Measurements, Statistics and Errors

CERN Accelerator School 2008

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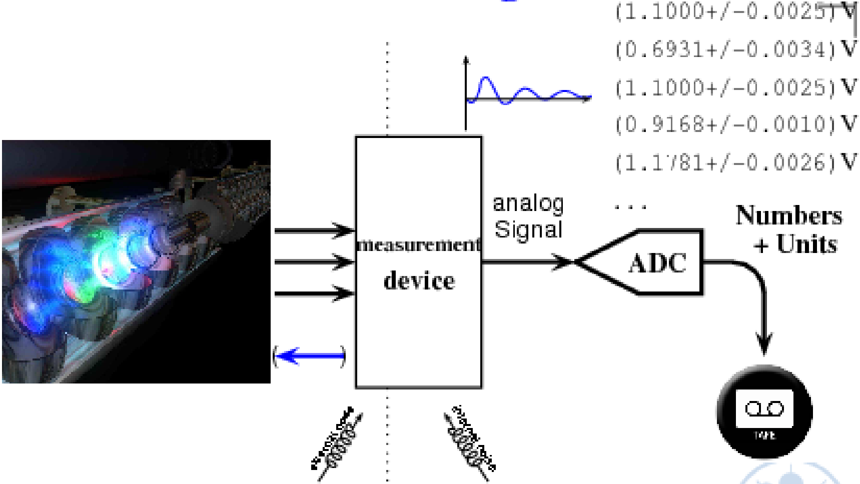
Contents

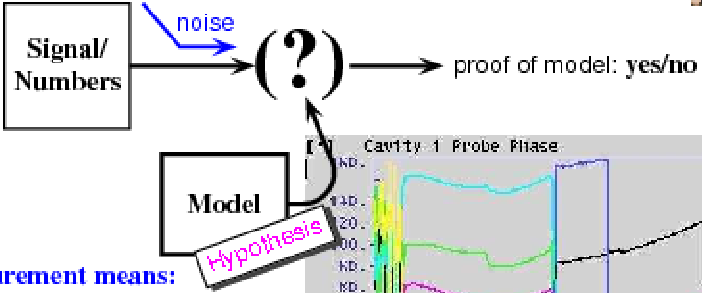
1. Measurement

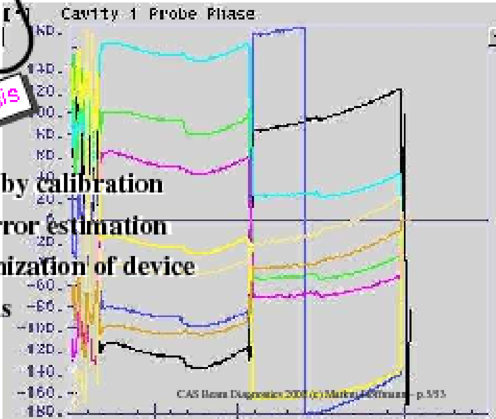
The Standard Signal Path



- The documentation about the measurement device ⟶ significance numbers + units.

Gnosis





Measurement means:

- Abstraction from raw data by calibration
- Significance of values by Error estimation
- Reduction of noise by optimization of device
- Model + simulate the signals
- proof or falsify model
- adapt model

Signals

Signal =
Voltage as a function of time $U(t)$

discrete	continous
analog	digital
causal	non-causal

Definition:

- Time: $t \in \mathbb{R}$ (sometimes $\in \mathbb{R}_0^+$)
- Amplitude: $s(t) \in \mathbb{R}$
- Power: $s^2(t) \in \mathbb{R}_0^+$ (constants are renormed to 1, e.g. $P = \frac{U^2}{R}$ or $P = I^2 R$.)

Energy-Signal:

$$\int_{-\infty}^{\infty} s^2(t)dt < \infty$$

(Problems with sin(), cos(), rect() and stochastic/noise signals)

Power-Signal:

$$\lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} s^2(t)dt < \infty$$

Errors

1. systematic (deterministic) error
2. unsystematic (statistic) error (noise)

systematic error

- due to characteristics of the measurement device. (ADC/DAC: *offset, gain, linearity-errors*)
- Improved by improvements of apparatus.
- Limits: Quantum mechanics

accuracy

statistic error

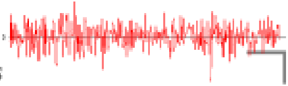
- unforseen fluctuations, random, stochastic, noise
- it is possible to estimate the extend
- can be reduced by statistical methods (averaging), multiple measurements

precision

Definition is context depentant: accuracy of 100 devices can be a matter of precision !

2. Noise

Noise



acoustic $\Longrightarrow$ electronics

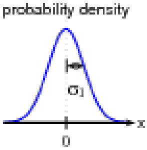
Noise sources:

- Brownian movement of charges (*thermal noise*)
- Variations of the number of charges involved in conduction (*Shottky noise/shot noise, flicker noise*)
- quantum effects (*zero point fluctuations*)

Noise classes:

- **white** noise $\longrightarrow$ **flat** spectrum
- **pink** noise $\longrightarrow$ low-passed spectrum
- **blue** noise $\longrightarrow$ high-passed spectrum

Amplitude density distribution:



$\longrightarrow$ **Gauß**

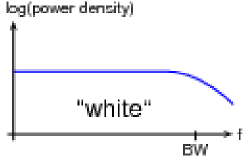
if many indep. sources contribute

$\longrightarrow$ stochastic signal

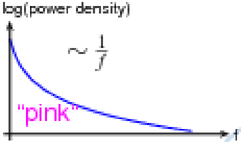
Sensor Noise

Sources:

Spectrum:



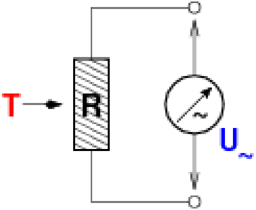
- Thermal noise (Johnson-/Nyquist-noise)
- Shot noise (Shottky-Noise)
  (quants of the elementary charge)



- $1/f^{\alpha}$ noise
- (e.g. flicker noise)

Thermal Noise

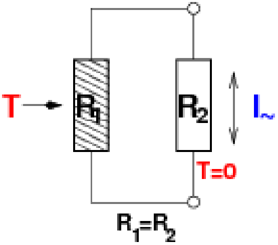
Brownian motion of charge carriers



effective value — measurement bandwidth

Noise Voltage: $U_{rms} = \sqrt{4kT \cdot R \cdot B}$

Note: Thermal noise is only emitted by structures with electromagnetic losses (resistors, R), which also absorb power!



$$I_{rms} = \sqrt{4kT\frac{B}{R}}$$

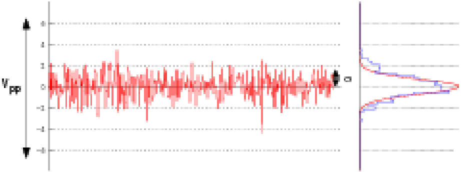
Reileigh-Jeans approximation, high T, small ν

independent from R!

Noise Power: $P_n = \frac{1}{4} U_{rms} I_{rms} = kTB$ $\sim U^2$ $\sim I^2$

received from R_1 (power matching)

Amplitude Characteristics

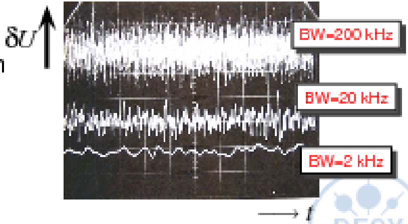


$$U_\sigma = U_{\rm rms} \sim \sqrt{\rm BW}$$

white noise gaussian distribution

$$\bar{U}_{\rm mean} = 0$$

$$U_{\rm peak} = \infty$$



The Physics behind

- **Power Spectral Density** $\rho(\nu)$:

$$\rho = kT \quad ; \quad P = \int_{BW} \rho \, d\nu$$

$$[\frac{W}{Hz} = J] \stackrel{T=290K}{=} 4 \cdot 10^{-21} \frac{W}{Hz}$$

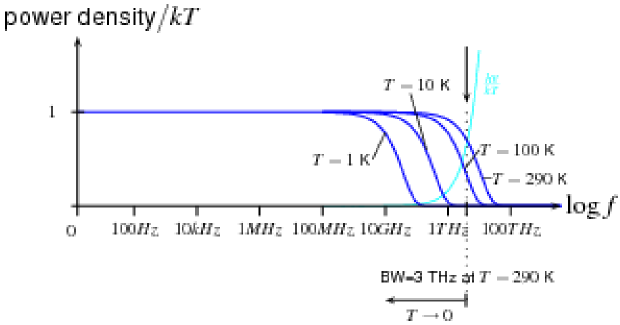
Problem: total power = ∞!

- derived from **Planck's law**:

$$\Rightarrow \boxed{\rho = kT \frac{\frac{h\nu}{kT}}{e^{\frac{h\nu}{kT}} - 1}} \quad \approx kT$$

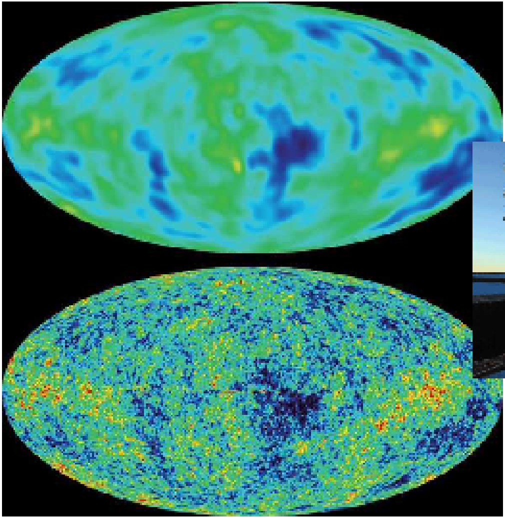
(**+ zero point vacuum fluctuations** $\frac{1}{2}h\nu$)

Thermal Noise Spectrum

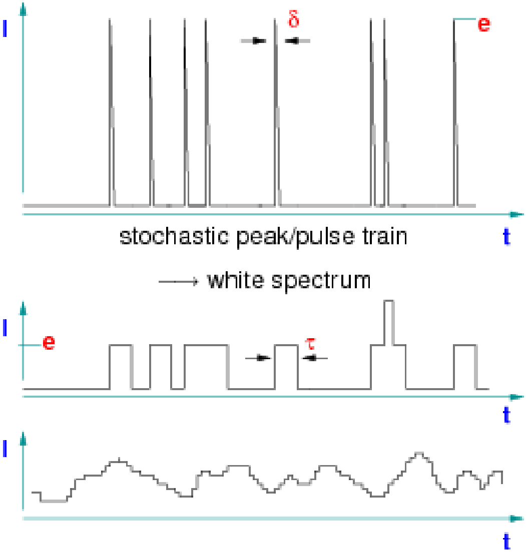


- For frequencies >10 GHz you can gain more than proportionally by **cooling**!
- The **zero-point noise** *can not deliver power* to a load. But it can be amplified to a noticable magnitude.

Cosmic Noise



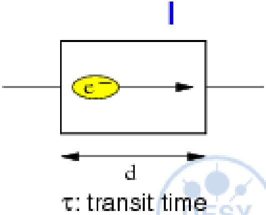
Shot(tky) Noise



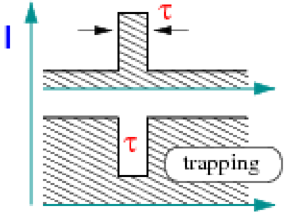
no current,
no shot noise!

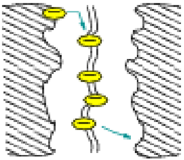
$$P_s \sim \langle I \rangle^2 = 2eI\Delta f$$

$$\delta I \sim \sqrt{2eI\Delta f}$$

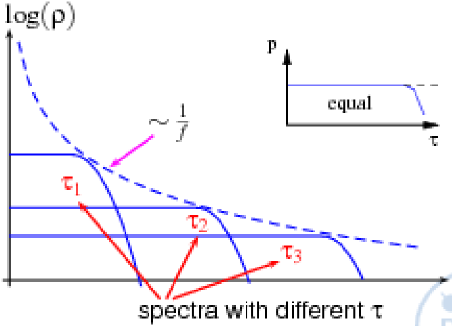


Flicker Noise



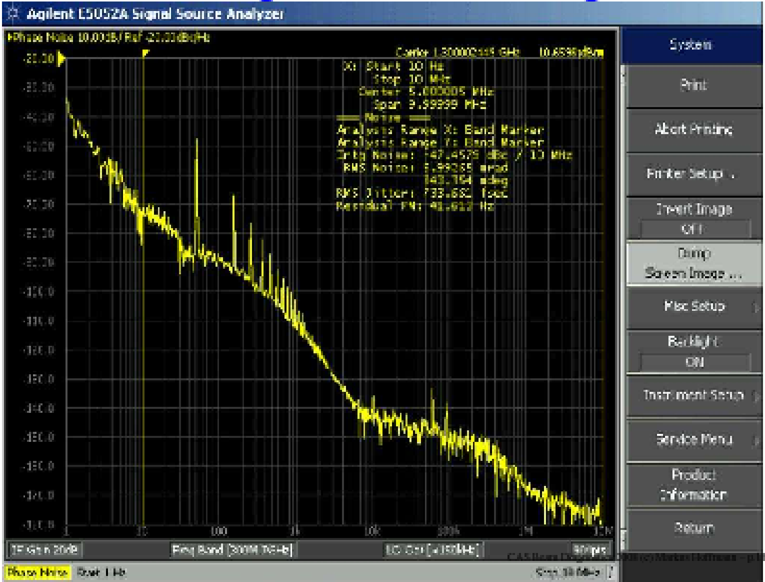


- Markow Process
- ⟶ Lorentz spectrum



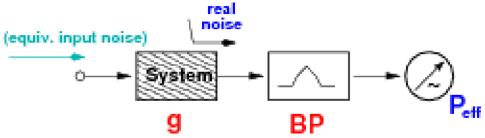
Noise Spectrum – Example

Phase noise spectrum of the FLASH Master Oscillator



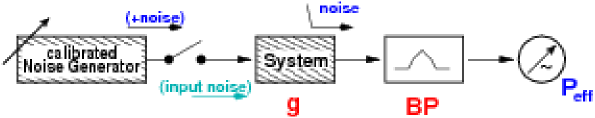
System Noise Measurement

Input Noise:



$$\frac{P_{\text{eff}}}{g_{\text{ain}}} =: P_{\text{inputnoise}} \qquad \longrightarrow \text{projected to the input}$$

- $\underbrace{\text{SNR} = 1}_{\text{At the output}}$ if input signal level = inputnoise level.



1. open input, measure $P_{\text{eff},1}$
2. close input, inject additional noise (P_r) so that $P_{\text{eff},2} = 2P_{\text{eff},1}$
3. read P_r from generator, $\rightarrow P_r$ = input noise.

Signal to Noise Ratio

Power ratio, inside bandwidth!

$$\text{SNR} := \frac{\bar{P}_{\text{signal}}}{\bar{P}_{\text{noise}}} = \left(\frac{\hat{A}_{\text{signal,rms}}}{\hat{A}_{\text{noise,rms}}}\right)^2$$

Measurement:

$$\frac{S+N}{N} = \frac{S}{N} + 1$$

$$\bar{P} := \int_{\text{BW}} \rho(\nu)\, d\nu$$

$$\begin{aligned} \text{SNR(dB)} \quad &:= \quad 10 \log_{10}\left(\frac{\bar{P}_{\text{signal}}}{\bar{P}_{\text{noise}}}\right) = 20 \log_{10}\left(\frac{\hat{A}_{\text{signal,rms}}}{\hat{A}_{\text{noise,rms}}}\right) \\ &= \quad P_{\text{signal}}[\text{dBm}] - P_{\text{noise}}[\text{dBm}] \end{aligned}$$

Units: [SNR(dB)]=dBc (="dB below carrier")

Noise Figures

noise factor:

$$F = \frac{P_r}{kT_0B} + 1 \quad \left(= \frac{SNR_{in}}{SNR_{out}} \right)$$

290 K

- Relation of input noise + noise of the source to noise of the source only
- number of additional kTB units of noise necessary on the input to double the output noise

noise figure:

$$F' = 10 \log F \quad [\mathrm{dB}]$$

- is a measure of the system itself
- The figure is independent from T and B!

Examples

1. the system adds no additional noise: $F = 1, F' = 0$ dB (The input noise equals thermal noise on the input. If the real temperature is lower than 290 K, F can be smaller.)
2. a very noisy system reduces the SNR from 100 (20dB) to 10 (10dB): $F = 10, F' = 10$ dB

Data sheet of a signal analyzer (AGILENT N8973A)

A Data Sheet

For NFA models that either have a serial prefix less than US4446 or are fitted with the Noise Figure RF board, N8972-60061.

Instrument's own noise figure

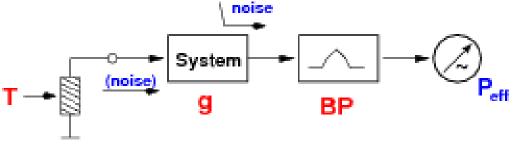
Frequency	Noise figure	Noise figure over a limited temperature range of 23° C ± 3° C
10 MHz to < 500 MHz	< 4.0 dB + (0.0025 * freq in MHz)	< 4.4 dB + (0.0025 * freq in MHz)
500 MHz to < 2.3 GHz	< 7.4 dB + (0.00135 * freq in MHz)	< 6.3 dB + (0.00135 * freq in MHz)
2.3 GHz to 3.0 GHz	< 4.9 dB + (0.0015 * freq in MHz)	< 2.3 dB + (0.0015 * freq in MHz)
> 3.0 GHz to 13.2 GHz	< 12.0 dB	< 10.5 dB
> 13.2 GHz to 26.5 GHz	< 15.0 dB	< 12.5 dB

For NFA models that either have a serial prefix greater than US4446 or are fitted with the Noise Figure RF board, N8972-60061.

Instrument's own noise figure

Frequency	Noise figure	Noise figure over a limited temperature range of 23° C ± 3° C
10 MHz to 3.0 GHz	< 4.5 dB + (0.00174 * freq in MHz)	< 4.4 dB + (0.00117 * freq in MHz)
> 3.0 GHz to 13.2 GHz	< 12.0 dB	< 10.5 dB
> 13.2 GHz to 26.5 GHz	< 16.0 dB	< 12.5 dB

Noise Temperature



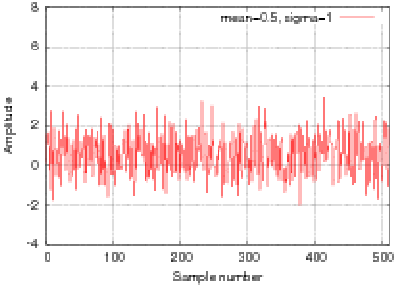
Put a resistor T_{noise} so that P_{eff} doubles:

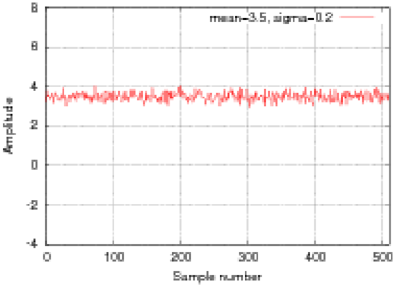
$$T_{\text{noise}} := \underbrace{(F-1)\cdot 290\,\text{K}}_{\text{for white noise}} \quad [\text{K}]$$

- for an ideal noise-free system $\rightarrow T_{\text{noise}} = 0$ K.
- is not necessarily identical with the real temperature the system is on.
- A low-noise amplifier can have $T_{\text{noise}} = 20$ K $\rightarrow F = 1.06, F' = 0.28$ dB. (This is really good!)

3. Statistics

Mean and Standard Deviation





$$\hat{x} := \frac{1}{N}\sum_{i=0}^{N-1} x_i$$

mean over N samples.

$$\sigma^2 := \frac{1}{N-1}\sum_{i=0}^{N-1}(x_i - \hat{x})^2$$

variance = (standard deviation)^2

"power of fluctuations"

RMS and the Running Statistics

$$\sigma_N^2 = \frac{1}{N-1}\left[\underbrace{\sum_{i=0}^{N-1} x_i^2}_{\text{sum of squares}} - \frac{1}{N}\underbrace{\left(\sum_{i=0}^{N-1} x_i\right)^2}_{\text{sum}^2}\right]$$

Signal to Noise Ratio (SNR): $\mathrm{SNR} = \frac{\hat{x}^2}{\sigma^2}$

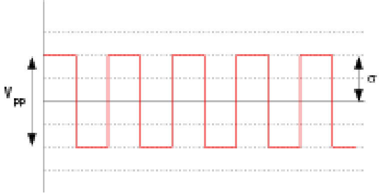
Coefficient of Variation (CV): $\mathrm{CV} = \frac{\sigma}{\hat{x}} \cdot 100\%$

Root Mean Square (RMS): $x_{\mathrm{rms}} := \sqrt{\frac{1}{N}\sum_{i=0}^{N-1} x_i^2}$

"Power of fluctuations plus power of DC component"

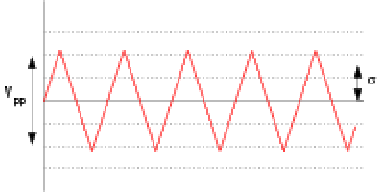
Standard Deviation of Common Waveforms

a. Square wave, $V_{pp} = 2\sigma$

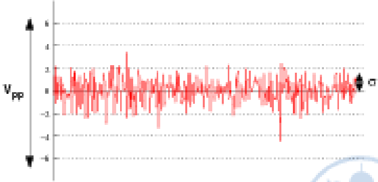


b. Sine wave, $V_{pp} = 2\sqrt{2}\sigma$

c. Triangle wave, $V_{pp} = \sqrt{12}\sigma$

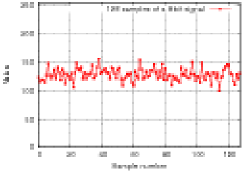


d. Random noise, $V_{pp} = 6 - 8\sigma$

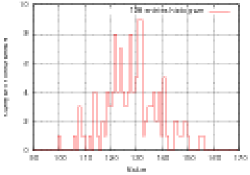


Histograms

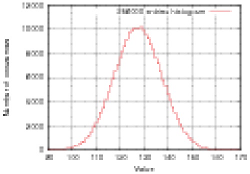
Snapshot of N samples



N small

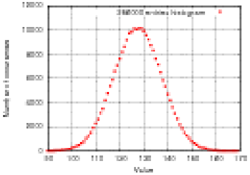


N large



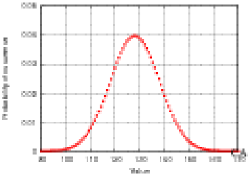
$$N = \sum_{i=0}^{M-1} H_i \qquad \hat{x} := \frac{1}{N} \sum_{i=0}^{M-1} i \cdot H_i \qquad \sigma^2 := \frac{1}{N-1} \sum_{i=0}^{M-1} (i - \hat{x})^2 H_i$$

Histogram

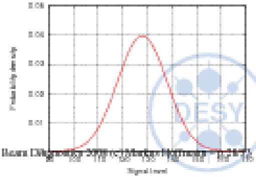


$N \to \infty$

Probability mass function

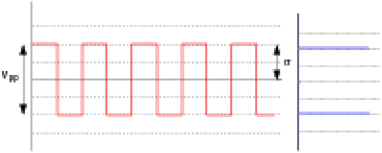


Probability density distribution

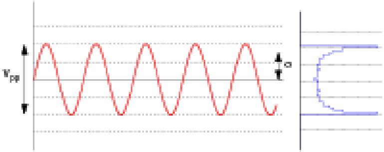


Probability Distribution Functions

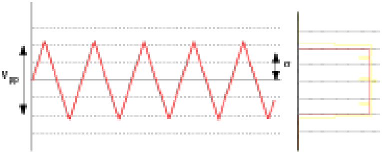
a. Square wave



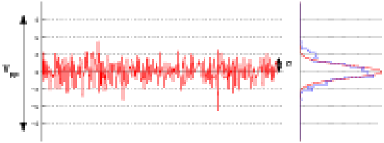
b. Sine wave



c. Triangle wave



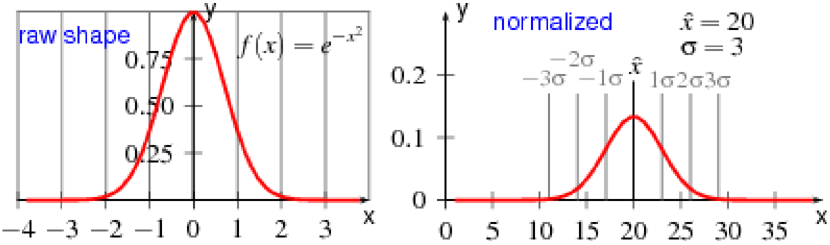
d. Random noise



The Normal Distribution

$$P(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\hat{x})^2}{2\sigma^2}}$$

Gauß function

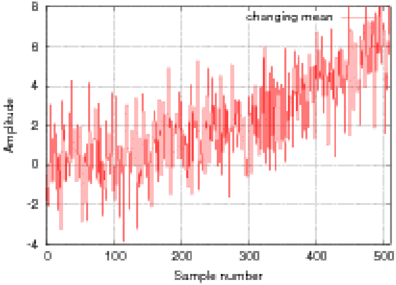


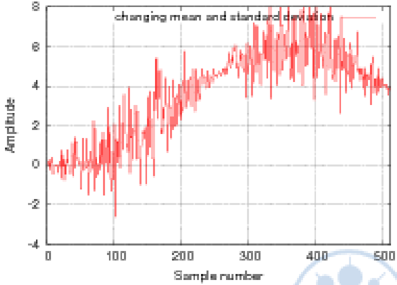
$$\int_{-\infty}^{+\infty} P(x)dx = 1$$

Underlying Process

typical error: $\Delta A = \frac{\sigma_N}{\sqrt{N}}$

- σ_N is an estimate of the standard deviation of the underlying process by N samples (e.g. extracted from the histogram).





- Extract the most information about the underlying process out of the sampled signal.

Propagation of Error

in general:

$$f = f(\alpha_1, \alpha_2, \ldots, \alpha_n)$$

a function of (model) parameters α_i with corresponding errors $\Delta\alpha_i$.

$$\Rightarrow \Delta f = \sqrt{\sum_i \left(\frac{\partial f}{\partial \alpha_i} \Delta\alpha_i \right)^2}$$

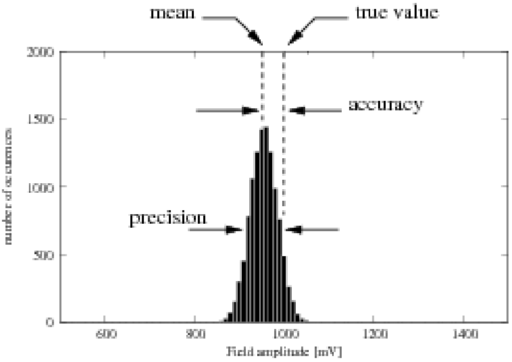
Example:

$$x(t) = v \cdot t + x_0$$

$$\Rightarrow \Delta x = \sqrt{\left(\frac{\partial x}{\partial v} \Delta v \right)^2 + \left(\frac{\partial x}{\partial x_0} \Delta x_0 \right)^2 + \left(\frac{\partial x}{\partial t} \Delta t \right)^2}$$

$$= \sqrt{(\Delta v \cdot t)^2 + (\Delta x_0)^2}$$

Accuracy and Precision



- measure, estimate, predict
- "true value"
- measured value
- systematic error
- random noise

- accuracy is a measure of calibration
- precision is a maeasure of statistics

Question: measuring with 100 different devices. ⟶ accuracy or precision?

The Central Limit Theorem

- generation of randum numbers (noise)
  - white noise RND $:= [0;1[$
  - gaussian noise:
    "the sum of random numbers (any distribution) becomes gaussian distributed"
    better:

$$x = \sqrt{-2\log_{10}(\mathrm{RND}_1)} \cdot \cos(2\pi \mathrm{RND}_2)$$

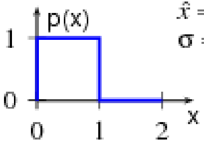
$$\hat{x} = 0, \quad \sigma = 1$$

- pseudo-random:

$$\mathrm{RND} = (as + b) \bmod c$$

seed (→ s)

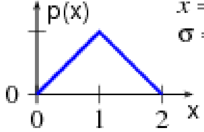
$x = \mathrm{RND}$



$\hat{x} = 0.5$

$\sigma = \frac{1}{\sqrt{12}} \approx 0.29$

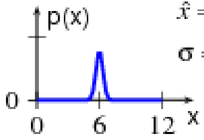
$x = \mathrm{RND} + \mathrm{RND}$



$\hat{x} = 1$

$\sigma = \frac{1}{\sqrt{6}} \approx 0.4$

$x = \mathrm{RND} + \cdots + \mathrm{RND}$ (12 ×)



$\hat{x} = 6$

$\sigma = 1$

Chi Square Distribution

X_i are k independent, *normally distributed* random variables with $\bar{X}_i = 0$, $\sigma^2_{X_i} = 1$:

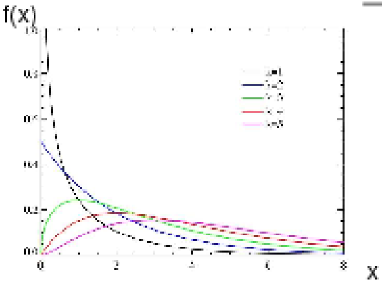
$$Q := \sum_{i=1}^{k} X_i^2$$

is distributed according to **„Chi Square"**

$$Q \sim \chi^2_k$$

k: number of degrees of freedom

For $k \gg 1$: Q becomes Gaussian distributed.



Probability distribution:

$$f(x;k) = \begin{cases} \dfrac{x^{\frac{k}{2}-1}}{2^{\frac{k}{2}}\Gamma\left(\frac{k}{2}\right)} e^{-\frac{x}{2}} & \text{for } x > 0, \\ 0 & \text{for } x \leq 0. \end{cases}$$

Chi Square Test

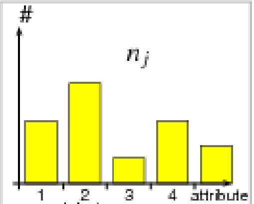
Null-hypothesis: is true if your alternative hypothesis (=your model) cannot be supported.

Statistical Test: falsify the Null-hypothesis!

χ^2-Test: the statistics of samples of data (derived from model) has a χ^2distribution if the null-hypothesis is true.

(χ^2distribution: Or gauss or any, if the pro-papility distribution approximates large # of samples)

example



histogram

probability, that x is in j

Theory: $n_{j0} = F_0(x)_j \cdot N$

$$\Rightarrow \quad \chi^2 = \sum_{j=1}^{m} \frac{(n_j - n_{j0})^2}{n_{j0}}$$

χ^2 distributed, $m-1$ degrees of freedom if N large

If χ^2 is larger than a **significance level** α, the hypothesis will be rejected.

from tables (out of the scope)

4. Advanced Concepts

Power Spectral Density

averaged power levels related to intervals of noise frequency

average over many time intervals T

$$\rho_x(\nu) := \lim_{T\to\infty} \frac{1}{T} \langle \left| \int_{-\frac{T}{2}}^{\frac{T}{2}} X(t) e^{2\pi i\nu t} dt \right|^2 \rangle \quad \textbf{power spectrum of } X(t)$$

Unit: $[\frac{W}{Hz}]$, $[\frac{dBm}{Hz}]$, $[\frac{dBc}{Hz}]$, or $[\frac{rad^2}{Hz}]$ (phase noise). Amplitude: $[\frac{rad}{\sqrt{Hz}}]$

Wiener-Khinchin theorem:

$\rho_x(\nu) =$ **Fourier Transformation** of **Autocorrelation** function of $X(t)$

$$\rho_x(\nu) = \int_{-\infty}^{\infty} g_x(\tau) e^{2\pi i\nu\tau} d\tau \quad \textbf{fourier transform of}$$

$$g_x(t) := \langle \int_{-\infty}^{\infty} x(\tau) x(t+\tau) d\tau \rangle \quad \textbf{average autocorrelation function}$$

The Fourier Transformation

time domain		frequency domain
$s(t)$	$\longrightarrow$	$S(\omega)$
signal		spectrum

Given $f : D \rightarrow \mathbb{C}$, where $D \subseteq \mathbb{R}$, the **Fourier tansformation** of f is:

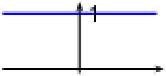
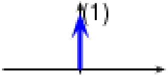
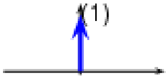
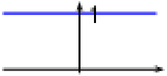
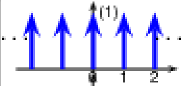
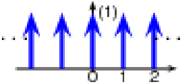
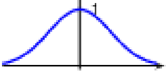
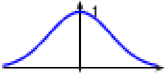
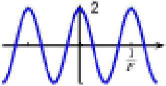
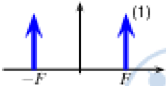
$$F(\omega) := \int_D f(t)\, e^{-i\omega t}\, dt$$

and the **back transformation**

$$f(t) := \int_{-\infty}^{\infty} F(\omega)\, e^{+i\omega t}\, d\omega \quad .$$

- **Note:** We use $\omega = 2\pi\nu$ to get rid of the constants.

Fourier Transformation Examples

$s(t)$	time domain	$S(f)$	frequency domain $\lvert S \rvert$
1		$\delta(f)$	
$\delta(t)$		1	
$\varpi(t)$		$\varpi(f)$	
$e^{-\pi t^2}$		$e^{-\pi f^2}$	
$2\cos(2\pi F t)$		$\delta(f+F)+$ $\delta(f-F)$	

Calculation with Fourier Transforms

$$x(t) \xrightarrow{\text{FT}} X(\omega) \xrightarrow{\text{FT}} x(-t)$$

Symmetry:

$$\mathbf{FT}^2\{x(t)\} = x(-t)$$

Linearity:

$$\mathbf{FT}\{c_1x_1(t) + c_2x_2(t)\} = c_1X_1(\omega) + c_2X_2(\omega)$$

Scaling:

$$\mathbf{FT}\{x(\lambda t)\} = \frac{1}{|\lambda|}X\left(\frac{\omega}{\lambda}\right)$$

For a real input, the transformation produces a complex spectrum which is symmetrical:

$$X(\omega) = X^*(-\omega)$$

complex conjugate

$$X(\text{cos-like}) = \text{real}$$
$$X(\text{sin-like}) = \text{imaginary}$$

Time-Shift:

$$\mathbf{FT}\{x(t + t_0)\} = e^{j\omega t_0}X(\omega)$$

Convolution:

$$\mathbf{FT}\{x_1(t) * x_2(t)\} = X_1(\omega) \cdot X_2(\omega) \quad ; \quad \mathbf{FT}\{x_1(t) \cdot x_2(t)\} = X_1(\omega) * X_2(\omega)$$

Autocorrelation

Given two functions $f, g : D \to \mathbb{C}$, where $D \subseteq \mathbb{R}$, the **cross correlation** of f with g:

$$(f \circ g)(t) := K \int_D f(\tau) g(t \overset{!}{+} \tau) d\tau$$

Autocorrelation

$$A_g(t) := g \circ g = K \int_D g(\tau) g(t + \tau) d\tau$$

Detect a known waveform in a noisy background, e.g. echoes.

Symmetry:

$$A_g(-t) = A_g^*(t)$$

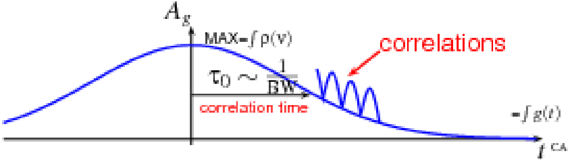
Convolution:

$$f[n] \circ g[n] = f[n] * g[\overset{!}{-} n]$$

Peak: $|A_g(t)| \leq A_g(0)$

Periodicity:

$$g(t)\text{periodic} \Leftrightarrow A_g(t)\text{periodic}$$



Power Spectral Density (2)

- **Spectrum of the autocorrelation function:**

$$s(t) * s(-t) \overset{\mathbf{FT}}{\longleftrightarrow} S(\omega) \cdot S^*(\omega) = |S(\omega)|^2 = \rho(\omega) \quad [\mathrm{W/Hz} = \mathrm{J}]$$

Energy spectrum

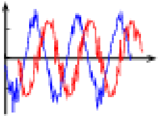
- we loose information about phase (or time/shift/location), Time invariance
- white noise part is contained in $g_x(0)$,
- relation to the **variance** and RMS of $x(t)$:

$$\langle x^2 \rangle = \int_{\nu_1}^{\nu_2} \rho_x(\nu) d\nu = P_{\mathrm{BW}} = (\mathrm{RMS}_x)^2 =: x_{\mathrm{eff}}^2 =: \sigma_x^2|_{[\nu_1,\nu_2]}$$

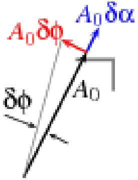
Wiener-Khinchin:

mean square value in time = mean square value in frequency

- analog and digital measurement techniques possible.



Complex Noise
(Amplitude & Phase Noise)



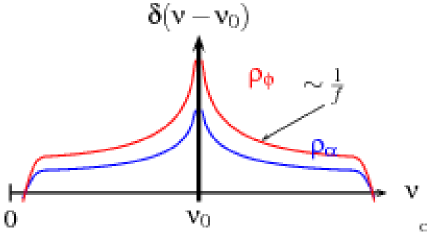
carrier frequency

Harmonic Signal with Noise: $x(t) = A_0\,(1+\delta\alpha(t))\,e^{2\pi i\nu_0 t + \delta\phi(t)}$

amplitude noise   phase noise

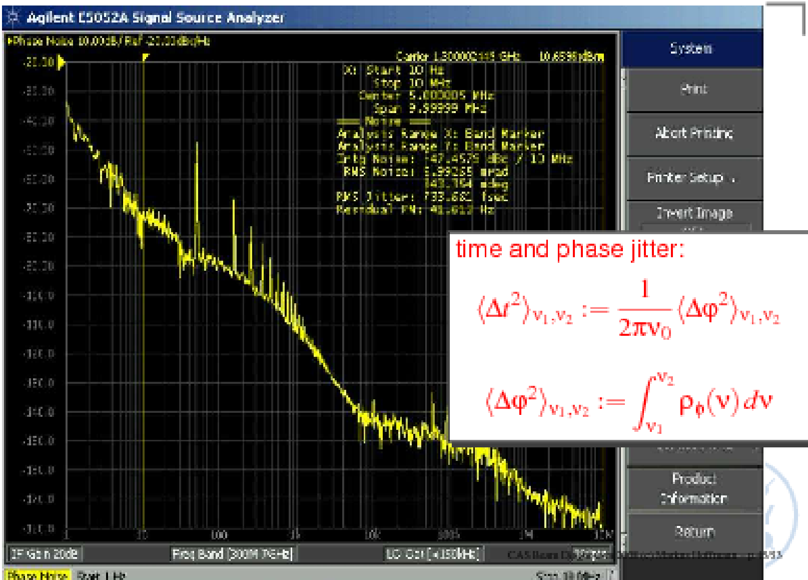
carrier δ peak

$$\rho(\nu) = A_0^2\left(\delta(\nu-\nu_0) + \rho_\alpha(\nu-\nu_0) + \rho_\phi(\nu-\nu_0) + \mathcal{O}^n(\rho_\alpha,\rho_\phi)\right)$$



Phase- and Time-Jitter

Phase noise spectrum of the FLASH Master Oscillator



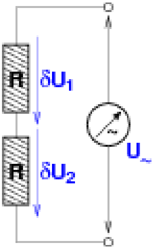
time and phase jitter:

$$\langle \Delta t^2 \rangle_{\nu_1,\nu_2} := \frac{1}{2\pi\nu_0} \langle \Delta\varphi^2 \rangle_{\nu_1,\nu_2}$$

$$\langle \Delta\varphi^2 \rangle_{\nu_1,\nu_2} := \int_{\nu_1}^{\nu_2} \rho_\phi(\nu)\, d\nu$$

Noise Propagation

- Adding noise sources:



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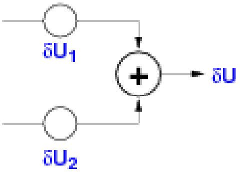
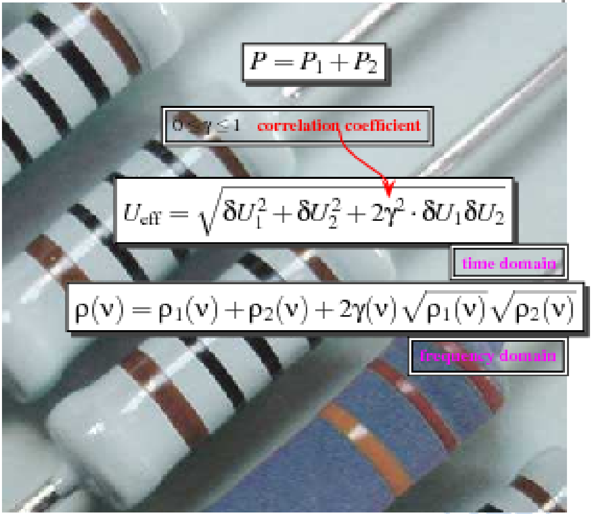




$$P = P_1 + P_2$$

$0 \leq \gamma \leq 1$ correlation coefficient

$$U_{\text{eff}} = \sqrt{\delta U_1^2 + \delta U_2^2 + 2\gamma^2 \cdot \delta U_1 \delta U_2}$$

time domain

$$\rho(\nu) = \rho_1(\nu) + \rho_2(\nu) + 2\gamma(\nu)\sqrt{\rho_1(\nu)}\sqrt{\rho_2(\nu)}$$

frequency domain

Noise Propagation (2)

- Amplifier/Attenuator:

$$y(t) = g\,x(t)$$

$$P_y = g^2 P_x$$

$$U_{\mathrm{eff},y} = g \cdot \delta U_x$$

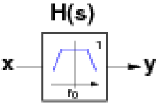
$$\rho_y(\nu) = g^2 \rho_x(\nu)$$

$$\rho_{\alpha,y}(\nu) = \rho_{\alpha,x}(\nu)$$

$$\rho_{\phi,y}(\nu) = \rho_{\phi,x}(\nu)$$

The power of the phase noise is not(!) amplified!

- LTI-Filter:



$$y(t) = h(t) * x(t)$$

transfer function

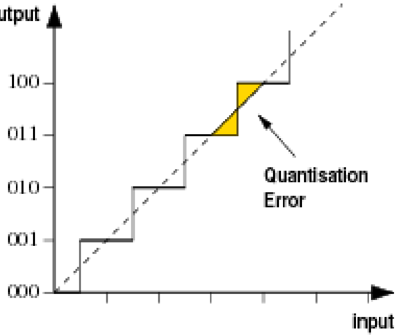
$$\rho_y(\nu) = |H(\nu)|^2 \rho_x(\nu)$$

$$\rho_{\alpha,y}(\nu) = \left|\frac{H(\nu+\nu_0)}{H(\nu_0)}\right|^2 \rho_{\alpha,x}(\nu)$$

$$\rho_{\phi,y}(\nu) = \left|\frac{H(\nu+\nu_0)}{H(\nu_0)}\right|^2 \rho_{\phi,x}(\nu)$$

Quantisation Noise

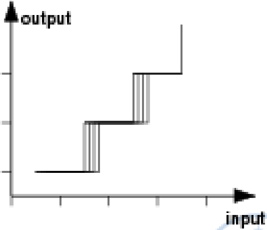
Transfer-function of ADC:



- Quantisation-Error
  $|A| < 0.5\,\mathrm{LSB}$

$$\mathrm{RMS}(\Delta A) \approx \sqrt{12}\,\mathrm{LSB}$$

- missing codes
- code transition noise



For a full scale $\sin()$-signal:

$$\mathrm{SNR} = 6.02\,n + 1.76\mathrm{dB} + 10\,\log\left(\frac{f_s}{2\,\mathrm{BW}}\right)$$

increases with lower BW.

$\longrightarrow$doubling the sampling frequency increases SNR by 3dB (same signal BW)

$\longrightarrow$"oversampling"

Quantisation Noise Spectra

$$\mathrm{SNR} = 6.02n + 1.76\mathrm{dB} + 10\log\left(\frac{f_s}{2\,\mathrm{BW}}\right)$$

it is assumed that the noise is equally distributed over the full BW.

! This is often not the case **!**

Mostly the noise is correlated with the input signal!

- The lower the signal, the more correlation!
- In case of strong correlation the noise is concentrated at the various harmonics of the input signal, just where you dont want them.
- dithering and broad input signal spectrum randomizes the quantisation noise.

5. Applications

Stochastic Signals in Accelerator Diagnostics

Use the noise which is there anyway

- **Tune measurements** from shottky noise of the beam itself (witout increasing the emittances), esp. hadron beams)
- Synchrotron light emmission –> noise source, beam exitation

Apply additional and/or artificial (pseudo random) noise

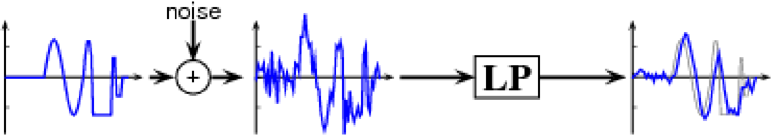
- dithering methods
- beam size blow ups (nbunch lengthening)
- decorrelate signals to avoid
  - systematic errors,
  - interference and
  - resonnant excitations.
- **transfer function** measurements
- **Stochastic cooling**

in many other cases the stochasic part of the signal is unwanted (noise)

→ Noise Filtering Techniques

Fighting the Noise

- How can we reconstruct a signal to which noise has been added?
- One Idea is to use a Low-Pass Filter:



Problems are: latency, dissipation, high frequency cut-off and still noise in the low frequencies.

- ⟶ solution to this: **the Kalman Filter**
- ⟶ other solution: (non-causal)

noisy Signal ⟶ Transformation ⟶ remove noise ⟶ inverse Transformation ⟶ Signal

Measurements, Statistics and Errors

The art of measurement is always also the art of error treatment!

The End