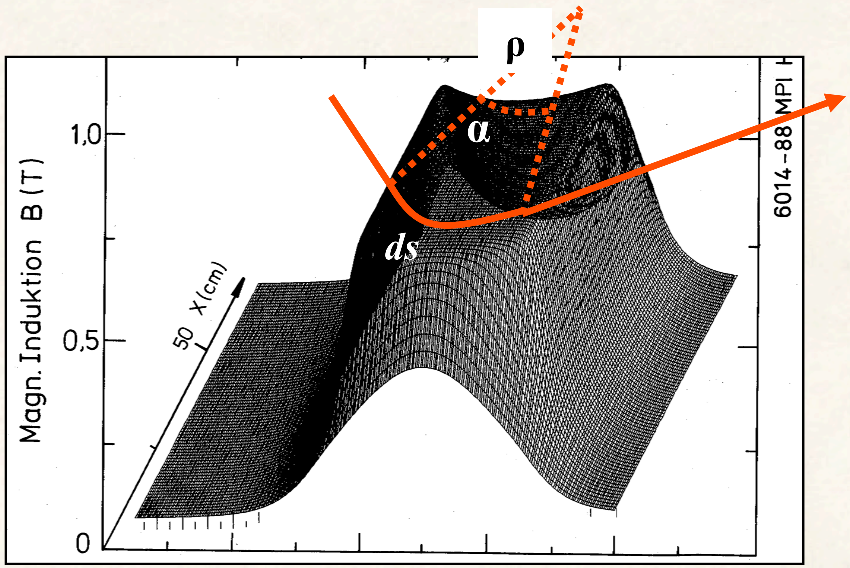
Bρ= beam rigidity

1.) Circular Orbit:

„... defining the geometry"

$$\alpha = \frac{ds}{\rho} \approx \frac{dl}{\rho}$$

$$\alpha = \frac{B*dl}{B*\rho}$$



field map of a storage ring dipole magnet

The angle swept out in one revolution must be 2π, so

$$\alpha = \frac{\int B dl}{B*\rho} = 2\pi \quad \rightarrow \quad \int B dl = 2\pi * \frac{p}{q}$$

... for a full circle

The strength of the dipoles B and the size of the storage ring ρ define the maximum momentum (i.e. energy) of the particles that can be carried in the machine. $B*\rho = p/q$

Nota bene: $\frac{\Delta B}{B} \approx 10^{-4}$ *is usually required !!*

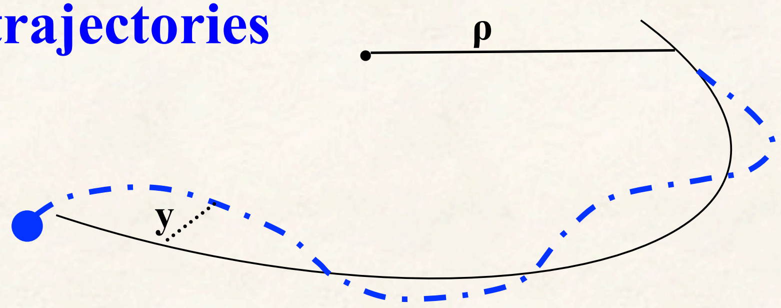
Example LHC:

7000 GeV Proton storage ring
dipole magnets N = 1232
l = 15 m
q = +1 e

$$\int B\, dl \approx N\, l\, B = 2\pi\ p/e$$

$$B \approx \frac{2\pi\ \ 7000\ \ 10^9\, eV}{1232\ \ 15\, m\ \ 3\ 10^8\, \frac{m}{s}\ \ e} = 8.3\, Tesla$$

2.) Focusing Forces: single particle trajectories



$$x''+K*x=0$$

$K=-k+1/\rho^2$ **hor. plane**

$K=k$ **vert. plane**

dipole magnet $\frac{1}{\rho}=\frac{B}{p/q}$

quadrupole magnet $k=\frac{g}{p/q}$

Example: HERA Ring:

Bending radius: ρ = 580 m

Quadrupol Gradient: g = 110 T/m

k = $33.64*10^{-3}$ /m²

$1/\rho^2$ = $2.97*10^{-6}$ /m²

For first estimates in large accelerators the weak focusing term $1/\rho^2$ can in general be neglected

Solution for a focusing magnet

$$y(s)=y_0*\cos(\sqrt{K}*s)+\frac{y'_0}{\sqrt{K}}*\sin(\sqrt{K}*s)$$

$$y'(s)=-y_0*\sqrt{K}*\sin(\sqrt{K}*s)+y'_0*\cos(\sqrt{K}*s)$$

Or written more convenient in matrix form:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M \begin{pmatrix} x \\ x' \end{pmatrix}_0$$

Hor. focusing Quadrupole Magnet

$$M_{QF} = \begin{pmatrix} \cos(\sqrt{K} * l) & \frac{1}{\sqrt{K}} \sin(\sqrt{K} * l) \\ -\sqrt{K} \sin(\sqrt{K} * l) & \cos(\sqrt{K} * l) \end{pmatrix}$$

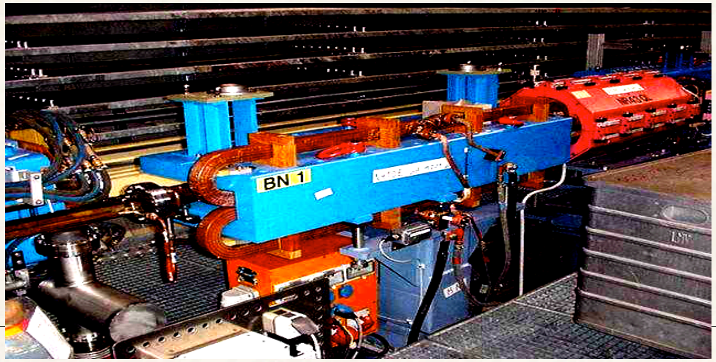
Hor. defocusing Quadrupole Magnet

$$M_{QD} = \begin{pmatrix} \cosh(\sqrt{K} * l) & \frac{1}{\sqrt{K}} \sinh(\sqrt{K} * l) \\ \sqrt{K} \sinh(\sqrt{K} * l) & \cosh(\sqrt{K} * l) \end{pmatrix}$$

Drift space

$$M_{Drift} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$$

$$M_{lattice} = M_{QF1} * M_{D1} * M_{QD} * M_{D1} * M_{QF2} \ldots$$



3.) Transfer Matrix … *yes we had the topic already*

… as Function of Twissparameters

general solution of Hill´s equation

$$x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}\cos\{\psi(s)+\phi\}$$

$$x'(s) = \frac{-\sqrt{\varepsilon}}{\sqrt{\beta(s)}}\left[\alpha(s)\cos\{\psi(s)+\phi\}+\sin\{\psi(s)+\phi\}\right]$$

remember the trigonometrical gymnastics: sin(a + b) = … etc

$$x(s) = \sqrt{\varepsilon}\sqrt{\beta_s}\left(\cos\psi_s\cos\phi - \sin\psi_s\sin\phi\right)$$

$$x'(s) = \frac{-\sqrt{\varepsilon}}{\sqrt{\beta_s}}\left[\alpha_s\cos\psi_s\cos\phi - \alpha_s\sin\psi_s\sin\phi + \sin\psi_s\cos\phi + \cos\psi_s\sin\phi\right]$$

starting at a point s(0) = s_0, where we put Ψ(0) = 0

$$\cos\phi = \frac{x_0}{\sqrt{\varepsilon\beta_0}} \quad ,$$

$$\sin\phi = -\frac{1}{\sqrt{\varepsilon}}\left(x'_0\sqrt{\beta_0}+\frac{\alpha_0 x_0}{\sqrt{\beta_0}}\right)$$

inserting above …

$$x(s) = \sqrt{\frac{\beta_s}{\beta_0}}\left\{\cos\psi_s + \alpha_0 \sin\psi_s\right\} x_0 + \left\{\sqrt{\beta_s\beta_0}\,\sin\psi_s\right\} x_0'$$

$$x'(s) = \frac{1}{\sqrt{\beta_s\beta_0}}\left\{(\alpha_0 - \alpha_s)\cos\psi_s - (1+\alpha_0\alpha_s)\sin\psi_s\right\} x_0 + \sqrt{\frac{\beta_0}{\beta_s}}\left\{\cos\psi_s - \alpha_s \sin\psi_s\right\} x_0'$$

which can be expressed ... for convenience ... in matrix form

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M \begin{pmatrix} x \\ x' \end{pmatrix}_0$$

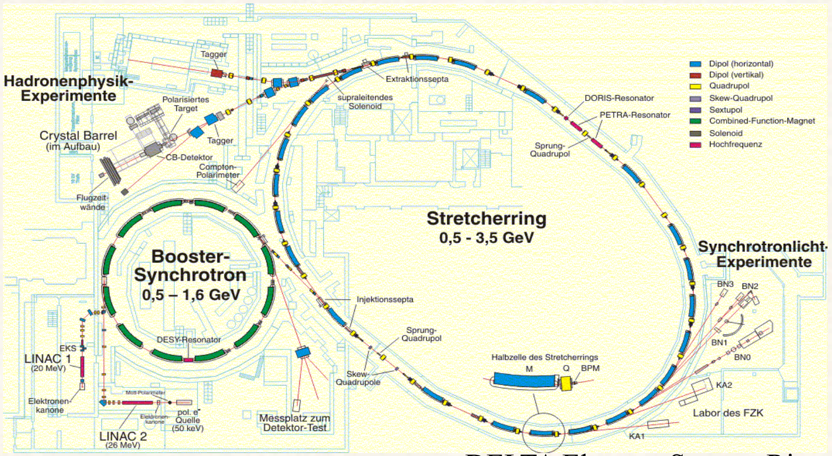
$$M = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}\left(\cos\psi_s + \alpha_0 \sin\psi_s\right) & \sqrt{\beta_s\beta_0}\,\sin\psi_s \\ \frac{(\alpha_0 - \alpha_s)\cos\psi_s - (1+\alpha_0\alpha_s)\sin\psi_s}{\sqrt{\beta_s\beta_0}} & \sqrt{\frac{\beta_0}{\beta s}}\left(\cos\psi_s - \alpha_s \sin\psi_s\right) \end{pmatrix}$$

we can calculate the single particle trajectories between two locations in the ring,

- **if we know the α β γ at these positions.**
- **and nothing but the α β γ at these positions.**
- **... !**

4.) Periodic Lattices

$$M = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}\left(\cos\psi_s + \alpha_0 \sin\psi_s\right) & \sqrt{\beta_s\beta_0}\,\sin\psi_s \\ \frac{(\alpha_0 - \alpha_s)\cos\psi_s - (1+\alpha_0\alpha_s)\sin\psi_s}{\sqrt{\beta_s\beta_0}} & \sqrt{\frac{\beta_0}{\beta s}}\left(\cos\psi_s - \alpha_s \sin\psi_s\right) \end{pmatrix}$$



DELTA Electron Storage Ring

„This **rather formidable looking matrix** simplifies considerably if we **consider one complete revolution** …“
M. Sands

$$M(s) = \begin{pmatrix} \cos\psi_{turn} + \alpha_s \sin\psi_{turn} & \beta_s \sin\psi_{turn} \\ -\gamma_s \sin\psi_{turn} & \cos\psi_{turn} - \alpha_s \sin\psi_{turn} \end{pmatrix}$$

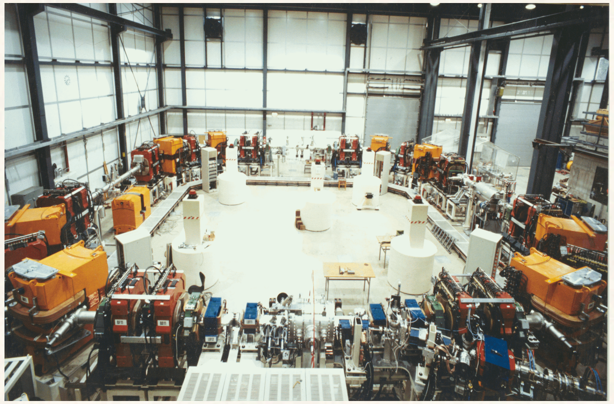
$$\psi_{turn} = \int_s^{s+L} \frac{ds}{\beta(s)}$$

ψ_{turn} = phase advance per period

Tune: **Phase advance per turn in units of 2π**

$$Q = \frac{1}{2\pi}\oint \frac{ds}{\beta(s)}$$

Stability Criterion:



Question: **what will happen, if we do not make too many mistakes and your particle performs one complete turn ?**

Matrix for 1 turn:

$$M = \begin{pmatrix} \cos\psi_{turn} + \alpha_s \sin\psi_{turn} & \beta_s \sin\psi_{turn} \\ -\gamma_s \sin\psi_{turn} & \cos\psi_{turn} - \alpha_s \sin\psi_{turn} \end{pmatrix} = \cos\psi \cdot \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\mathbf{1}} + \sin\psi \underbrace{\begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}}_{\mathbf{J}}$$

Matrix for N turns:

$$M^N = \left(1 \cdot \cos\psi + J \cdot \sin\psi\right)^N = 1 \cdot \cos N\psi + J \cdot \sin N\psi$$

The motion for N turns remains bounded, if the elements of M^N remain bounded

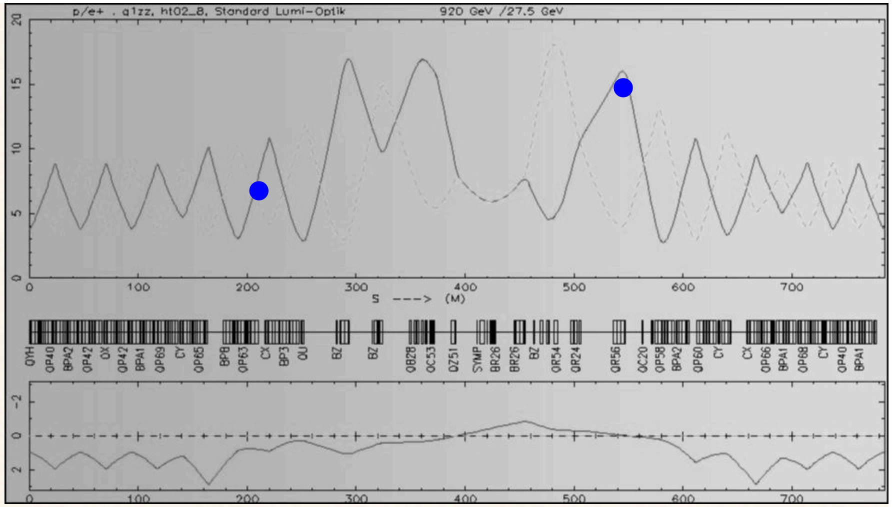
$$\psi = real \quad \leftrightarrow \quad \left|\cos\psi\right| \le 1 \quad \leftrightarrow \quad Tr(M) \le 2$$

5.) Transformation of α, β, γ

consider two positions in the storage ring: s_0 , s

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M * \begin{pmatrix} x \\ x' \end{pmatrix}_{s0}$$

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$



Betafunction in a Storage Ring

since ε = const (Liouville):

$$\varepsilon = \beta_s x'^2 + 2\alpha_s x x' + \gamma_s x^2$$

$$\varepsilon = \beta_0 x_0'^2 + 2\alpha_0 x_0 x_0' + \gamma_0 x_0^2$$

... remember W = CS´-SC´ = 1

$$\begin{pmatrix} x \\ x' \end{pmatrix}_0 = M^{-1} * \begin{pmatrix} x \\ x' \end{pmatrix}_s$$

$$M^{-1} = \begin{pmatrix} m_{22} & -m_{12} \\ -m_{21} & m_{11} \end{pmatrix}$$

→

$$x_0 = m_{22} x - m_{12} x'$$

$$x_0' = -m_{21} x + m_{11} x'$$

... inserting into ε

$$\varepsilon = \beta_0 (m_{11} x' - m_{21} x)^2 + 2\alpha_0 (m_{22} x - m_{12} x')(m_{11} x' - m_{21} x) + \gamma_0 (m_{22} x - m_{12} x')^2$$

sort via x, x´and compare the coefficients to get ....

The Twiss parameters α, β, γ can be transformed through the lattice via the matrix elements defined above.

$$\beta(s) = m_{11}^2 \beta_0 - 2m_{11}m_{12}\alpha_0 + m_{12}^2 \gamma_0$$
$$\alpha(s) = -m_{11}m_{21}\beta_0 + (m_{12}m_{21} + m_{11}m_{22})\alpha_0 - m_{12}m_{22}\gamma_0$$
$$\gamma(s) = m_{21}^2 \beta_0 - 2m_{21}m_{22}\alpha_0 + m_{22}^2 \gamma_0$$

in matrix notation:

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_{s2} = \begin{pmatrix} m_{11}^2 & -2m_{11}m_{12} & m_{12}^2 \\ -m_{11}m_{21} & m_{12}m_{21} + m_{22}m_{11} & -m_{12}m_{22} \\ m_{12}^2 & -2m_{22}m_{21} & m_{22}^2 \end{pmatrix} * \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_{s1}$$

!

1.) this expression is important

2.) given the twiss parameters α, β, γ at any point in the lattice we can transform them and calculate their values at any other point in the ring.

3.) the transfer matrix is given by the focusing properties of the lattice elements, the elements of M are just those that we used to calculate single particle trajectories.

... and here starts the lattice design !!!

Most simple example: drift space

$$M_{drift} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} = \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix}$$

particle coordinates

$$\begin{pmatrix} x \\ x' \end{pmatrix}_l = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} x \\ x' \end{pmatrix}_0 \quad \rightarrow \quad \begin{aligned} x(l) &= x_0 + l * x_0' \\ x'(l) &= x_0' \end{aligned}$$

transformation of twiss parameters:

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_l = \begin{pmatrix} 1 & -2l & l^2 \\ 0 & 1 & -l \\ 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_0$$

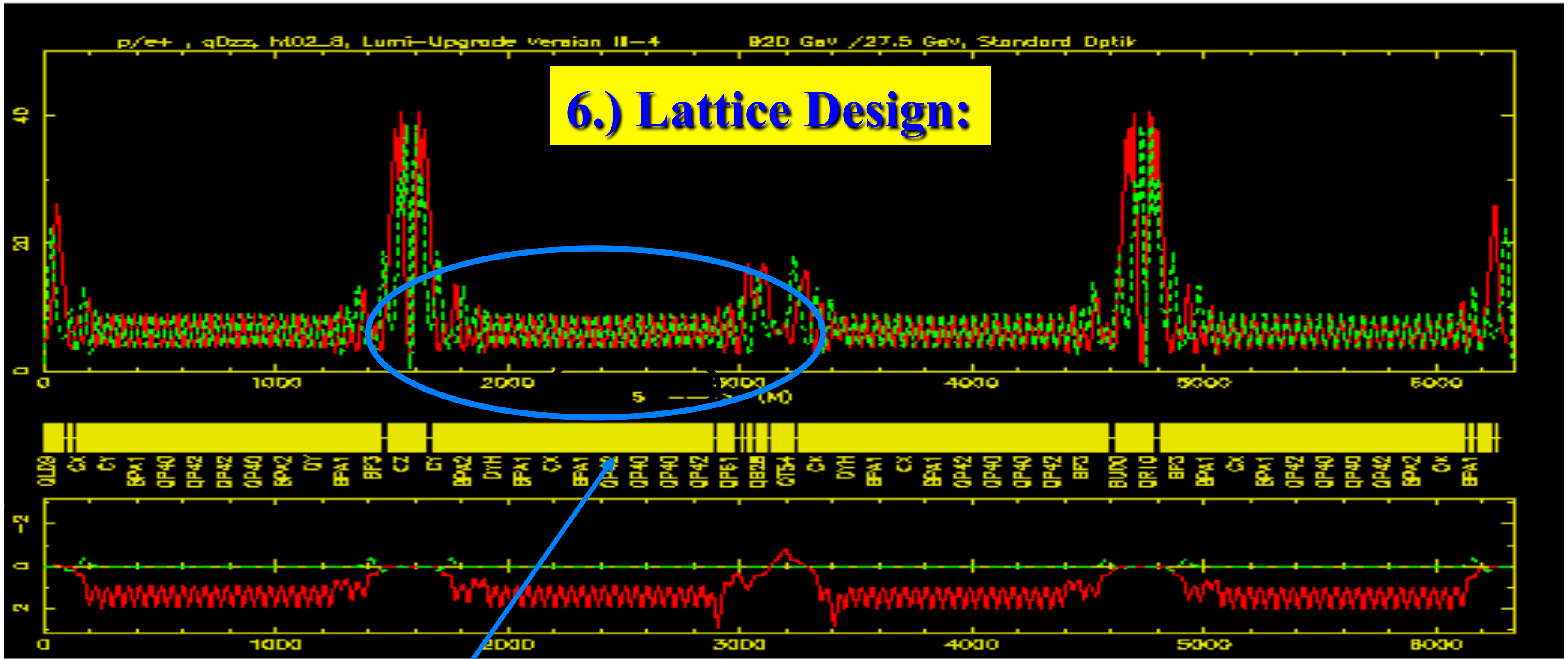
$$\beta(s) = \beta_0 - 2l * \alpha_0 + l^2 * \gamma_0$$

Stability ...?

$$trace(M) = 1 + 1 = 2$$

→ ***A periodic solution doesn 't exist in a lattice built exclusively out of drift spaces.***

6.) Lattice Design:



Arc: regular **(periodic) magnet structure:**

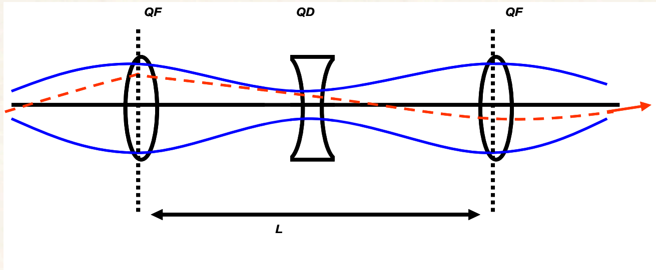
bending magnets → define the energy of the ring
main focusing & tune control, chromaticity correction,
multipoles for higher order corrections

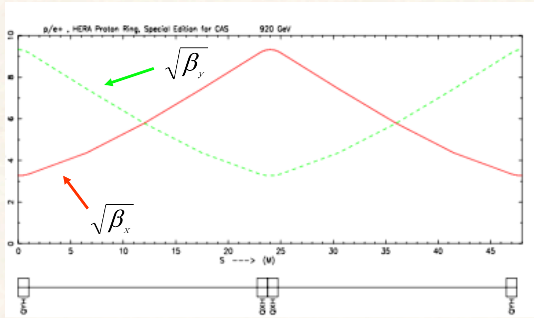
Straight sections: **drift spaces for injection, dispersion suppressors,**
low beta insertions, RF cavities, etc....
... and the high energy experiments if they cannot be avoided

Periodic Solution of a FoDo Cell

A magnet structure consisting of focusing and defocusing quadrupole lenses in alternating order with nothing in between.

(Nothing = elements that can be neglected on first sight: drift, bending magnets, RF structures ... and especially experiments...)





Output of the optics program:

Starting point for the calculation: in the middle of a focusing quadrupole, Phase advance per cell $\mu = 45°$, → calculate the twiss parameters for a periodic solution

Nr	*Type*	*Length*	*Strength*	β_x	α_x	φ_x	β_z	α_z	φ_z
		m	*1/m2*	*m*		*1/2π*	*m*		*1/2π*
0	*IP*	*0,000*	*0,000*	*11,611*	*0,000*	*0,000*	*5,295*	*0,000*	*0,000*
1	*QFH*	*0,250*	*-0,541*	*11,228*	*1,514*	*0,004*	*5,488*	*-0,781*	*0,007*
2	*QD*	*3,251*	*0,541*	*5,488*	*-0,781*	*0,070*	*11,228*	*1,514*	*0,066*
3	*QFH*	*6,002*	*-0,541*	*11,611*	*0,000*	*0,125*	*5,295*	*0,000*	*0,125*
4	*IP*	*6,002*	*0,000*	*11,611*	*0,000*	*0,125*	*5,295*	*0,000*	*0,125*

QX= 0,125 *QZ= 0,125* → $\mathbf{0.125 * 2\pi = 45^0}$

Can we understand what the optics code is doing ?

matrices

$$M_{QF} = \begin{pmatrix} \cos(\sqrt{K} * l_q) & \frac{1}{\sqrt{K}} \sin(\sqrt{K} * l_q) \\ -\sqrt{K} \sin(\sqrt{K} * l_q) & \cos(\sqrt{K} * l_q) \end{pmatrix}, \qquad M_{Drift} = \begin{pmatrix} 1 & l \\ 0 & 1_d \end{pmatrix}$$

strength and length of the FoDo elements

K = +/- 0.54102 m^{-2}
lq = 0.5 m
ld = 2.5 m

The matrix for the complete cell is obtained by multiplication of the element matrices

$$M_{FoDo} = M_{qfh} * M_{ld} * M_{qd} * M_{ld} * M_{qfh}$$

Putting the numbers in and multiplying out ...

$$M_{FoDo} = \begin{pmatrix} 0.707 & 8.206 \\ -0.061 & 0.707 \end{pmatrix}$$

The transfer matrix for 1 period gives us all the information that we need !

1.) is the motion stable?

$$trace(M_{FoDo}) = 1.415 \rightarrow \quad \underline{\underline{< 2}}$$

2.) Phase advance per cell

$$M(s) = \begin{pmatrix} \cos\psi_{cell} + \alpha_s \sin\psi_{cell} & \beta_s \sin\psi_{cell} \\ -\gamma_s \sin\psi_{cell} & \cos\psi_{cell} - \alpha_s \sin\psi_{cell} \end{pmatrix}$$

$$\cos\psi_{cell} = \frac{1}{2} trace(M) = 0.707$$

$$\psi_{cell} = \cos^{-1}\left(\frac{1}{2} trace(M)\right) \underline{\underline{= 45}}$$

3.) hor β-function

$$\beta = \frac{m_{12}}{\sin\psi_{cell}} = \underline{\underline{11.611\, m}}$$

4.) hor α-function

$$\alpha = \frac{m_{11} - \cos\psi_{cell}}{\sin\psi_{cell}} = \underline{\underline{0}}$$

Can we do a bit easier ?
We can ... in thin lens approximation !

Matrix of a focusing quadrupole magnet:

$$M_{QF} = \begin{pmatrix} \cos(\sqrt{K} * l) & \frac{1}{\sqrt{K}} \sin(\sqrt{K} * l) \\ -\sqrt{K} \sin(\sqrt{K} * l) & \cos(\sqrt{K} * l) \end{pmatrix}$$

If the focal length *f* is much larger than the length of the quadrupole magnet,

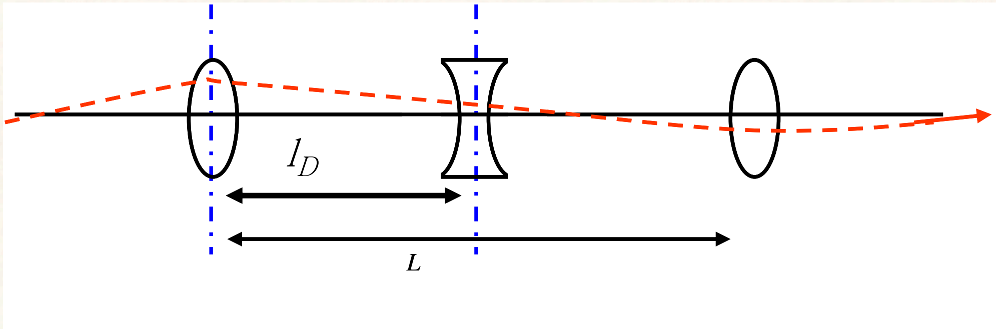
$$f = 1/kl_Q >> l_Q$$

the transfer matrix can be aproximated using ♪

$$kl_q = const, \; l_q \rightarrow 0$$

$$M = \begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix}$$

7.) FoDo in thin lens approximation



$$l_D = L/2$$
$$\tilde{f} = 2f$$

Calculate the matrix for a half cell, starting in the middle of a foc. quadrupole:

$$M_{halfCell} = M_{QD/2} * M_{lD} * M_{QF/2}$$

$$M_{halfCell} = \begin{pmatrix} 1 & 0 \\ 1/\tilde{f} & 1 \end{pmatrix} * \begin{pmatrix} 1 & l_D \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} 1 & 0 \\ -1/\tilde{f} & 1 \end{pmatrix}$$

note: $\tilde{f}$ denotes the focusing strength of half a quadrupole, so $\tilde{f} = 2f$

$$M_{halfCell} = \begin{pmatrix} 1 - l_D/\tilde{f} & l_D \\ -l_D/\tilde{f}^2 & 1 + l_D/\tilde{f} \end{pmatrix}$$

for the second half cell set f → -f

FoDo in thin lens approximation

Matrix for the complete FoDo cell

$$M = \begin{pmatrix} 1+ l_D/\tilde{f} & l_D \\ -l_D/\tilde{f}^2 & 1- l_D/\tilde{f} \end{pmatrix} * \begin{pmatrix} 1- l_D/\tilde{f} & l_D \\ -l_D/\tilde{f}^2 & 1+ l_D/\tilde{f} \end{pmatrix}$$

$$M = \begin{pmatrix} 1-\frac{2l_D^2}{\tilde{f}^2} & 2l_D(1+\frac{l_D}{\tilde{f}}) \\ 2(\frac{l_D^2}{\tilde{f}^3}-\frac{l_D}{\tilde{f}^2}) & 1-2\frac{l_D^2}{\tilde{f}^2} \end{pmatrix}$$

Now we know, that the phase advance is related to the transfer matrix by

$$\cos\psi_{cell} = \frac{1}{2} trace(M) = \frac{1}{2} * (2 - \frac{4l_d^2}{\tilde{f}^2}) = 1 - \frac{2l_d^2}{\tilde{f}^2}$$

After some beer and with a little bit of trigonometric gymnastics

$$\cos(x) = \cos^2(x/2) - \sin^2(x/2) = 1 - 2\sin^2(x/2)$$

we can calculate the phase advance as a function of the FoDo parameter …

$$\cos\psi_{cell} = 1 - 2\sin^2(\psi_{cell}/2) = 1 - \frac{2l_d^2}{\tilde{f}^2}$$

$$\sin(\psi_{cell}/2) = l_d/\tilde{f} = \frac{L_{cell}}{2\tilde{f}}$$

$$\sin(\psi_{cell}/2) = \frac{L_{cell}}{4f}$$

The ratio between cell length L_{cell} and focal length of the quadrupoles f determines the phase advance ψ_{cell}

Example:
45-degree Cell

$L_{Cell} = l_{QF} + l_D + l_{QD} + l_D = 0.5m+2.5m+0.5m+2.5m = 6m$

$1/f = k*l_Q = 0.5m*0.541\ m^{-2} = 0.27\ m^{-1}$

$$\sin(\psi_{cell}/2) = \frac{L_{cell}}{4f} = 0.405$$

$$\rightarrow \quad \psi_{cell} = 47.8^\circ$$

$$\rightarrow \quad \beta = 11.4\ m$$

Remember:
Exact calculation yields:

$$\rightarrow \quad \psi_{cell} = 45^\circ$$

$$\rightarrow \quad \beta = 11.6\ m$$

8.) Stability in a FoDo structure


SPS Lattice

$$M_{FoDo} = \begin{pmatrix} 1 - \frac{2 l_D^2}{\tilde{f}^2} & 2 l_D (1 + \frac{l_D}{\tilde{f}}) \\ 2(\frac{l_D^2}{\tilde{f}^3} - \frac{l_D}{\tilde{f}^2}) & 1 - 2\frac{l_D^2}{\tilde{f}^2} \end{pmatrix}$$

Stability requires:

$$|Trace(M)| < 2$$

$$|Trace(M)| = \left| 2 - \frac{4 l_d^2}{\tilde{f}^2} \right| < 2$$

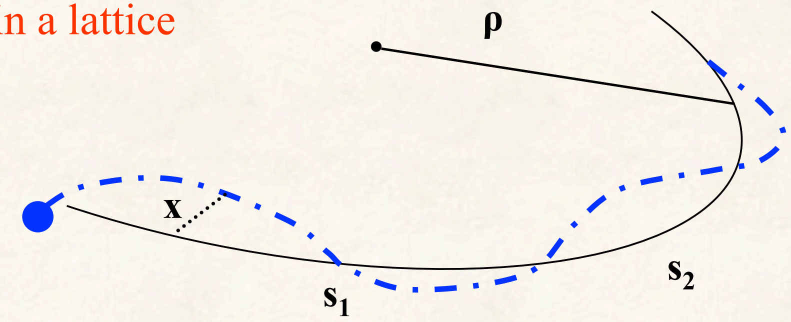
$$\rightarrow f > \frac{L_{cell}}{4}$$

For stability the focal length has to be larger than a quarter of the cell length
... don't focus to strong !

Transformation Matrix in Terms of the Twiss Parameters

Transformation of the coordinate vector (x,x´) in a lattice

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = M_{s1,s2} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$



General solution of the equation of motion

$$x(s) = \sqrt{\varepsilon * \beta(s)} * \cos(\psi(s) + \varphi)$$

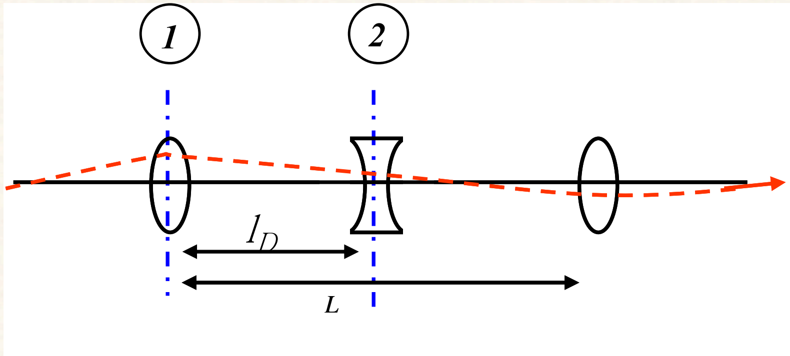
$$x'(s) = \sqrt{\varepsilon / \beta(s)} * \{ \alpha(s)\cos(\psi(s) + \varphi) + \sin(\psi(s) + \varphi) \}$$

Transformation of the coordinate vector (x,x´)
expressed as a function of the twiss parameters

$$M_{1\to 2} = \begin{pmatrix} \sqrt{\frac{\beta_2}{\beta_1}}(\cos\psi_{12} + \alpha_1 \sin\psi_{12}) & \sqrt{\beta_1\beta_2}\sin\psi_{12} \\ \frac{(\alpha_1 - \alpha_2)\cos\psi_{12} - (1 + \alpha_1\alpha_2)\sin\psi_{12}}{\sqrt{\beta_1\beta_2}} & \sqrt{\frac{\beta_1}{\beta_2}}(\cos\psi_{12} - \alpha_2 \sin\psi_{12}) \end{pmatrix}$$

Transfer Matrix for half a FoDo cell:

$$M_{halfcell} = \begin{pmatrix} 1 - l_D/\tilde{f} & l_D \\ -l_D/\tilde{f}^2 & 1 + l_D/\tilde{f} \end{pmatrix}$$



Compare to the twiss parameter form of M

$$M_{1\rightarrow 2} = \begin{pmatrix} \sqrt{\frac{\beta_2}{\beta_1}}(\cos\psi_{12} + \alpha_1 \sin\psi_{12}) & \sqrt{\beta_1\beta_2}\sin\psi_{12} \\ \frac{(\alpha_1 - \alpha_2)\cos\psi_{12} - (1 + \alpha_1\alpha_2)\sin\psi_{12}}{\sqrt{\beta_1\beta_2}} & \sqrt{\frac{\beta_1}{\beta_2}}(\cos\psi_{12} - \alpha_2 \sin\psi_{12}) \end{pmatrix}$$

In the middle of a foc (defoc) quadrupole of the FoDo we allways have $\alpha = 0$, and the half cell will lead us from β_{max} to β_{min}

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{\check{\beta}}{\hat{\beta}}}\cos\frac{\psi_{cell}}{2} & \sqrt{\check{\beta}\hat{\beta}}\sin\frac{\psi_{cell}}{2} \\ \frac{-1}{\sqrt{\hat{\beta}\check{\beta}}}\sin\frac{\psi_{cell}}{2} & \sqrt{\frac{\hat{\beta}}{\check{\beta}}}\cos\frac{\psi_{cell}}{2} \end{pmatrix}$$

Solving for β_{max} and β_{min} and remembering that …. $\sin\frac{\psi_{cell}}{2} = \frac{l_d}{\tilde{f}} = \frac{L}{4f}$

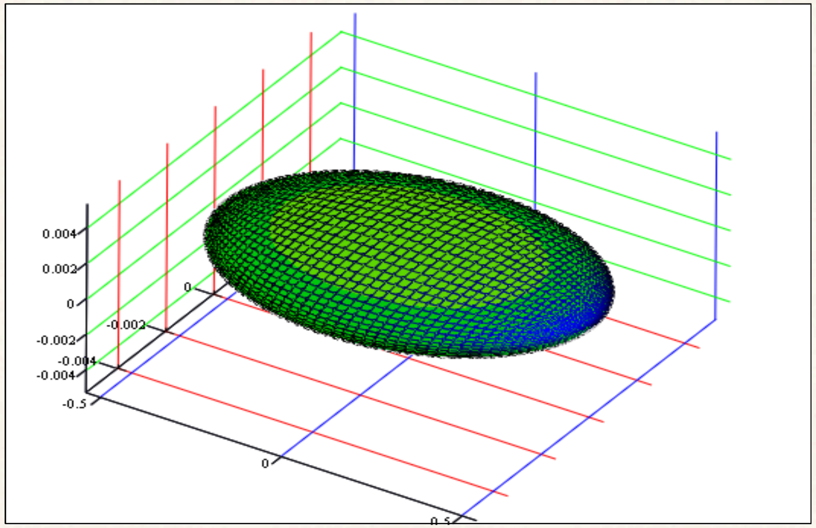
$$\frac{m_{22}}{m_{11}} = \frac{\hat{\beta}}{\breve{\beta}} = \frac{1 + l_d/\tilde{f}}{1 - l_d/\tilde{f}} = \frac{1+\sin(\psi_{cell}/2)}{1-\sin(\psi_{cell}/2)}$$

$$\frac{m_{12}}{m_{21}} = \hat{\beta}\breve{\beta} = \tilde{f}^2 = \frac{l_d^2}{\sin^2(\psi_{cell}/2)}$$

$\rightarrow$

$$\hat{\beta} = \frac{(1+\sin\frac{\psi_{cell}}{2})L}{\sin\psi_{cell}} \quad !$$

$$\breve{\beta} = \frac{(1-\sin\frac{\psi_{cell}}{2})L}{\sin\psi_{cell}} \quad !$$



The maximum and minimum values of the β-function are solely determined by the phase advance and the length of the cell.

Longer cells lead to larger β

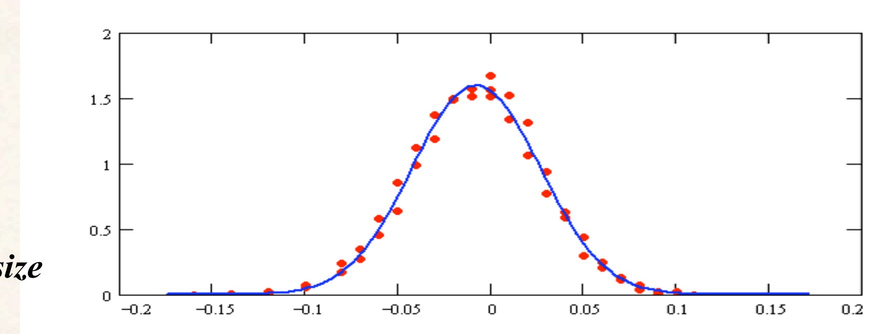
typical shape of a proton bunch in a FoDo Cell

9.) Beam dimension:

Optimisation of the FoDo Phase advance

In both planes a gaussian particle distribution is assumed, given by the beam emittance ε and the β-function

$$\sigma = \sqrt{\varepsilon\beta}$$



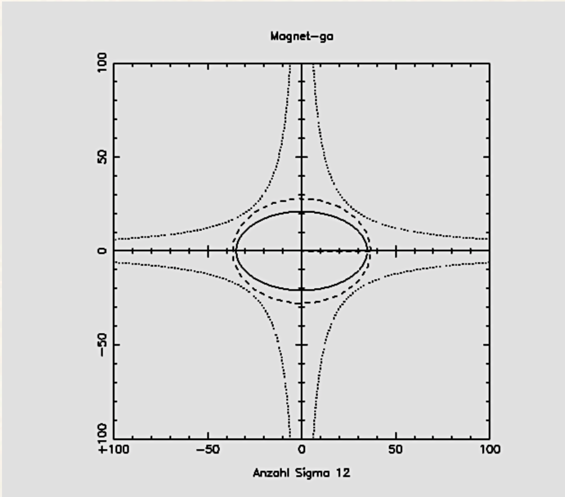
HERA beam size

In general proton beams are *„round"* in the sense that

$$\varepsilon_x \approx \varepsilon_y$$

So for highest aperture we have to minimise the β-function in both planes:

$$r^2 = \varepsilon_x\beta_x + \varepsilon_y\beta_y$$



typical beam envelope, vacuum chamber and pole shape in a foc. Quadrupole lens in HERA

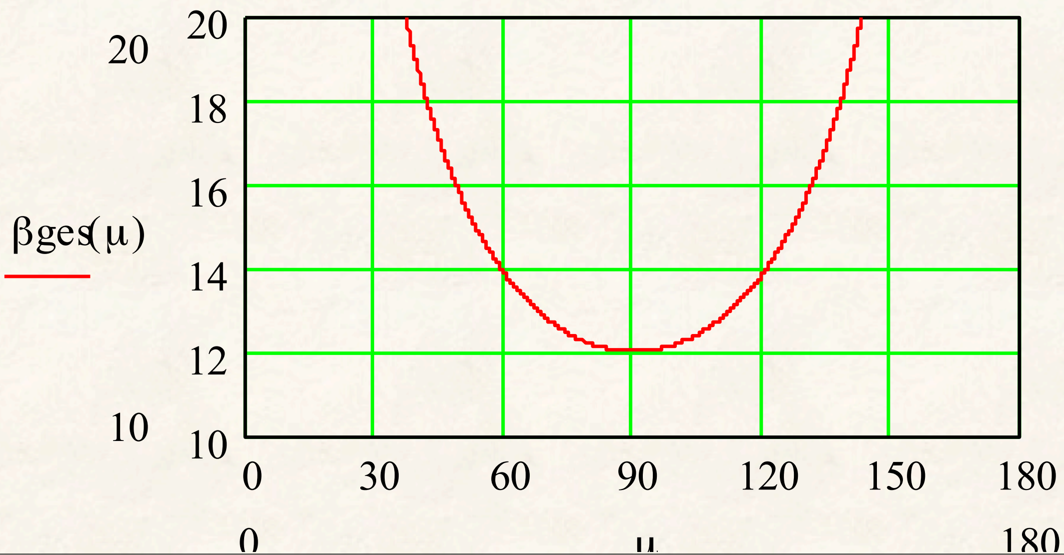
9.) Optimisation of the FoDo Phase advance

$$r^2 = \varepsilon_x \beta_x + \varepsilon_y \beta_y$$

search for the phase advance μ that results in a minimum of the sum of the beta's

$$\hat{\beta} + \breve{\beta} = \frac{(1+\sin\frac{\psi_{cell}}{2})L}{\sin\psi_{cell}} + \frac{(1-\sin\frac{\psi_{cell}}{2})L}{\sin\psi_{cell}}$$

$$\hat{\beta} + \breve{\beta} = \frac{2L}{\sin\psi_{cell}} \qquad \frac{d}{d\psi_{cell}}\left(2L/\sin\psi_{cell}\right) = 0$$



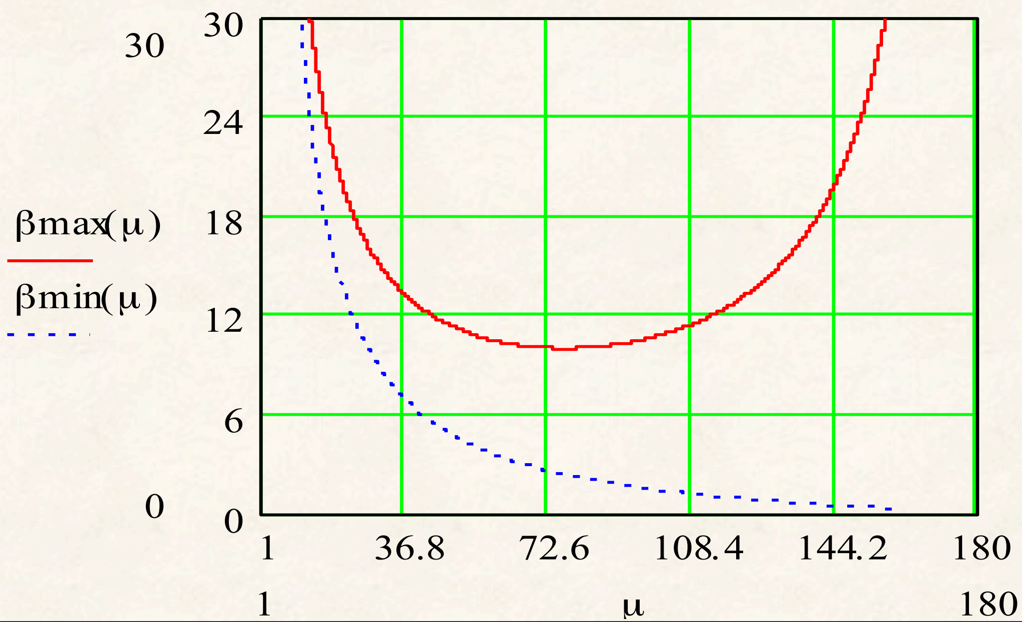
$$\frac{L}{\sin^2\psi_{cell}} * \cos\psi_{cell} = 0 \quad \rightarrow \quad \psi_{cell} = 90^\circ$$

Electrons are different

electron beams are usually flat, $\varepsilon_y \approx 2 - 10\ \%\ \varepsilon_x$

→ ***optimise only*** β_{hor}

$$\frac{d}{d\psi_{cell}}(\hat{\beta}) = \frac{d}{d\psi_{cell}} \frac{L(1+\sin\frac{\psi_{cell}}{2})}{\sin\psi_{cell}} = 0 \quad \rightarrow \quad \psi_{cell} = 76^\circ$$



red curve: β_{max}
blue curve: β_{min}
as a function of the phase advance ψ

The „ not so ideal world "

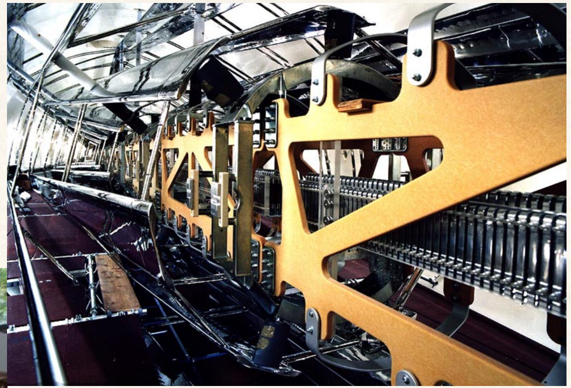
13.) The „ Δp / p ≠ 0" Problem

ideal accelerator: *all particles will see the same accelerating voltage.*

→ $\Delta p / p = 0$

„nearly ideal" accelerator: Cockroft Walton or van de Graaf

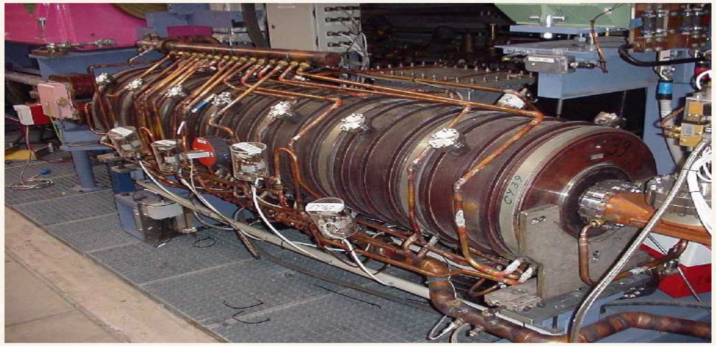
$\Delta p / p \approx 10^{-5}$



Vivitron, Straßbourg, inner structure of the acc. section

MP Tandem van de Graaf Accelerator at MPI for Nucl. Phys. Heidelberg

RF Acceleration ↔ AC voltage

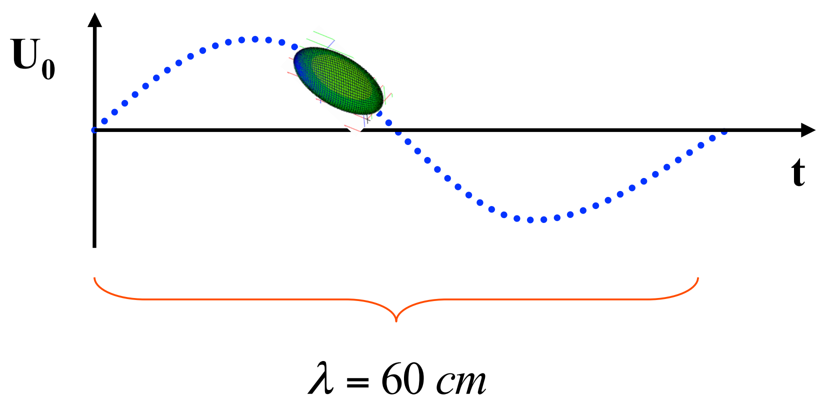


RF cavities of an electron ring

Problem: panta rhei !!!

(Heraklit: 540-480 v. Chr.)

Example:



$$\left.\begin{aligned} \nu &= 500\,MHz \\ c &= \lambda \nu \end{aligned}\right\} \quad \lambda = 60\ cm$$

$$\sin(90^o) = 1$$

$$\sin(84^o) = 0.994$$

$$\frac{\Delta U}{U} = 6.0\ \ 10^{-3}$$

typical momentum spread of an electron bunch $\Delta p/p \approx 1*10^{-3}$

By definition, the RF systems installed for the quest of higher beam energies, lead unavoidable to a considerable momentum spread in the beam.

Dispersive and Chromatic Effects: $\Delta p/p \neq 0$

Are there any Problems ???

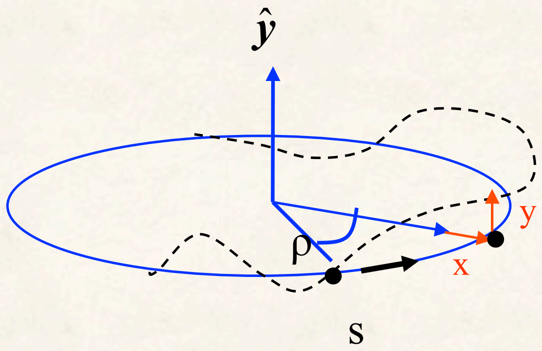
Sure there are !!!

... font colors for pedagogical reasons

14.) Dispersion: trajectories for $\Delta p / p \neq 0$

Question: do you remember last session, page 12 ? … sure you do

Force acting on the particle



$$F = m \frac{d^2}{dt^2}(x+\rho) - \frac{mv^2}{x+\rho} = e B_y v$$

remember: x ≈ mm , ρ ≈ m … → develop for small x

$$m \frac{d^2 x}{dt^2} - \frac{mv^2}{\rho}\left(1 - \frac{x}{\rho}\right) = eB_y v$$

consider only linear fields, and change independent variable: t → s

$$B_y = B_0 + x \frac{\partial B_y}{\partial x}$$

$$x'' - \frac{1}{\rho}\left(1 - \frac{x}{\rho}\right) = \frac{e B_0}{mv} + \frac{e\, x\, g}{mv}$$

$p = p_0 + \Delta p$

… but now take a small momentum error into account !!!

Dispersion:

develop for small momentum error $\Delta p << p_0 \Rightarrow \frac{1}{p_0 + \Delta p} \approx \frac{1}{p_0} - \frac{\Delta p}{p_0^2}$

$$x'' - \frac{1}{\rho} + \frac{x}{\rho^2} \approx \underbrace{\frac{e\ B_0}{p_0}}_{-\frac{1}{\rho}} - \frac{\Delta p}{p_0^2} eB_0 + \underbrace{\frac{xeg}{p_0}}_{k*x} - \underbrace{xeg\frac{\Delta p}{p_0^2}}_{\approx 0}$$

$$x'' + \frac{x}{\rho^2} \approx \frac{\Delta p}{p_0} * \underbrace{\frac{(-eB_0)}{p_0}}_{\frac{1}{\rho}} + k*x = \frac{\Delta p}{p_0} * \frac{1}{\rho} + k*x$$

$$x'' + \frac{x}{\rho^2} - kx = \frac{\Delta p}{p_0}\frac{1}{\rho} \longrightarrow x'' + x(\frac{1}{\rho^2} - k) = \frac{\Delta p}{p_0}\frac{1}{\rho}$$

Momentum spread of the beam adds a term on the r.h.s. of the equation of motion.
→ inhomogeneous differential equation.

Dispersion:

$$x'' + x\left(\frac{1}{\rho^2} - k\right) = \frac{\Delta p}{p} \cdot \frac{1}{\rho}$$

general solution:

$$x(s) = x_h(s) + x_i(s)$$

$$\begin{cases} x_h''(s) + K(s) \cdot x_h(s) = 0 \\ x_i''(s) + K(s) \cdot x_i(s) = \dfrac{1}{\rho} \cdot \dfrac{\Delta p}{p} \end{cases}$$

Normalise with respect to $\Delta p/p$:

$$D(s) = \frac{x_i(s)}{\Delta p / p}$$

Dispersion function D(s)

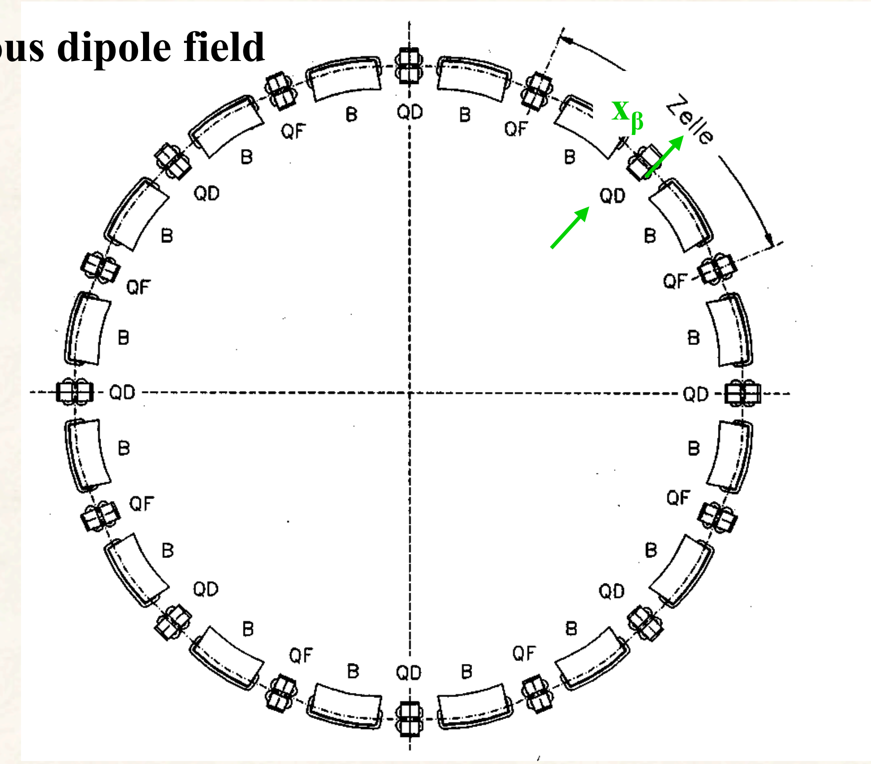
* is that special orbit, an ideal particle would have for $\Delta p/p = 1$

* the orbit of any particle is the sum of the well known x_β and the dispersion

* as D(s) is just another orbit it will be subject to the focusing properties of the lattice

Dispersion

Example: homogeneous dipole field



bit for Δp/p > 0

$$ D(s) \cdot \frac{\Delta p}{p} $$

Matrix formalism:

$$x(s) = x_\beta(s) + D(s) \cdot \frac{\Delta p}{p}$$

$$x(s) = C(s) \cdot x_0 + S(s) \cdot x_0' + D(s) \cdot \frac{\Delta p}{p}$$

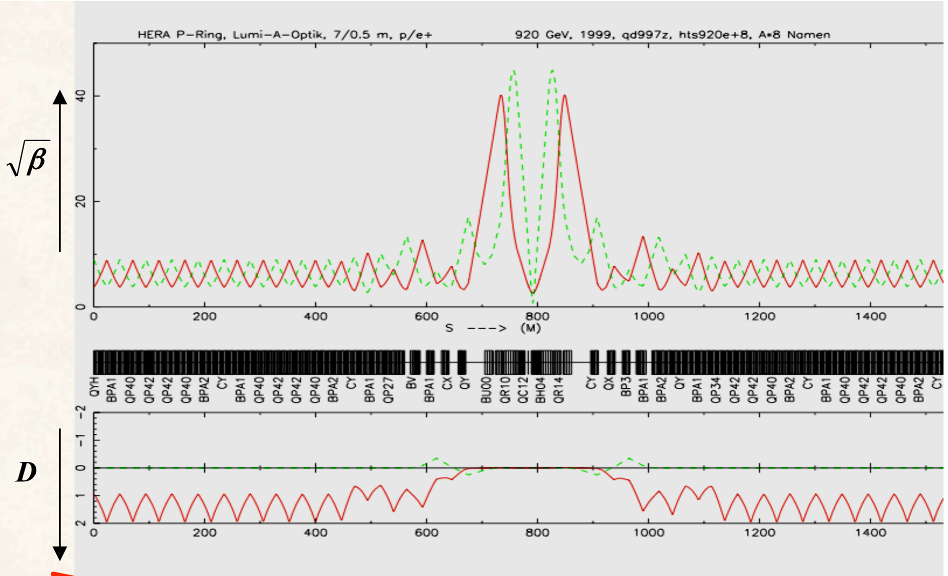
$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_0 + \frac{\Delta p}{p} \begin{pmatrix} D \\ D' \end{pmatrix}$$

$$C = \cos(\sqrt{|k|}s) \qquad S = \frac{1}{\sqrt{|k|}} \sin(\sqrt{|k|}s)$$

$$C' = \frac{dC}{ds} \qquad S' = \frac{dS}{ds}$$

or expressed as 3x3 matrix

$$\begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_s = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_0$$



Example

$$x_\beta = 1 \ldots 2\,mm$$
$$D(s) \approx 1 \ldots 2\,m$$
$$\Delta p/p \approx 1 \cdot 10^{-3}$$

Amplitude of Orbit oscillation gets additional contribution
→ overall beam size :

$$\sigma = \sqrt{\varepsilon\beta + D^2 \left(\frac{\Delta p}{p}\right)^2}$$

!

Whenever we want to get smallest beam sizes the dispersion must vanish (e.g. at the IP)

Calculate D, D´: ... takes a couple of sunny Sunday evenings !

$$D(s) = S(s) \int_{s0}^{s1} \frac{1}{\rho} C(\tilde{s}) d\tilde{s} - C(s) \int_{s0}^{s1} \frac{1}{\rho} S(\tilde{s}) d\tilde{s}$$

11.) Résumé

1.) Dipole strength:

$$\int B\,ds = N * B_0 * l_{eff} = 2\pi \frac{p}{q}$$

2.Transfer Matrix in Twiss form:

$$M = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}\left(\cos\psi_s + \alpha_0 \sin\psi_s\right) & \sqrt{\beta_s\beta_0}\,\sin\psi_s \\ \frac{(\alpha_0 - \alpha_s)\cos\psi_s - (1 + \alpha_0\alpha_s)\sin\psi_s}{\sqrt{\beta_s\beta_0}} & \sqrt{\frac{\beta_0}{\beta s}}\left(\cos\psi_s - \alpha_s \sin\psi_s\right) \end{pmatrix}$$

for periodic structures:

$$M(s) = \begin{pmatrix} \cos\psi_{turn} + \alpha_s \sin\psi_{turn} & \beta_s \sin\psi_{turn} \\ -\gamma_s \sin\psi_{turn} & \cos\psi_{turn} - \alpha_s \sin\psi_{turn} \end{pmatrix}$$

3.) Stability condition

$$Trace(M) < 2$$

4.) Transformation of Twiss parameters

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_{s2} = \begin{pmatrix} m_{11}^2 & -2m_{11}m_{12} & m_{12}^2 \\ -m_{11}m_{21} & m_{12}m_{21} + m_{22}m_{11} & -m_{12}m_{22} \\ m_{12}^2 & -2m_{22}m_{21} & m_{22}^2 \end{pmatrix} * \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_{s1}$$

5.) Thin lens approximation

$$M_{QF} = \begin{pmatrix} 1 & 0 \\ \frac{1}{f_Q} & 1 \end{pmatrix}, \qquad f_Q = \frac{1}{k_Q l_Q}$$

6.) Phase advance per cell:

$$\sin(\psi_{cell}/2) = \frac{L_{cell}}{4f}$$

(thin lens approx)

7.) Beta-function in a FoDo cell

$$\hat{\beta} = \frac{(1+\sin\frac{\psi_{cell}}{2})L}{\sin\psi_{cell}}$$

$$\breve{\beta} = \frac{(1-\sin\frac{\psi_{cell}}{2})L}{\sin\psi_{cell}}$$

(thin lens approx)

8.) Dispersion:

$$D(s) = \frac{x_i(s)}{\Delta p/p}$$

9.) Overall beam size:

$$\sigma = \sqrt{\varepsilon\beta + D^2\left(\frac{\Delta p}{p}\right)^2}$$

10.) Tune (rough estimate)

$$Q = N * \frac{\psi_{period}}{2\pi} = \frac{1}{2\pi} * \oint \frac{ds}{\beta(s)} \approx \frac{1}{2\pi} * \frac{2\pi\bar{R}}{\bar{\beta}} = \frac{\bar{R}}{\bar{\beta}}$$

$$Q \approx \frac{\bar{R}}{\bar{\beta}}$$

= average radius / average β-function

11.) Stability in a FoDo cell

$$f_Q > \frac{L_{Cell}}{4}$$

(thin lens approx)

Appendix I:

Stability criterion …. proof for the disbelieving collegues !!

Matrix for 1 turn:

$$M = \begin{pmatrix} \cos\psi_{turn} + \alpha_s \sin\psi_{turn} & \beta_s \sin\psi_{turn} \\ -\gamma_s \sin\psi_{turn} & \cos\psi_{turn} - \alpha_s \sin\psi_{turn} \end{pmatrix} = \cos\psi \cdot \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\mathbf{I}} + \sin\psi \underbrace{\begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}}_{\mathbf{J}}$$

Matrix for 2 turns:

$$\boldsymbol{M}^2 = \left(\boldsymbol{I} \cos\psi_1 + \boldsymbol{J} \sin\psi_1\right)\left(\boldsymbol{I} \cos\psi_2 + \boldsymbol{J} \sin\psi_2\right)$$

$$= \boldsymbol{I}^2 \cos\psi_1 \cos\psi_2 + \boldsymbol{IJ} \cos\psi_1 \sin\psi_2 + \boldsymbol{JI} \sin\psi_1 \cos\psi_2 + \boldsymbol{J}^2 \sin\psi_1 \sin\psi_2$$

now …

$$\boldsymbol{I}^2 = \boldsymbol{I}$$

$$\boldsymbol{I}\,\boldsymbol{J} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}$$

$$\boldsymbol{J}\,\boldsymbol{I} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} * \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}$$

$$\Big\} \quad \boldsymbol{I}\,\boldsymbol{J} = \boldsymbol{J}\,\boldsymbol{I}$$

$$\boldsymbol{J}^2 = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} * \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} = \begin{pmatrix} \alpha^2 - \gamma\beta & \alpha\beta - \beta\alpha \\ -\gamma\alpha + \alpha\gamma & \alpha^2 - \gamma\beta \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\boldsymbol{I}$$

$$\boldsymbol{M}^2 = \boldsymbol{I} \cos(\psi_1 + \psi_2) + \boldsymbol{J} \sin(\psi_1 + \psi_2)$$

$$\boldsymbol{M}^2 = \boldsymbol{I} \cos(2\psi) + \boldsymbol{J} \sin(2\psi)$$

Appendix II: Dipole Errors / Quadrupole Misalignment

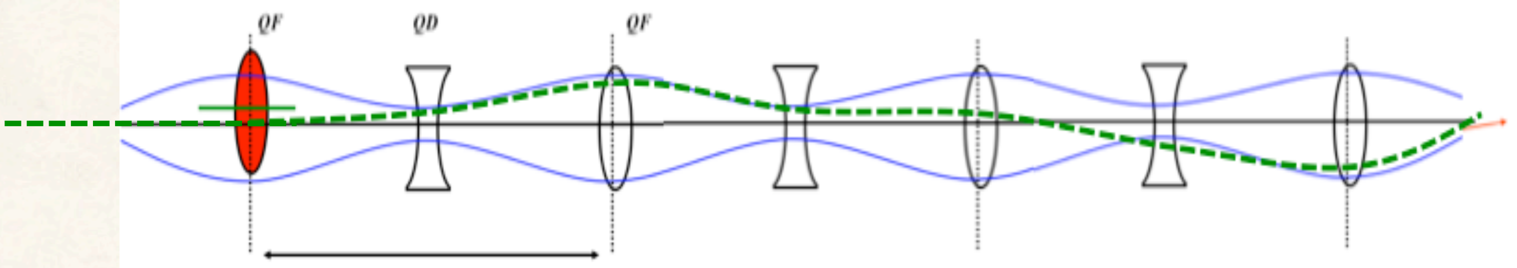
The Design Orbit is defined by the strength and arrangement of the dipoles.
Under the influence of dipole imperfections and quadrupole misalignments we obtain a "Closed Orbit" which is hopefully still closed and not too far away from the design.

Dipole field error: $$\theta = \frac{dl}{\rho} = \frac{\int B\,dl}{B\rho}$$

Quadrupole offset: $$g = \frac{dB}{dx} \rightarrow \quad \Delta x \cdot g \quad = \Delta x \frac{dB}{dx} = \Delta B$$

misaligned quadrupoles (or orbit offsets in quadrupoles) create dipole effects that lead to a distorted "closed orbit"

normalised to p/e: $$\Delta x \cdot k = \Delta x \cdot \frac{g}{B\rho} = \frac{1}{\rho} \qquad \begin{pmatrix} x \\ x' \end{pmatrix}_i = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ x' \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{l}{\rho} \end{pmatrix}$$

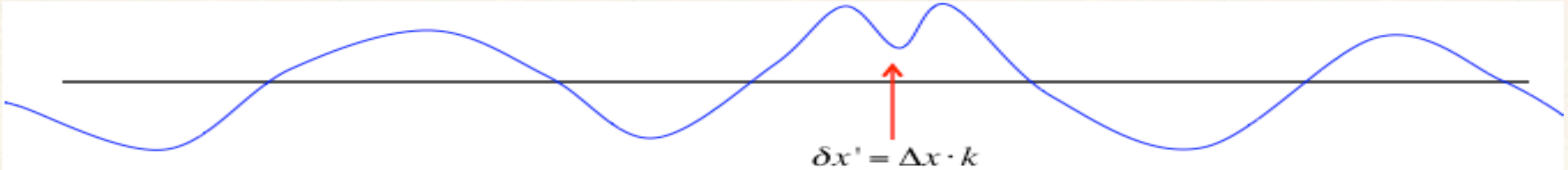


In a Linac – starting with a perfect orbit – the misaligned quadrupole creates an oscillation that is transformed from now on downstream via

$$\begin{pmatrix} x \\ x' \end{pmatrix}_f = M \begin{pmatrix} x \\ x' \end{pmatrix}_i$$

... and in a circular machine ??

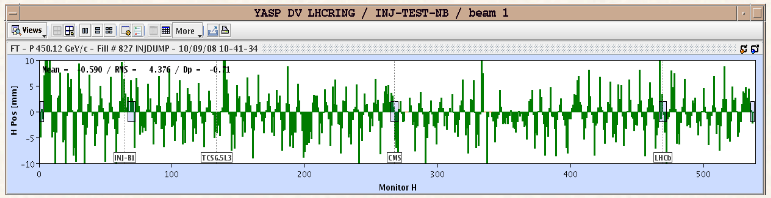
we have to obey the periodicity condition.
The orbit is closed !! ... even under the influence of a orbit kick.



Calculation of the new closed orbit:
the general orbit will always be a solution of Hill, so ...

$$x(s) = a \cdot \sqrt{\beta} \cos(\psi(s) + \varphi$$

We set at the location of the error s=0, $\Psi(s)=0$
and require as 1st boundary condition:
periodic amplitude



$$x(s+L) = x(s)$$

$$a \cdot \sqrt{\beta(s+L)} \cdot \cos(\psi(s) + 2\pi Q - \varphi) = a \cdot \sqrt{\beta(s)} \cdot \cos(\psi(s) - \varphi)$$

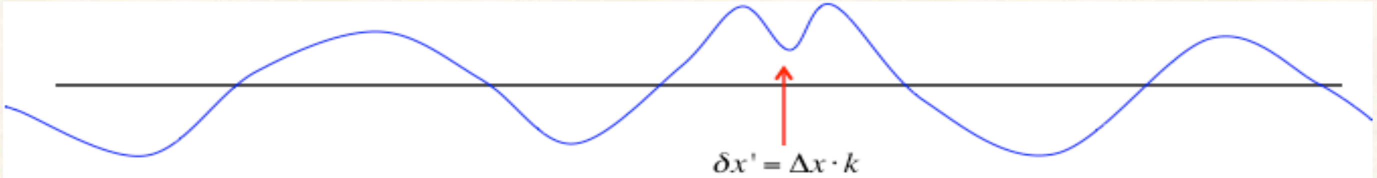
$$\cos(2\pi Q - \varphi) = \cos(-\varphi) = \cos(\varphi)$$

$$\rightarrow \varphi = \pi Q$$

$$\beta(s+L) = \beta(s)$$
$$\psi(s=0) = 0$$
$$\psi(s+L) = 2\pi Q$$

Misalignment error in a circular machine

2nd boundary condition: x' (s+L) + δx'= x'(s)
we have to close the orbit



$$x(s) = a\cdot\sqrt{\beta}\cos(\psi(s)-\varphi)$$

$$x'(s) = a\cdot\sqrt{\beta}\left(-\sin(\psi(s)-\varphi)\right)\psi' + \frac{\beta'(s)}{2\sqrt{\beta}}a\cdot\cos(\psi(s)-\varphi)$$

$$x'(s) = -a\cdot\frac{1}{\sqrt{\beta}}\left(\sin(\psi(s)-\varphi)\right) + \frac{\beta'(s)}{2\sqrt{\beta}}a\cdot\cos(\psi(s)-\varphi)$$

$$\psi(s) = \int\frac{1}{\beta(s)}ds$$

$$\psi'(s) = \frac{1}{\beta(s)}$$

boundary condition: x' (s+L) + δx'= x'(s)

$$-a\cdot\frac{1}{\sqrt{\beta(\tilde{s}+L)}}\left(\sin(2\pi Q-\varphi)\right) + \frac{\beta'(\tilde{s}+L)}{2\beta(\tilde{s}+L)}\sqrt{\beta(\tilde{s}+L)}\,a\cdot\cos(2\pi Q-\varphi) + \frac{\Delta\tilde{s}}{\rho} =$$

$$= -a\cdot\frac{1}{\sqrt{\beta(\tilde{s})}}\left(\sin(-\varphi)\right) + \frac{\beta'(\tilde{s})}{2\beta(\tilde{s})}\sqrt{\beta(\tilde{s})}\,a\cdot\cos(-\varphi)$$

Nota bene: $\tilde{s}$ refers to the location of the kick

Misalignment error in a circular machine

Now we use: $\beta(s+L) = \beta(s)$, $\varphi=\pi Q$

$$\frac{-a}{\sqrt{\beta(\tilde{s})}}\left(\sin(\pi Q)+\frac{\beta'(\tilde{s})}{2\beta(\tilde{s})}\sqrt{\beta(\tilde{s})}\,a\cdot\cos(\pi Q)+\frac{\Delta\tilde{s}}{\rho}\right. = \frac{a}{\sqrt{\beta(\tilde{s})}}\left(\sin(\pi Q)+\frac{\beta'(\tilde{s})}{2\beta(\tilde{s})}\sqrt{\beta(\tilde{s})}\,a\cdot\cos(\pi Q)\right.$$

$$\Rightarrow 2\,a\cdot\frac{\sin(\pi Q)}{\sqrt{\beta(\tilde{s})}}=\frac{\Delta\tilde{s}}{\rho} \quad \Rightarrow$$

$$a=\frac{\Delta\tilde{s}}{\rho}\cdot\sqrt{\beta(\tilde{s})}\,\frac{1}{2\sin(\pi Q)}$$

! this is the amplitude of the orbit oscillation resulting from a single kick

inserting in the equation of motion

$$x(s)=a\cdot\sqrt{\beta}\cos(\psi(s)+\varphi$$

$$x(s)=\frac{\Delta\tilde{s}}{\rho}\cdot\frac{\sqrt{\beta(\tilde{s})}\sqrt{\beta(s)}\cos(\psi(s)-\varphi)}{2\sin(\pi Q)}$$

! the distorted orbit depends on the kick strength,
! the local β function
! the β function at the observation point

!!! there is a resoncance denominator
→ watch your tune !!!

Misalignment error in a circular machine

For completeness:

if we do not set $\psi(s=0)=0$ we have to write a bit more but finally we get:

$$x(s) = \frac{\sqrt{\beta(s)}}{2\sin(\pi Q)} * \oint \sqrt{\beta(\tilde{s})}\, \frac{1}{\rho(\tilde{s})} \cos\left(\left|\psi(\tilde{s}) - \psi(s)\right| - \pi Q\right) d\tilde{s}$$

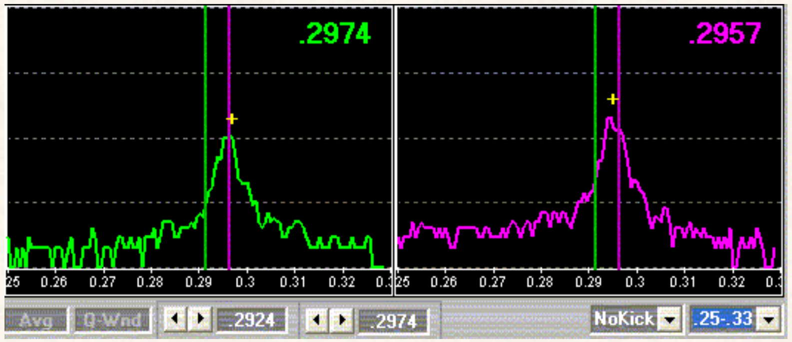
Reminder: LHC

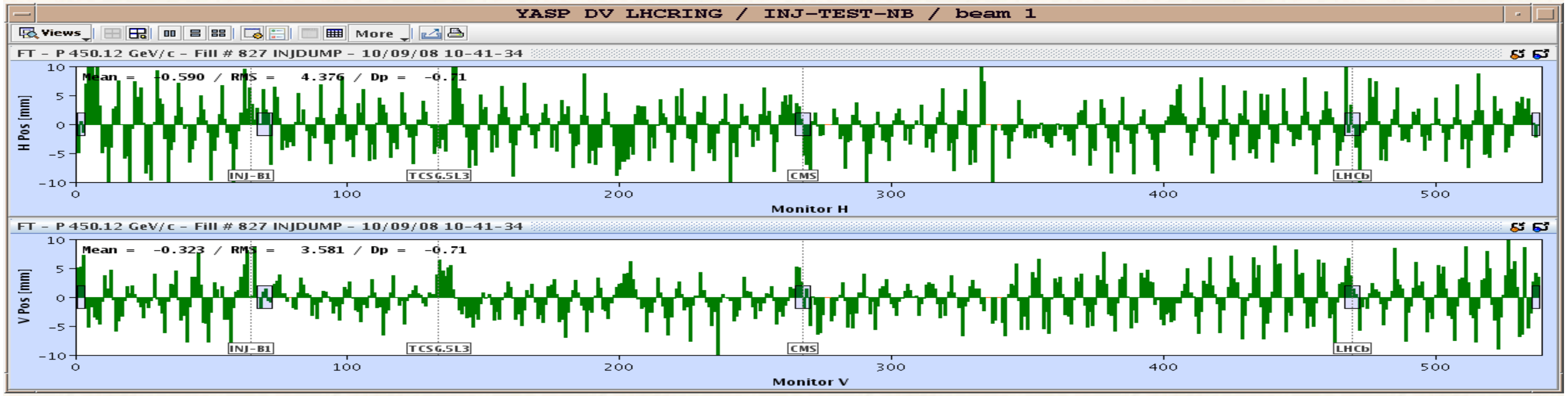
Tune: $Q_x = 64.31$, $Q_y = 59.32$

Relevant for beam stability:

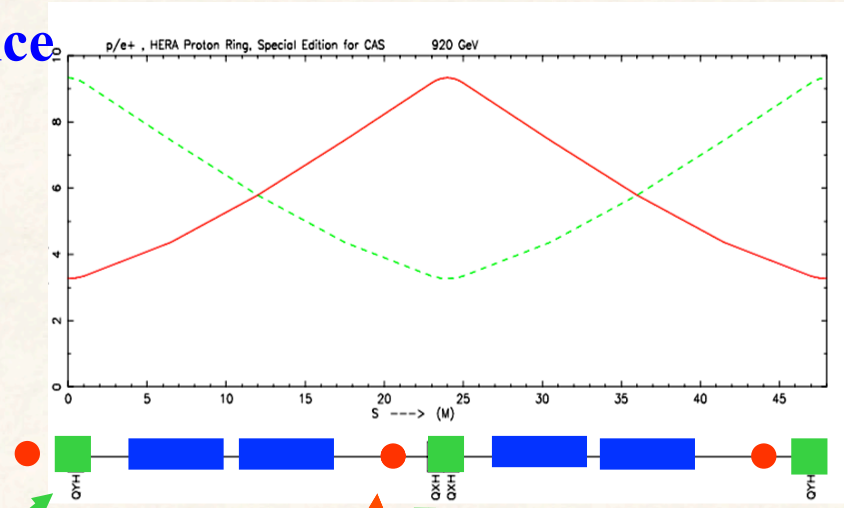
non integer part

avoid integer tunes





Orbit distortions in a periodic lattice



field error of a dipole/distorted quadrupole

$$\rightarrow \delta(mrad) = \frac{ds}{\rho} = \frac{\int B ds}{p/e}$$

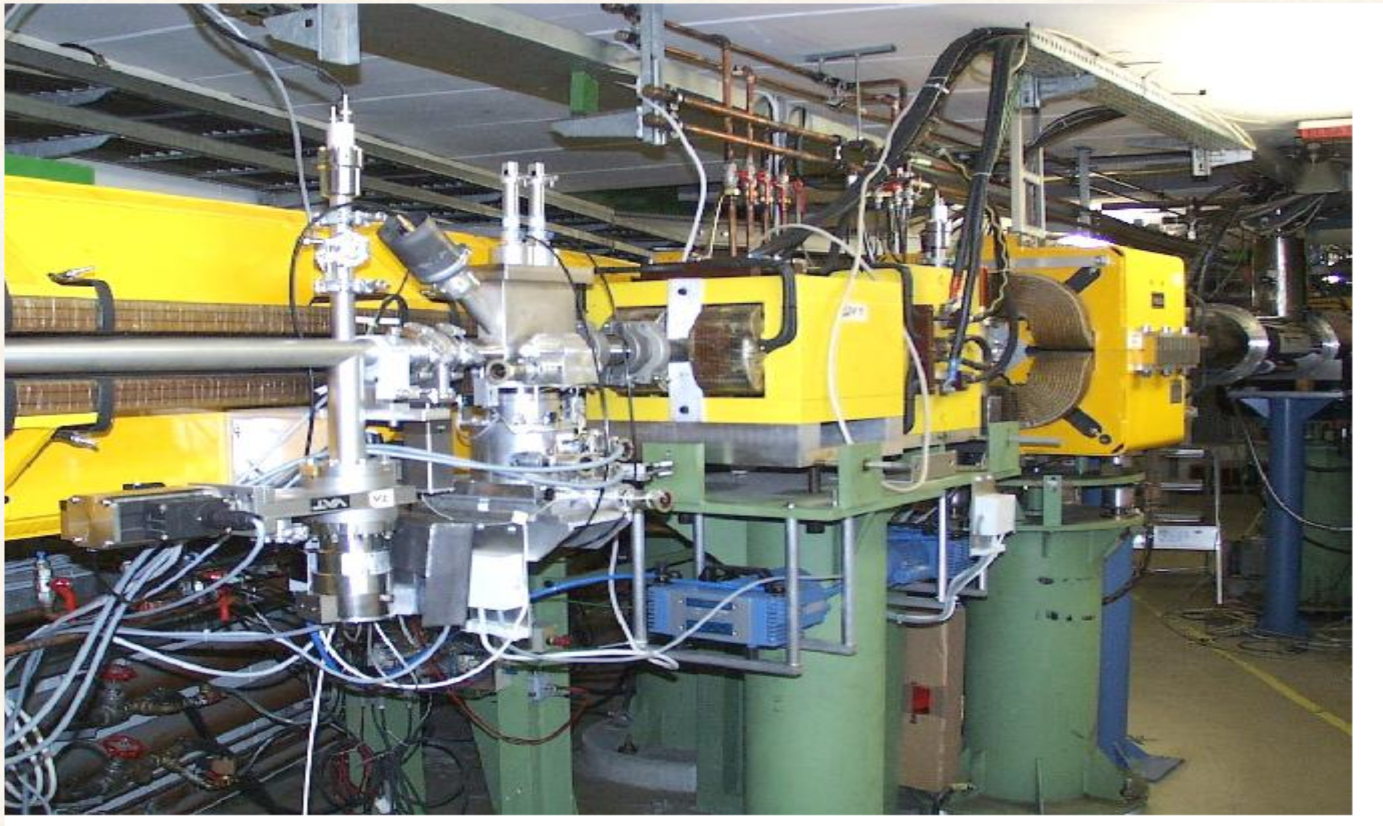
the particle will follow a new closed trajectory, the distorted orbit:

$$x(s) = \frac{\sqrt{\beta(s)}}{2\sin(\pi Q)} * \oint \sqrt{\beta(\tilde{s})} \frac{1}{\rho(\tilde{s})} \cos(|\psi(\tilde{s}) - \psi(s)| - \pi Q) d\tilde{s}$$

* the orbit amplitude will be large if the β function at the location of the kick is large
  $\beta(\tilde{s})$ indicates the sensitivity of the beam → here orbit correctors should be placed in the lattice

* the orbit amplitude will be large at places where in the lattice **β(s)** is large → here beam position monitors should be installed

Orbit Correctors and Beam Instrumentation in a Storage Ring



Elsa ring, Bonn

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