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Theory of Electromagnetic Fields

Part II: Standing Waves

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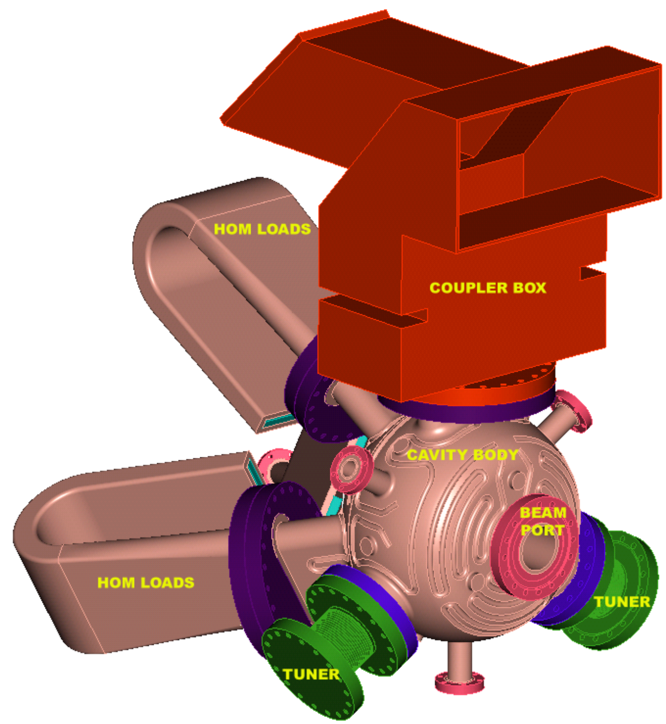
In the previous lecture, we saw that:

- Maxwell's equations have wave-like solutions for the electric and magnetic fields in free space.
- Electromagnetic waves can be generated by oscillating electric charges.
- Expressions for the energy density and energy flow in an electromagnetic field may be obtained from Poynting's theorem.

In this lecture, we shall see how the waves may be "captured" as standing waves in a region of free space bounded by conducting materials – an electromagnetic cavity.

By applying the boundary conditions on the fields (which we derive in the first part of this lecture), we shall see how the electromagnetic field patterns are determined by the geometry of the cavity.



So far, we have considered electromagnetic fields only in materials that have infinite extent in all directions.

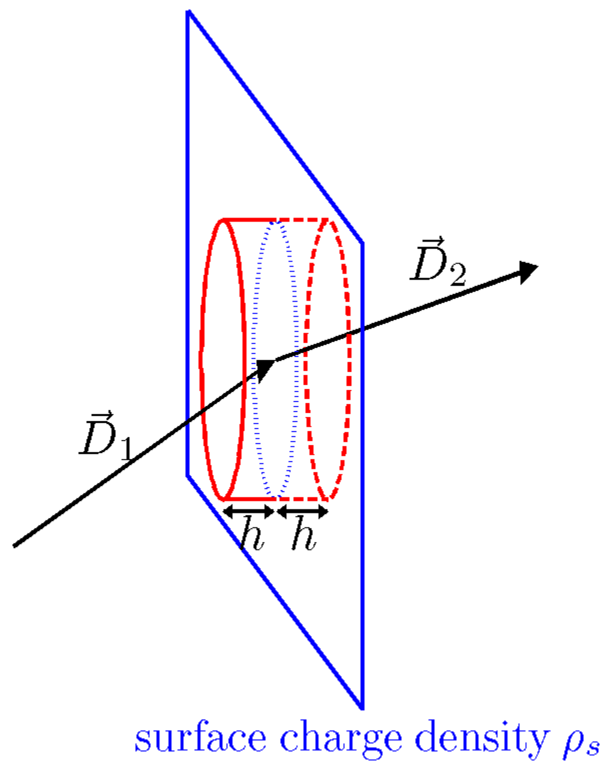
In realistic electromagnetic systems, we have to consider the behaviour of fields at the interfaces between materials with different properties.

We can derive “boundary conditions” on the electric and magnetic fields (i.e. relationships between the electric and magnetic fields on either side of a boundary) from Maxwell’s equations.

These boundary conditions are important for understanding the behaviour of electromagnetic fields in accelerator components.

Consider the electric displacement at a boundary between two different materials. We need make no assumptions about the properties of the materials.

To be completely general, we will assume that there is some surface charge density ρ_s (charge per unit area) on the boundary.



We will apply Gauss' theorem to the divergence of the electric displacement, integrated over a pillbox crossing the boundary.

Take Maxwell's equation:

$$\nabla \cdot \vec{D} = \rho \tag{1}$$

Integrate over the volume of the pillbox, and apply Gauss' theorem:

$$\int_V \nabla \cdot \vec{D}\, dV = \oint_S \vec{D} \cdot d\vec{S} = \int_V \rho\, dV \tag{2}$$

Now we take the limit in which the height of the pillbox becomes zero. If the flat ends of the pillbox have (small) area A, then:

$$-D_{1n}A + D_{2n}A = \rho_s A \tag{3}$$

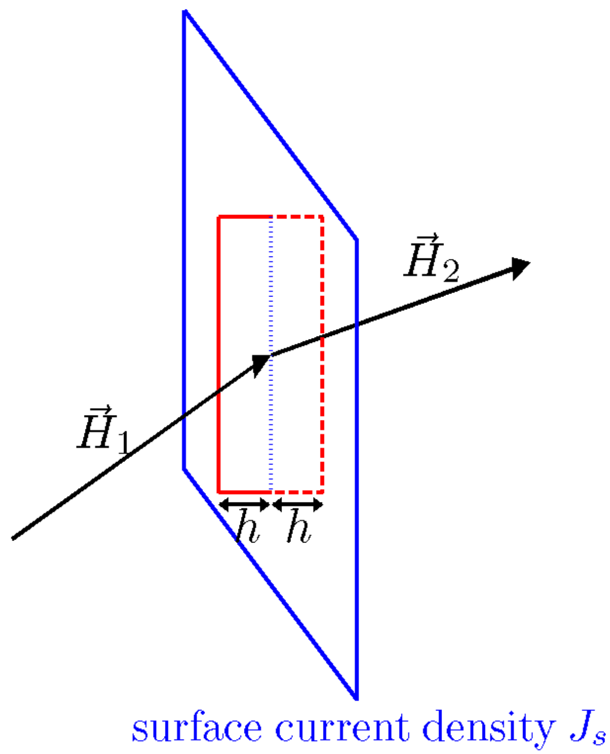
Dividing by the area A, we arrive at:

$$D_{2n} - D_{1n} = \rho_s \tag{4}$$

Now consider the magnetic intensity at a boundary between two different materials.

We will assume that there is some surface current density $\vec{J}_s$ (current per unit length) on the boundary.

We will apply Stokes' theorem to the curl of the magnetic intensity, integrated over a loop crossing the boundary.



Take Maxwell's equation:

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \tag{5}$$

Integrate over the surface bounded by the loop, and apply Stokes' theorem to obtain:

$$\int_S \nabla \times \vec{H} \cdot d\vec{S} = \oint_C \vec{H} \cdot d\vec{l} = \int_S \vec{J} \cdot d\vec{S} + \frac{\partial}{\partial t} \int_S \vec{D} \cdot d\vec{S} \tag{6}$$

Now take the limit where the lengths of the narrow edges of the loop become zero:

$$H_{1t} l - H_{2t} l = J_{s\perp} l \tag{7}$$

or:

$$H_{1t} - H_{2t} = J_{s\perp} \tag{8}$$

where $J_{s\perp}$ represents a surface current density perpendicular to the direction of the tangential component of $\vec{H}$ that is being matched.

Boundary conditions on the normal component of the magnetic field $\vec{B}$, and on the tangential component of the electric field $\vec{E}$ can be obtained using the same arguments.

The general conditions on electric and magnetic fields at the boundary between two materials can be summarised as follows:

Boundary condition:	Derived from...	...applied to:
$D_{2n} - D_{1n} = \rho_s$	$\nabla \cdot \vec{D} = \rho$	pillbox
$H_{2t} - H_{1t} = -J_{s\perp}$	$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$	loop
$B_{2n} = B_{1n}$	$\nabla \cdot \vec{B} = 0$	pillbox
$E_{2t} = E_{1t}$	$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$	loop

Static electric fields cannot persist inside a conductor. This is simply because the free charges within the conductor will re-arrange themselves to cancel any electric field; this can result in a surface charge density, ρ_s.

We have seen that electromagnetic waves can pass into a conductor, but the field amplitudes fall exponentially with decay length given by the skin depth, δ:

$$\delta \approx \sqrt{\frac{2}{\omega\mu\sigma}} \tag{9}$$

As the conductivity increases, the skin depth gets smaller.

Since both static and oscillating electric fields vanish within a good conductor, we can write the boundary conditions at the surface of a good conductor:

$$\begin{aligned} E_{1t} &\approx 0 & \qquad E_{2t} &\approx 0 \\ D_{1n} &\approx -\rho_s & \qquad D_{2n} &\approx 0 \end{aligned}$$

Lenz's law states that a changing magnetic field will induce currents in a conductor that will act to oppose the change. In other words, currents are induced that will tend to cancel the magnetic field in the conductor.

This means that a good conductor will tend to exclude magnetic fields. Thus the boundary conditions on *oscillating* magnetic fields at the surface of a good conductor can be written:

$$\begin{aligned} B_{1n} &\approx 0 & \qquad B_{2n} &\approx 0 \\ H_{1t} &\approx J_{s\perp} & \qquad H_{2t} &\approx 0 \end{aligned}$$

We can consider an "ideal" conductor as having infinite conductivity. In that case, we would expect the boundary conditions to become:

$$\begin{aligned} B_{1n} &= 0 & \qquad B_{2n} &= 0 \\ E_{1t} &= 0 & E_{2t} &= 0 \\ D_{1n} &= -\rho_s & D_{2n} &= 0 \\ H_{1t} &= J_{s\perp} & H_{2t} &= 0 \end{aligned}$$

Strictly speaking, the boundary conditions on the magnetic field apply only to oscillating fields, and not to static fields.

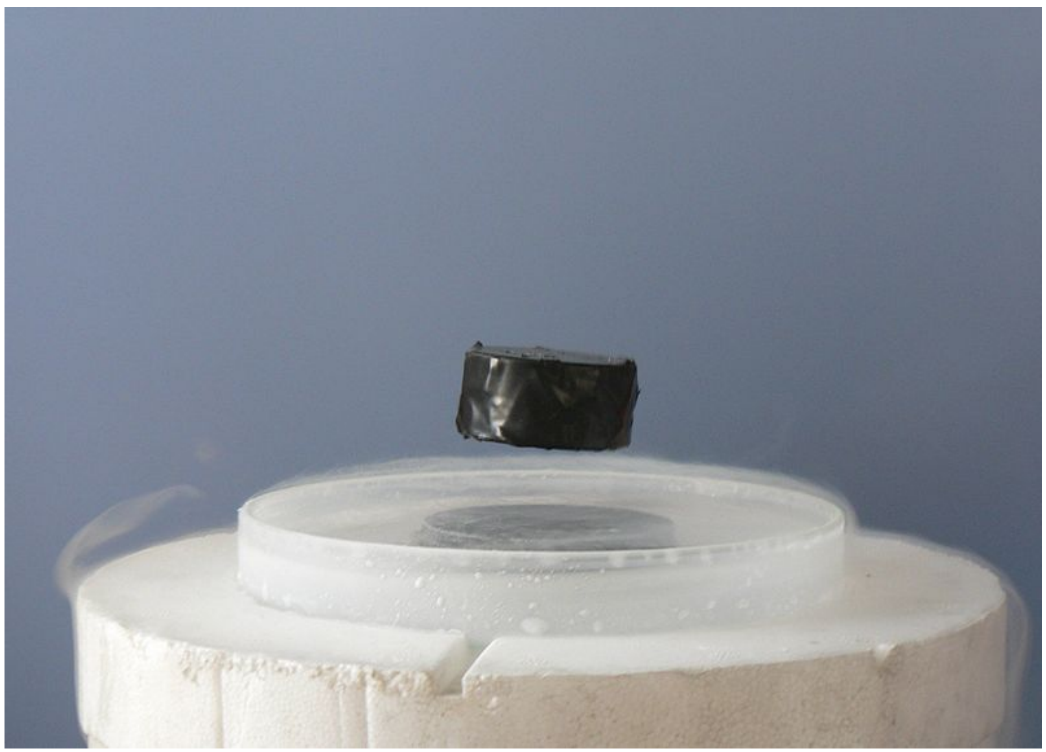
But it turns out that for (some) superconductors, static magnetic fields are excluded as well as oscillating magnetic fields. This is not expected for classical "ideal" conductors.

Although superconductors have infinite conductivity, they cannot be understood in terms of classical theories in the limit $\sigma \to \infty$.

Superconductivity is a quantum phenomenon: one aspect of this is the Meissner effect, which refers to the expulsion of all magnetic fields (static as well as oscillating) from within a superconductor.

In fact, even in a superconductor, the magnetic field is not completely excluded from the material but penetrates a small distance (the London penetration depth, typically around 100 nm) into the material.

As long as the applied magnetic field is not too large, a sample of material cooled below its critical temperature will expel any magnetic field as it undergoes the phase transition to superconductivity: when this happens, a magnet placed on top of the sample will start to levitate.

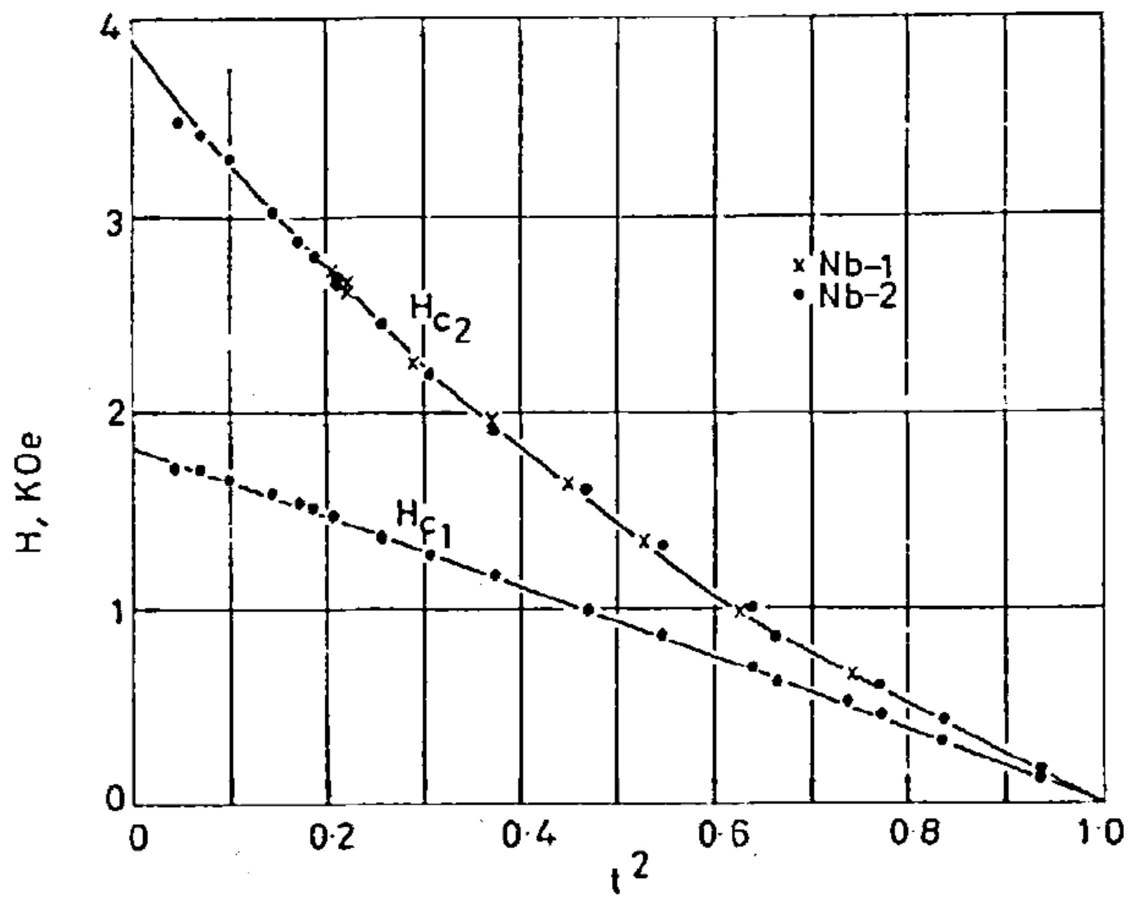


The Meissner effect allows us to classify superconductors into two distinct classes:

- **Type I superconductors:** above a certain critical field H_c (which depends on the temperature), superconductivity is abruptly destroyed.

- **Type II superconductors:** above one critical field value H_{c1}, the magnetic field starts to penetrate, but the electrical resistance remains zero. Above a second, higher critical field value H_{c2}, superconductivity is abruptly destroyed.

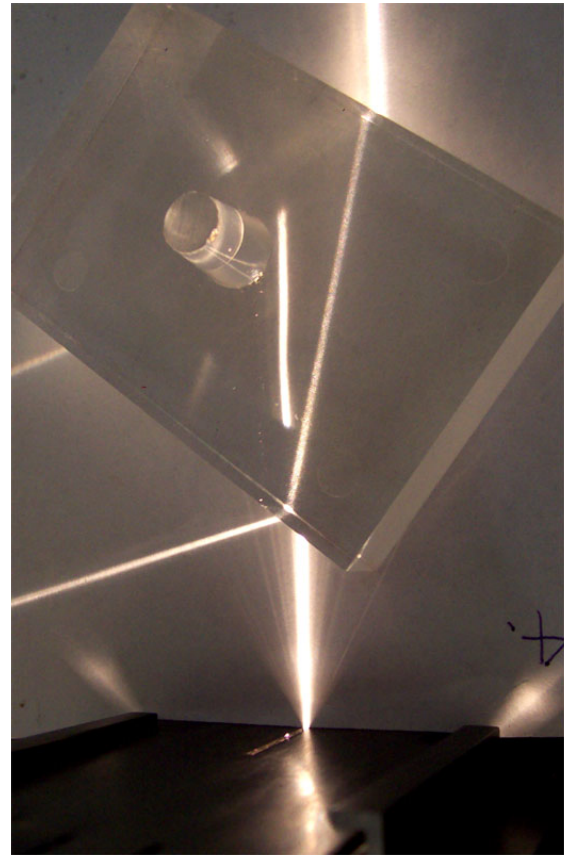
Critical fields in niobium (Type II Superconductor)



R.A. French, "Intrinsic Type-2 Superconductivity in Pure Niobium," *Cryogenics*, **8**, 301 (1968). Note: $t = T/T_c$. The critical temperature for niobium is $T_c = 9.2$ K.

When an electromagnetic wave is incident on a boundary between two materials, part of the energy in the wave will be transmitted across the boundary, and some of the energy will be reflected.

The relationships between the directions and intensities of the incident, transmitted and reflected waves can be derived from the boundary conditions on the fields.



Applied to waves, the boundary conditions on the fields lead to the familiar laws of reflection and refraction, and describe phenomena such as total internal reflection, and polarisation by reflection.

The relationships between the amplitudes of the incident, transmitted and reflected waves can be summed up in a set of formulae, known as Fresnel's equations.

We do not go through the derivations, but present the results on the following slides. The equations depend on the polarisation of the wave, i.e. the orientation of the electric field with respect to the plane of incidence.

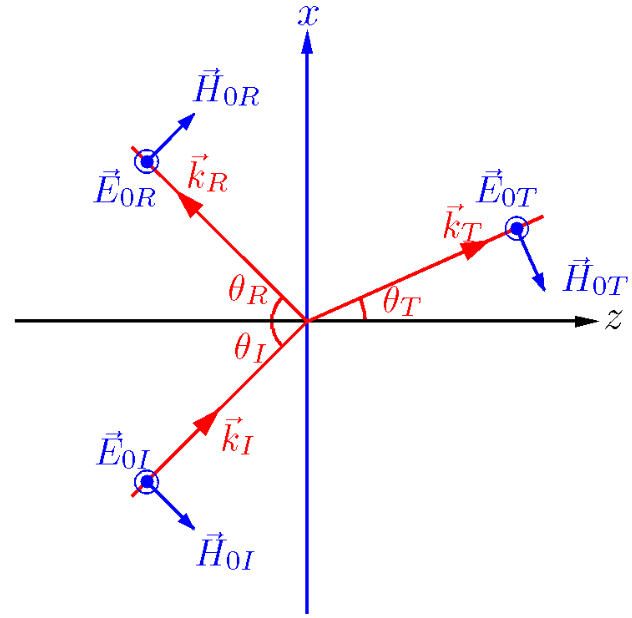
Note the definitions of the *refractive index*, n, and *impedance*, Z of a material:

$$n = \frac{c}{v} = \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}}, \quad \text{and} \quad Z = \sqrt{\frac{\mu}{\varepsilon}}, \tag{10}$$

where μ and ε are the *absolute* permeability and permittivity of the material, c the speed of light in free space, and v the speed of light in the material.

The angles of transmission, θ_T, and incidence, θ_I, are related by Snell's law:

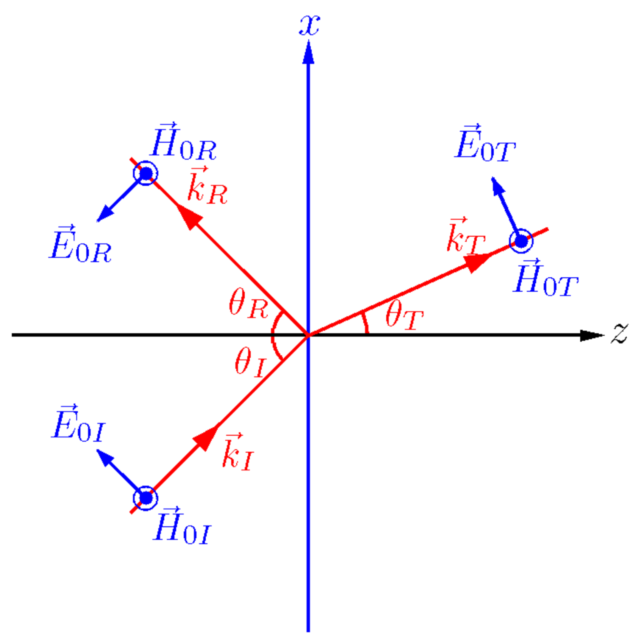
$$\frac{\sin\theta_I}{\sin\theta_T} = \frac{n_2}{n_1} \qquad (11)$$



For a wave in which the electric field is normal to the plane of incidence (i.e. parallel to the boundary), Fresnel's equations are:

$$\left(\frac{E_{0R}}{E_{0I}}\right)_\perp = \frac{Z_2\cos\theta_I - Z_1\cos\theta_T}{Z_2\cos\theta_I + Z_1\cos\theta_T} \qquad (12)$$

$$\left(\frac{E_{0T}}{E_{0I}}\right)_\perp = \frac{2Z_2\cos\theta_I}{Z_2\cos\theta_I + Z_1\cos\theta_T} \qquad (13)$$



For a wave in which the electric field is parallel to the plane of incidence (i.e. normal to the boundary), Fresnel's equations are:

$$\left(\frac{E_{0R}}{E_{0I}}\right)_{\parallel} = \frac{Z_2 \cos\theta_T - Z_1 \cos\theta_I}{Z_2 \cos\theta_T + Z_1 \cos\theta_I} \tag{14}$$

$$\left(\frac{E_{0T}}{E_{0I}}\right)_{\parallel} = \frac{2Z_2 \cos\theta_I}{Z_2 \cos\theta_T + Z_1 \cos\theta_I} \tag{15}$$

Fresnel's equations have important consequences when applied to conductors; but to understand this, we first need to derive the wave impedance of a conductor.

First, note that:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \quad \text{so for a wave:} \quad \vec{k} \times \vec{E} = \omega \vec{B}. \tag{16}$$

Hence, the wave impedance can be written:

$$Z = \frac{E_0}{H_0} = \frac{\omega\mu}{k}. \tag{17}$$

Recall that, for a conductor, the wave vector is complex (the imaginary part describes the attenuation of the wave). In fact, for a good conductor:

$$k \approx (1+i)\sqrt{\frac{\omega\mu\sigma}{2}}, \qquad \sigma \gg \omega\varepsilon. \tag{18}$$

Thus, we find:

$$Z \approx (1-i)\sqrt{\frac{\mu}{\varepsilon}}\sqrt{\frac{\omega\varepsilon}{2\sigma}}, \qquad \sigma \gg \omega\varepsilon. \tag{19}$$

If the permeability and permittivity of the conductor are close to the permeability and permittivity of free space, then it follows that:

$$|Z| \ll Z_0, \qquad \sigma \gg \omega\varepsilon. \tag{20}$$

Finally, putting $|Z_2| \ll Z_1$ into Fresnel's equations, we find, for a good conductor:

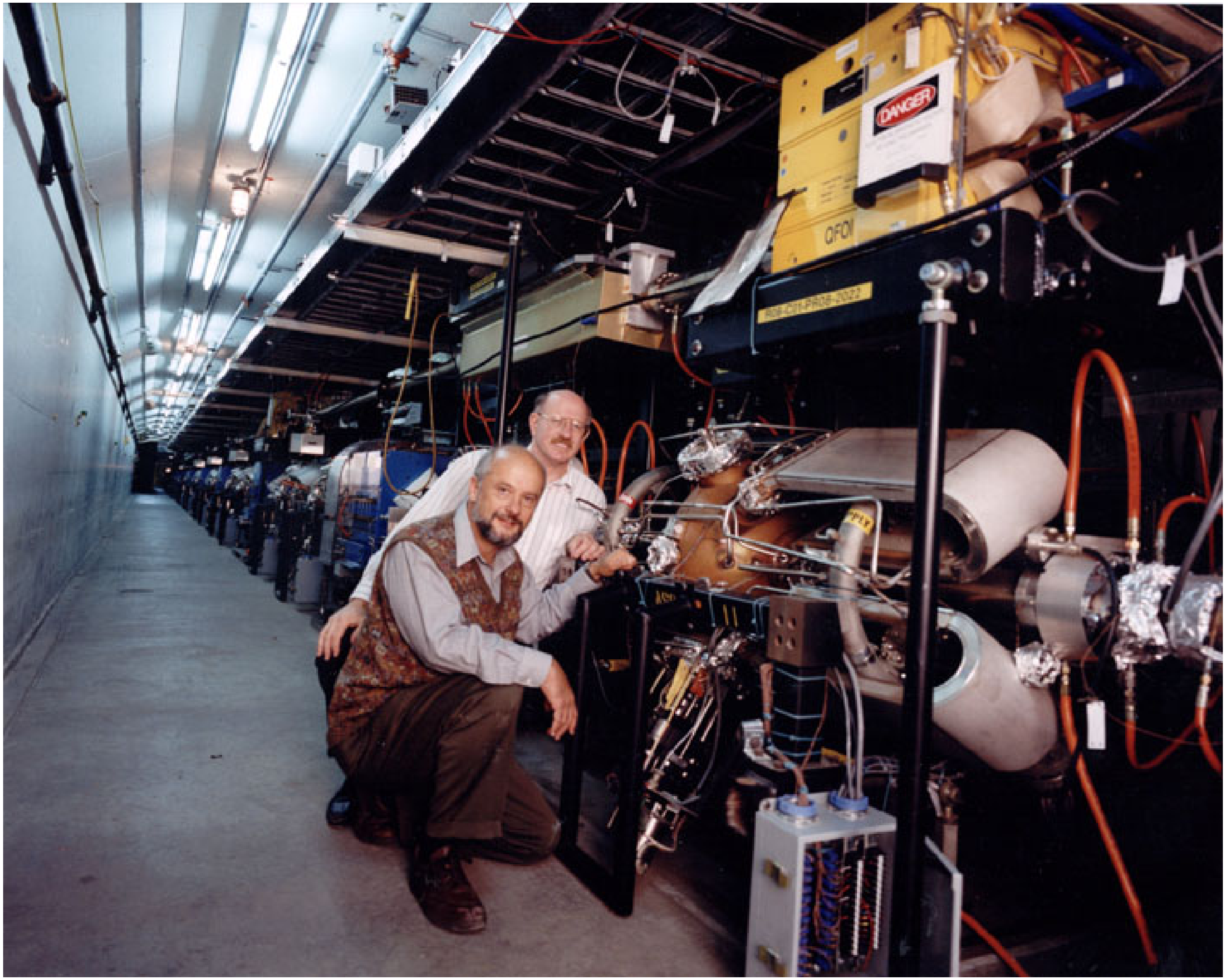
$$\frac{E_{0R}}{E_{0I}} \approx 1, \qquad \text{and} \qquad \frac{E_{0T}}{E_{0I}} \approx 0. \tag{21}$$

In other words, nearly all the energy in the wave is reflected from the surface of a good conductor, and very little is transmitted into the material.

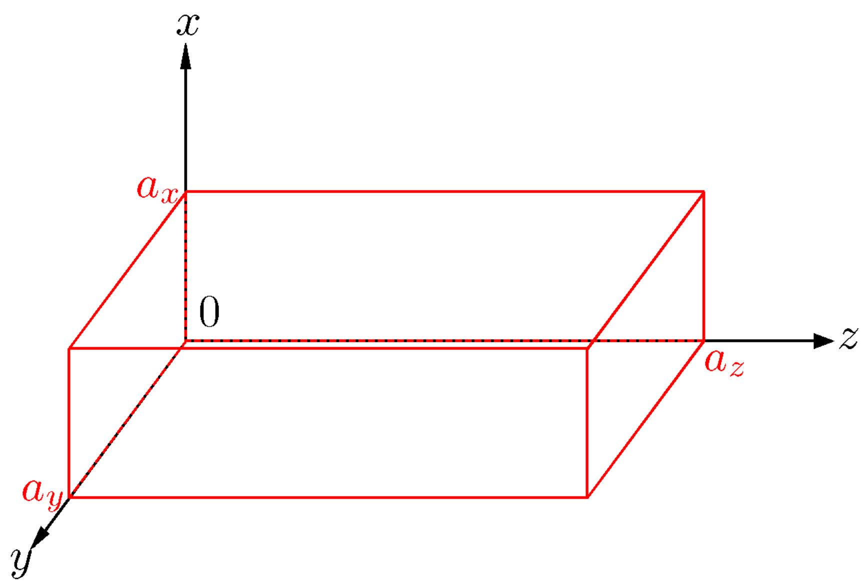
This simple phenomenon allows us to "store" electromagnetic waves in metal boxes.

In the next half of this lecture, we will develop the formulae used to describe electromagnetic waves in conducting cavities.

RF Cavities in PEP-II



We consider first a rectangular cavity with perfectly conducting walls, containing a perfect vacuum.



The wave equation for the electric field inside the cavity is:

$$\nabla^2 \vec{E} - \frac{1}{c^2}\ddot{\vec{E}} = 0, \tag{22}$$

where c is the speed of light in a vacuum. There is a similar equation for the magnetic field $\vec{B}$.

We are looking for solutions to the wave equations for $\vec{E}$ and $\vec{B}$ that also satisfy Maxwell's equations, and also satisfy the boundary conditions for the fields at the walls of the cavity.

If the walls of the cavity are perfectly conducting, then the boundary conditions are:

$$E_t = 0, \tag{23}$$

$$B_n = 0, \tag{24}$$

where E_t is the component of the electric field tangential to the wall, and B_n is the component of the magnetic field normal to the wall.

Plane wave solutions will not satisfy the boundary conditions.

However, we can look for solutions of the form:

$$\vec{E}(x, y, z, t) = \vec{E}(x, y, z)e^{-i\omega t}. \tag{25}$$

Substituting into the wave equation, we find that the spatial dependence satisfies:

$$\nabla^2\vec{E} + \frac{\omega^2}{c^2}\vec{E} = 0. \tag{26}$$

The full solution can be derived using the method of separation of variables: for details, see Appendix A.

However, it is sufficient to quote the result: it is possible to verify the solution simply by substitution into the wave equation.

The electric field in the rectangular cavity is given by:

$$E_x = E_{x0} \cos k_x x \sin k_y y \sin k_z z \, e^{-i\omega t}, \tag{27}$$
$$E_y = E_{y0} \sin k_x x \cos k_y y \sin k_z z \, e^{-i\omega t}, \tag{28}$$
$$E_z = E_{z0} \sin k_x x \sin k_y y \cos k_z z \, e^{-i\omega t}. \tag{29}$$

To satisfy the wave equation, we require:

$$k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2}. \tag{30}$$

To satisfy Maxwell's equation $\nabla \cdot \vec{E} = 0$, we require:

$$k_x E_{x0} + k_y E_{y0} + k_z E_{z0} = 0. \tag{31}$$

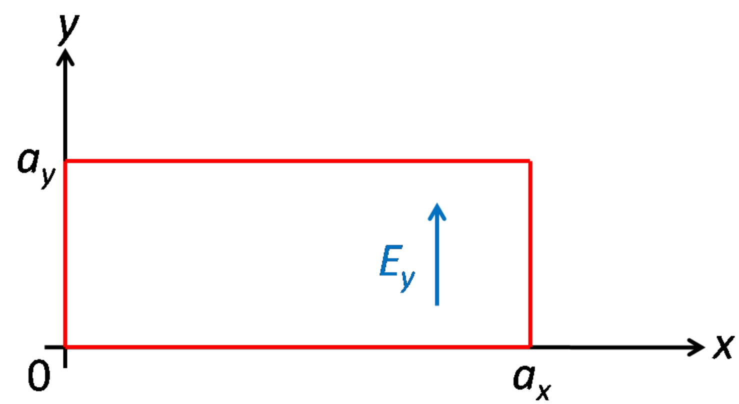
We also need to satisfy the boundary conditions, in particular that the tangential component of the electric field vanishes at the walls of the cavity. This imposes additional constraints on k_x, k_y and k_z.

Consider:

$$E_y = E_{y0} \sin k_x x \, \cos k_y y \, \sin k_z z \, e^{-i\omega t}. \tag{32}$$

The boundary conditions require that $E_y = 0$ at $x = 0$ and $x = a_x$, for all y, z, and t.

These conditions are satisfied if $k_x a_x = m_x \pi$, where m_x is an integer.



To satisfy *all* the boundary conditions, we require:

$$k_x = \frac{m_x \pi}{a_x}, \qquad k_y = \frac{m_y \pi}{a_y}, \qquad k_z = \frac{m_z \pi}{a_z}, \tag{33}$$

where m_x, m_y and m_z are integers.

The magnetic field can be obtained from the electric field, using Maxwell's equation:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}. \tag{34}$$

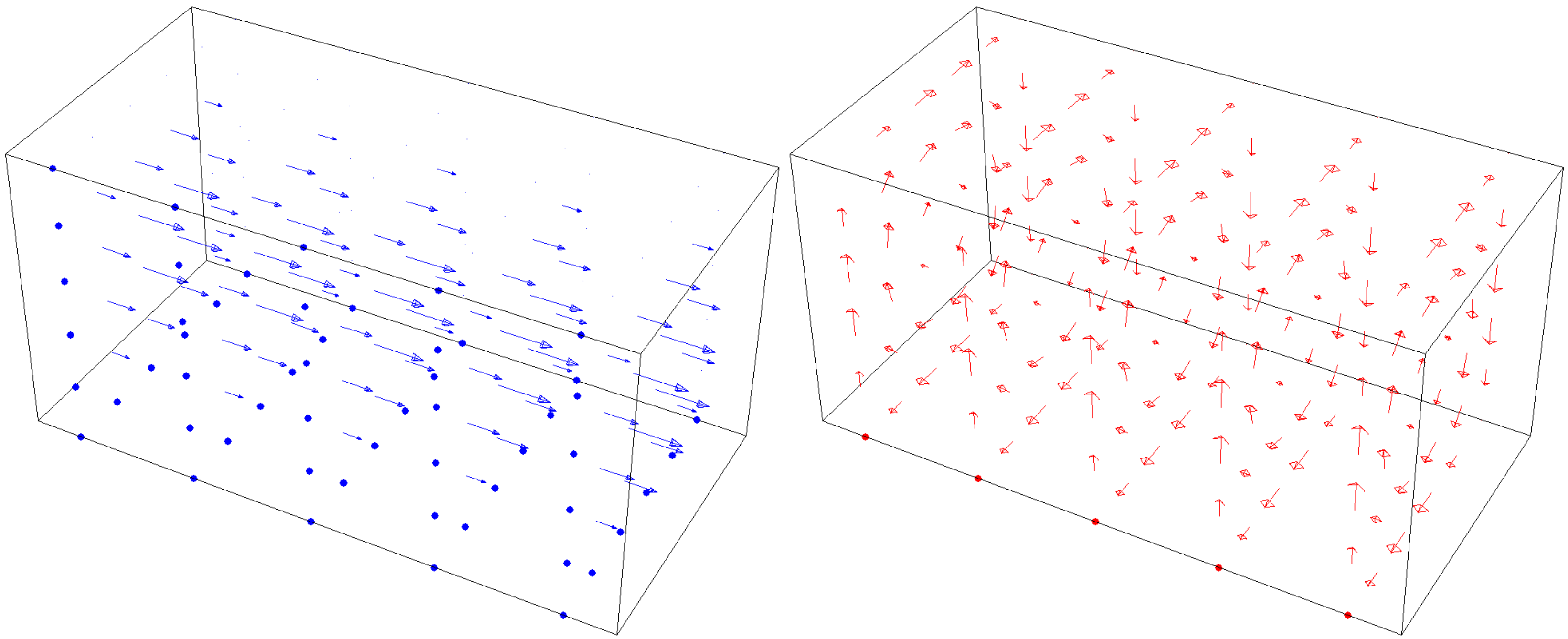
This gives:

$$\begin{aligned} B_x &= \frac{i}{\omega}(E_{y0}k_z - E_{z0}k_y)\sin k_x x \, \cos k_y y \, \cos k_z z \, e^{-i\omega t}, & (35) \\ B_y &= \frac{i}{\omega}(E_{z0}k_x - E_{x0}k_z)\cos k_x x \, \sin k_y y \, \cos k_z z \, e^{-i\omega t}, & (36) \\ B_z &= \frac{i}{\omega}(E_{x0}k_y - E_{y0}k_x)\cos k_x x \, \cos k_y y \, \sin k_z z \, e^{-i\omega t}. & (37) \end{aligned}$$

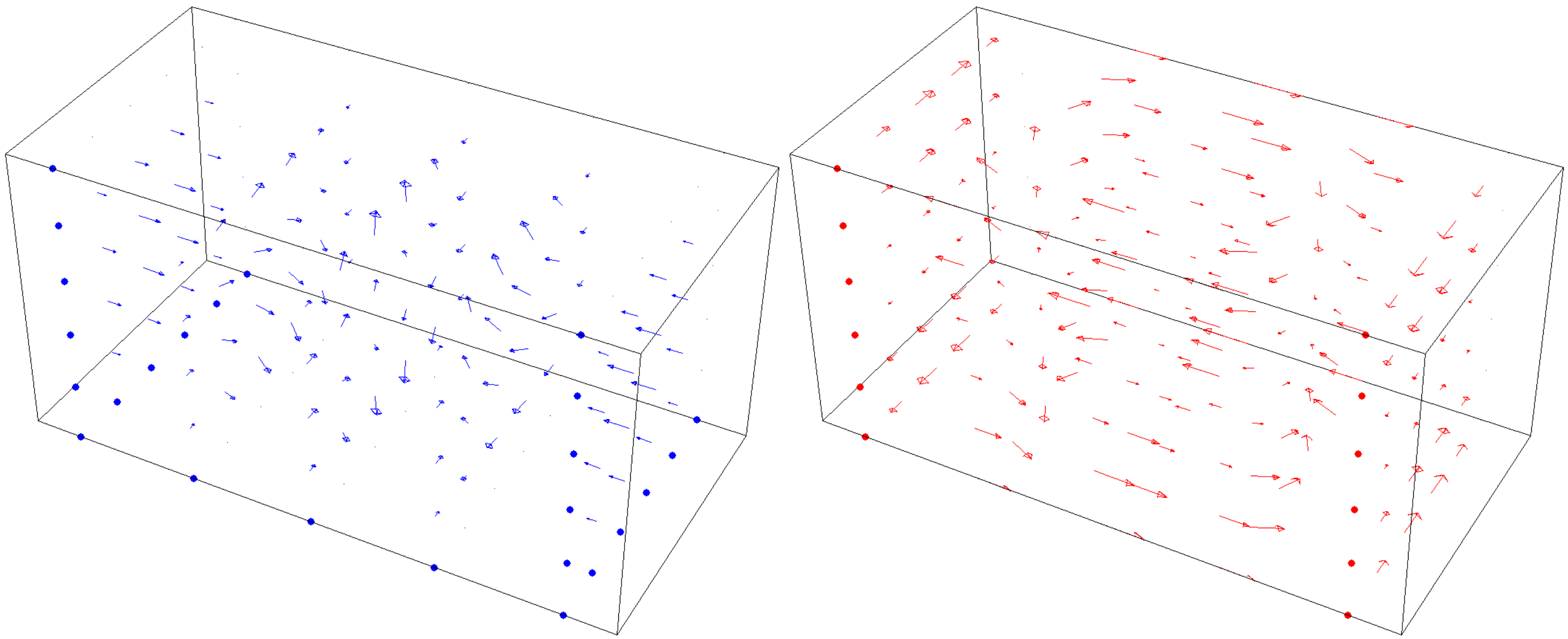
It is left as an exercise for the student to show that these fields satisfy the boundary condition on the magnetic field at the walls of the cavity, and also satisfy the remaining Maxwell's equations:

$$\nabla \cdot \vec{B} = 0, \quad \text{and} \quad \nabla \times \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}. \tag{38}$$

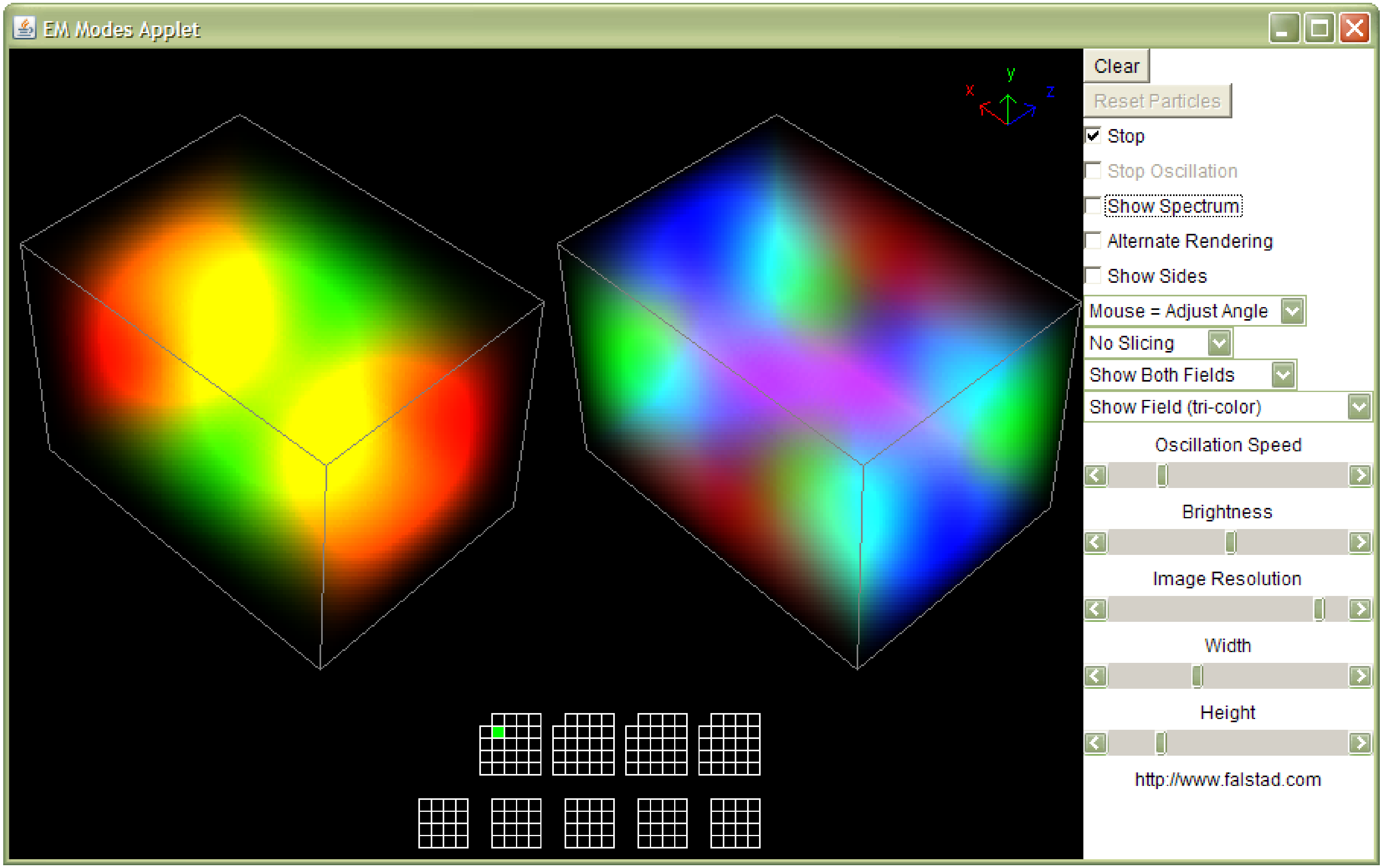
Rectangular cavity mode $(m_x, m_y, m_z) = (1, 1, 0)$



Rectangular cavity mode $(m_x, m_y, m_z) = (1, 1, 1)$



Rectangular cavity modes

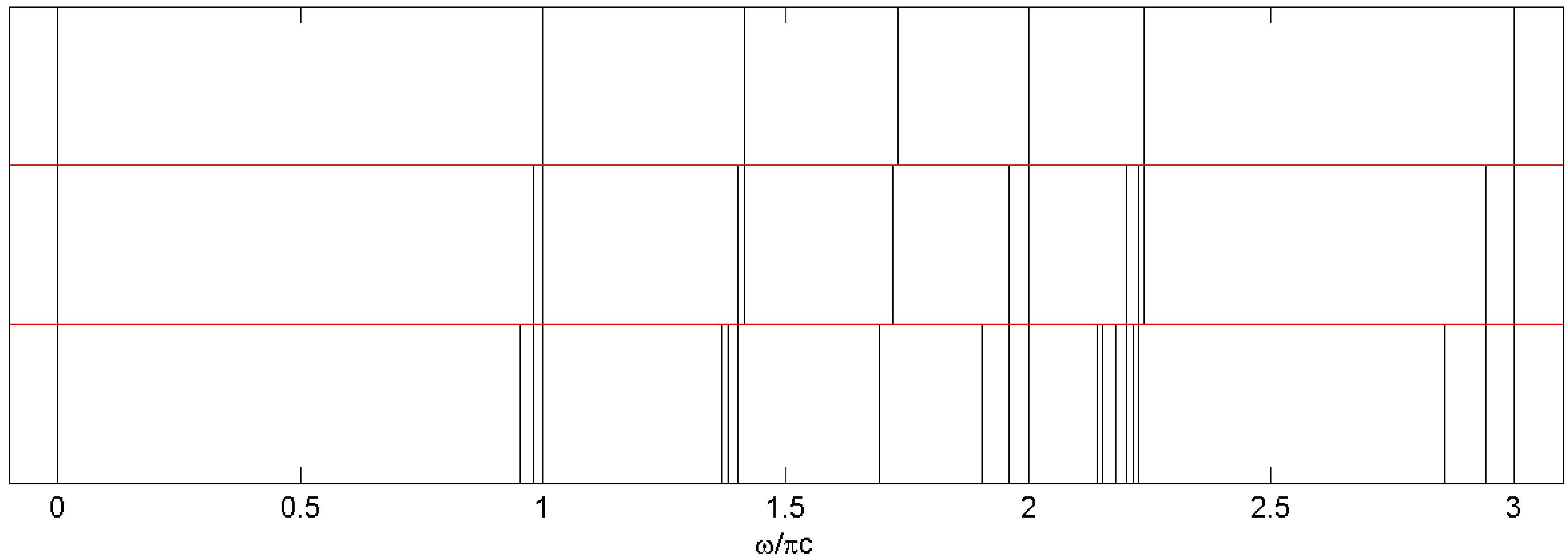


Note that the frequency of oscillation of the wave in the cavity is determined by the *mode numbers* m_x, m_y and m_z:

$$\omega = \pi c \sqrt{\left(\frac{m_x}{a_x}\right)^2 + \left(\frac{m_y}{a_y}\right)^2 + \left(\frac{m_z}{a_z}\right)^2}. \tag{39}$$

For a cubic cavity ($a_x = a_y = a_z$), there will be a high degree of degeneracy, i.e. there will generally be several different sets of mode numbers leading to different field patterns, but all with the same frequency of oscillation.

The degeneracy can be broken by making the side lengths different...



Top: all side lengths equal.
Middle: two side lengths equal.
Bottom: all side lengths different.

Note: we show all modes, including those with two (or three) mode numbers equal to zero, even though such modes will have zero amplitude.

Note that the standing wave solution represents an oscillation that will continue indefinitely: there is no mechanism for dissipating the energy.

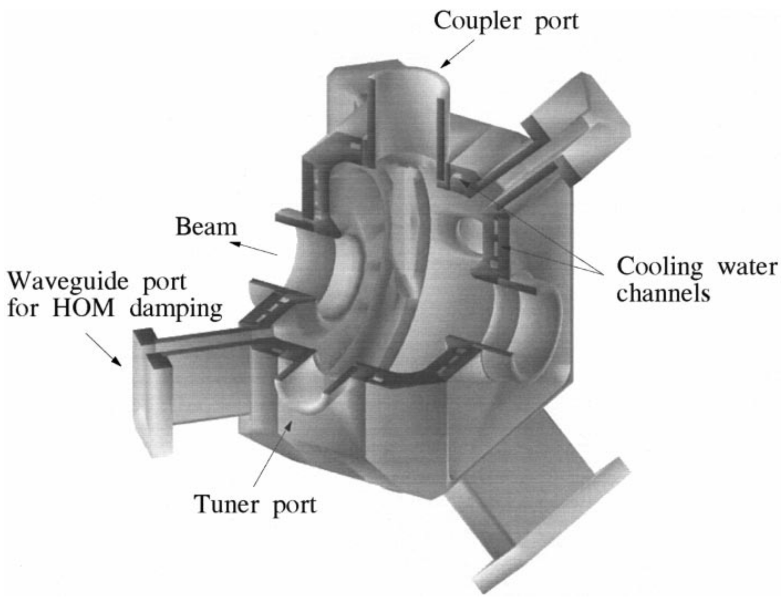
In practice, the walls of the cavity will not be perfectly conducting, and the boundary conditions will vary slightly from those we have assumed.

The electric and (oscillating) magnetic fields on the walls will induce currents, which will dissipate the energy.

However, if a field is generated in the cavity corresponding to one of the modes we have calculated, the fields on the wall will be small, and the dissipation will be slow: such modes (with integer values of m_x, m_y and m_z) will have a high "quality factor", compared to other field patterns inside the cavity.

A mode with a high quality factor is called a "resonant mode".

Example: mode spectrum in a 714 MHz cavity



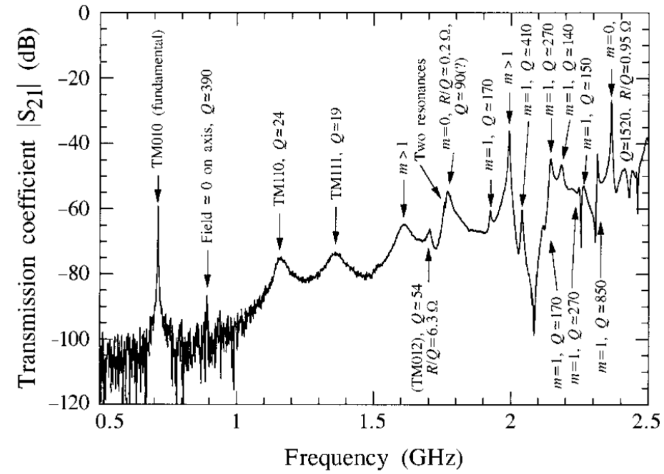


Figure 4
Higher-order-mode spectrum of the cold-model cavity. The transmission coefficient between two probes, located at both end plates, was observed.

S. Sakanaka, F. Hinode, K. Kubo and J. Urakawa,
"Construction of 714 MHz HOM-free accelerating cavities,"
J. Synchrotron Rad. (1998) 5, 386-388.

Let us calculate the energy stored in the fields in a resonant mode.

The energy density in an electric field is:

$$U_E = \frac{1}{2}\vec{D} \cdot \vec{E}. \tag{40}$$

Therefore, the total energy stored in the electric field in a resonant mode is:

$$\mathcal{E}_E = \frac{1}{2}\varepsilon_0 \int \vec{E}^2 \, dV, \tag{41}$$

where the volume integral extends over the entire volume of the cavity.

In any resonant mode, we have:

$$\int_0^{a_x} \cos^2 k_x x \, dx = \int_0^{a_x} \sin^2 k_x x \, dx = \frac{1}{2}, \tag{42}$$

where $k_x = m_x \pi / a_x$, and m_x is a non-zero integer.

We have similar results for the y and z directions, so we find (for m_x, m_y and m_z all non-zero integers):

$$\mathcal{E}_E = \frac{1}{16} \varepsilon_0 (E_{x0}^2 + E_{y0}^2 + E_{z0}^2) \cos^2 \omega t. \tag{43}$$

The energy varies as the square of the field amplitude, and oscillates sinusoidally in time.

Now let us calculate the energy in the magnetic field. The energy density is:

$$U_B = \frac{1}{2}\vec{B} \cdot \vec{H}. \tag{44}$$

Using:

$$k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2}, \quad \text{and} \quad E_{x0}k_x + E_{y0}k_y + E_{z0}k_z = 0, \tag{45}$$

we find, after some algebraic manipulation (and noting that the magnetic field is 90° out of phase with the electric field):

$$\mathcal{E}_B = \frac{1}{16}\frac{1}{\mu_0 c^2}(E_{x0}^2 + E_{y0}^2 + E_{z0}^2)\sin^2 \omega t. \tag{46}$$

As in the case of the electric field, the numerical factor is correct if the mode numbers m_x, m_y and m_z are non-zero integers.

Finally, using $1/c^2 = \mu_0\varepsilon_0$, we have (for m_x, m_y and m_z non-zero integers):

$$\mathcal{E}_E + \mathcal{E}_B = \frac{1}{16}\varepsilon_0(E_{x0}^2 + E_{y0}^2 + E_{z0}^2). \tag{47}$$

The total energy in the cavity is constant over time, although the energy "oscillates" between the electric field and the magnetic field.

The power flux in the electromagnetic field is given by the Poynting vector:

$$\vec{S} = \vec{E} \times \vec{H}. \tag{48}$$

Since the electric and magnetic fields in the cavity are 90° out of phase (if the electric field varies as $\cos \omega t$, then the magnetic field varies as $\sin \omega t$), averaging the Poynting vector over time at any point in the cavity gives zero: this is again consistent with conservation of energy.

In practice, some of the energy stored in a cavity will be dissipated in the walls. The rate of energy dissipation for a given mode is measured by the quality factor, Q:

$$P_d = -\frac{d\mathcal{E}}{dt} = \frac{\omega}{Q}\mathcal{E}. \qquad (49)$$

For a mode with a longitudinal electric field component $E_{z0} = V_0/L$ (where L is the length of the cavity), we define the *shunt impedance*, R_s:

$$R_s = \frac{V_0^2}{P_d}. \qquad (50)$$

Combining the above equations, we see that:

$$\frac{R_s}{Q} = \frac{V_0^2}{P_d} \cdot \frac{P_d}{\omega\mathcal{E}} = \frac{V_0^2}{\omega\mathcal{E}}. \qquad (51)$$

Consider a mode with $B_z = 0$. Such modes have only transverse components of the magnetic field, and are called TM modes.

Using equation (37), we see that the electric field in TM modes obeys:

$$k_y E_{x0} = k_x E_{y0}. \tag{52}$$

We also have, from (31):

$$k_x E_{x0} + k_y E_{y0} + k_z E_{z0} = 0. \tag{53}$$

These relations allow us to write the energy stored in the cavity purely in terms of the mode numbers and the peak longitudinal electric field:

$$\mathcal{E} = \frac{\varepsilon_0}{16} \left(\frac{k_x^2 + k_y^2 + k_z^2}{k_x^2 + k_y^2} \right) E_{z0}^2. \tag{54}$$

Combining equations (51) and (54), we see that:

$$\frac{R_s}{Q} = \frac{16}{\varepsilon_0} \left(\frac{k_x^2 + k_y^2}{k_x^2 + k_y^2 + k_z^2} \right) \frac{L^2}{\omega}. \tag{55}$$

For a TM mode in a rectangular cavity, the quantity R_s/Q depends only on the length of the cavity and the mode numbers.

In fact, this result generalises: for TM modes, R_s/Q depends only on the geometry of the cavity, and the mode numbers. This is of practical significance since, to optimise the design of a cavity for accelerating a beam, the goal is to maximise R_s/Q for the accelerating mode, and minimise this quantity for all other modes.

Most cavities in accelerators are closer to a cylindrical than a rectangular geometry. It is worth looking at the solutions to Maxwell's equations, subject to the usual boundary conditions, for a cylinder with perfectly conducting walls.

We can find the modes in just the same way as we did for a rectangular cavity: that is, we find solutions to the wave equations for the electric and magnetic fields using separation of variables; then find the "allowed" solutions by imposing the boundary conditions.

The algebra is more complicated this time, because we have to work in cylindrical polar coordinates. We will not go through the derivation in detail: the solutions for the fields can be checked by taking the appropriate derivatives.

In cylindrical polar coordinates (r, θ, z) the TM modes can be expressed as:

$$E_r = -E_0 \frac{k_z}{k_r} J_n'(k_r r) \cos n\theta \, \sin k_z z \, e^{-i\omega t} \tag{56}$$

$$E_\theta = E_0 \frac{n k_z}{k_r^2 r} J_n(k_r r) \sin n\theta \, \sin k_z z \, e^{-i\omega t} \tag{57}$$

$$E_z = E_0 J_n(k_r r) \cos n\theta \, \cos k_z z \, e^{-i\omega t} \tag{58}$$

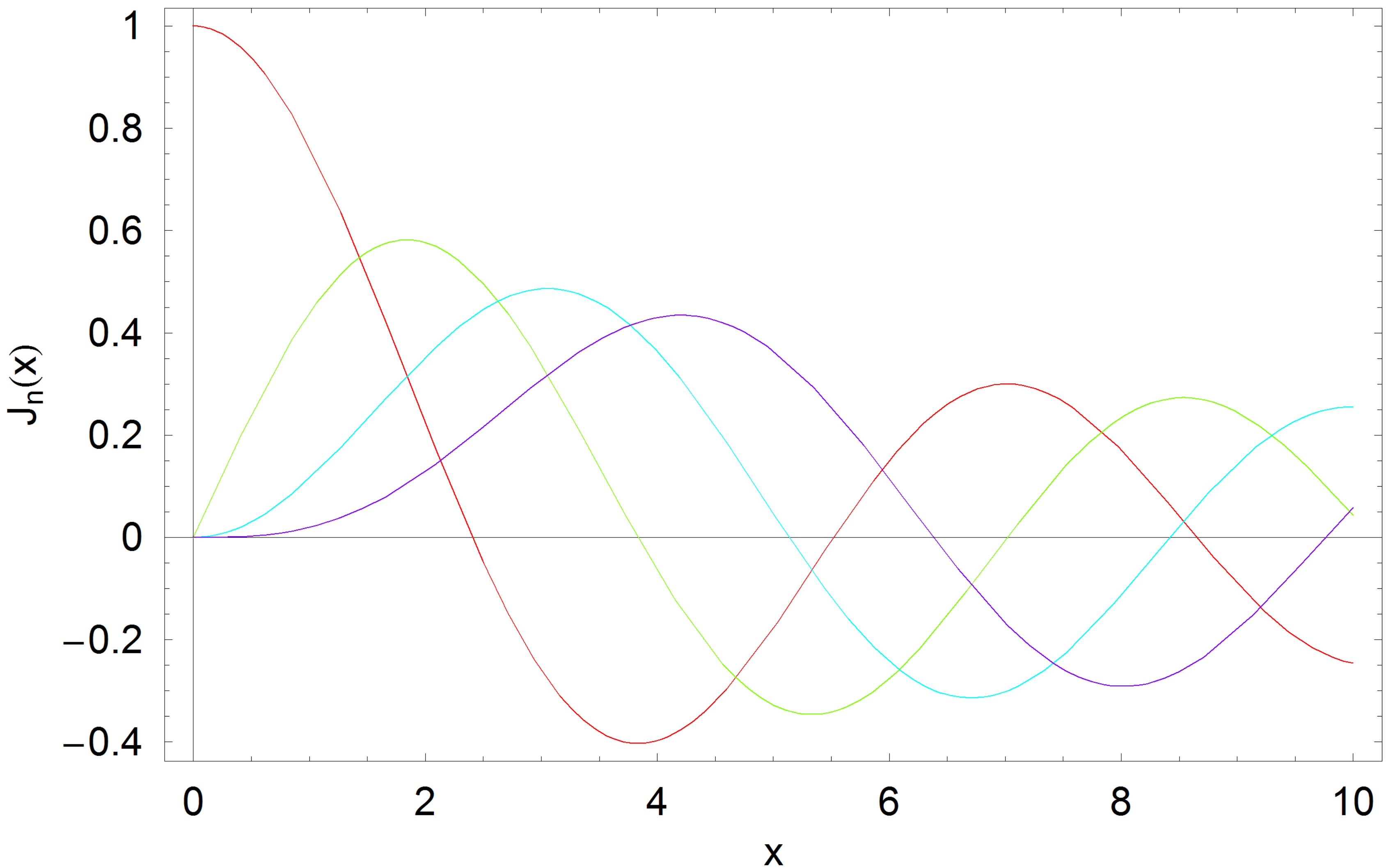
$$B_r = i E_0 \frac{n\omega}{c^2 k_r^2 r} J_n(k_r r) \sin n\theta \, \cos k_z z \, e^{-i\omega t} \tag{59}$$

$$B_\theta = i E_0 \frac{\omega}{c^2 k_r} J_n'(k_r r) \cos n\theta \, \cos k_z z \, e^{-i\omega t} \tag{60}$$

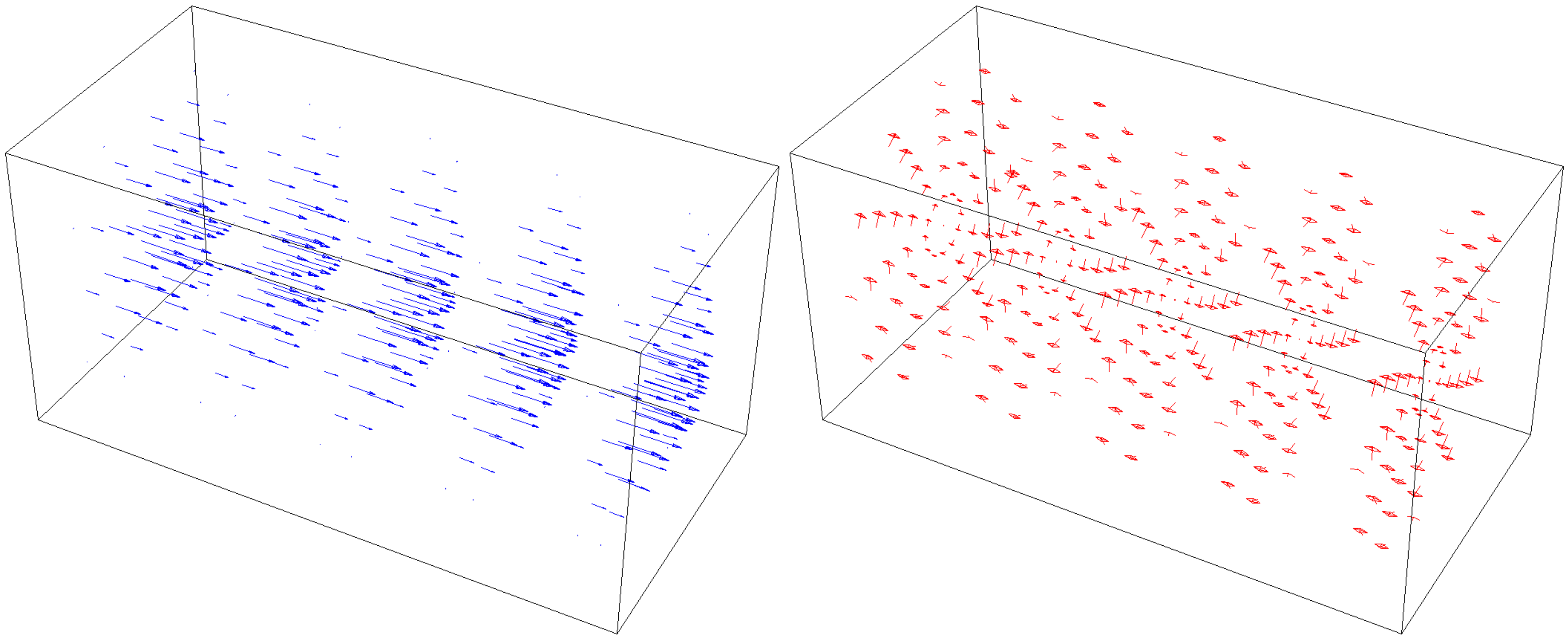
$$B_z = 0 \tag{61}$$

Because of the longitudinal electric field, TM modes are good for acceleration.

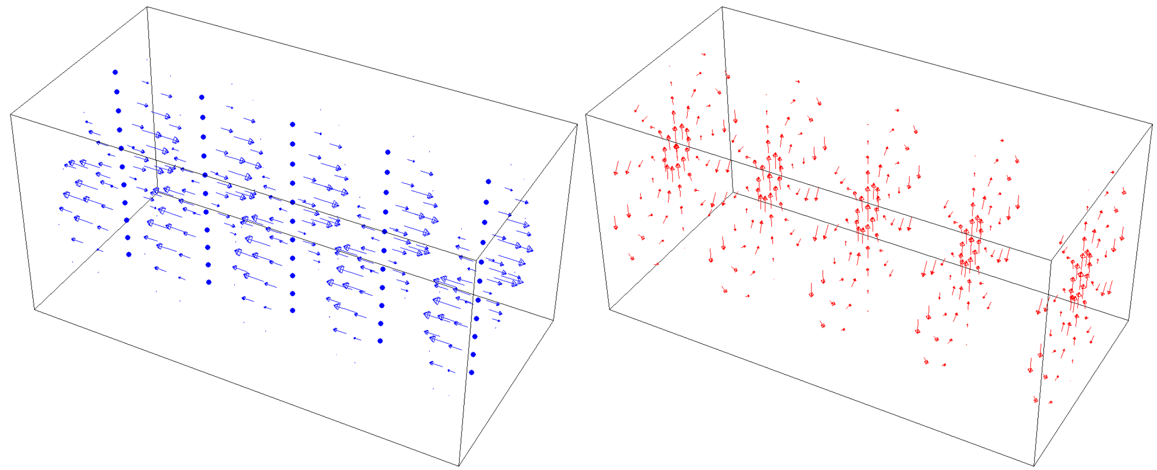
The functions $J_n(x)$ are Bessel functions:



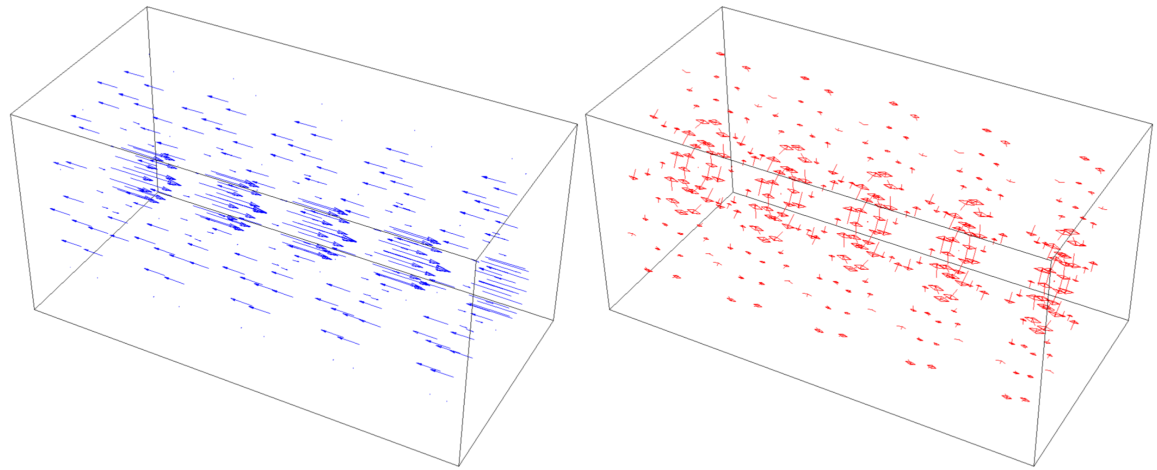
The "fundamental" accelerating mode in a cylindrical cavity is the TM_{010} (mode numbers refer to the azimuthal, radial, and longitudinal coordinates, respectively):



For more information on modes in cylindrical cavities, including TE modes, see Appendix B.

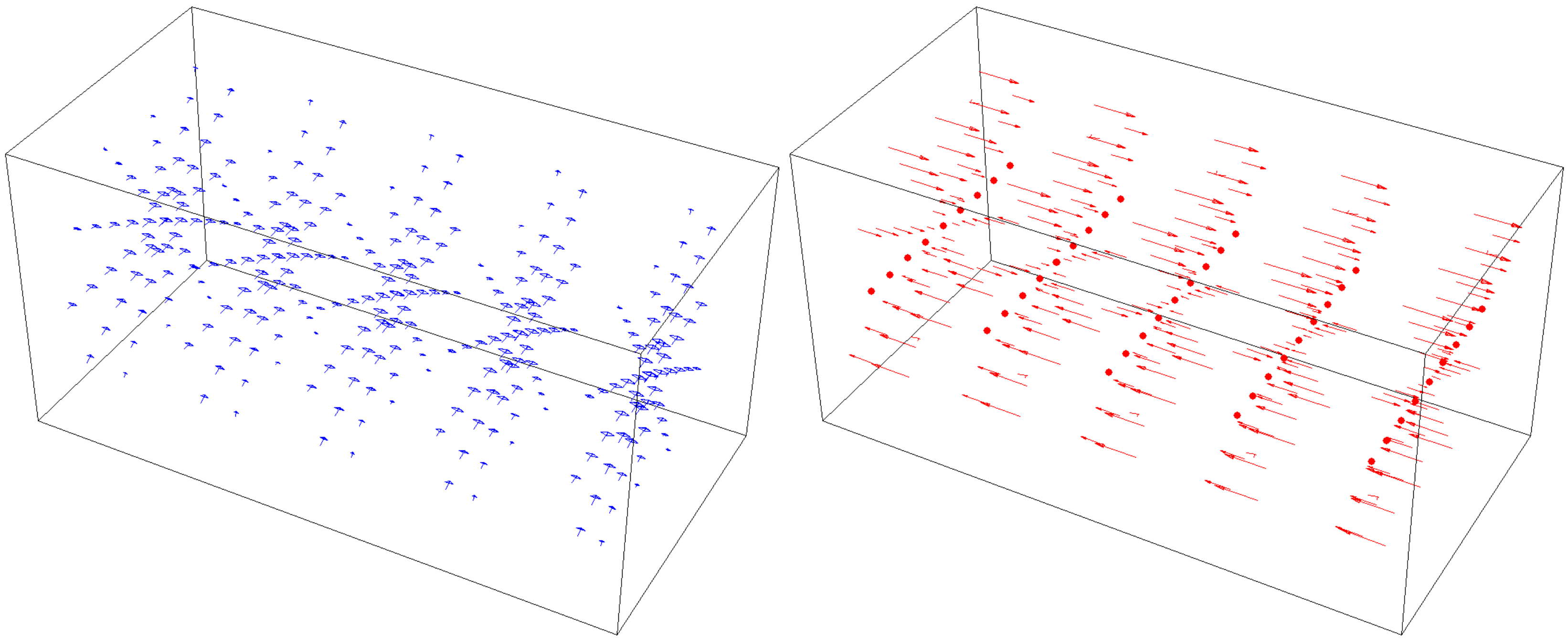


TM110



TM020

TE110 mode in a cylindrical ("crab") cavity



Summary of Part II: Standing waves

We have shown that:

- Maxwell's equations lead to relationships on the electric and magnetic fields on either side of a boundary between two materials.
- Applied to the surface of a good conductor, the boundary conditions imply that the normal component of the magnetic field and the tangential component of the electric field both vanish.
- The boundary conditions allow us to find expressions for the reflection and transmission coefficients for waves at a boundary.
- Good conductors have a very low wave impedance: this means that nearly all the energy in a wave striking the surface of a good conductor is reflected.
- Applied to electromagnetic fields in cavities, the boundary conditions impose constraints on the "patterns" and oscillation frequencies of electric and magnetic fields that can exist as waves within the cavity.

A Mathematica 5.2 Notebook for generating plots of field modes in cavities and waveguides can be downloaded from:

- pcwww.liv.ac.uk/~awolski/CAS2010/CavityModes.nb

Animations showing the field modes in particular cases can be downloaded from:

- pcwww.liv.ac.uk/~awolski/CAS2010/RectangularCavityModes.zip
- pcwww.liv.ac.uk/~awolski/CAS2010/CylindricalCavityModes.zip
- pcwww.liv.ac.uk/~awolski/CAS2010/RectangularWaveguideModes.zip

Consider the x component E_x, and look for solutions of the form:

$$E_x = X(x)Y(y)Z(z)e^{-i\omega t} \tag{62}$$

Substitute into the wave equation (26):

$$YZ\frac{\partial^2 X}{\partial x^2} + XZ\frac{\partial^2 Y}{\partial y^2} + XY\frac{\partial^2 Z}{\partial z^2} + \frac{\omega^2}{c^2}XYZ = 0 \tag{63}$$

and divide by XYZ:

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = -\frac{\omega^2}{v^2} \tag{64}$$

This must be true for all x, y and z.

Each term on the l.h.s. is independent of the other terms, and must therefore be constant. Therefore, we write:

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -k_x^2, \quad \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -k_y^2, \quad \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = -k_z^2. \tag{65}$$

Consider the equation for $Z(z)$:

$$\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = -k_z^2. \tag{66}$$

The general solution is:

$$Z(z) = Z_C \cos k_z z + Z_S \sin k_z z. \tag{67}$$

However, to satisfy the boundary condition $E_x = 0$ at $z = 0$, we must have:

$$Z_C = 0. \tag{68}$$

Also, to satisfy the boundary conditon $E_x = 0$ at $z = a_z$, we must have:

$$k_z = \frac{m_z \pi}{a_z}, \tag{69}$$

where m_z is an integer.

Solving the wave equation for $E_x = X(x)Y(y)Z(z)e^{-i\omega t}$ with the boundary conditions $E_x = 0$ at $z = 0$ and at $z = a_z$ gives:

$$Z(z) = Z_S \sin k_z z, \qquad \text{where } k_z = \frac{m_z \pi}{a_z}. \tag{70}$$

Similarly, we find:

$$Y(y) = Y_S \sin k_y y, \qquad \text{where } k_y = \frac{m_y \pi}{a_y}. \tag{71}$$

where m_y is an integer.

Hence, we can write:

$$E_x = (X_C \cos k_x x + X_S \sin k_x x) \sin k_y y \sin k_z z \, e^{-i\omega t}, \tag{72}$$

and following the same procedure, we find:

$$E_y = \sin k'_x x \, (Y_C \cos k'_y y + Y_S \sin k'_y y) \sin k'_z z \, e^{-i\omega t}, \tag{73}$$

$$E_z = \sin k''_x x \sin k''_y y \, (Z_C \cos k''_z z + Z_S \sin k''_z z) \, e^{-i\omega t}. \tag{74}$$

Now, as well as satisfying the wave equation, the electric field must satisfy Maxwell's equations, thus we require that $\nabla \cdot \vec{E} = 0$.

Applying this condition to the above expressions for the field components, we find that we must have:

$$k_x = k'_x = k''_x, \tag{75}$$

and similarly for the y and z directions.

Also, we must have:

$$X_S = Y_S = Z_S = 0. \tag{76}$$

One set of modes (not the most general solution) we can write down is as follows:

$$E_r = -iB_0 \frac{n\omega}{k_r^2 r} J_n(k_r r) \sin n\theta \sin k_z z \, e^{-i\omega t} \tag{77}$$

$$E_\theta = -iB_0 \frac{\omega}{k_r} J_n'(k_r r) \cos n\theta \sin k_z z \, e^{-i\omega t} \tag{78}$$

$$E_z = 0 \tag{79}$$

$$B_r = B_0 \frac{k_z}{k_r} J_n'(k_r r) \cos n\theta \cos k_z z \, e^{-i\omega t} \tag{80}$$

$$B_\theta = -B_0 \frac{n k_z}{k_r^2 r} J_n(k_r r) \sin n\theta \cos k_z z \, e^{-i\omega t} \tag{81}$$

$$B_z = B_0 J_n(k_r r) \cos n\theta \sin k_z z \, e^{-i\omega t} \tag{82}$$

Note that $J_n(x)$ is a Bessel function of order n, and $J_n'(x)$ is the derivative of $J_n(x)$.

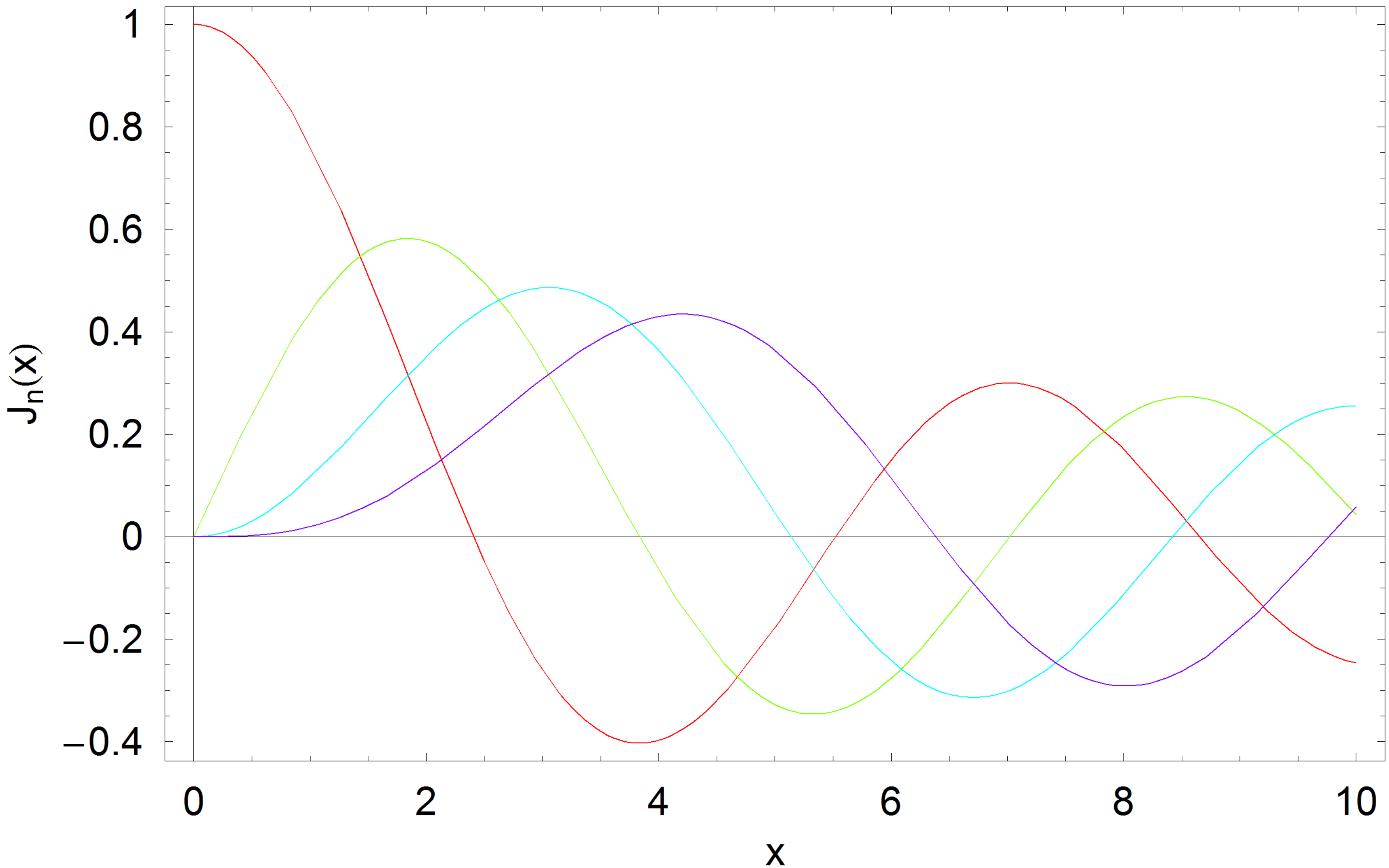
The Bessel functions are solutions of the differential equation:

$$y'' + \frac{y'}{x} + \left(1 - \frac{n^2}{x^2}\right) y = 0. \tag{83}$$

This equation appears when we separate variables in finding a solution to the wave equation.

Because of the dependence of the fields on the azimuthal angle θ, we require that n is an integer: n provides an azimuthal index in specifying a mode.

Bessel functions:



From the boundary conditions, E_θ and B_r must both vanish on the curved wall of the cavity, i.e. when $r = a$, where a is the radius of the cylinder.

Therefore, we have a constraint on k_r:

$$J_n'(k_r a) = 0, \tag{84}$$

or:

$$k_r = \frac{p'_{n,m}}{a}, \tag{85}$$

where p'_{nm} is the mth zero of the derivative of the nth order Bessel function.

This equation is analogous to the conditions we had for the rectangular cavity, e.g. $k_x = m_x \pi / a_x$.

We can use the integer m as a radial index in specifying a mode.

We also need to have $B_z = E_r = E_\theta = 0$ on the flat ends of the cavity.

Assuming the flat ends of the cavity are at $z = 0$ and $z = L$, these boundary conditions are satisfied if:

$$\sin k_z L = 0, \qquad \text{therefore} \quad k_z = \frac{\ell\pi}{L}, \tag{86}$$

where ℓ is an integer.

ℓ provides a longitudinal index in specifying a mode.

Also, we find that:

$$\nabla^2 \vec{E} = -(k_r^2 + k_z^2)\vec{E}, \tag{87}$$

so from the wave equation:

$$\nabla^2 \vec{E} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = 0, \tag{88}$$

we must have:

$$k_r^2 + k_z^2 = \frac{\omega^2}{c^2}. \tag{89}$$

Similar equations hold for the magnetic field, $\vec{B}$.

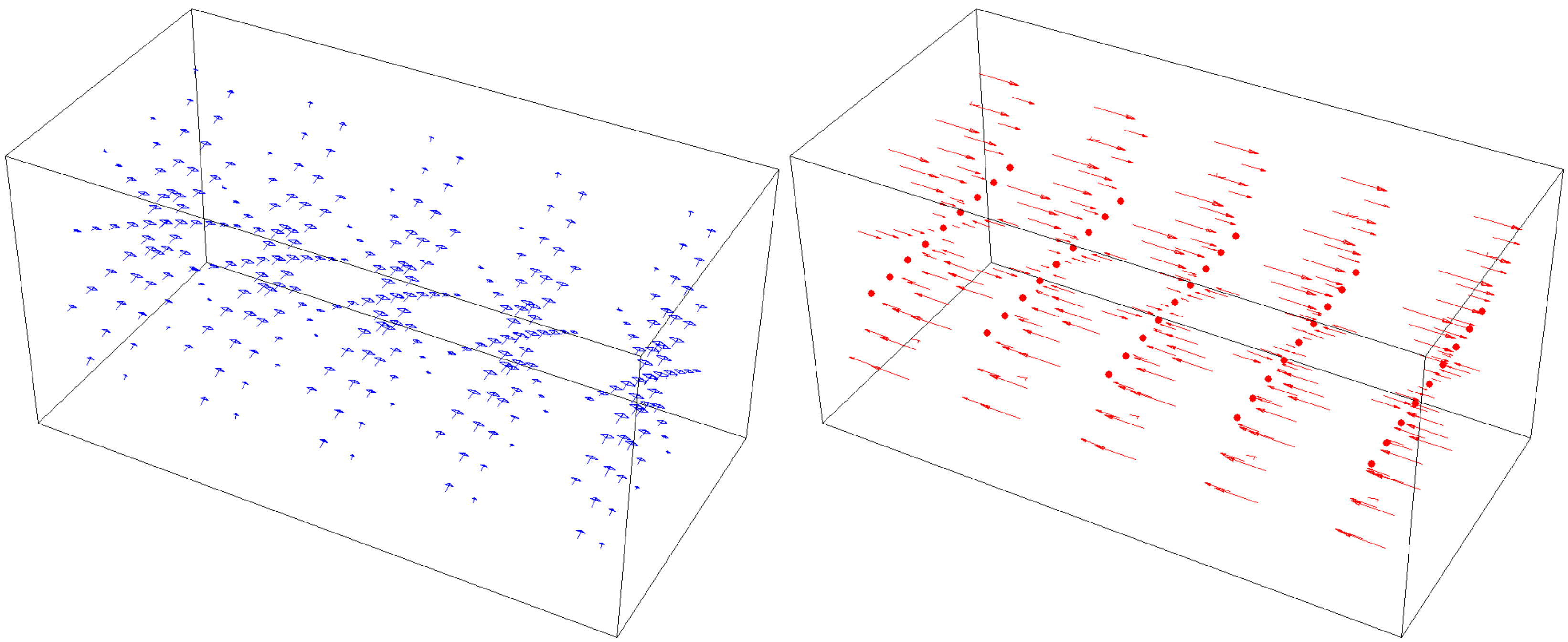
In the modes that we are considering, the longitudinal component of the electric field $E_z = 0$, i.e. the electric field is purely transverse.

These modes are known as "TE" modes. A specific mode of this type, with indices n (azimuthal), m (radial), and ℓ (longitudinal) is commonly written as $\mathrm{TE}_{nm\ell}$.

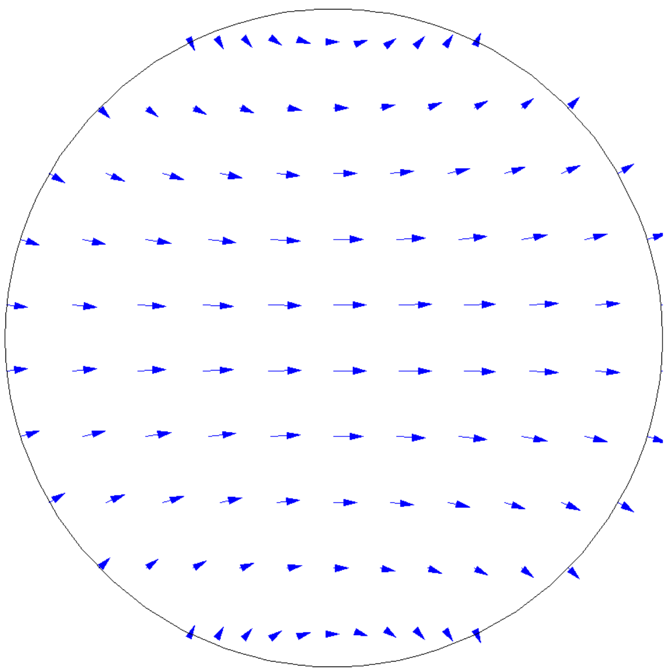
The frequency of mode $\mathrm{TE}_{mn\ell}$ depends on the dimensions of the cavity, and is given by:

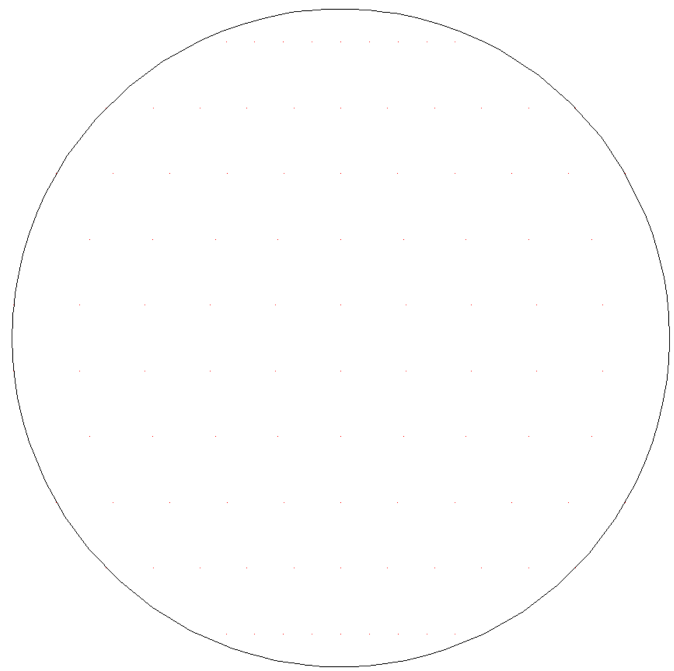
$$\omega_{mn\ell} = c\sqrt{k_r^2 + k_z^2} = c\sqrt{\left(\frac{p'_{nm}}{a}\right)^2 + \left(\frac{\ell\pi}{L}\right)^2}. \qquad (90)$$

Cylindrical cavity mode TE_{110}:



Cylindrical cavity mode TE_{110}:





TE modes are useful for giving a transverse deflection to a beam in an accelerator, but are not much use for providing acceleration.

Fortunately, cylindrical cavities allow another set of modes that have non-zero longitudinal electric field:

$$E_r = -E_0 \frac{k_z}{k_r} J_n'(k_r r) \cos n\theta \sin k_z z \, e^{-i\omega t} \tag{91}$$

$$E_\theta = E_0 \frac{n k_z}{k_r^2 r} J_n(k_r r) \sin n\theta \sin k_z z \, e^{-i\omega t} \tag{92}$$

$$E_z = E_0 J_n(k_r r) \cos n\theta \cos k_z z \, e^{-i\omega t} \tag{93}$$

$$B_r = iE_0 \frac{n\omega}{c^2 k_r^2 r} J_n(k_r r) \sin n\theta \cos k_z z \, e^{-i\omega t} \tag{94}$$

$$B_\theta = iE_0 \frac{\omega}{c^2 k_r} J_n'(k_r r) \cos n\theta \cos k_z z \, e^{-i\omega t} \tag{95}$$

$$B_z = 0 \tag{96}$$

In these modes, the magnetic field is purely transverse (zero longitudinal component); therefore, they are referred to as "TM" modes.

As before, for physical fields, n must be an integer.

The boundary conditions on the fields give:

$$k_r = \frac{p_{nm}}{a}, \qquad \text{and} \qquad k_z = \frac{\ell\pi}{L}, \tag{97}$$

where p_{nm} is the mth zero of the nth order Bessel function $J_n(x)$.

The frequency of a mode $\mathrm{TM}_{mn\ell}$ is given by:

$$\omega_{mn\ell} = c\sqrt{k_r^2 + k_z^2} = c\sqrt{\left(\frac{p_{nm}}{a}\right)^2 + \left(\frac{\ell\pi}{L}\right)^2}. \tag{98}$$

The lowest frequency accelerating mode in a cylindrical cavity is the TM_{010} mode ($n = 0$, $m = 1$, $\ell = 0$).

The fields in the TM_{010} mode are given by:

$$E_r = 0 \tag{99}$$

$$E_\theta = 0 \tag{100}$$

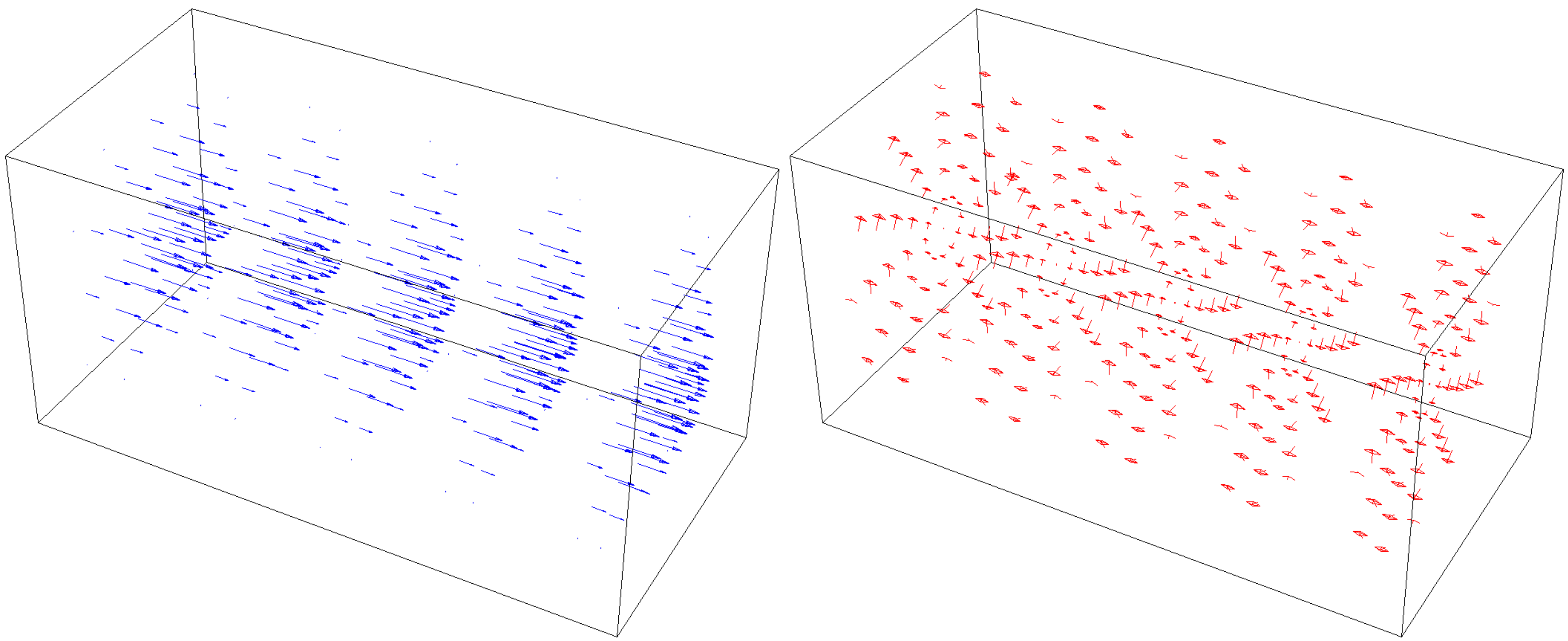
$$E_z = E_0 J_0\left(p_{01}\frac{r}{a}\right) e^{-i\omega t} \tag{101}$$

$$B_r = 0 \tag{102}$$

$$B_\theta = -i\frac{E_0}{c} J_1\left(p_{01}\frac{r}{a}\right) e^{-i\omega t} \tag{103}$$

$$B_z = 0 \tag{104}$$

Cylindrical cavity mode TM_{010}:



The frequency of the TM_{010} mode is determined by the radius of the cavity, not by its length:

$$\omega_{010} = c\sqrt{k_r^2 + k_z^2} = p_{01}\frac{c}{a}. \qquad (105)$$

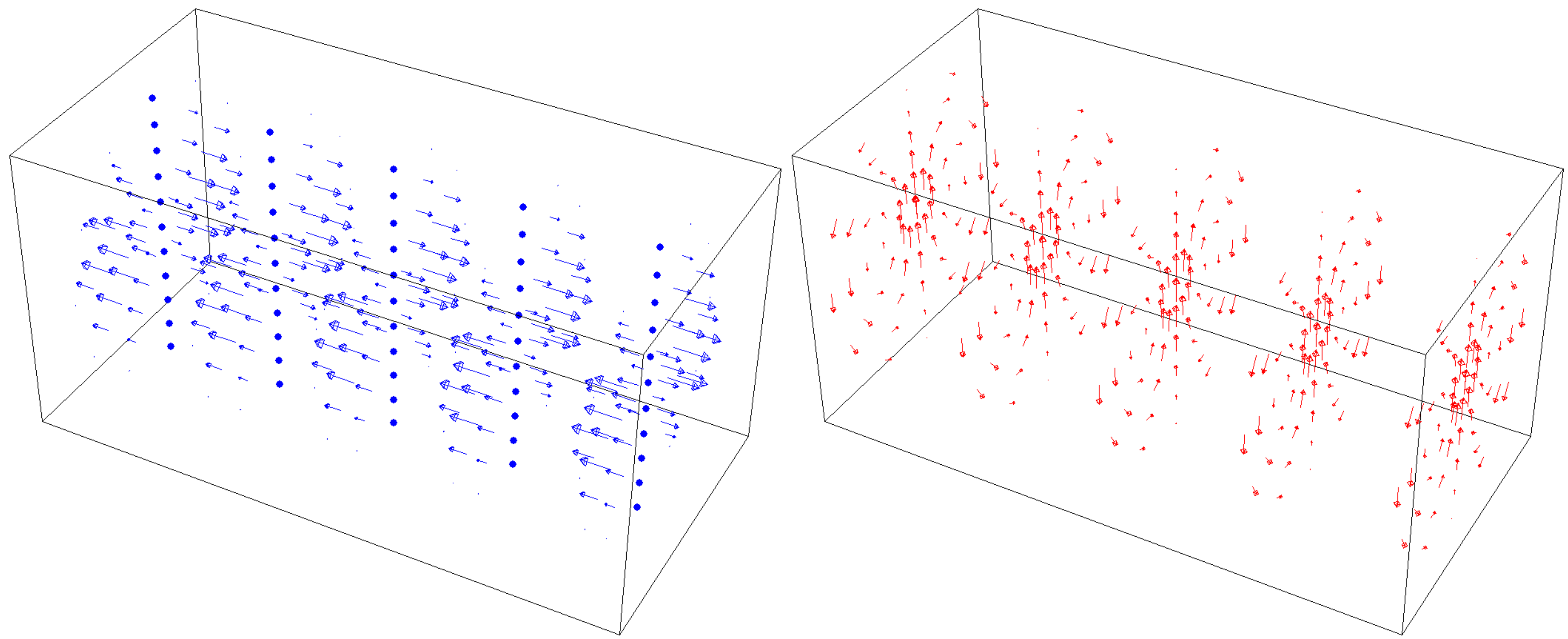
Note that:

$$p_{01} \approx 2.40483. \qquad (106)$$

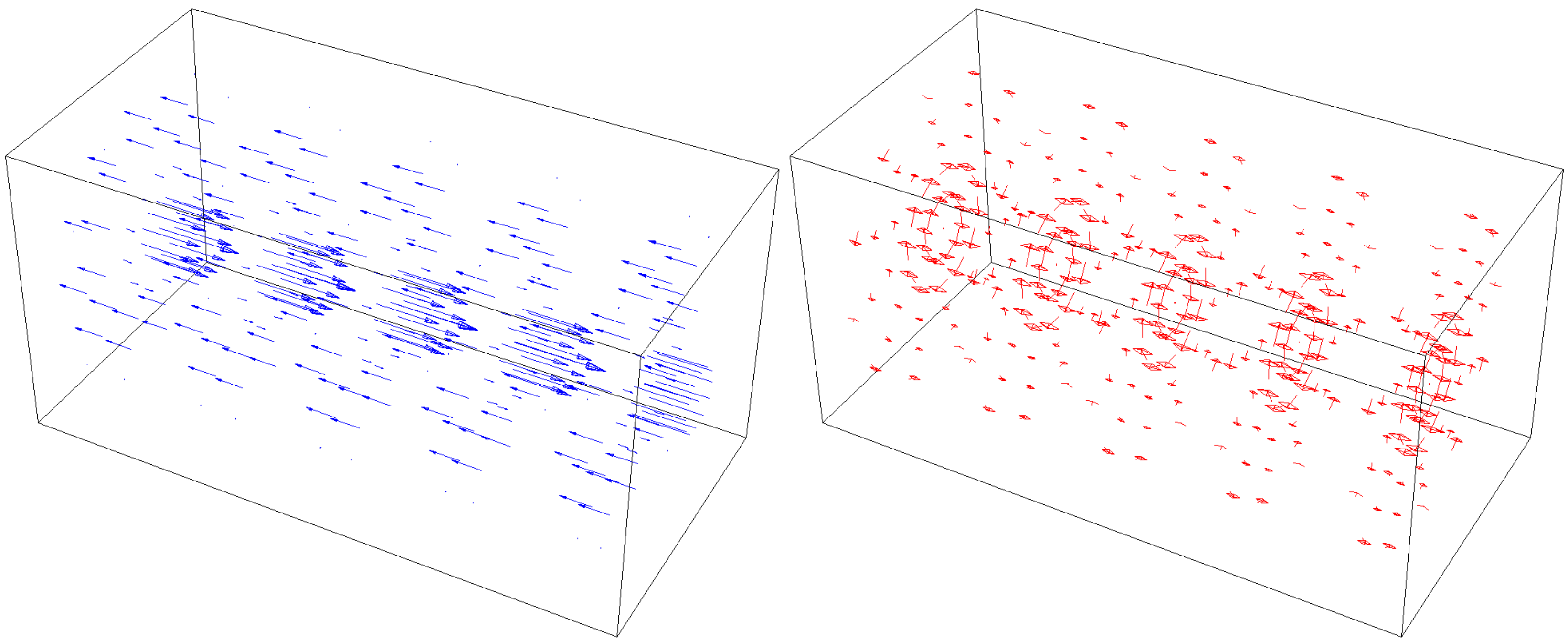
However, to get the maximum acceleration from the cavity, the time taken for a particle to pass through the cavity should be one half of the rf period, i.e. π/ω.

Therefore, for best efficiency, the length of the cavity should be $\pi c/\omega = \lambda/2$, where λ is the wavelength of an electromagnetic wave with angular frequency ω in free space.

Cylindrical cavity mode TM_{110}:



Cylindrical cavity mode TM_{020}:



1. By considering the boundary conditions on the electric and magnetic fields at a boundary between two different materials, show that, for an electromagnetic plane wave striking the boundary at normal incidence, the amplitudes of the transmitted and reflected waves are given (relative to the amplitude of the incident wave) by:

$$\frac{E_{0R}}{E_{0I}} = \frac{Z_2 - Z_1}{Z_2 + Z_1}, \quad \text{and} \quad \frac{E_{0T}}{E_{0I}} = \frac{2Z_2}{Z_2 + Z_1},$$

   where Z_1 is the impedance of the material on the incident side, and Z_2 is the impedance of the material on the transmitted side.

   Show that the energy in the wave is conserved at the boundary.

2. Calculate the radius of a cylindrical cavity for which the frequency of the fundamental accelerating mode TM_{010} is 500 MHz. What is the optimal length of the cavity for accelerating an ultra-relativistic beam? If the peak electric field in the cavity is 5 MV/m, what is the energy stored in the cavity?

   You are given that: $\int_0^1 r J_0(p_{01}r)^2 dr = \int_0^1 r J_1(p_{01}r)^2 dr \approx 0.1348$.