13.) The „ $\Delta p / p \neq 0$“ Problem

ideal accelerator: ***all particles will see the same accelerating voltage.***

$\rightarrow \Delta p / p = 0$

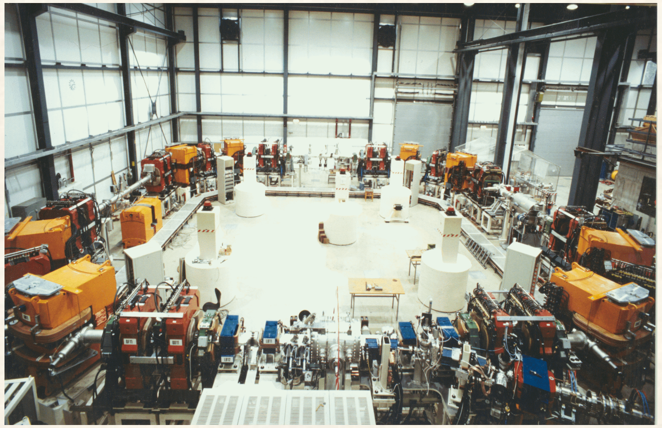
„nearly ideal“ accelerator: van de Graaf

$\Delta p / p \approx 10^{-5}$

„not-at-all ideal“ accelerator: RF structures

$\Delta p / p \approx 10^{-3}$

14.) Gradient Errors



Remember: Matrix in Twiss Form

Transfer Matrix from point „0" in the lattice to point „s":

$$M(s)=\begin{pmatrix} \sqrt{\dfrac{\beta_s}{\beta_0}}(\cos\psi_s+\alpha_0\sin\psi_s) & \sqrt{\beta_s\beta_0}\,\sin\psi_s \\ \dfrac{(\alpha_0-\alpha_s)\cos(\psi_s-(1+\alpha_0\alpha_s)\sin\psi_s}{\sqrt{\beta_s\beta_0}} & \sqrt{\dfrac{\beta_0}{\beta_s}}(\cos(\psi_s-\alpha_0\sin\psi_s) \end{pmatrix}$$

For one complete turn the Twiss parameters have to obey periodic bundary conditions:

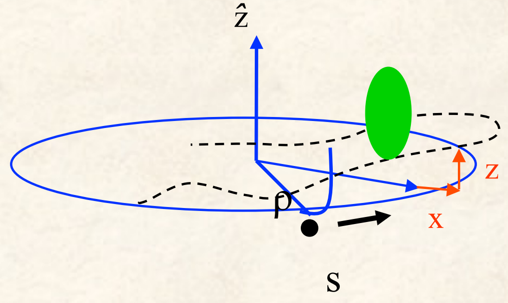
$$\beta(s+L)=\beta(s)$$
$$\alpha(s+L)=\alpha(s)$$
$$\gamma(s+L)=\gamma(s)$$

$$\boxed{M(s)=\begin{pmatrix} \cos\psi_{turn}+\alpha_s\sin\psi_{turn} & \beta_s\sin\psi_{turn} \\ -\gamma_s\sin\psi_s & \cos\psi_{turn}-\alpha_s\sin\psi_{turn} \end{pmatrix}}$$

Introduce Quadrupole Error in the Lattice

optics perturbation described by thin lens quadrupole

$$M_{dist} = M_{\Delta k} \cdot M_0 = \underbrace{\begin{pmatrix} 1 & 0 \\ \Delta k ds & 1 \end{pmatrix}}_{\text{quad error}} \cdot \underbrace{\begin{pmatrix} \cos\psi_{turn} + \alpha \sin\psi_{turn} & \beta \sin\psi_{turn} \\ -\gamma \sin\psi_{turn} & \cos\psi_{turn} - \alpha \sin\psi_{turn} \end{pmatrix}}_{\text{ideal storage ring}}$$



$$M_{dist} = \begin{pmatrix} \cos\psi_0 + \alpha \sin\psi_0 & \beta \sin\psi_0 \\ \Delta k ds (\cos\psi_0 + \alpha \sin\psi_0) - \gamma \sin\psi_0 & \Delta k ds \, \beta \sin\psi_0 + \cos\psi_0 - \alpha \sin\psi_0 \end{pmatrix}$$

rule for getting the tune

$$Trace(M) = 2\cos\psi = 2\cos\psi_0 + \Delta k ds \beta \sin\psi_0$$

Quadrupole error → Tune Shift

$$\boxed{\psi = \psi_0 + \Delta\psi} \longrightarrow \cos(\psi_0 + \Delta\psi) = \cos\psi_0 + \frac{\Delta k ds\, \beta \sin\psi_0}{2}$$

remember the old fashioned trigonometric stuff and assume that the error is small !!!

$$\cos\psi_0 \underbrace{\cos\Delta\psi}_{\approx 1} - \sin\psi_0 \underbrace{\sin\Delta\psi}_{\approx \Delta\psi} = \cos\psi_0 + \frac{k ds\, \beta \sin\psi_0}{2}$$

$$\Delta\psi = \frac{k ds\, \beta}{2}$$

and referring to Q instead of ψ:

$$\psi = 2\pi Q$$

$$\boxed{\Delta Q = \int_{s0}^{s0+l} \frac{\Delta k(s)\beta(s)ds}{4\pi}}$$

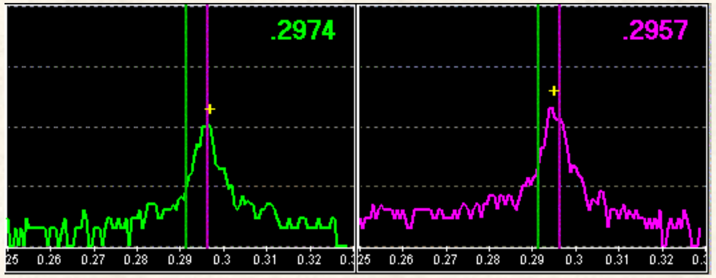
! the tune shift is proportional to the β-function at the quadrupole

!! field quality, power supply tolerances etc are much tighter at places where β is large

!!! mini beta quads: $\beta \approx 1900$ m
arc quads: $\beta \approx 80$ m

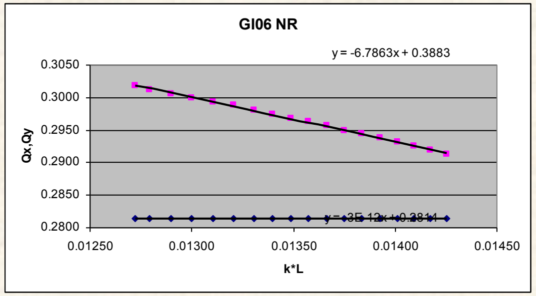
!!!! β is a measure for the sensitivity of the beam

A quadrupol error leads to a shift of the tune
... and this can be used to measure the β-function



$$\Delta Q = \int_{s0}^{s0+l} \frac{\Delta k \beta(s)}{4\pi} ds \approx \frac{\Delta k l_{quad} \bar{\beta}}{4\pi}$$

Example: measurement of β in a storage ring: tune spectrum



Beyond that: without proof (e.g. CERN-94-01)
A quadrupole error will always lead to a tune shift, but in addition to a change of the beta–function.

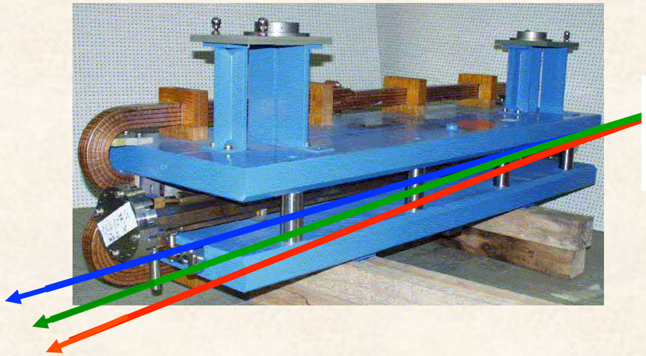
$$\Delta\beta(s) = \frac{\beta(s)}{2\sin(2\pi Q)} \oint \beta(\tilde{s}) \Delta k(\tilde{s}) \cos(2\left|\psi(s) - \psi(\tilde{s})\right| - \pi Q)\, d\tilde{s}$$

As before the effect of the error depends on the β-function at the observation point as well as at the place of the error itself, on the error strength and there is again a resonance denominator
→ half integer tunes are forbidden.

15.) Chromaticity:
A Quadrupole Error for $\Delta p/p \neq 0$

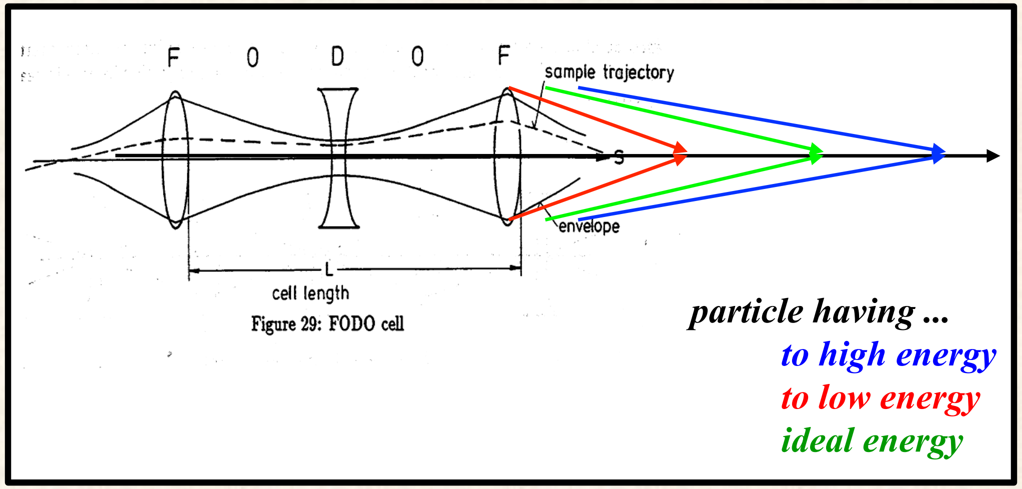
Influence of external fields on the beam: *prop. to magn. field & prop. zu 1/p*

dipole magnet $\alpha = \dfrac{\int B\, dl}{p/e}$



$$x_D(s) = D(s)\,\frac{\Delta p}{p}$$

focusing lens $k = \dfrac{g}{p/e}$



Chromaticity: Q'

$$k = \frac{g}{p/e} \qquad\qquad p = p_0 + \Delta p$$

in case of a momentum spread:

$$\boldsymbol{k = \frac{eg}{p_0 + \Delta p} \approx \frac{e}{p_0}(1 - \frac{\Delta p}{p_0})\, g = k_0 + \Delta k}$$

$$\Delta k = -\frac{\Delta p}{p_0} k_0$$

… which acts like a quadrupole error in the machine and leads to a tune spread:

$$\boldsymbol{\Delta Q = -\frac{1}{4\pi} \frac{\Delta p}{p_0} k_0 \beta(s) ds}$$

definition of chromaticity:

$$\boldsymbol{\Delta Q = Q' \frac{\Delta p}{p} \quad ; \qquad Q' = -\frac{1}{4\pi} \oint k(s)\beta(s) ds}$$

… what is wrong about Chromaticity:

Q' is a number indicating the size of the tune spot in the working diagram,
Q' is always created if the beam is focussed
→ *it is determined by the focusing strength k of all quadrupoles*

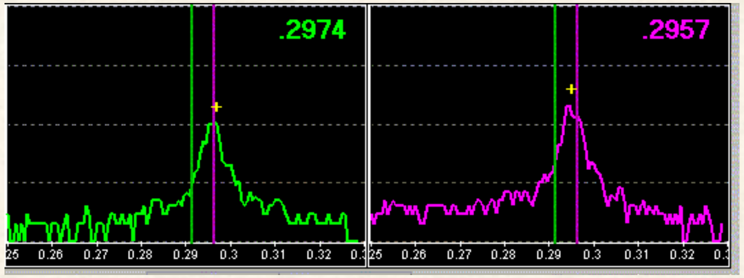
$$Q' = -\frac{1}{4\pi}\oint \beta(s)k(s)\,ds$$

k = quadrupole strength
β = betafunction indicates the beam size … and even more the sensitivity of the beam to external fields

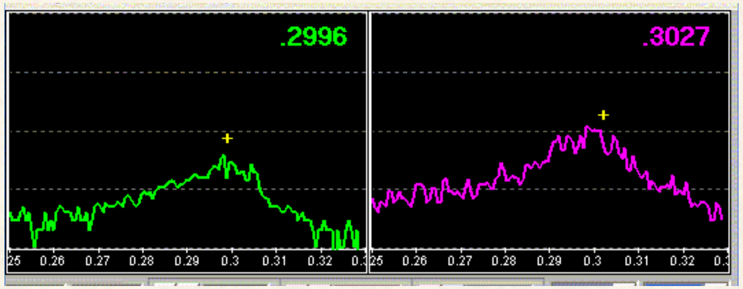
Example: LHC

$$\left.\begin{array}{l} Q' = -250 \\ \Delta p/p = +/- 0.2 * 10^{-3} \\ \Delta Q = 0.256 \ldots 0.36 \end{array}\right\}$$

→ *Some particles get very close to resonances and are lost*
in other words: the tune is not a point
it is a pancake



Ideal situation: cromaticity well corrected (Q' ≈ 1)

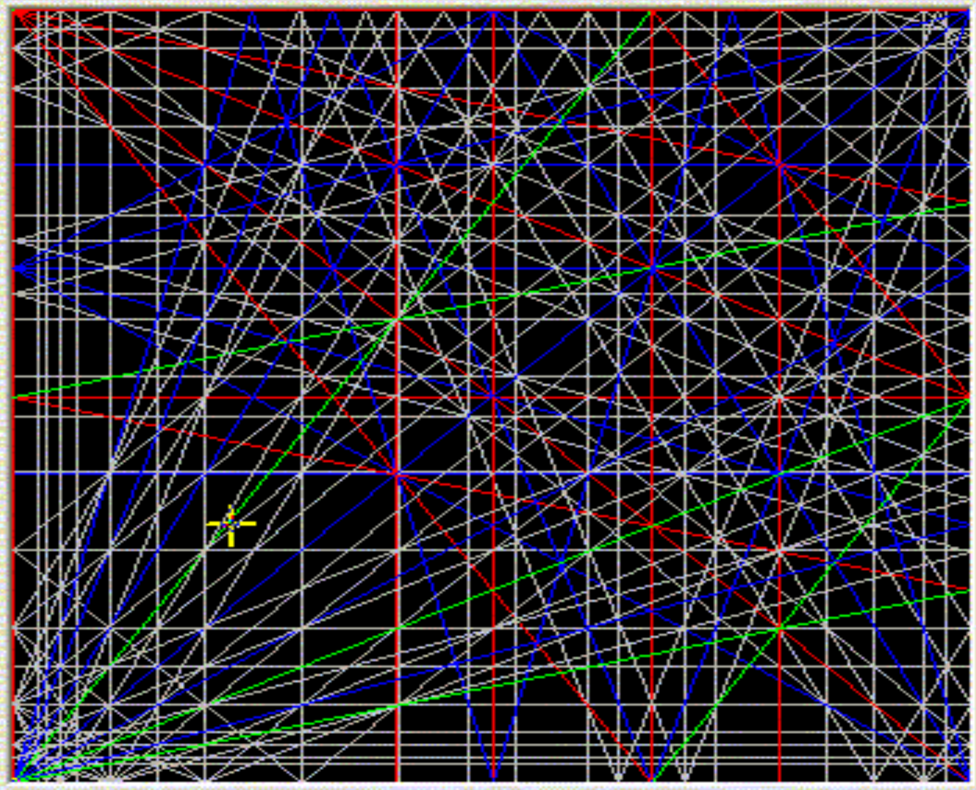


Tune signal for a nearly uncompensated cromaticity (Q' ≈ 20)

Tune and Resonances

$m*Q_x+n*Q_y+l*Q_s = integer$

Tune diagram up to 3rd order



… and up to 7th order

Homework for the operateurs:
find a nice place for the tune
where against all probability
the beam will survive

Correction of Q'

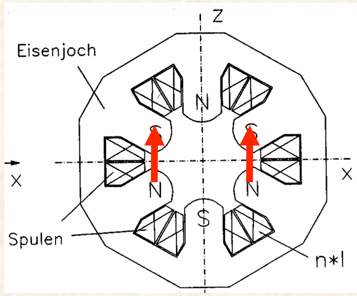
1.) sort the particles acording to their momentum $$x_D(s) = D(s)\frac{\Delta p}{p}$$

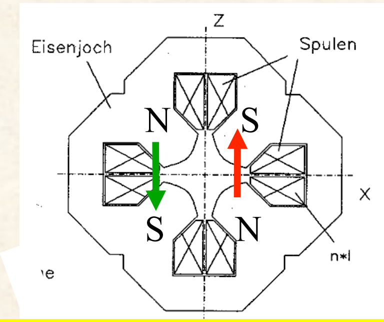
2.) apply a magnetic field that rises quadratically with x (sextupole field)

$$\left.\begin{aligned} B_x &= \tilde{g}xy \\ B_y &= \frac{1}{2}\tilde{g}(x^2 - y^2) \end{aligned}\right\} \qquad \frac{\partial B_x}{\partial y} = \frac{\partial B_y}{\partial x} = \tilde{g}x$$

linear rising „gradient":

Sextupole Magnets:





normalised quadrupole strength:

$$k_{sext} = \frac{\tilde{g}x}{p/e} = m_{sext}.x$$

$$k_{sext} = m_{sext}.D\frac{\Delta p}{p}$$

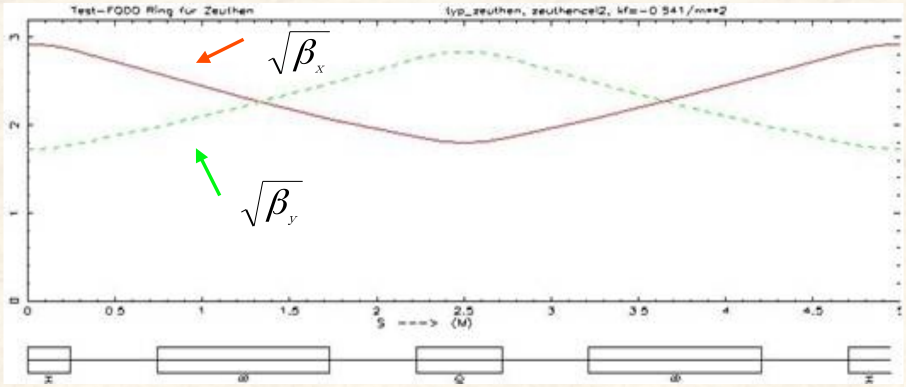
cor

$$Q'_{cell\_x} = \frac{-1}{4\pi}\left\{k_{qf}\hat{\beta}_x l_{qf} - k_{qd}\breve{\beta}_x l_{qd}\right\} + \frac{1}{4\pi}\sum_{F\,sext} k_2^F l_{sext} D_x^F \beta_x^F - \frac{1}{4\pi}\sum_{D\,sext} k_2^D l_{sext} D_x^D \beta_x^D$$

$$Q'_{cell\_y} = \frac{-1}{4\pi}\left\{-k_{qf}\breve{\beta}_y l_{qf} + k_{qd}\hat{\beta}_y l_{qd}\right\} - \frac{1}{4\pi}\sum_{F\,sext} k_2^F l_{sext} D_x^F \beta_y^F + \frac{1}{4\pi}\sum_{D\,sext} k_2^D l_{sext} D_x^D \beta_y^D$$

Chromaticity in a FODO Cell

$$Q' = \frac{-1}{4\pi} * \oint k(s)\beta(s)\, ds$$



β-Function in a single FoDo

$$\hat{\beta} = \frac{(1+\sin\frac{\psi_{cell}}{2})L}{\sin\psi_{cell}} \qquad \breve{\beta} = \frac{(1-\sin\frac{\psi_{cell}}{2})L}{\sin\psi_{cell}}$$

$$Q' = \frac{-1}{4\pi} * \frac{\hat{\beta}-\breve{\beta}}{f_Q}$$

$$Q' = \frac{-1}{4\pi} * \frac{1}{f_Q} * \left\{ \frac{L(1+\sin\frac{\psi_{cell}}{2}) - L(1-\sin\frac{\psi_{cell}}{2})}{\sin\psi_{cell}} \right\}$$

using some TLC transformations ... Q' can be expressed in a very simple form:

$$Q' = \frac{-1}{4\pi} * \frac{1}{f_Q} * \frac{2L \sin\frac{\psi_{cell}}{2}}{\sin\psi_{cell}}$$

$$Q' = \frac{-1}{4\pi} * \frac{1}{f_Q} * \frac{L \sin\frac{\psi_{cell}}{2}}{\sin\frac{\psi_{cell}}{2}\cos\frac{\psi_{cell}}{2}}$$

remember ... $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$

$$Q'_{cell} = \frac{-1}{4\pi f_Q} * \frac{L \tan\frac{\psi_{cell}}{2}}{\sin\frac{\psi_{cell}}{2}}$$

putting ... $\sin\frac{\psi_{cell}}{2} = \frac{L}{4 f_Q}$

$$Q'_{cell} = \frac{-1}{\pi} * \tan\frac{\psi_{cell}}{2}$$

and so we have to power the sextupoles properly ...

$$\Delta Q'_x = \frac{-1}{4\pi} * \left\{ k_2^F l_s D_x^F \beta_x^F - k_2^D l_s D_x^D \beta_x^D \right\}$$

$$\Delta Q'_y = \frac{-1}{4\pi} * \left\{ -k_2^F l_s D_x^F \beta_y^F + k_2^D l_s D_x^D \beta_y^D \right\}$$


INSERTIONS

5.) Lattice Design: Insertions

... the most complicated one: the drift space

Question to the auditorium: what will happen to the beam parameters α, β, γ if we stop focusing for a while …?

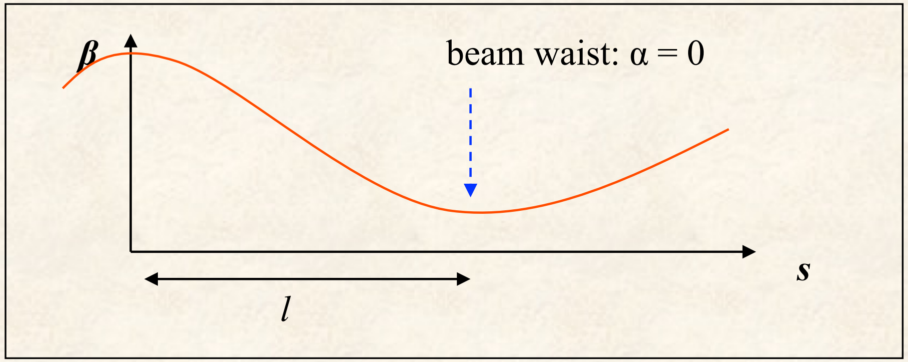
$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_S = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC'+S'C & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} * \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_0$$

transfer matrix for a drift: $M = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$

$$\beta(s) = \beta_0 - 2\alpha_0 s + \gamma_0 s^2$$
$$\alpha(s) = \alpha_0 - \gamma_0 s$$
$$\gamma(s) = \gamma_0$$

„0“ refers to the position of the last lattice element
„s“ refers to the position in the drift

location of the waist:



given the initial conditions $\alpha_0, \beta_0, \gamma_0$: where is the point of smallest beam dimension in the drift … or at which location occurs the beam waist ?

beam waist:

$$\alpha(s) = 0 \quad \rightarrow \quad \alpha_0 = \gamma_0 * s$$

$$l = \frac{\alpha_0}{\gamma_0}$$

beam size at that position:

$$\gamma(l) = \gamma_0$$
$$\alpha(l) = 0$$

$$\rightarrow \quad \gamma(l) = \frac{1+\alpha^2(l)}{\beta(l)} = \frac{1}{\beta(l)}$$

$$\beta(l) = \frac{1}{\gamma_0}$$

β-Function in a Drift:

let's assume we are at a symmetry point in the center of a drift.

$$\beta(s) = \beta_0 - 2\alpha_0 s + \gamma_0 s^2$$

as $\alpha_0 = 0, \quad \rightarrow \quad \gamma_0 = \frac{1+\alpha_0^2}{\beta_0} = \frac{1}{\beta_0}$

and we get for the β function in the neighborhood of the symmetry point

$$\beta(s) = \beta_0 + \frac{s^2}{\beta_0}$$

! ! !

Nota bene:

1.) this is very bad !!!

2.) this is a direct consequence of the conservation of phase space density (... in our words: ε = const) … and there is no way out.

3.) Thank you, Mr. Liouville !!!

Joseph Liouville, 1809-1882

A bit more in detail: β-Function in a Drift

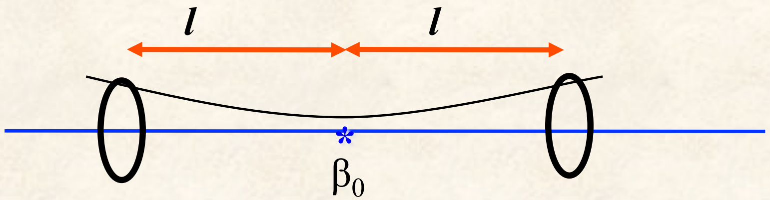
If we cannot fight against Liouvuille theorem ... at least we can optimise

Optimisation of the beam dimension:

$$\beta(l) = \beta_0 + \frac{l^2}{\beta_0}$$

Find the β at the center of the drift that leads to the lowest maximum β at the end:

$$\frac{d\hat{\beta}}{d\beta_0} = 1 - \frac{l^2}{\beta_0^2} = 0$$



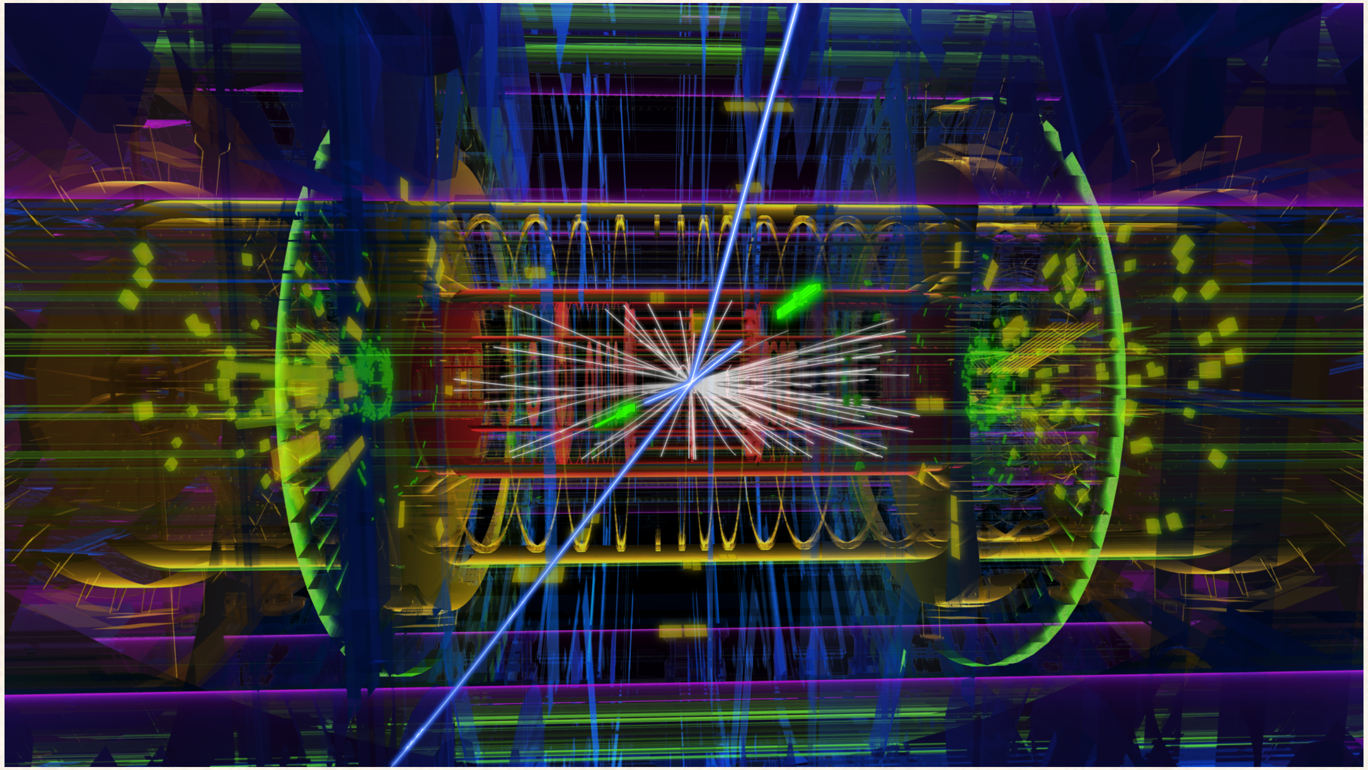
$$\rightarrow \beta_0 = l$$

$$\rightarrow \hat{\beta} = 2\beta_0$$

If we choose $\beta_0 = \ell$ we get the smallest β at the end of the drift and the maximum β is just twice the distance ℓ

... and why all that ??

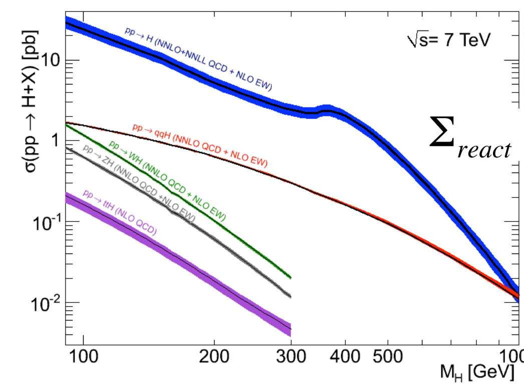
High Light of the HEP-Year 2012 / 13 naturally the HIGGS



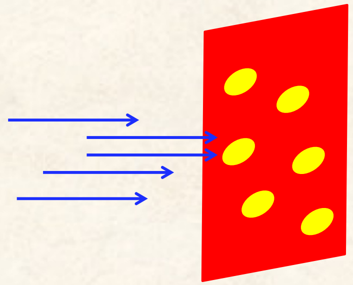
ATLAS event display: Higgs => two electrons & two muons

Problem: Our particles are VERY small !!

Overall cross section of the Higgs:



$$\Sigma_{react} \approx 1\,pb$$



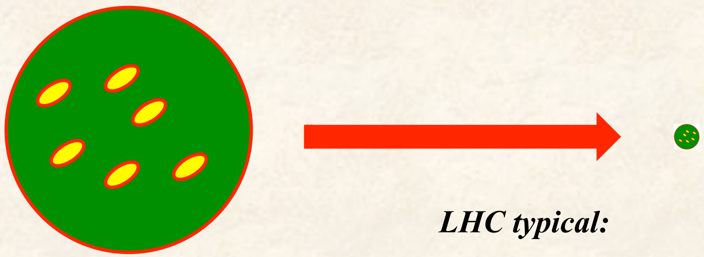
$$1b \;= 10^{-24}\,cm^2$$

The particles are indeed "very small"

$$1pb = \quad 10^{-12} * 10^{-24}\,cm^2 = 1/mio * 1/mio * 1/mio * 1/mio * 1/mio * 1/10000\;mm^2$$

The only chance we have:
compress the transverse beam size … at the IP

During collider run we had in Run 1 …
1400 bunches circulating,
with 800 Mio proton collisions per second
in the experiments and collected
only 450 Higgs particles in three years.

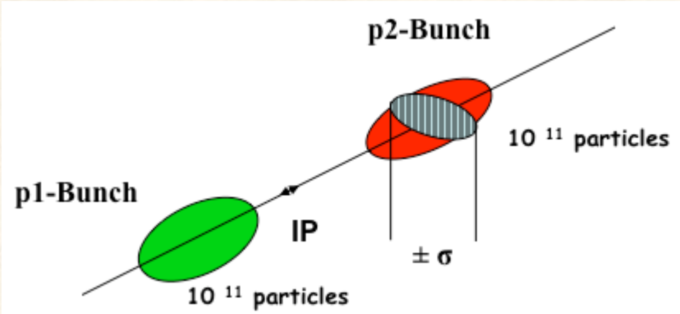


LHC typical:

σ = 0.1 mm → 16 µm

6.) Luminosity & Minibeta Insertion

$$R = L * \Sigma_{react}$$



$$L = \frac{1}{4\pi e^2 f_0 n_b} * \frac{I_{p1} I_{p2}}{\sigma_x \sigma_y}$$

Example: Luminosity run at LHC

$\beta_{x,y} = 0.55\ m$ $\qquad f_0 = 11.245\ kHz$

$\varepsilon_{x,y} = 5 * 10^{-10}\ rad\ m$ $\qquad n_b = 2808$

$\sigma_{x,y} = 17\ \mu m$

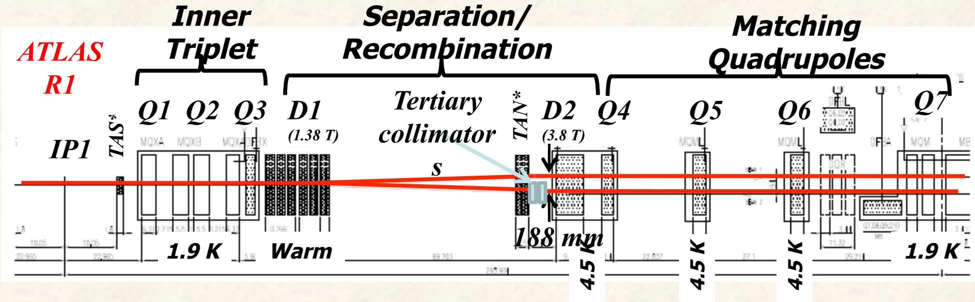
$I_p = 584\ mA$

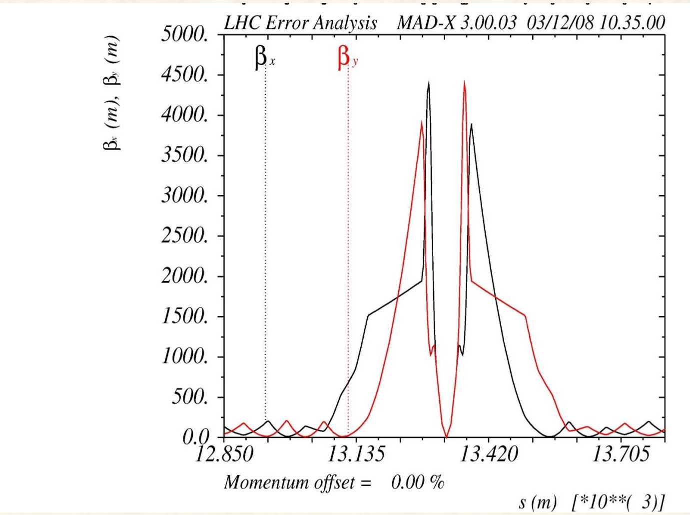
$$L = 1.0 * 10^{34}\ {}^{1}\!/_{cm^2 s}$$

production rate of events is determined by the cross section Σ_{react} and the luminosity which is given by the design of the accelerator

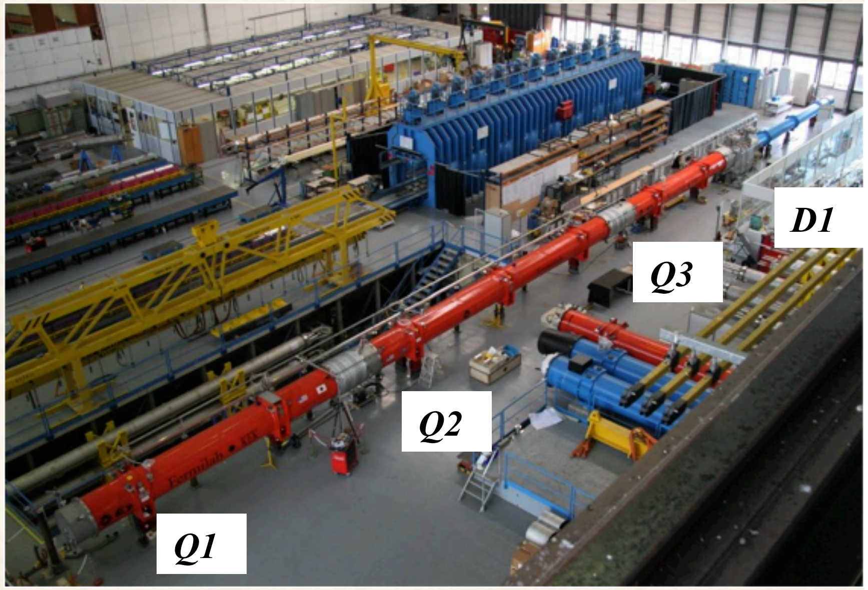
The LHC Mini-Beta-Insertions

$$\beta(s) = \beta_0 + \frac{s^2}{\beta_0}$$





LHC mini β optics



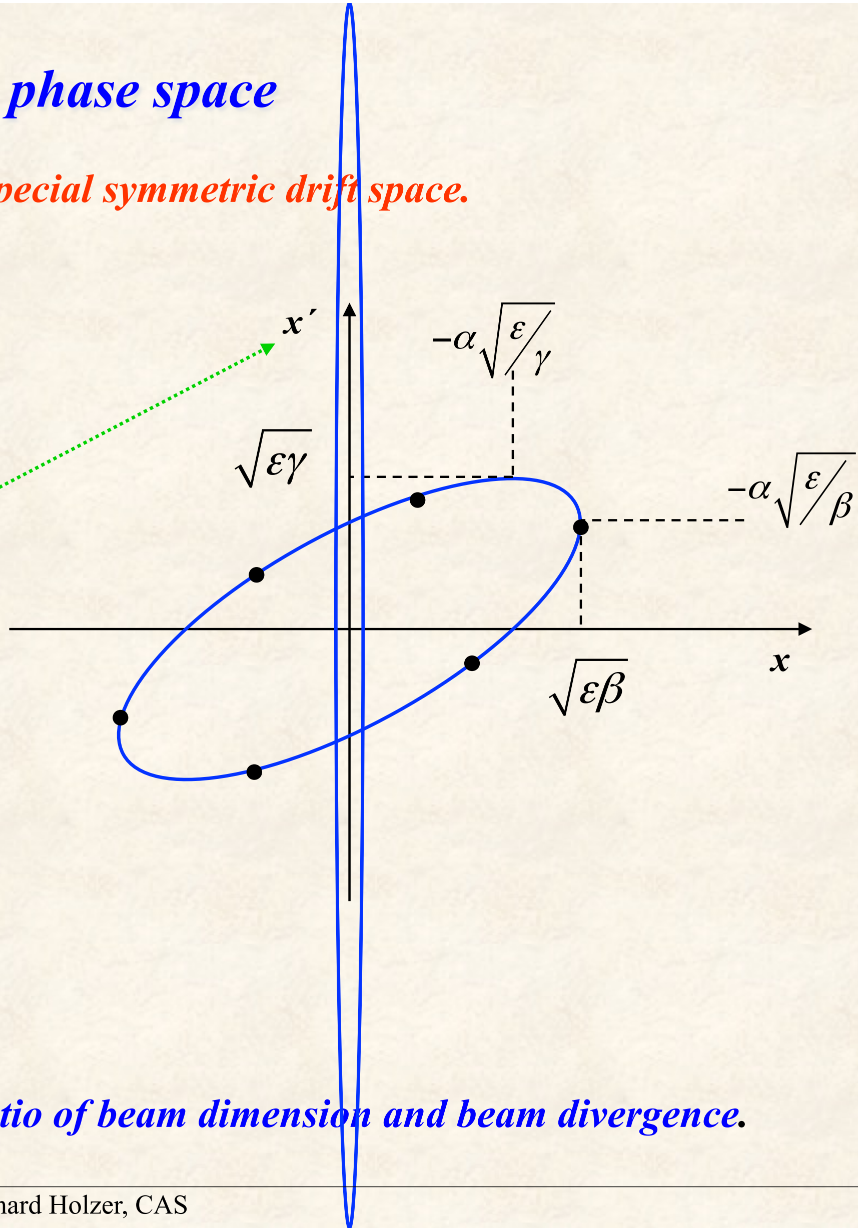
Mini-β Insertions: A look into phase space

A mini-β insertion is always a kind of special symmetric drift space.
→greetings from Liouville

$$\alpha^* = 0$$

$$\gamma^* = \frac{1+\alpha^2}{\beta} = \frac{1}{\beta^*}$$

$$\sigma'^* = \sqrt{\frac{\varepsilon}{\beta^*}}$$



$$\beta^* = \frac{\sigma^*}{\sigma'^*}$$

at a symmetry point β is just the ratio of beam dimension and beam divergence.

Mini-β Insertions: Phase advance

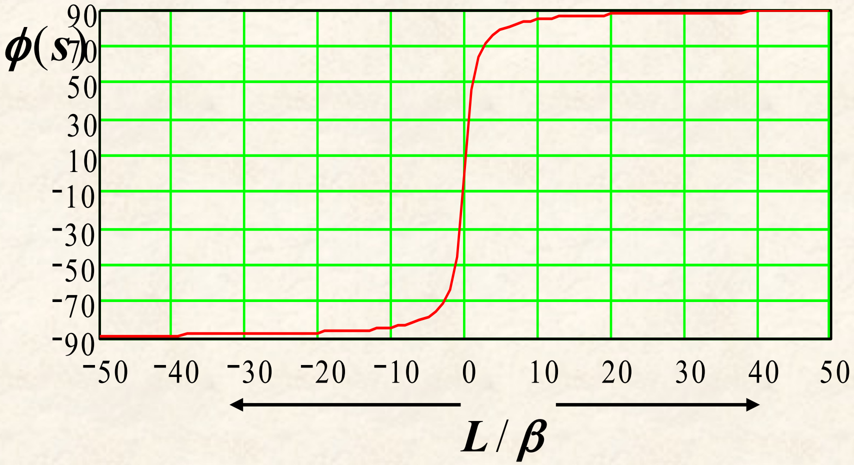
By definition the phase advance is given by:

$$\Phi(s) = \int \frac{1}{\beta(s)} ds$$

Now in a mini β insertion:

$$\beta(s) = \beta_0 \ (1 + \frac{s^2}{\beta_0^2})$$

$$\rightarrow \Phi(s) = \frac{1}{\beta_0} \int_0^L \frac{1}{1 + s^2/\beta_0^2} ds = \arctan \frac{L}{\beta_0}$$



Consider the length of the drift spaces on both sides of the IP:
the phase advance of a mini β insertion is always close to π
in other words: the tune will increase by half an integer.

Are there any problems ?

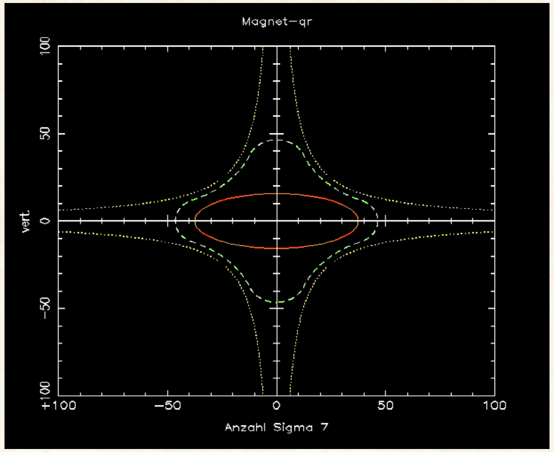
sure there are...

* ***large β values** at the doublet quadrupoles → **large contribution to chromaticity Q'** … and no local correction (... why not ???)*

$$Q' = \frac{-1}{4\pi}\oint K(s)\beta(s)ds$$

* ***aperture of mini β quadrupoles** limit the luminosity*

beam envelope at the first mini β quadrupole lens in the HERA proton storage ring



* ***field quality** and magnet stability most critical at the high β sections*
effect of a quad error:

$$\Delta Q = \int_{s0}^{s0+l} \frac{\Delta K(s)\beta(s)ds}{4\pi}$$

$$\Delta\beta(s) = \frac{\beta(s)}{2\sin(2\pi Q)}\oint \beta(\tilde{s})\Delta k(\tilde{s})\cos(2\left|\psi(s)-\psi(\tilde{s})\right| - \pi Q)\,d\tilde{s}$$

→ keep distance „s" to the first mini β quadrupole as small as possible

Mini-β Insertions: some guide lines

* *calculate the periodic solution in the arc*

* *introduce the drift space needed for the insertion device (detector ...)*

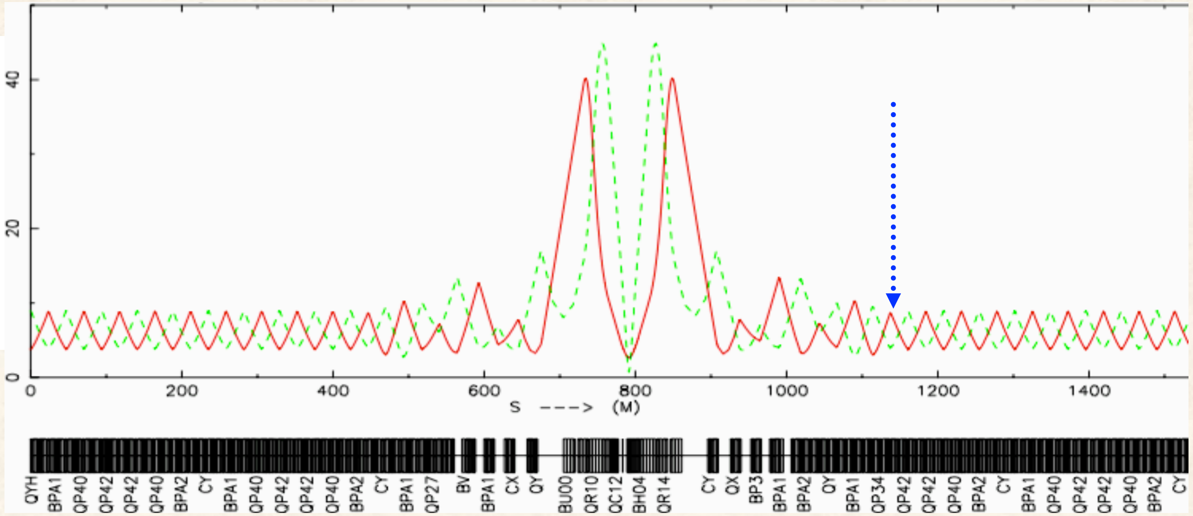
* *put a quadrupole doublet (or triplet ??) as close as possible*

* *introduce additional quadrupole lenses to match the beam parameters to the values at the beginning of the arc structure*

parameters to be optimised & matched to the periodic solution:

$$\alpha_x,\ \beta_x \qquad D_x,\ D_x'$$
$$\alpha_y,\ \beta_y \qquad Q_x,\ Q_y$$

8 individually powered quad magnets are needed to match the insertion (... at least)



7.) Dispersion Suppressors

There are two rules of paramount importance about dispersion:

! it is nasty
!! it is not easy to get rid of it.

remember: oscillation amplitude for a particle with momentum deviation

$$x(s) = x_\beta(s) + D(s) * \frac{\Delta p}{p}$$

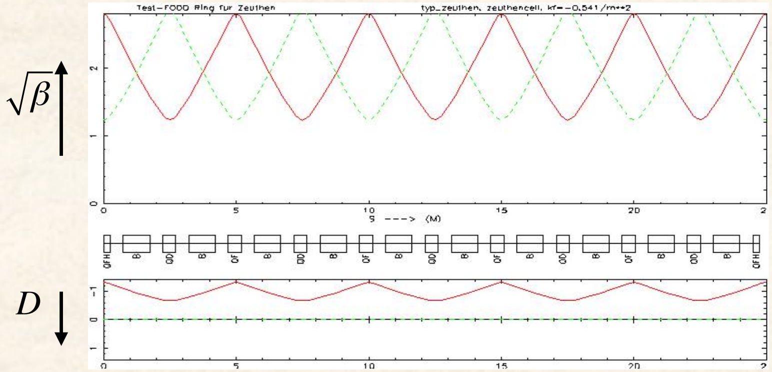
beam size at the IP

$$\sigma^* = 17\mu m$$

dispersion trajectory

$$\bar{D} = 1.5m \qquad \frac{\Delta p}{p} \approx 1.1 * 10^{-4} \qquad x_D = 165\mu m$$

Dispersion in a FoDo cell with Dipoles:
(proof see appendix)



$$D(s) = S(s) * \int \frac{1}{\rho(\tilde{s})} C(\tilde{s}) d\tilde{s} - C(s) * \int \frac{1}{\rho(\tilde{s})} S(\tilde{s}) d\tilde{s}$$

$$\hat{D} = \frac{\ell^2}{\rho} * \frac{\left(1 + \frac{1}{2}\sin\frac{\psi_{cell}}{2}\right)}{\sin^2\frac{\psi_{cell}}{2}} \qquad \breve{D} = \frac{\ell^2}{\rho} * \frac{\left(1 - \frac{1}{2}\sin\frac{\psi_{cell}}{2}\right)}{\sin^2\frac{\psi_{cell}}{2}}$$

Dispersion Suppressor Schemes

There are some locations in the ring, where the dispersion has to vanish,

* at the IP, to avoid unnecessary increase of the beam size
* at the RF to avoid unwanted coupling between transv. and long. oscillations
* at the injection / extraction points etc

The way the trick goes: ... we turn it the other way round

Starting from D = D' = 0, we create dispersion in combining the dispersive effect of the dipoles and the phase advance at their location (defined by the quadrupoles) in such a way to get the D, D' values of the periodic arc

Three major schemes:

1.) The straight forward one: Quadrupole based Dispersion suppressor

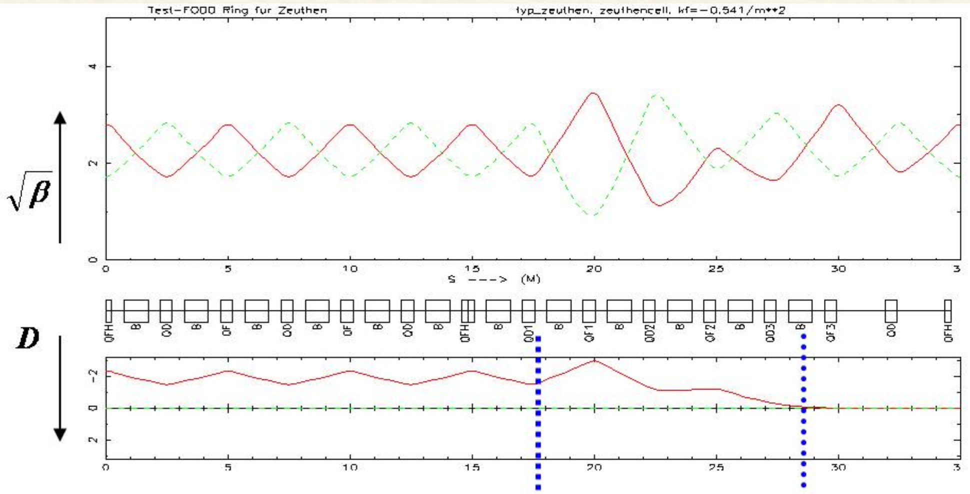
use additional quadrupole lenses to match the optical parameters ... including the D(s), D´(s) terms

**** Dispersion suppressed by 2 quadrupole lenses,***

**** β and α restored to the values of the periodic solution by 4 additional quadrupoles***

$$D(s),\ D'(s), \qquad \left.\begin{matrix} \beta_x(s), \alpha_x(s) \\ \beta_y(s), \alpha_y(s) \end{matrix}\right\} \rightarrow$$ ***6 additional quadrupole lenses required***

Dispersion Suppressor Quadrupole Scheme



periodic FoDo structure

matching section including 6 additional quadrupoles

dispersion free section, regular FoDo without dipoles

Advantage:

! easy,

! flexible: it works for any phase advance per cell

! does not change the geometry of the storage ring,

! can be used to match between different lattice structures (i.e. phase advances)

Disadvantage:

! additional power supplies needed (→ expensive)

! requires stronger quadrupoles

! due to higher β values: more aperture required

8.) The Missing Bend Dispersion Suppressor

… turn it the other way round:

Start with $D(s)=0, \quad D'(s)=0$

and create dispersion – using dipoles - in such a way, that it fits exactly the conditions at the centre of the first regular arc cells:

$$D(s)=S(s)*\int\frac{1}{\rho(\tilde{s})}C(\tilde{s})\,d\tilde{s}-C(s)*\int\frac{1}{\rho(\tilde{s})}S(\tilde{s})\,d\tilde{s} \quad \rightarrow \quad \hat{D}=\frac{\ell^2}{\rho}*\frac{\left(1+\frac{1}{2}\sin\frac{\psi_{cell}}{2}\right)}{\sin^2\frac{\psi_{cell}}{2}}, \quad D'=0$$

Depending on the phase advance, add at the end of the arc:

m cells without **dipoles**

followed by

n regular arc cells.

The Missing Bend Dispersion Suppressor

conditions for the (missing) dipole field scheme:

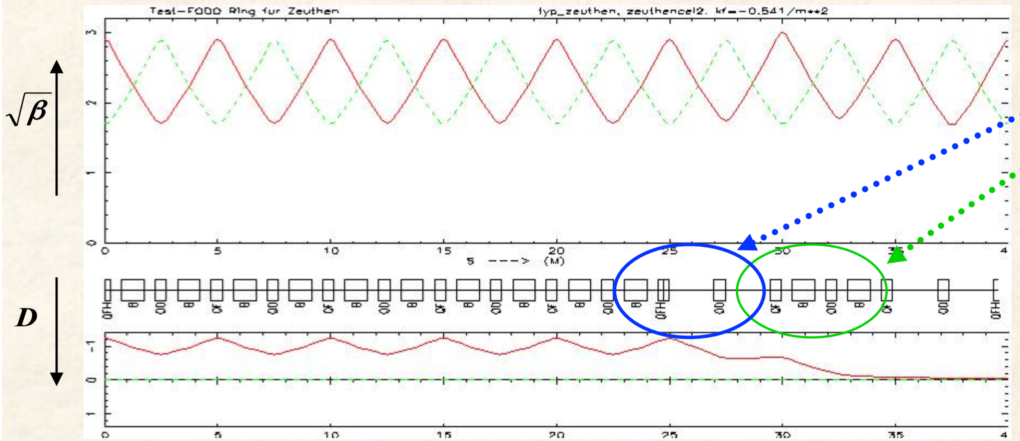
$$\frac{2m+n}{2}\Phi_C = (2k+1)\frac{\pi}{2}$$

$$\sin\frac{n\Phi_C}{2} = \frac{1}{2}\ ,\ k = 0,2\ \ldots\ or$$

$$\sin\frac{n\Phi_C}{2} = \frac{-1}{2}\ ,\ k = 1,3\ \ldots$$

Cooking Recipe:

At the end of the arc we add m cells without dipoles followed by n regular arc cells.



Example:

phase advance in the arc $\Phi_C = 60°$
number of suppr. cells $m = 1$
number of regular cells $n = 1$

9.) The Half Bend Dispersion Suppressor

at the end of the arc cells we add a number of "n" additional cells, with different dipole strength.

depending on the phase advance per cell different possibilities exist with the

general condition for vanishing dispersion

$$2 * \delta_{\text{supr}} * \sin^2\left(\frac{n\Phi_c}{2}\right) = \delta_{\text{arc}}$$

δ_{arc} = dipole strength in the arc

δ_{supr} = dipole strength in the suppressor cells

n = number of suppressor cells

Φ_c = phase advance of the cells

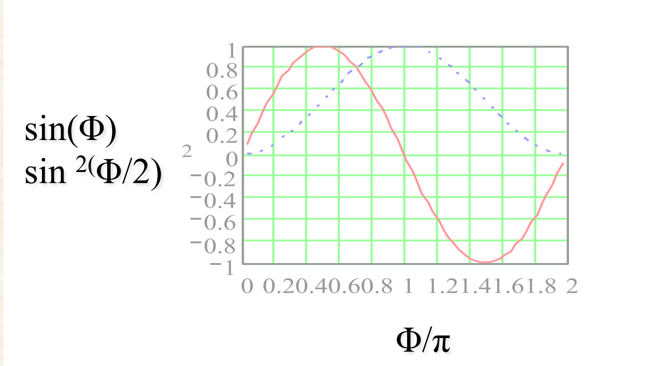
(proof see appendix)

so if we require $\boldsymbol{\delta}_{\text{supr}} = \frac{1}{2} * \boldsymbol{\delta}_{\text{arc}}$ ***which means we iunstall dipoles of half the arc strength***

we get $\sin^2\left(\frac{\boldsymbol{n\Phi_c}}{2}\right) = 1$

and equivalent for D'=0 $\sin(\boldsymbol{n}\Phi_{\boldsymbol{c}}) = 0 \qquad \boldsymbol{n}\Phi_{\boldsymbol{c}} = \boldsymbol{k} * \boldsymbol{\pi}, \qquad \boldsymbol{k} = 1,3, \ldots$

The Half Bend Dispersion Suppressor



combining these two conditions

$$\sin^2\left(\frac{n\Phi_c}{2}\right) = 1\,, \qquad \sin(n\Phi_c) = 0$$

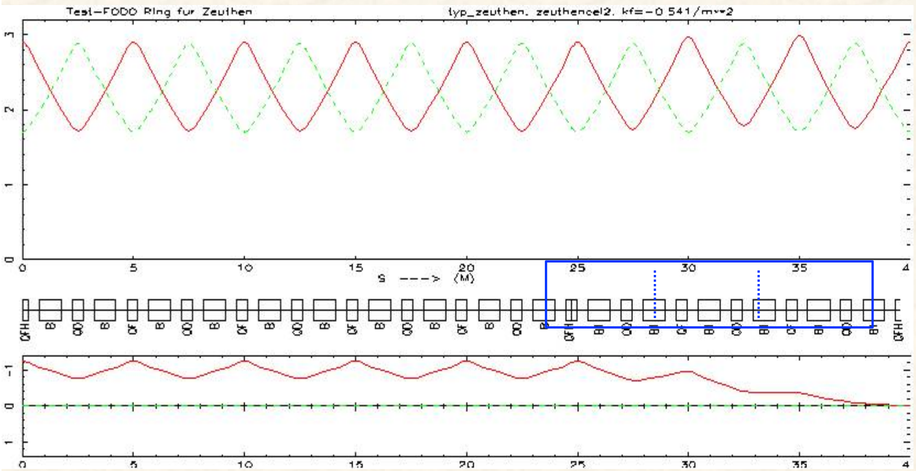
the phase advance in the n suppressor cells has to accumulate to a odd multiple of π

$$n\Phi_c = k * \pi, \qquad k = 1,3, \ldots$$

Example:

phase advance in the arc Φ_C = 60°
number of suppr. cells n = 3

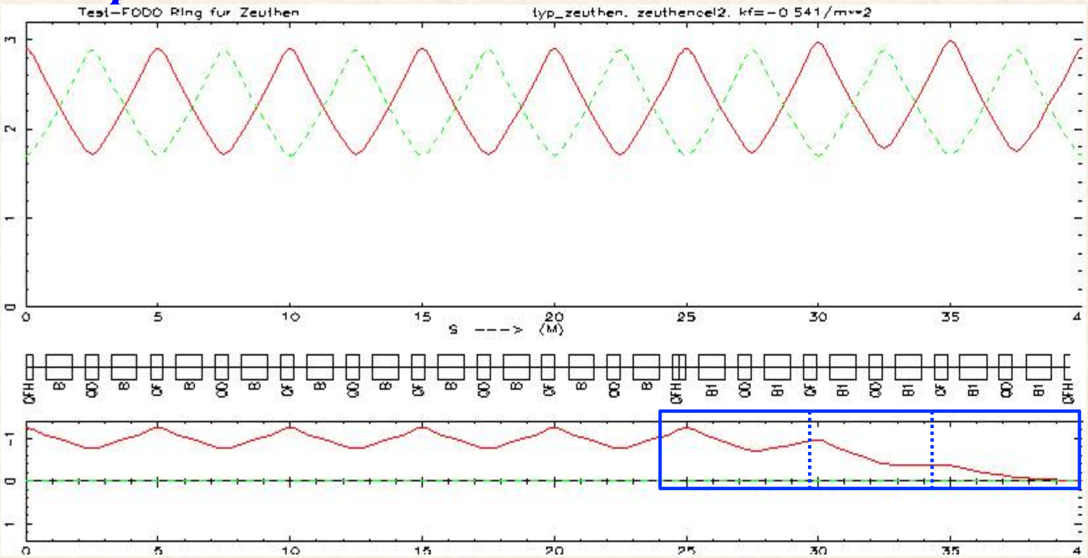
phase advance in the arc Φ_C = 90°
number of suppr. cells n = 2



strength of suppressor dipoles is half as strong as that of arc dipoles,

$$\delta_{suppr} = 1/2\ \delta_{arc}$$

Dispersion Suppressor: dipole based schemes



Advantage:

- *elegant*
- *does not affect the beam optics*
- *no additional quadrupoles required*
- *no impact on aperture*
- *lower field dipoles are often required anyway*

Disadvantage:

- *changes the geometry of the machine*

  think first, digg later
- *requires different dipoles*

  (e.g. shorter, or weaker)
- *limits the achievable beam momentum*

  $B*\rho=p/q$
- *works for specific phase advances per cell*

Resume ‘

1.) Dispersion in a FoDo cell:

small dispersion ↔ large bending radius
short cells
strong focusing

$$\hat{D} = \frac{\ell^2}{\rho} * \frac{\left(1 + \frac{1}{2}\sin\frac{\psi_{cell}}{2}\right)}{\sin^2\frac{\psi_{cell}}{2}} \qquad \breve{D} = \frac{\ell^2}{\rho} * \frac{\left(1 - \frac{1}{2}\sin\frac{\psi_{cell}}{2}\right)}{\sin^2\frac{\psi_{cell}}{2}}$$

2.) Chromaticity of a cell:

small Q’ ↔ weak focusing
small β

$$Q'_{total} = \frac{-1}{4\pi}\oint\{K(s) - mD(s)\}\beta(s)ds$$

3.) Position of a waist at the cell end:

α_0, β_0 = values at the end of the cell

$$l = \frac{\alpha_0}{\gamma_0} \qquad \beta(l) = 1/\gamma_0$$

4.) β function in a drift

$$\beta(s) = \beta_0 - 2\alpha_0 s + \gamma_0 s^2$$

5.) Mini β insertion

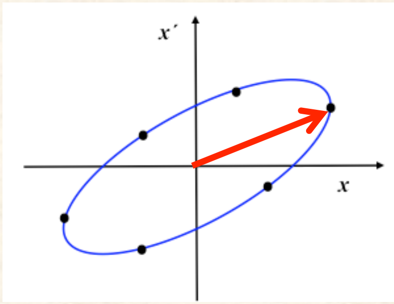
$$\beta(s) = \beta_0 + \frac{s^2}{\beta_0}$$

Appendix I: The Beam Matrix and the Mini-Beta

„Once more unto the breach dear friends:“

Transformation of Twiss parameters

just because it is mathematical more elegant ...



let ‘s define a beam matrix: $B_0 = \begin{pmatrix} \beta_0 & -\alpha_0 \\ \alpha_0 & \gamma_0 \end{pmatrix}$

and a orbit vector: $X_0 = \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}, \quad X^T = \begin{pmatrix} x_0, & x'_0 \end{pmatrix}$

the product $\underline{X_0{}^T * B_0^{-1} * X_0} = (x_0, x'_0) * \begin{pmatrix} \gamma_0 & \alpha_0 \\ \alpha_0 & \beta_0 \end{pmatrix} * \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix} = \gamma_0 x_0^2 + 2\alpha_0 x_0 x'_0 + \beta_0 x_0^{'2} = \varepsilon$

... is constant

transformation of the orbit vector: $X_1 = M * X_0$

and so we get: $\varepsilon = X_0{}^T * B_0^{-1} * X_0 = X_0^T \; \underline{M^T (M^T)^{-1}} \; B_0^{-1} \; \underline{M^{-1} M} \; X_0$

$$= X_0^T \; M^T \left\{ (M^T)^{-1} \; B_0^{-1} \; M^{-1} \right\} M \; X_0$$

Transformation of Twiss parameters

and using $A^T B^T = (BA)^T$ ***and*** $A^{-1}B^{-1} = (BA)^{-1}$

$$\varepsilon = X_0^T \; M^T \left\{ (M^T)^{-1} (MB_0)^{-1} \right\} M \; X_0$$

$$= X_0^T \; M^T \left\{ MB_0 M^T \right\}^{-1} M \; X_0$$

$$= \underbrace{(MX_0)^T}_{} \left\{ MB_0 M^T \right\}^{-1} \underbrace{MX_0}_{}$$

$$= X_1^{\;T} \left\{ MB_0 M^T \right\}^{-1} \; X_1$$

but we know already that

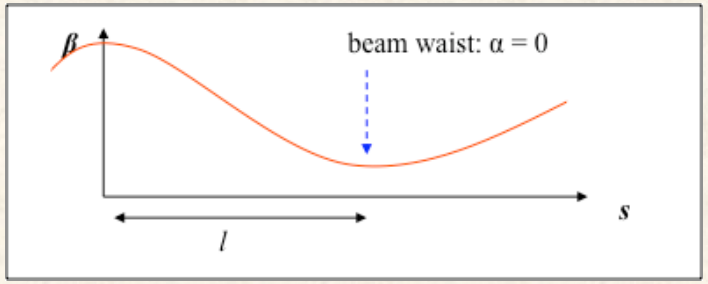
$$\varepsilon = const = \quad X_0^{\;T} * B_0^{-1} * X_0 \;\; = \;\; X_1^{\;T} * B_1^{-1} * X_1$$

and in the end and after all we learn that ... ***in full equivalence to ...***

$$B_1 = \begin{pmatrix} \beta_1 & -\alpha_1 \\ \alpha_1 & \gamma_1 \end{pmatrix} = M * B_0 * M^T$$

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_{s2} = \begin{pmatrix} m_{11}^2 & -2m_{11}m_{12} & m_{12}^2 \\ -m_{11}m_{21} & m_{12}m_{21} + m_{22}m_{11} & -m_{12}m_{22} \\ m_{12}^2 & -2m_{22}m_{21} & m_{22}^2 \end{pmatrix} * \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_{s1}$$

Transformation of Twiss parameters



Example again the drift space

... starting from $\alpha_0 = 0$

$$M = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$$

$$B_0 = \begin{pmatrix} \beta_0 & -\alpha_0 \\ -\alpha_0 & \gamma_0 \end{pmatrix} = \begin{pmatrix} \beta_0 & 0 \\ 0 & 1/\beta_0 \end{pmatrix}$$

Beam parameters after the drift: $B_1 = M B_0 M^T = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} \beta_0 & 0 \\ 0 & 1/\beta_0 \end{pmatrix} * \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}$

$$= \begin{pmatrix} \beta_0 + \frac{s^2}{\beta_0} & \frac{s}{\beta_0} \\ \frac{s}{\beta_0} & \frac{1}{\beta_0} \end{pmatrix} \longrightarrow \beta_1 = \beta_0 + \frac{s^2}{\beta_0}$$

Appendix II: Dispersion

… solution of the inhomogenious equation of motion

the dispersion function is given by

$$D(s) = S(s) * \int \frac{1}{\rho(\tilde{s})} C(\tilde{s}) d\tilde{s} - C(s) * \int \frac{1}{\rho(\tilde{s})} S(\tilde{s}) d\tilde{s}$$

proof:

$$D'(s) = S'(s) * \int \frac{1}{\rho(\tilde{s})} C(\tilde{s}) d\tilde{s} + S(s) * \frac{C(\tilde{s})}{\rho(\tilde{s})} - C'(s) * \int \frac{1}{\rho(\tilde{s})} S(\tilde{s}) d\tilde{s} - C(s) \frac{S(\tilde{s})}{\rho(\tilde{s})}$$

$$D'(s) = S'(s) * \int \frac{C}{\rho} d\tilde{s} - C'(s) * \int \frac{S}{\rho} d\tilde{s}$$

$$D''(s) = S''(s) * \int \frac{C}{\rho} d\tilde{s} + S' \frac{C}{\rho} - C''(s) * \int \frac{S}{\rho} d\tilde{s} - C' \frac{S}{\rho}$$

$$D''(s) = S''(s) * \int \frac{C}{\rho} d\tilde{s} - C''(s) * + \frac{1}{\rho})\underbrace{(CS' - SC')}_{= \det(M) = 1}$$

$$D''(s) = S''(s) * \int \frac{C}{\rho} d\tilde{s} - C''(s) * \int \frac{S}{\rho} d\tilde{s} + \frac{1}{\rho}$$

now the principal trajectories S and C fulfill the homogeneous equation

$$S''(s) = -K * S \quad , \qquad C''(s) = -K * C$$

and so we get:

$$D''(s) = -K * S(s) * \int \frac{C}{\rho} d\tilde{s} + K * C(s) * \int \frac{S}{\rho} d\tilde{s} + \frac{1}{\rho}$$

$$D''(s) = -K * D(s) + \frac{1}{\rho}$$

$$D''(s) + K * D(s) = \frac{1}{\rho}$$

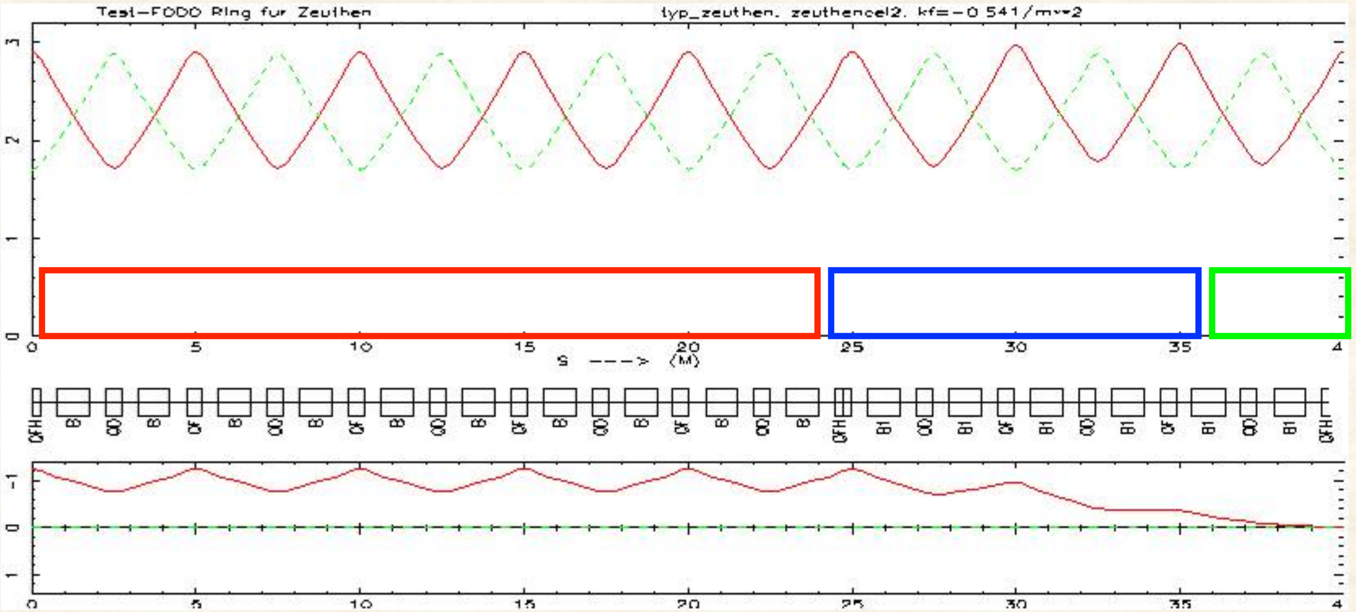
qed.

Appendix III: Dispersion Suppressors

... the calculation of the half bend scheme in full detail (for purists only)

1.) the lattice is split into 3 parts: ***(Gallia divisa est in partes tres)***

* periodic solution of the arc — periodic β, periodic dispersion D
* section of the dispersion suppressor — periodic β, dispersion vanishes
* FoDo cells without dispersion — periodic β, D = D′ = 0



2.) calculate the dispersion D in the periodic part of the lattice

transfer matrix of a periodic cell:

$$M_{0\to S} = \begin{pmatrix} \sqrt{\frac{\beta_S}{\beta_0}}(\cos\phi + \alpha_0 \sin\phi) & \sqrt{\beta_S \beta_0}\sin\phi \\ \frac{(\alpha_0 - \alpha_S)\cos\phi - (1+\alpha_0\alpha_S)\sin\phi}{\sqrt{\beta_S\beta_0}} & \sqrt{\frac{\beta_S}{\beta_0}}(\cos\phi - \alpha_S \sin\phi) \end{pmatrix}$$

for the transformation from one symmetriy point to the next (i.e. one cell) we have:
Φ_C = phase advance of the cell, $\alpha = 0$ at a symmetry point. The index "*c*" refers to the periodic solution of one cell.

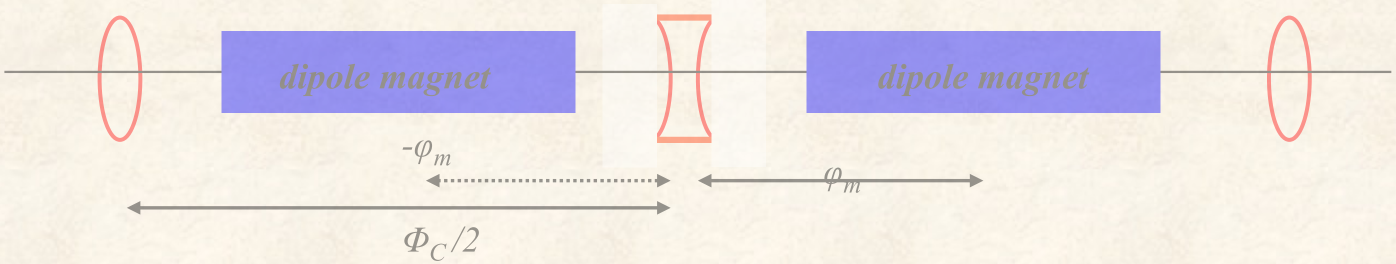
$$M_{Cell} = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\Phi_C & \beta_C \sin\Phi_C & D(l) \\ \frac{-1}{\beta_C}\sin\Phi_C & \cos\Phi_C & D'(l) \\ 0 & 0 & 1 \end{pmatrix}$$

The matrix elements D and D' are given by the C and S elements in the usual way:

$$D(l) = S(l) * \int_0^l \frac{1}{\rho(\tilde{s})} C(\tilde{s})\, d\tilde{s} - C(l) * \int_0^l \frac{1}{\rho(\tilde{s})} S(\tilde{s})\, d\tilde{s}$$

$$D'(l) = S'(l) * \int_0^l \frac{1}{\rho(\tilde{s})} C(\tilde{s})\, d\tilde{s} - C'(l) * \int_0^l \frac{1}{\rho(\tilde{s})} S(\tilde{s})\, d\tilde{s}$$

here the values C(l) and S(l) refer to the symmetry point of the cell (middle of the quadrupole) and the integral is to be taken over the dipole magnet where $\rho \neq 0$. For ρ = const the integral over C(s) and S(s) is approximated by the values in the middle of the dipole magnet.



Transformation of C(s) from the symmetry point to the center of the dipole:

$$C_m = \sqrt{\frac{\beta_m}{\beta_C}} \cos\Delta\Phi = \sqrt{\frac{\beta_m}{\beta_C}} \cos\left(\frac{\Phi_C}{2} \pm \varphi_m\right) \qquad S_m = \sqrt{\beta_m \beta_C} \sin\left(\frac{\Phi_C}{2} \pm \varphi_m\right)$$

where β_C is the periodic β function at the beginning and end of the cell, β_m its value at the middle of the dipole and φ_m the phase advance from the quadrupole lens to the dipole center.

Now we can solve the intergal for D and D':

$$D(l) = S(l) * \int_0^l \frac{1}{\rho(\tilde{s})} C(\tilde{s})\, d\tilde{s} - C(l) * \int_0^l \frac{1}{\rho(\tilde{s})} S(\tilde{s})\, d\tilde{s}$$

$$D(l) = \beta_C \sin\Phi_C * \frac{L}{\rho} * \sqrt{\frac{\beta_m}{\beta_C}} * \cos\left(\frac{\Phi_C}{2} \pm \varphi_m\right) - \cos\Phi_C * \frac{L}{\rho} \sqrt{\beta_m \beta_C} * \sin\left(\frac{\Phi_C}{2} \pm \varphi_m\right)$$

$$D(l)=\delta\sqrt{\beta_m\beta_C}\left\{\sin\Phi_C\left[\cos(\frac{\Phi_C}{2}+\varphi_m)+\cos(\frac{\Phi_C}{2}-\varphi_m)\right]-\right.$$

$$\left.-\cos\Phi_C\left[\sin(\frac{\Phi_C}{2}+\varphi_m)+\sin(\frac{\Phi_C}{2}-\varphi_m)\right]\right\}$$

I have put δ = L/ρ for the strength of the dipole

remember the relations $\cos x+\cos y=2\cos\frac{x+y}{2}*\cos\frac{x-y}{2}$

$\sin x+\sin y=2\sin\frac{x+y}{2}*\cos\frac{x-y}{2}$

$$D(l)=\delta\sqrt{\beta_m\beta_C}\left\{\sin\Phi_C*2\cos\frac{\Phi_C}{2}*\cos\varphi_m-\cos\Phi_C*2\sin\frac{\Phi_C}{2}*\cos\varphi_m\right\}$$

$$D(l)=2\delta\sqrt{\beta_m\beta_C}*\cos\varphi_m\left\{\sin\Phi_C*\cos\frac{\Phi_C}{2}*-\cos\Phi_C*\sin\frac{\Phi_C}{2}\right\}$$

remember: $\sin 2x=2\sin x*\cos x$

$\cos 2x=\cos^2 x-\sin^2 x$

$$D(l)=2\delta\sqrt{\beta_m\beta_C}*\cos\varphi_m\left\{2\sin\frac{\Phi_C}{2}*\cos^2\frac{\Phi_C}{2}-(\cos^2\frac{\Phi_C}{2}-\sin^2\frac{\Phi_C}{2})*\sin\frac{\Phi_C}{2}\right\}$$

$$D(l) = 2\delta\sqrt{\beta_m\beta_C} * \cos\varphi_m * \sin\frac{\Phi_C}{2}\left\{2\cos^2\frac{\Phi_C}{2} - \cos^2\frac{\Phi_C}{2} + \sin^2\frac{\Phi_C}{2}\right\}$$

$$D(l) = 2\delta\sqrt{\beta_m\beta_C} * \cos\varphi_m * \sin\frac{\Phi_C}{2}$$

in full analogy one derives the expression for D ‘:

$$\boldsymbol{D'(l)} = 2\boldsymbol{\delta}\sqrt{\boldsymbol{\beta_m} / \boldsymbol{\beta_c}} * \cos\boldsymbol{\varphi_m} * \cos\frac{\Phi_c}{2}$$

As we refer the expression for D and D ‘ to a periodic struture, namly a FoDo cell we require periodicity conditons:

$$\begin{pmatrix} D_C \\ D'_C \\ 1 \end{pmatrix} = M_C * \begin{pmatrix} D_C \\ D'_C \\ 1 \end{pmatrix}$$

and by symmetry: $D'_C = 0$

With these boundary conditions the Dispersion in the FoDo is determined:

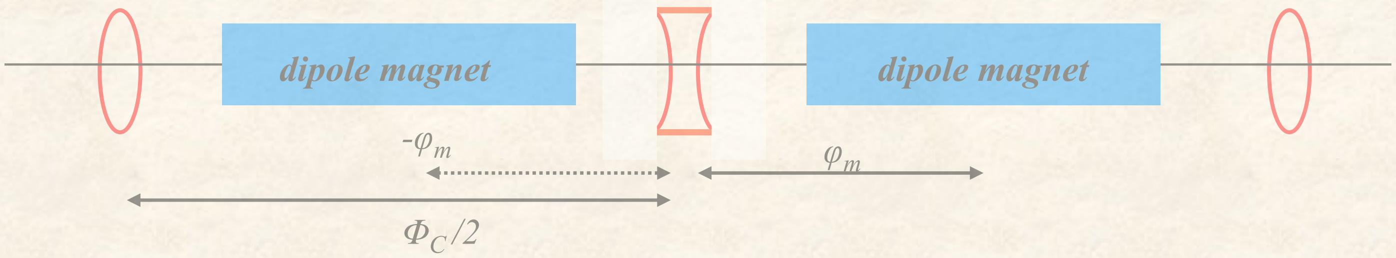
$$D_C * \cos\Phi_C + \delta\sqrt{\beta_m\beta_C} * \cos\varphi_m * 2\sin\frac{\Phi_C}{2} = D_C$$

(A1) $$D_C = \delta\sqrt{\beta_m \beta_C} * \cos\varphi_m / \sin\frac{\Phi_C}{2}$$

This is the value of the periodic dispersion in the cell evaluated at the position of the dipole magnets.

3.) Calculate the dispersion in the suppressor part:

We will now move to the second part of the dispersion suppressor: The section where ... starting from D=D ‘=0 the dispesion is generated ... or turning it around where the Dispersion of the arc is reduced to zero.
The goal will be to generate the dispersion in this section in a way that the values of the periodic cell that have been calculated above are obtained.



The relation for D, generated in a cell still holds in the same way:

$$D(l) = S(l) * \int_0^l \frac{1}{\rho(\tilde{s})} C(\tilde{s})\, d\tilde{s} - C(l) * \int_0^l \frac{1}{\rho(\tilde{s})} S(\tilde{s})\, d\tilde{s}$$

as the dispersion is generated in a number of n cells the matrix for these n cells is

$$M_n = M_C^n = \begin{pmatrix} \cos n\Phi_C & \beta_C \sin n\Phi_C & D_n \\ \frac{-1}{\beta_C}\sin n\Phi_C & \cos n\Phi_C & D'_n \\ 0 & 0 & 1 \end{pmatrix}$$

$$D_n = \beta_C \sin n\Phi_C * \delta_{\sup r} * \sum_{i=1}^{n} \cos(i\Phi_C - \frac{1}{2}\Phi_C \pm \varphi_m) * \sqrt{\frac{\beta_m}{\beta_C}} - $$
$$- \cos n\Phi_C * \delta_{\sup r} * \sum_{i=1}^{n} \sqrt{\beta_m \beta_C} * \sin(i\Phi_C - \frac{1}{2}\Phi_C \pm \varphi_m)$$

$$D_n = \sqrt{\beta_m \beta_C} * \sin n\Phi_C * \delta_{\sup r} * \sum_{i=1}^{n} \cos((2i-1)\frac{\Phi_C}{2} \pm \varphi_m) - \sqrt{\beta_m \beta_C} * \delta_{\sup r} * \cos n\Phi_C \sum_{i=1}^{n} \sin((2i-1)\frac{\Phi_C}{2} \pm \varphi_m)$$

remember: $\sin x + \sin y = 2\sin\frac{x+y}{2} * \cos\frac{x-y}{2}$ $\qquad \cos x + \cos y = 2\cos\frac{x+y}{2} * \cos\frac{x-y}{2}$

$$D_n = \delta_{\sup r} * \sqrt{\beta_m \beta_C} * \sin n\Phi_C * \sum_{i=1}^{n} \cos((2i-1)\frac{\Phi_C}{2}) * 2\cos\varphi_m - $$
$$-\delta_{\sup r} * \sqrt{\beta_m \beta_C} * \cos n\Phi_C \sum_{i=1}^{n} \sin((2i-1)\frac{\Phi_C}{2}) * 2\cos\varphi_m$$

$$D_n = 2\delta_{\sup r} * \sqrt{\beta_m \beta_C} * \cos\varphi_m \left\{ \sum_{i=1}^{n} \cos((2i-1)\frac{\Phi_C}{2}) * \sin n\Phi_C - \sum_{i=1}^{n} \sin((2i-1)\frac{\Phi_C}{2}) * \cos n\Phi_C \right\}$$

$$D_n = 2\delta_{\sup r} * \sqrt{\beta_m \beta_C} * \cos\varphi_m \left\{ \sin n\Phi_C \left\{ \frac{\sin\frac{n\Phi_C}{2} * \cos\frac{n\Phi_C}{2}}{\sin\frac{\Phi_C}{2}} \right\} - \cos n\Phi_C * \left\{ \frac{\sin\frac{n\Phi_C}{2} * \sin\frac{n\Phi_C}{2}}{\sin\frac{\Phi_C}{2}} \right\} \right\}$$

$$D_n = \frac{2\delta_{\sup r} * \sqrt{\beta_m \beta_C} * \cos\varphi_m}{\sin\frac{\Phi_C}{2}} \left\{ \sin n\Phi_C * \sin\frac{n\Phi_C}{2} * \cos\frac{n\Phi_C}{2} - \cos n\Phi_C * \sin^2\frac{n\Phi_C}{2} \right\}$$

set for more convenience $x = n\Phi_C/2$

$$D_n = \frac{2\delta_{\sup r} * \sqrt{\beta_m \beta_C} * \cos\varphi_m}{\sin\frac{\Phi_C}{2}} \left\{ \sin 2x * \sin x * \cos x - \cos 2x * \sin^2 x \right\}$$

$$D_n = \frac{2\delta_{\sup r} * \sqrt{\beta_m \beta_C} * \cos\varphi_m}{\sin\frac{\Phi_C}{2}} \left\{ 2\sin x \cos x * \cos x \sin x - (\cos^2 x - \sin^2 x)\sin^2 x \right\}$$

(A2)

$$D_n = \frac{2\delta_{\sup r} * \sqrt{\beta_m \beta_C} * \cos\varphi_m}{\sin\frac{\Phi_C}{2}} * \sin^2\frac{n\Phi_C}{2}$$

and in similar calculations:

$$D'_n = \frac{2\delta_{\sup r} * \sqrt{\beta_m \beta_C} * \cos\varphi_m}{\sin\frac{\Phi_C}{2}} * \sin n\Phi_C$$

This expression gives the dispersion generated in a certain number of *n* cells as a function of the dipole kick δ in these cells.
At the end of the dispersion generating section the value obtained for D(s) and D '(s) has to be equal to the value of the periodic solution:

→equating *(A1)* and *(A2)* gives the conditions for the matching of the periodic dispersion in the arc to the values D = D '= 0 after the suppressor.

$$D_n = \frac{2\delta_{\sup r} * \sqrt{\beta_m \beta_C} * \cos\varphi_m}{\sin\frac{\Phi_C}{2}} * \sin^2\frac{n\Phi_C}{2} = \delta_{arc}\sqrt{\beta_m \beta_C} * \frac{\cos\varphi_m}{\sin\frac{\Phi_C}{2}}$$

$$\left.\begin{array}{l} \rightarrow 2\delta_{\sup r}\sin^2(\dfrac{n\Phi_C}{2}) = \delta_{arc} \\ \rightarrow \sin(n\Phi_C) = 0 \end{array}\right\} \delta_{\sup r} = \frac{1}{2}\delta_{arc}$$

and at the same time the phase advance in the arc cell has to obey the relation:

$$n\Phi_C = k*\pi,\ k = 1,3,\ ...$$

Appendix IV: Dispersion in a FoDo Cell

equation of motion

$$x'' + K(s) * x = 0 \qquad K = -k + 1/\rho^2$$

single particle trajectory considering both planes

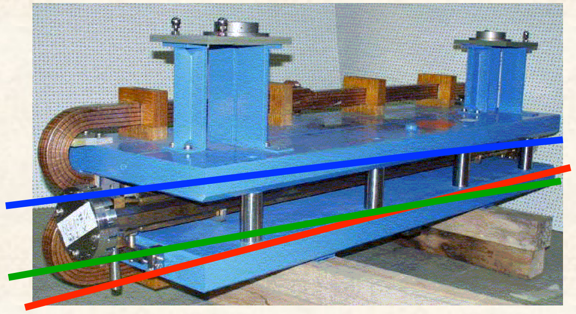
$$\begin{pmatrix} x(s) \\ x'(s) \\ y(s) \\ y'(s) \end{pmatrix} = M * \begin{pmatrix} x(s_0) \\ x'(s_0) \\ y(s_0) \\ y'(s_0) \end{pmatrix}$$

APS Light Source

e.g. matrix for a quadrupole lens:

$$M_{foc} = \begin{pmatrix} \cos(\sqrt{|k|}s & \frac{1}{\sqrt{|k|}}\sin(\sqrt{|k|}s & 0 & 0 \\ -\sqrt{|k|}\sin(\sqrt{|k|}s & \cos(\sqrt{|k|}s & 0 & 0 \\ 0 & 0 & \cosh(\sqrt{|k|}s & \frac{1}{\sqrt{|k|}}\sinh(\sqrt{|k|}s \\ 0 & 0 & \sqrt{|k|}\sinh(\sqrt{|k|}s & \cosh(\sqrt{|k|}s \end{pmatrix} = \begin{pmatrix} C_x & S_x & 0 & 0 \\ C'_x & S'_x & 0 & 0 \\ 0 & 0 & C_y & S_y \\ 0 & 0 & C'_y & S'_y \end{pmatrix}$$

Dispersion



momentum error:

$$\Delta p / p \neq 0$$

$$x'' + x\left(\frac{1}{\rho^2} - k\right) = \frac{\Delta p}{p} \cdot \frac{1}{\rho}$$

general solution:

$$x(s) = x_h(s) + x_i(s)$$

$$D(s) = \frac{x_i(s)}{\Delta p / p}$$

$$x(s) = x_\beta(s) + D(s) \cdot \frac{\Delta p}{p}$$

$$\begin{pmatrix} x \\ x' \\ \frac{\Delta p}{p} \end{pmatrix}_s = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} x \\ x' \\ \frac{\Delta p}{p} \end{pmatrix}_0$$

Dispersion

the dispersion function D(s) is (...obviously) defined by the focusing properties of the lattice and is given by:

$$D(s) = S(s) * \int \frac{1}{\rho(\tilde{s})} C(\tilde{s}) d\tilde{s} - C(s) * \int \frac{1}{\rho(\tilde{s})} S(\tilde{s}) d\tilde{s}$$

! weak dipoles → large bending radius → small dispersion

Example: Drift

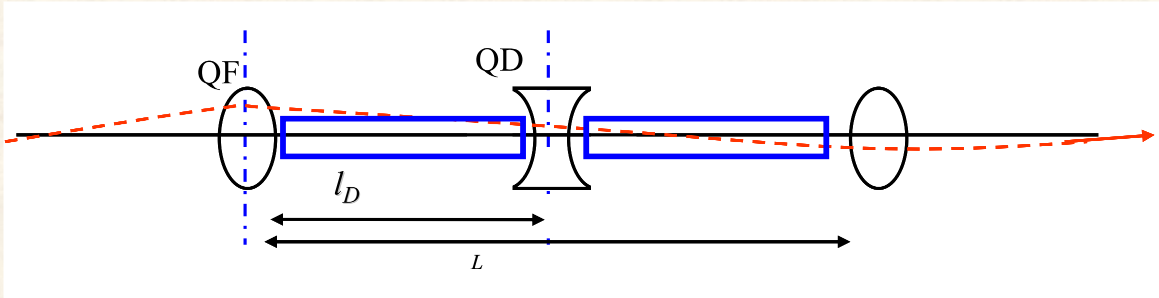
$$M_D = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} \qquad D(s) = S(s) * \int \underbrace{\frac{1}{\rho(\tilde{s})}}_{=0} C(\tilde{s}) d\tilde{s} - C(s) * \int \underbrace{\frac{1}{\rho(\tilde{s})}}_{=0} S(\tilde{s}) d\tilde{s}$$

$$\rightarrow M_D = \begin{pmatrix} 1 & l & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

...in similar way for quadrupole matrices,

!!! in a quite different way for dipole matrix (see appendix)

Dispersion in a FoDo Cell:



!! we have now introduced dipole magnets in the FoDo:

→ we still neglect the weak focusing contribution $1/\rho^2$

→ but take into account $1/\rho$ for the dispersion effect

assume: length of the dipole = l_D

Calculate the matrix of the FoDo half cell in thin lens approximation:

in analogy to the derivations of $\hat{\beta}, \check{\beta}$

* ***thin lens approximation***: $f = \frac{1}{k l_Q} >> l_Q$

* ***length of quad negligible*** $l_Q \approx 0, \rightarrow l_D = \frac{1}{2} L$

* ***start at half quadrupole*** $\frac{1}{\tilde{f}} = \frac{1}{2f}$

Matrix of the half cell

$$M_{HalfCell} = M_{\frac{QD}{2}} * M_B * M_{\frac{QF}{2}}$$

$$M_{HalfCell} = \begin{pmatrix} 1 & 0 \\ \frac{1}{\tilde{f}} & 1 \end{pmatrix} * \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} 1 & 0 \\ \frac{-1}{\tilde{f}} & 1 \end{pmatrix}$$

$$M_{HalfCell} = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} 1 - \frac{1}{\tilde{f}} & 1 \\ \frac{-1}{\tilde{f}} & 1 + \frac{1}{\tilde{f}} \end{pmatrix}$$

calculate the dispersion terms D, D´ from the matrix elements

$$D(s) = S(s) * \int \frac{1}{\rho(\tilde{s})} C(\tilde{s}) d\tilde{s} - C(s) * \int \frac{1}{\rho(\tilde{s})} S(\tilde{s}) d\tilde{s}$$

$$D(\ell) = \ell * \underbrace{\frac{1}{\rho}}_{S(s)} \int_0^\ell \underbrace{(1 - \frac{s}{\tilde{f}})}_{C(s)} ds - \underbrace{(1 - \frac{\ell}{\tilde{f}})}_{C(s)} \frac{1}{\rho} \int_0^\ell \underbrace{s\, ds}_{S(s)}$$

$$D(\ell) = \frac{\ell}{\rho}(\ell - \frac{\ell^2}{2\tilde{f}}) - (1 - \frac{\ell}{\tilde{f}}) * \frac{1}{\rho} * \frac{\ell^2}{2} = \frac{\ell^2}{\rho} - \frac{\ell^3}{2\tilde{f}\rho} - \frac{\ell^2}{2\rho} + \frac{\ell^3}{2\tilde{f}\rho}$$

$$D(\ell) = \frac{\ell^2}{2\rho}$$

in full analogy on derives for D´:

$$D'(s) = \frac{\ell}{\rho}\left(1 + \frac{\ell}{2\tilde{f}}\right)$$

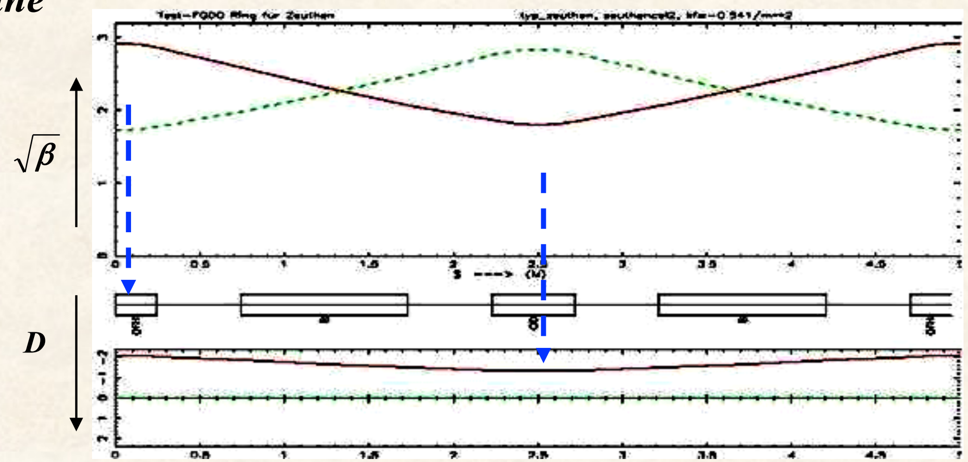
and we get the complete matrix including the dispersion terms D, D´

$$M_{halfCell} = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 - \frac{\ell}{\tilde{f}} & \ell & \frac{\ell^2}{2\rho} \\ \frac{-\ell}{\tilde{f}^2} & 1 + \frac{\ell}{\tilde{f}} & \frac{\ell}{\rho}(1 + \frac{\ell}{2\tilde{f}}) \\ 0 & 0 & 1 \end{pmatrix}$$

Dispersion in a FoDo Cell:

***boundary conditions** for the transfer from the center of the foc. to the center of the defoc. quadrupole*

$$\begin{pmatrix} \check{D} \\ 0 \\ 1 \end{pmatrix} = M_{1/2} * \begin{pmatrix} \hat{D} \\ 0 \\ 1 \end{pmatrix}$$

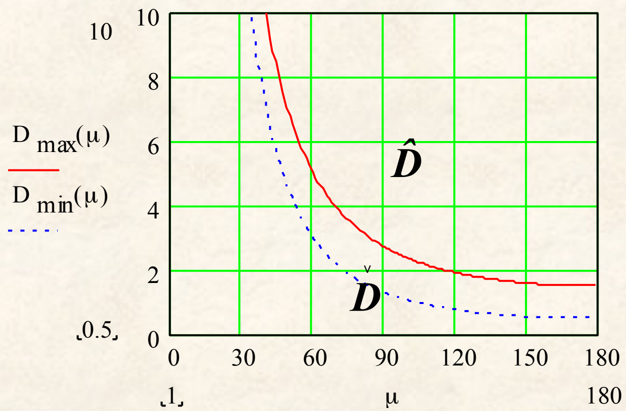


$$\rightarrow \quad \breve{D} = \hat{D}\left(1 - \frac{\ell}{\tilde{f}}\right) + \frac{\ell^2}{2\rho} \qquad \rightarrow \quad 0 = -\frac{\ell}{\tilde{f}^2} * \hat{D} + \frac{\ell}{\rho}\left(1 + \frac{\ell}{2\tilde{f}}\right)$$

$$\hat{D} = \frac{\ell^2}{\rho} * \frac{\left(1 + \frac{1}{2}\sin\frac{\psi_{cell}}{2}\right)}{\sin^2\frac{\psi_{cell}}{2}} \qquad \check{D} = \frac{\ell^2}{\rho} * \frac{\left(1 - \frac{1}{2}\sin\frac{\psi_{cell}}{2}\right)}{\sin^2\frac{\psi_{cell}}{2}}$$

where ψ_{cell} denotes the phase advance of the full cell and $l/f = \sin(\psi/2)$

Dispersion in a FoDo Cell



Nota bene:

! small dispersion needs strong focusing

→ large phase advance

!! ↔ there is an optimum phase for small β

!!! ...do you remember the stability criterion?

½ trace = cos ψ ↔ ψ < 180°

!!!! … life is not easy

latest news: TLEP **E=175 GeV**

C_0 = 100km, ψ=90^0

l = 50m, ρ = 9400m, D = 14cm

$$\hat{D} = \frac{\ell^2}{\rho} * \frac{(1+\frac{1}{2}\sin\frac{\psi_{cell}}{2})}{\sin^2\frac{\psi_{cell}}{2}}$$