Introduction to Transverse Beam Optics

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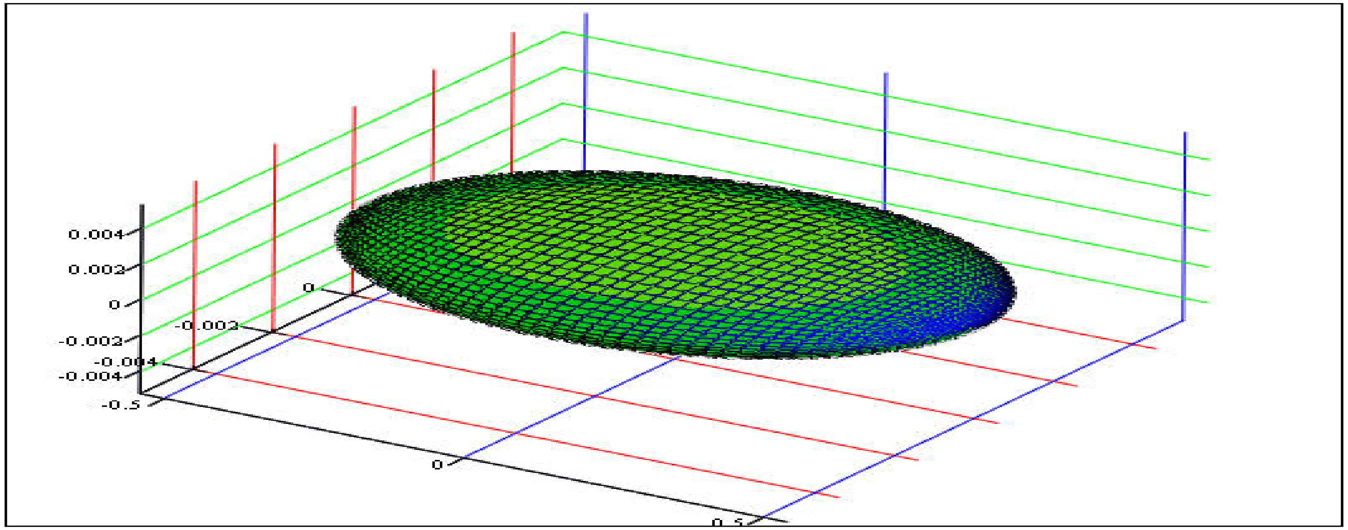
II.) ε & β

... don't worry: it's still the "ideal world"



particle bunch in a storage ring

Reminder of Part I

Equation of Motion:

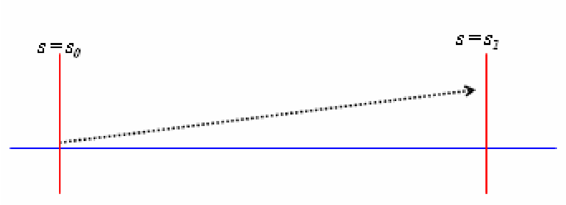
$$x'' + K\,x = 0$$

$K = 1/\rho^2 - k$ … *hor. plane:*

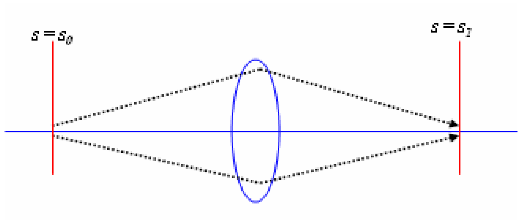
$K = k$ … *vert. Plane:*

Solution of Trajectory Equations

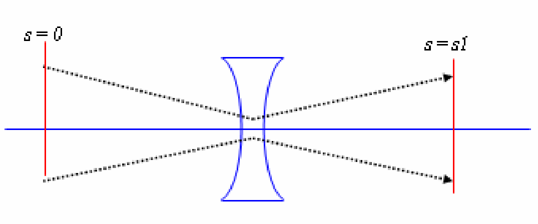
$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s1} = M * \begin{pmatrix} x \\ x' \end{pmatrix}_{s0}$$



$$M_{drift} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$$



$$M_{foc} = \begin{pmatrix} \cos(\sqrt{|K|}\,l) & \frac{1}{\sqrt{|K|}}\sin(\sqrt{|K|}\,l) \\ -\sqrt{|K|}\sin(\sqrt{|K|}\,l) & \cos(\sqrt{|K|}\,l) \end{pmatrix}$$



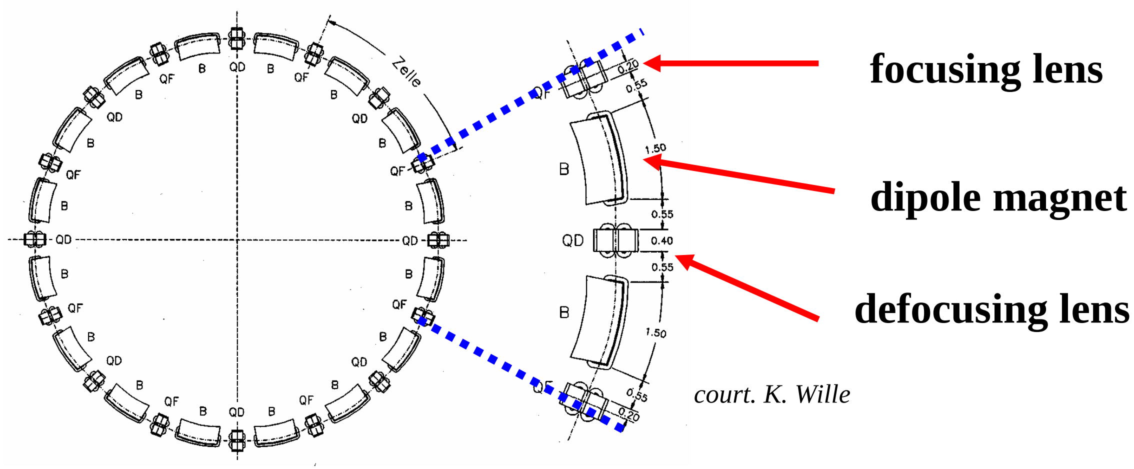
$$M_{defoc} = \begin{pmatrix} \cosh(\sqrt{|K|}\,l & \frac{1}{\sqrt{|K|}}\sinh(\sqrt{|K|}\,l \\ \sqrt{|K|}\sinh(\sqrt{|K|}\,l & \cosh(\sqrt{|K|}\,l \end{pmatrix}$$

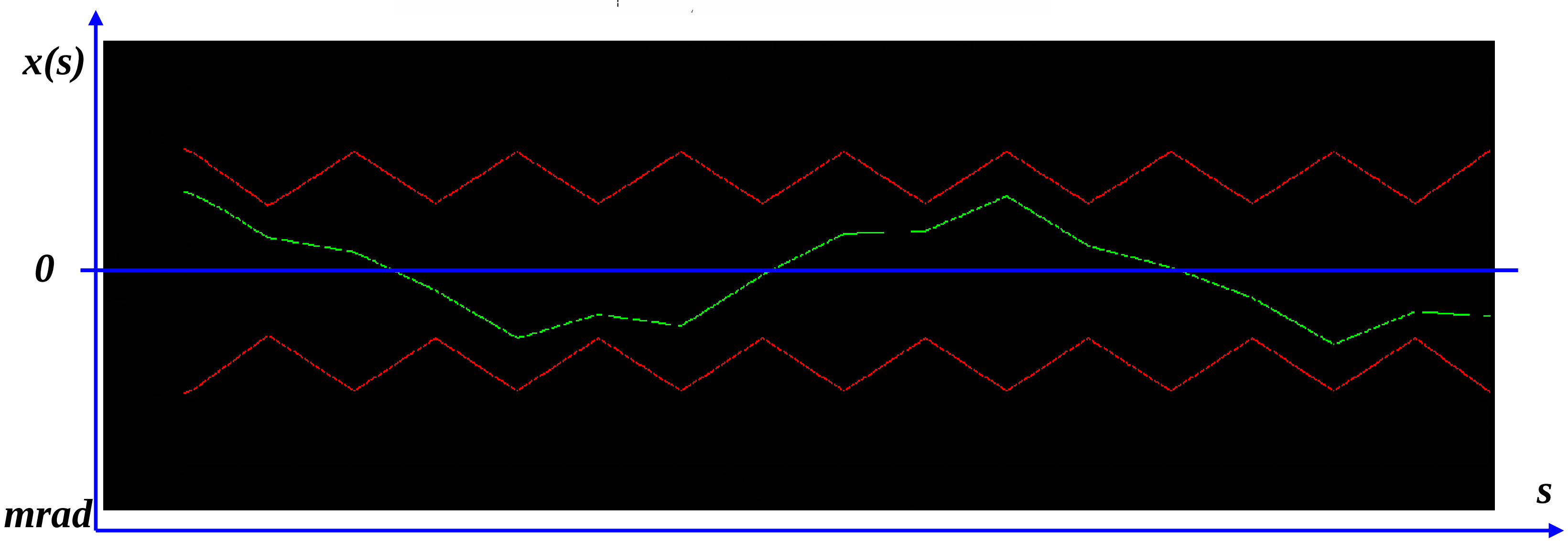
Transformation through a system of lattice elements

combine the single element solutions by multiplication of the matrices

$$M_{total} = M_{QF} * M_{D} * M_{QD} * M_{Bend} * M_{D*.....}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s2} = M(s2,s1) * \begin{pmatrix} x \\ x' \end{pmatrix}_{s1}$$

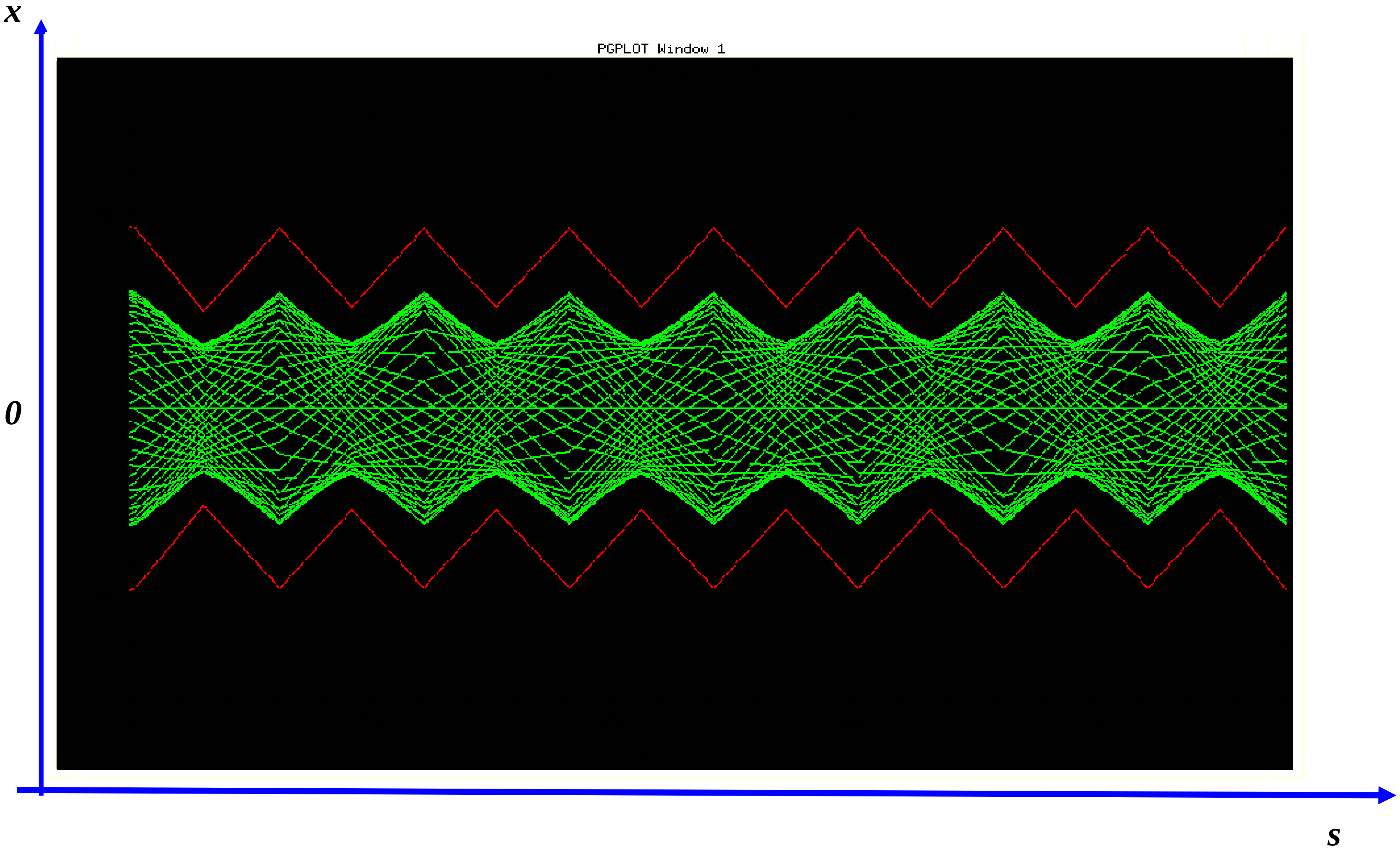




typical values
in a strong
foc. machine:
x ≈ mm, x´ ≤ mrad

Question: ***what will happen, if the particle performs a second turn ?***

... or a third one or ... 10^{10} turns



Astronomer Hill:

differential equation for motions with periodic focusing properties
„Hill's equation"

Example: particle motion with periodic coefficient

equation of motion: $$x''(s) - k(s)x(s) = 0$$

restoring force $\neq$ ***const,***
k(s) = depending on the position s
k(s+L) = k(s), periodic function

we expect a kind of quasi harmonic oscillation: amplitude & phase will depend on the position s in the ring.

The Beta Function

General solution of Hill´s equation:

(i) $$x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)} \cdot \cos(\psi(s) + \phi)$$

ε, Φ = integration constants determined by initial conditions

β(s) periodic function given by focusing properties of the lattice ↔ quadrupoles

$$\beta(s + L) = \beta(s)$$

Inserting (i) into the equation of motion …

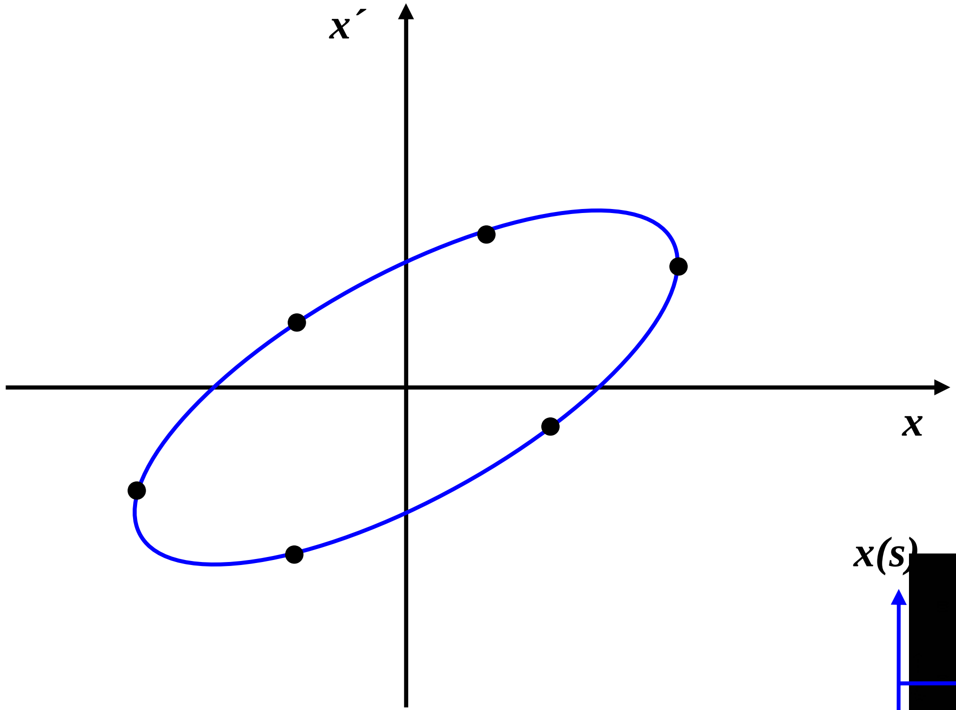
$$\psi(s) = \int_0^s \frac{ds}{\beta(s)}$$

Ψ(s) = „phase advance“ of the oscillation between point „0“ and „s“ in the lattice.
For one complete revolution: number of oscillations per turn „Tune“

$$Q_y = \frac{1}{2\pi} \cdot \oint \frac{ds}{\beta(s)}$$

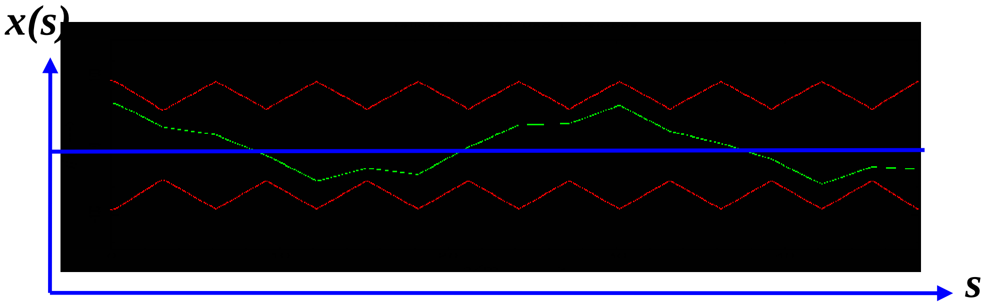
9.) Beam Emittance and Phase Space Ellipse

$$\varepsilon = \gamma(s)\, x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)\, x'^2(s)$$



Liouville: *in reasonable storage rings area in phase space is constant.*

$A = \pi * \varepsilon = const$



ε beam emittance = woozilycity of the particle ensemble, intrinsic beam parameter, cannot be changed by the foc. properties.

Scientifiquely spoken: area covered in transverse x, x´ phase space … and it is constant !!!

Phase Space Ellipse

particel trajectory: $x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}\cos\{\psi(s)+\phi\}$

max. Amplitude: $\hat{x}(s) = \sqrt{\varepsilon\beta}$ → *x´ at that position …?*

… put $\hat{x}(s)$ *into* $\varepsilon = \gamma(s)\, x^2(s) + 2\alpha(s) x(s) x'(s) + \beta(s)\, x'^2(s)$ *and solve for x´*

$$\varepsilon = \gamma \cdot \varepsilon\beta + 2\alpha\sqrt{\varepsilon\beta} \cdot x' + \beta x'^2$$

$$\rightarrow \quad x' = -\alpha \cdot \sqrt{\varepsilon / \beta}$$

* *A high β-function means a large beam size and a small beam divergence.* **!**
  … et vice versa !!!

* *In the middle of a quadrupole β = maximum,*
  α = zero } $x' = 0$

… and the ellipse is flat

Phase Space Ellipse

$$\varepsilon = \gamma(s)\, x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)\, x'^2(s)$$

$$\alpha(s) = \frac{-1}{2}\beta'(s)$$

$$\gamma(s) = \frac{1+\alpha(s)^2}{\beta(s)}$$

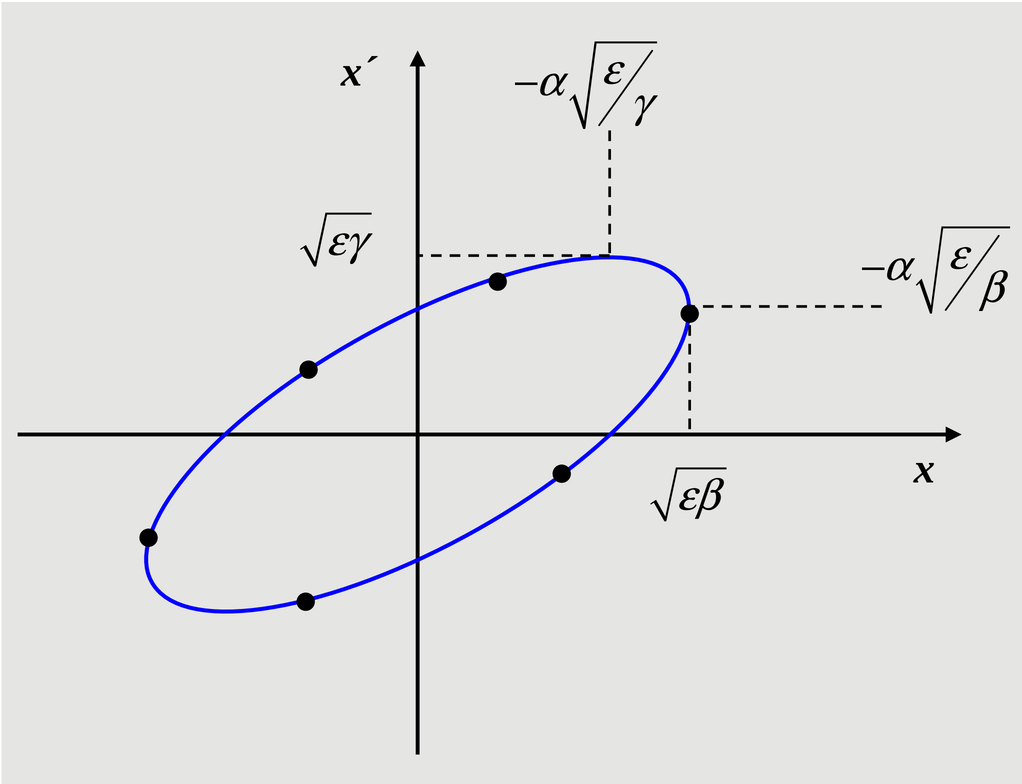
$$\longrightarrow \quad \varepsilon = \frac{x^2}{\beta} + \frac{\alpha^2 x^2}{\beta} + 2\alpha \cdot xx' + \beta \cdot x'^2$$

… solve for x´ $\quad x'_{1,2} = \dfrac{-\alpha \cdot x \pm \sqrt{\varepsilon\beta - x^2}}{\beta}$

… and determine $\hat{x}'$ ***via:*** $\quad \dfrac{dx'}{dx} = 0$

$$\longrightarrow \quad \hat{x}' = \sqrt{\varepsilon\gamma}$$

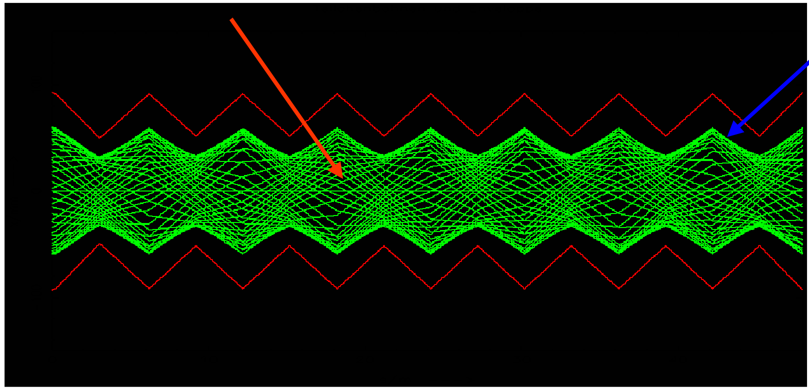
$$\longrightarrow \quad \hat{x} = \pm\alpha\sqrt{\varepsilon/\gamma}$$



shape and orientation of the phase space ellipse depend on the Twiss parameters β α γ

Emittance of the Particle Ensemble:

$$x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)} \cdot \cos(\Psi(s) + \phi) \qquad \hat{x}(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}$$

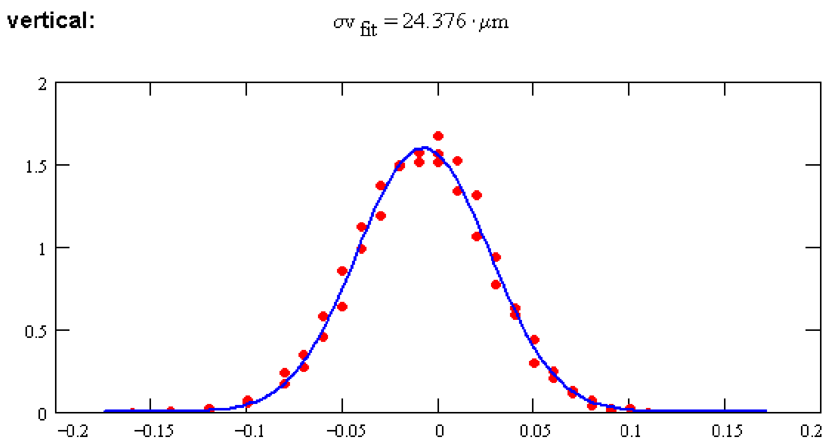


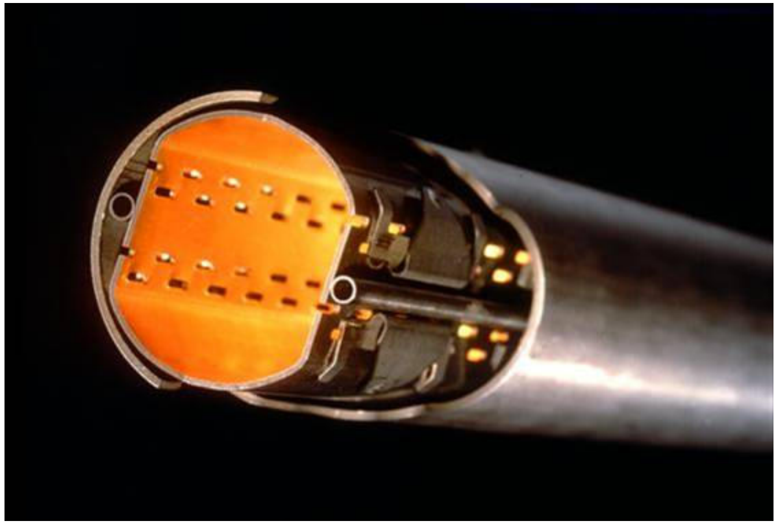
single particle trajectories, N ≈ 10 [11] per bunch

Gauß
Particle Distribution:

$$\rho(x) = \frac{N \cdot e}{\sqrt{2\pi}\sigma_x} \cdot e^{-\frac{1}{2}\frac{x^2}{\sigma_x{}^2}}$$

particle at distance 1 σ from centre ↔ 68.3 % of all beam particles

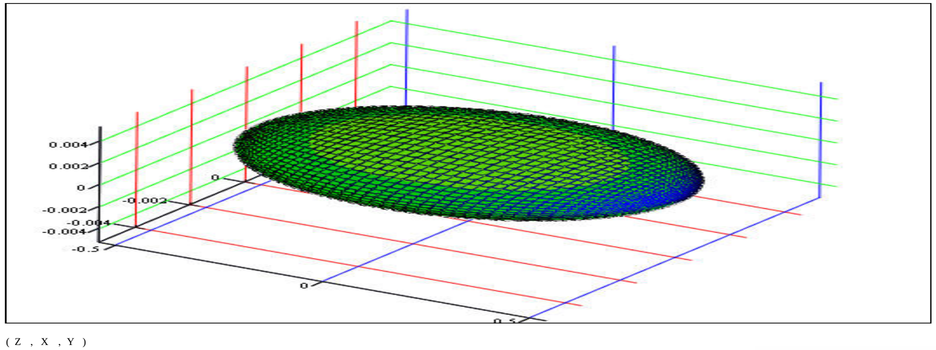




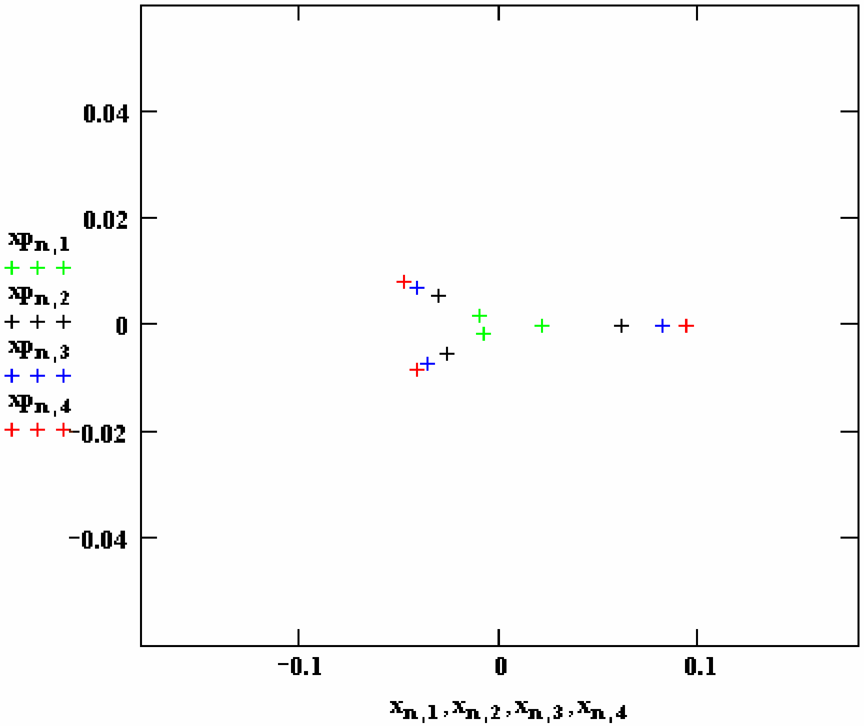
LHC: $\sigma = \sqrt{\varepsilon * \beta} = \sqrt{5 * 10^{-10}\,m * 180\,m} = 0.3\,mm$

aperture requirements: $r_0 = 10 * \sigma$

Emittance of the Particle Ensemble:



particle bunch



Example: HERA

beam parameters in the arc

$$\beta(x) \approx 80\, m$$

$$\varepsilon \approx 7*10^{-9}\ rad \cdot m \qquad (\leftrightarrow 1\sigma)$$

$$\sigma = \sqrt{\varepsilon\beta} \approx 0.75\, mm$$

10.) Transfer Matrix M … *yes we had the topic already*

general solution of Hill´s equation

$$x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}\cos\{\psi(s)+\phi\}$$

$$x'(s) = \frac{-\sqrt{\varepsilon}}{\sqrt{\beta(s)}}\left[\alpha(s)\cos\{\psi(s)+\phi\}+\sin\{\psi(s)+\phi\}\right]$$

remember the trigonometrical gymnastics: sin(a + b) = … etc

$$x(s) = \sqrt{\varepsilon}\sqrt{\beta_s}\left(\cos\psi_s\cos\phi-\sin\psi_s\sin\phi\right)$$

$$x'(s) = \frac{-\sqrt{\varepsilon}}{\sqrt{\beta_s}}\left[\alpha_s\cos\psi_s\cos\phi-\alpha_s\sin\psi_s\sin\phi+\sin\psi_s\cos\phi+\cos\psi_s\sin\phi\right]$$

starting at point s(0) = s_0***, where we put*** $\Psi(0) = 0$

$$\cos\phi = \frac{x_0}{\sqrt{\varepsilon\beta_0}} \quad ,$$

$$\sin\phi = -\frac{1}{\sqrt{\varepsilon}}\left(x'_0\sqrt{\beta_0}+\frac{\alpha_0 x_0}{\sqrt{\beta_0}}\right)$$

inserting above …

$$x(s)=\sqrt{\frac{\beta_s}{\beta_0}}\{\cos\psi_s+\alpha_0\sin\psi_s\}\,x_0+\left\{\sqrt{\beta_s\beta_0}\,\sin\psi_s\right\}x_0'$$

$$x'(s)=\frac{1}{\sqrt{\beta_s\beta_0}}\{(\alpha_0-\alpha_s)\cos\psi_s-(1+\alpha_0\alpha_s)\sin\psi_s\}\,x_0+\sqrt{\frac{\beta_0}{\beta_s}}\{\cos\psi_s-\alpha_s\sin\psi_s\}\,x_0'$$

which can be expressed ... for convenience ... in matrix form

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M \begin{pmatrix} x \\ x' \end{pmatrix}_0$$

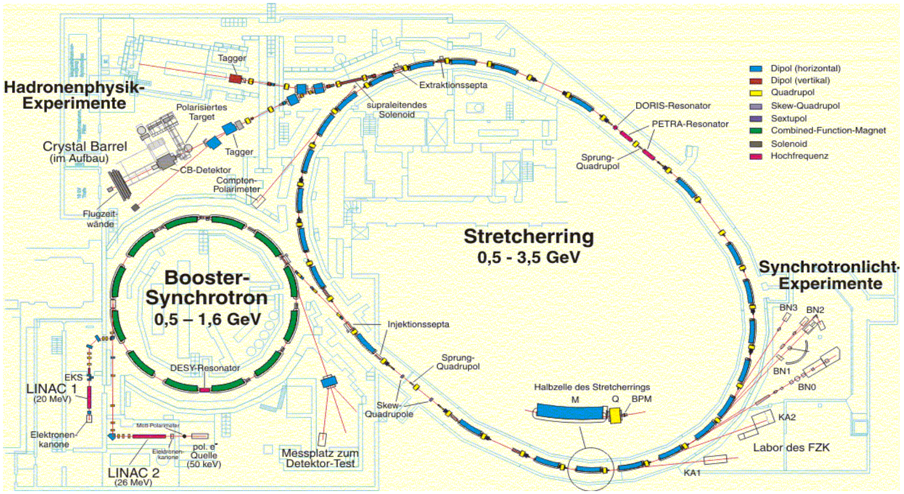
$$M=\begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}\left(\cos\psi_s+\alpha_0\sin\psi_s\right) & \sqrt{\beta_s\beta_0}\,\sin\psi_s \\ \frac{(\alpha_0-\alpha_s)\cos\psi_s-(1+\alpha_0\alpha_s)\sin\psi_s}{\sqrt{\beta_s\beta_0}} & \sqrt{\frac{\beta_0}{\beta s}}\left(\cos\psi_s-\alpha_s\sin\psi_s\right) \end{pmatrix}$$

* ***we can calculate the single particle trajectories between two locations in the ring, if we know the α β γ at these positions.***
* ***and nothing but the α β γ at these positions.***
* ***... !***

**** Äquivalenz der Matrizen***

11.) Periodic Lattices

$$M = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}\left(\cos\psi_s + \alpha_0 \sin\psi_s\right) & \sqrt{\beta_s \beta_0}\,\sin\psi_s \\ \frac{(\alpha_0 - \alpha_s)\cos\psi_s - (1+\alpha_0\alpha_s)\sin\psi_s}{\sqrt{\beta_s\beta_0}} & \sqrt{\frac{\beta_0}{\beta s}}\left(\cos\psi_s - \alpha_s \sin\psi_s\right) \end{pmatrix}$$



ELSA Electron Storage Ring

„This ***rather formidable looking matrix*** ***simplifies considerably if we*** ***consider one complete revolution …“***

$$M(s) = \begin{pmatrix} \cos\psi_{turn} + \alpha_s \sin\psi_{turn} & \beta_s \sin\psi_{turn} \\ -\gamma_s \sin\psi_{turn} & \cos\psi_{turn} - \alpha_s \sin\psi_{turn} \end{pmatrix}$$

$$\psi_{turn} = \int_s^{s+L} \frac{ds}{\beta(s)}$$

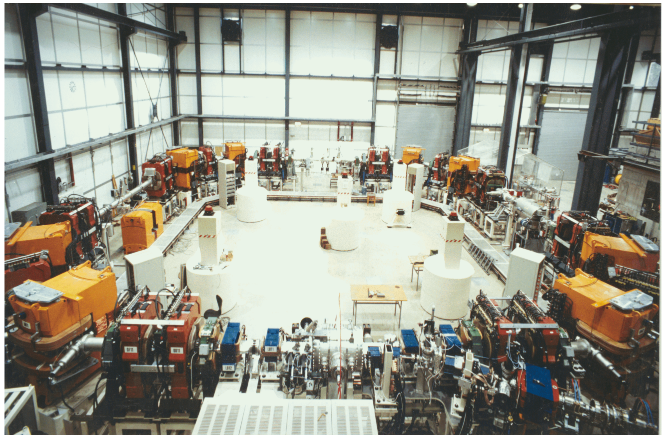
ψ_{turn} = phase advance per period

Tune: ***Phase advance per turn in units of 2π***

$$Q = \frac{1}{2\pi}\oint \frac{ds}{\beta(s)}$$

Stability Criterion:

Question: ***what will happen, if we do not make too many mistakes and your particle performs one complete turn ?***



Matrix for 1 turn:

$$M = \begin{pmatrix} \cos\psi_{turn} + \alpha_s \sin\psi_{turn} & \beta_s \sin\psi_{turn} \\ -\gamma_s \sin\psi_{turn} & \cos\psi_{turn} - \alpha_s \sin\psi_{turn} \end{pmatrix} = \cos\psi \cdot \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\mathbf{1}} + \sin\psi \underbrace{\begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}}_{\mathbf{J}}$$

Matrix for N turns:

$$M^N = (1 \cdot \cos\psi + J \cdot \sin\psi)^N = 1 \cdot \cos N\psi + J \cdot \sin N\psi$$

The motion for N turns remains bounded, if the elements of M^N remain bounded

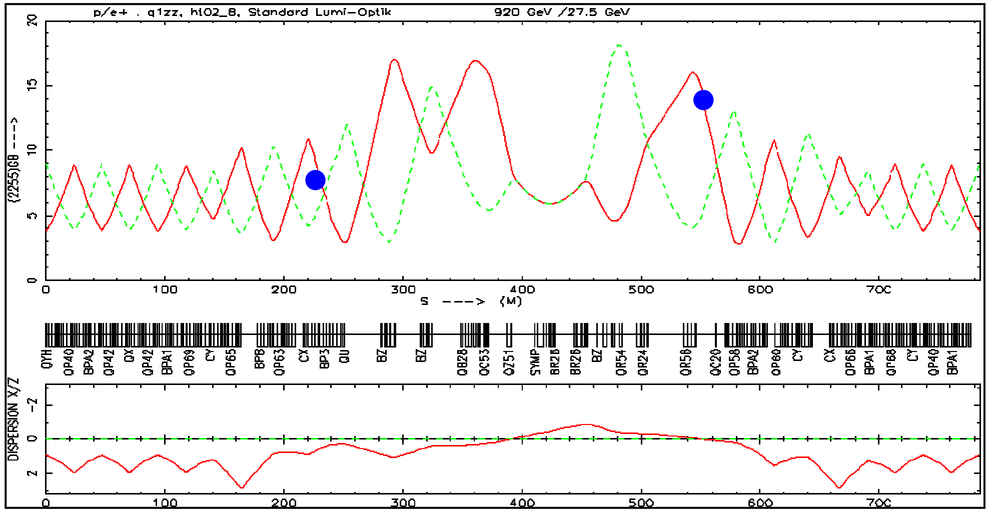
$$\psi = real \quad \leftrightarrow \quad |\cos\psi| \leq 1 \quad \leftrightarrow \quad Tr(M) \leq 2$$

12.) Transformation of α, β, γ

consider two positions in the storage ring: s_0 , s

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M * \begin{pmatrix} x \\ x' \end{pmatrix}_{s0}$$

$$M = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix}$$



Betafunction in a storage ring

since ε = const (Liouville):

$$\varepsilon = \beta_s x'^2 + 2\alpha_s x x' + \gamma_s x^2$$

$$\varepsilon = \beta_0 x_0'^2 + 2\alpha_0 x_0 x_0' + \gamma_0 x_0{}^2$$

... remember W = CS´-SC´ = 1

$$\begin{pmatrix} x \\ x' \end{pmatrix}_0 = M^{-1} * \begin{pmatrix} x \\ x' \end{pmatrix}_s$$

$$M^{-1} = \begin{pmatrix} S' & -S \\ -C' & C \end{pmatrix}$$

→

$$x_0 = S'x - Sx'$$

$$x_0' = -C'x + Cx'$$

... inserting into ε

$$\varepsilon = \beta_0 (Cx' - C'x)^2 + 2\alpha_0 (S'x - Sx')(Cx' - C'x) + \gamma_0 (S'x - Sx')^2$$

sort via x, x´and compare the coefficients to get ....

$$\beta(s) = C^2\beta_0 - 2SC\alpha_0 + S^2\gamma_0$$

$$\alpha(s) = -CC'\beta_0 + (SC' + S'C)\alpha_0 - SS'\gamma_0$$

$$\gamma(s) = C'^2\beta_0 - 2S'C'\alpha_0 + S'^2\gamma_0$$

in matrix notation:

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_s = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC' + CS' & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} \cdot \begin{pmatrix} \beta_0 \\ \alpha_0 \\ \gamma_0 \end{pmatrix}$$

!

1.) this expression is important

2.) given the twiss parameters α, β, γ at any point in the lattice we can transform them and calculate their values at any other point in the ring.

3.) the transfer matrix is given by the focusing properties of the lattice elements, the elements of M are just those that we used to calculate single particle trajectories.

4.) go back to point 1.)

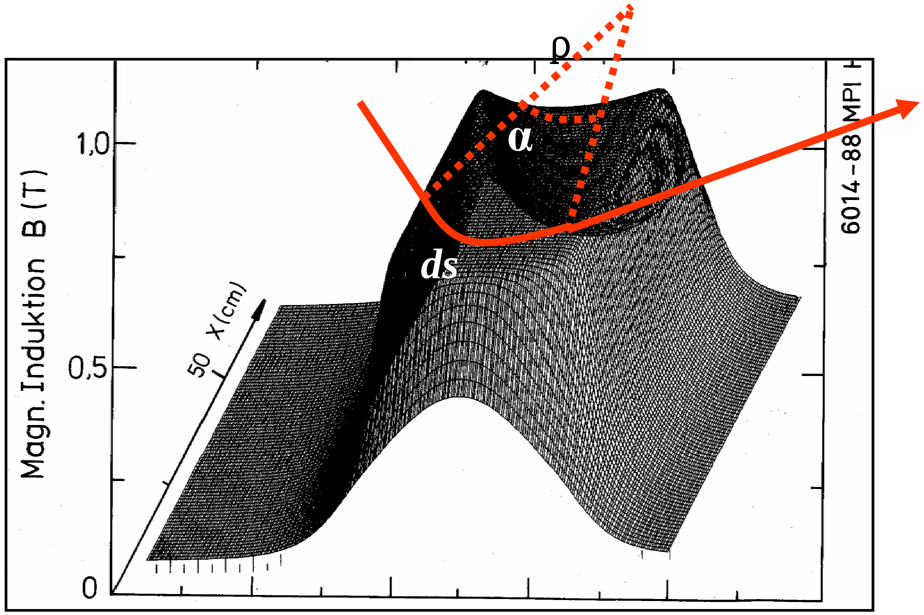
13.) Lattice Design:

„… how to build a storage ring“

$$B\,\rho = p/q$$

Circular Orbit: dipole magnets to define the geometry

$$\alpha = \frac{ds}{\rho} \approx \frac{dl}{\rho} = \frac{Bdl}{B\rho}$$



field map of a storage ring dipole magnet

The angle run out in one revolution must be 2π, so

… for a full circle $$\alpha = \frac{\int Bdl}{B\,\rho} = 2\pi \quad \rightarrow \quad \int Bdl = 2\pi\,\frac{p}{q}$$ *… defines the integrated dipole field around the machine.*

Nota bene: $\frac{\Delta B}{B} \approx 10^{-4}$ *is usually required !!*

Example LHC:

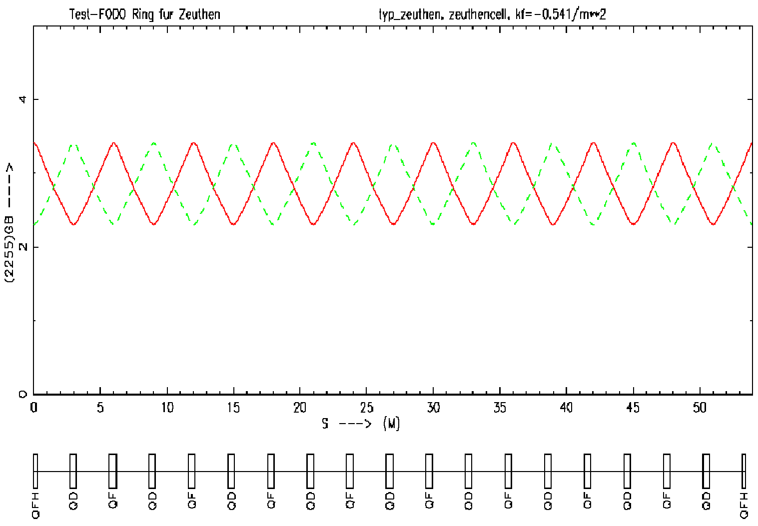
7000 GeV Proton storage ring
dipole magnets N = 1232
l = 15 m
q = +1 e

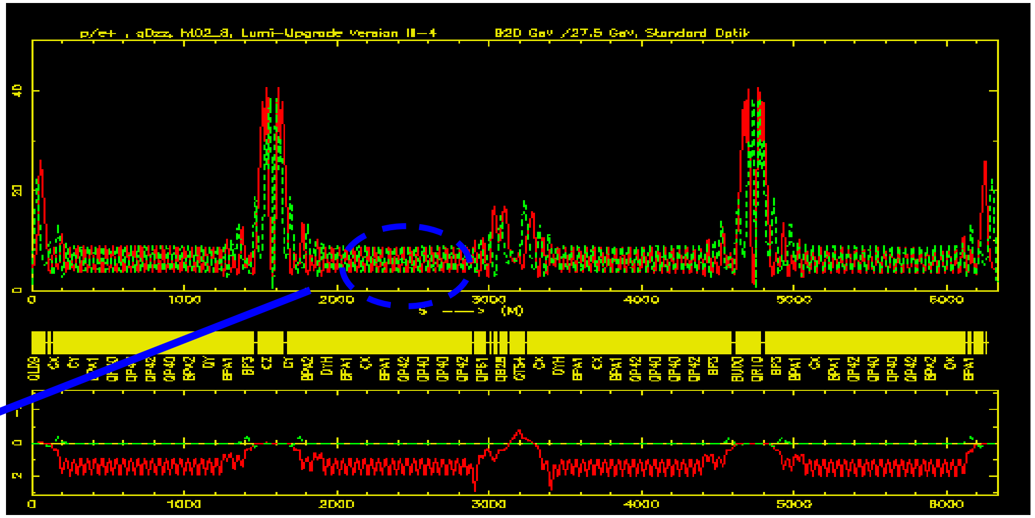
$$\int \boldsymbol{B\,dl} \approx \boldsymbol{N\ l\ B} = 2\pi\ \boldsymbol{p/e}$$

$$\boldsymbol{B} \approx \frac{2\pi\ \ 7000\ \ 10^{9}\,\boldsymbol{eV}}{1232\ \ 15\,\boldsymbol{m}\ \ 3\ 10^{8}\,\frac{\boldsymbol{m}}{\boldsymbol{s}}\ \ \ \boldsymbol{e}} = 8.3\,\boldsymbol{Tesla}$$

The FoDo-Lattice

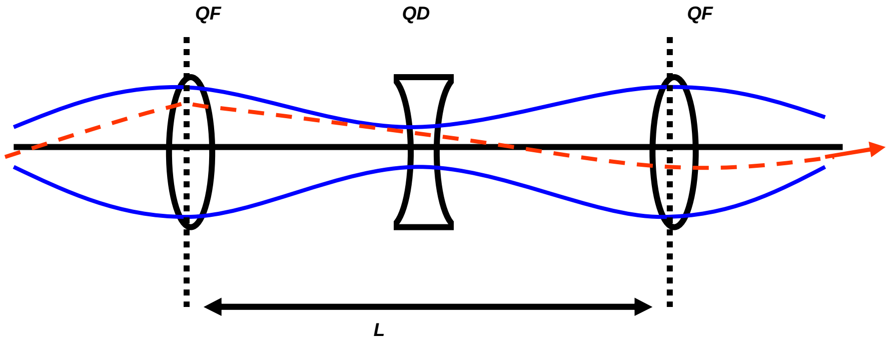
A magnet structure consisting of focusing and defocusing quadrupole lenses in alternating order with nothing in between.
(Nothing = elements that can be neglected on first sight: drift, bending magnets, RF structures ... and especially experiments...)

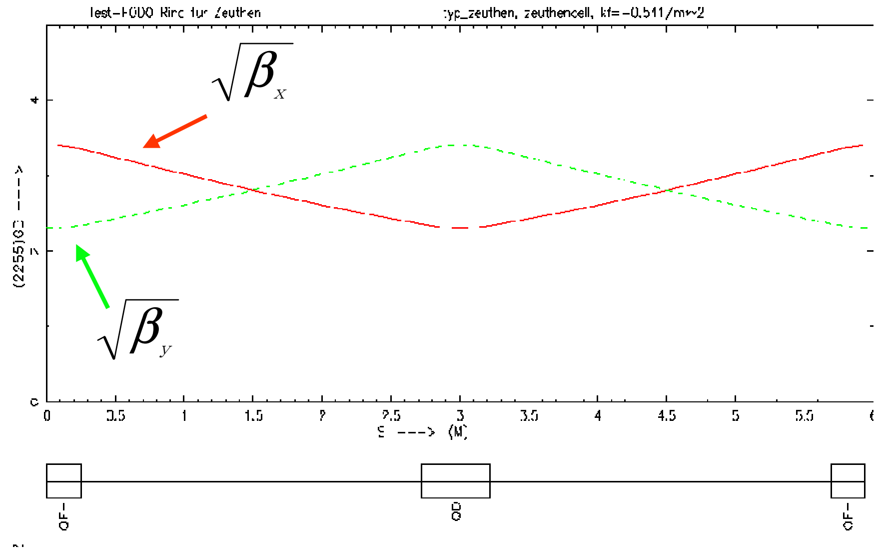




Starting point for the calculation: in the middle of a focusing quadrupole
Phase advance per cell $\mu = 45°$,
→ calculate the twiss parameters for a periodic solution

Periodic solution of a FoDo Cell





Output of the optics program:

Nr	Type	Length	Strength	β_x	α_x	ψ_x	β_y	α_y	ψ_y
		m	1/m2	m		1/2π	m		1/2π
0	IP	0,000	0,000	11,611	0,000	0,000	5,295	0,000	0,000
1	QFH	0,250	-0,541	11,228	1,514	0,004	5,488	-0,781	0,007
2	QD	3,251	0,541	5,488	-0,781	0,070	11,228	1,514	0,066
3	QFH	6,002	-0,541	11,611	0,000	0,125	5,295	0,000	0,125
4	IP	6,002	0,000	11,611	0,000	0,125	5,295	0,000	0,125

Q_X= 0,125 Q_Y= 0,125

$$0.125 * 2\pi = 45°$$

Can we understand, what the optics code is doing?

matrices

$$M_{foc} = \begin{pmatrix} \cos(\sqrt{|K|}\,l_q) & \frac{1}{\sqrt{|K|}}\sin(\sqrt{|K|}\,l_q) \\ -\sqrt{|K|}\sin(\sqrt{|K|}\,l_q) & \cos(\sqrt{|K|}\,l_q) \end{pmatrix}$$

$$M_{drift} = \begin{pmatrix} 1 & l_d \\ 0 & 1 \end{pmatrix}$$

strength and length of the FoDo elements

K = +/- 0.54102 m^{-2}
lq = 0.5 m
ld = 2.5 m

The matrix for the complete cell is obtained by multiplication of the element matrices

$$M_{FoDo} = M_{qfh} * M_{ld} * M_{qd} * M_{ld} * M_{qfh}$$

Putting the numbers in and multiplying out ...

$$M_{FoDo} = \begin{pmatrix} 0.707 & 8.206 \\ -0.061 & 0.707 \end{pmatrix}$$

The transfer matrix for one period gives us all the information that we need !

1.) is the motion stable?

$$trace(M_{FoDo}) = 1.415 \rightarrow \quad < 2$$

2.) Phase advance per cell

$$M(s) = \begin{pmatrix} \cos\psi + \alpha\sin\psi & \beta\sin\psi \\ -\gamma\sin\psi & \cos\psi - \alpha\sin\psi \end{pmatrix} \rightarrow$$

$$\cos(\psi) = \frac{1}{2}\boldsymbol{Trace}(\boldsymbol{M}) = 0.707$$

$$\psi = \boldsymbol{arc}\cos(\frac{1}{2}\boldsymbol{Trace}(\boldsymbol{M})) = 45°$$

3.) hor β-function

$$\beta = \frac{\boldsymbol{M}_{1,2}}{\sin\psi} = 11.611\,\boldsymbol{m}$$

4.) hor α-function

$$\alpha = \frac{\boldsymbol{M}_{1,1} - \cos\psi}{\sin\psi} = 0$$

III.) Acceleration and Momentum Spread

The „ not so ideal world "

Remember:

Beam Emittance and Phase Space Ellipse:

equation of motion: $$x''(s) - k(s)\, x(s) = 0$$

general solution of Hills equation: $$x(s) = \sqrt{\varepsilon}\,\sqrt{\beta(s)}\,\cos(\psi(s)+\varphi)$$

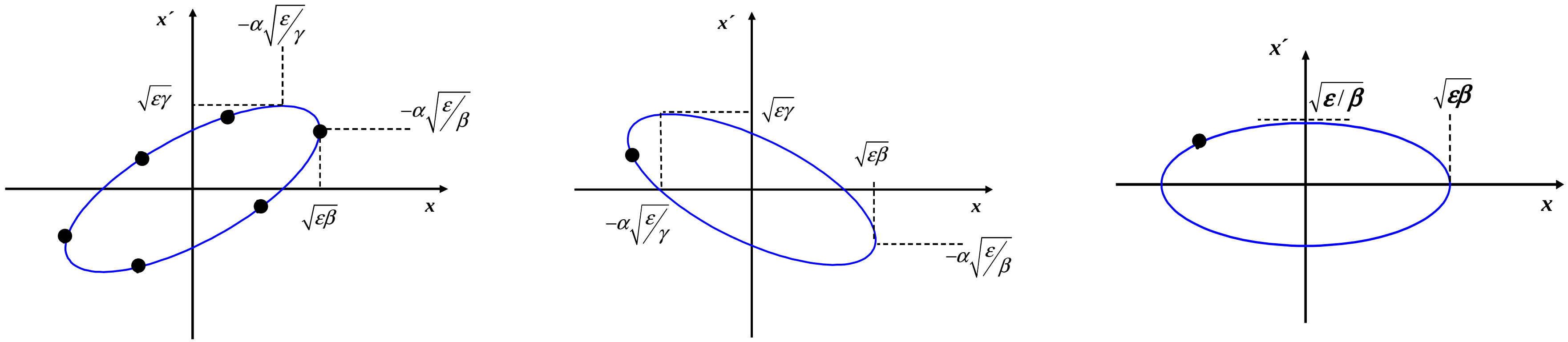
beam size: $$\sigma = \sqrt{\varepsilon\beta} \approx \text{"mm"}$$

$$\varepsilon = \gamma(s)\, x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)\, x'^2(s)$$

$$\alpha(s) = \frac{-1}{2}\beta'(s)$$

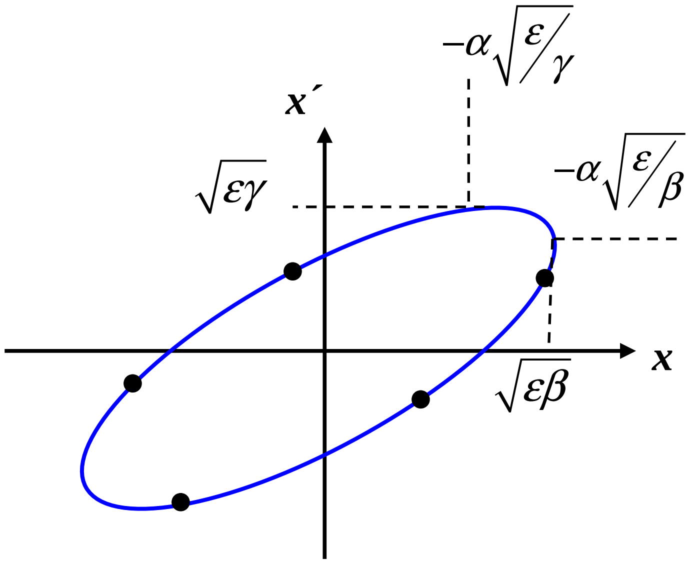
$$\gamma(s) = \frac{1+\alpha(s)^2}{\beta(s)}$$

**** ε is a constant of the motion … it is independent of „s“***
**** parametric representation of an ellipse in the x x‘ space***
**** shape and orientation of ellipse are given by α, β, γ***



14.) Liouville during Acceleration

$$\varepsilon = \gamma(s)\, x^2(s) + 2\alpha(s) x(s) x'(s) + \beta(s)\, x'^2(s)$$



***Beam Emittance** corresponds to the area covered in the x, x´ Phase Space Ellipse*

***Liouville:** Area in phase space is constant.*

But so sorry ... $\varepsilon \neq const$!

Classical Mechanics:

***phase space** = diagram of the two canonical variables*
position** & **momentum
x p_x

$$p_j = \frac{\partial L}{\partial \dot{q}_j} \quad ; \quad L = T - V = kin.\,Energy - pot.\,Energy$$

According to Hamiltonian mechanics:
phase space diagram relates the variables q and p

q = position = x
p = momentum = γmv = mcγβ$_x$

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \quad ; \quad \beta_x = \frac{\dot{x}}{c}$$

Liouvilles Theorem: $$\int p\,dq = const$$

for convenience (i.e. because we are lazy bones) we use in accelerator theory:

$$x' = \frac{dx}{ds} = \frac{dx}{dt}\frac{dt}{ds} = \frac{\beta_x}{\beta}$$

where $\beta_x = v_x / c$

$$\int p\,dq = mc\int \gamma\beta_x dx$$

$$\int p\,dq = mc\gamma\beta\underbrace{\int x' dx}_{\varepsilon}$$

$$\Rightarrow \quad \varepsilon = \int x' dx \propto \frac{1}{\beta\gamma}$$

the beam emittance shrinks during acceleration $\varepsilon \sim 1/\gamma$

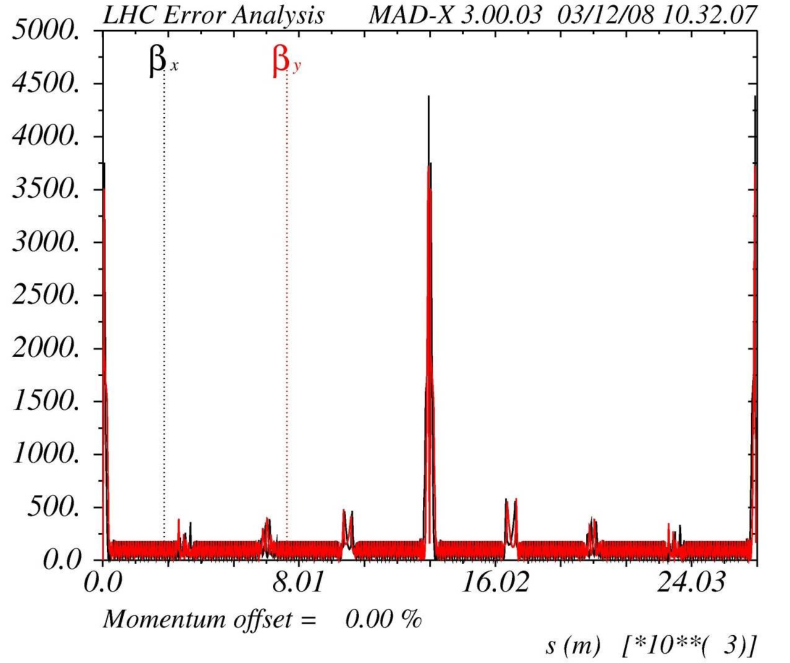
Nota bene:

1.) A proton machine … or an electron linac … needs the highest aperture at injection energy !!!
as soon as we start to accelerate the beam size shrinks as $\gamma^{-1/2}$ in both planes.

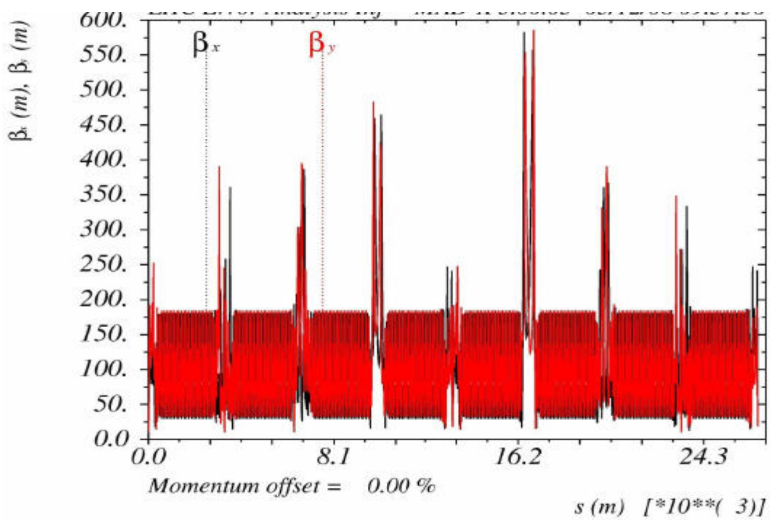
$$\sigma = \sqrt{\varepsilon\beta}$$

2.) At lowest energy the machine will have the major aperture problems,
→ here we have to minimise $\hat{\beta}$

3.) we need different beam optics adopted to the energy:
A Mini Beta concept will only be adequate at flat top.



LHC mini beta optics at 7000 GeV

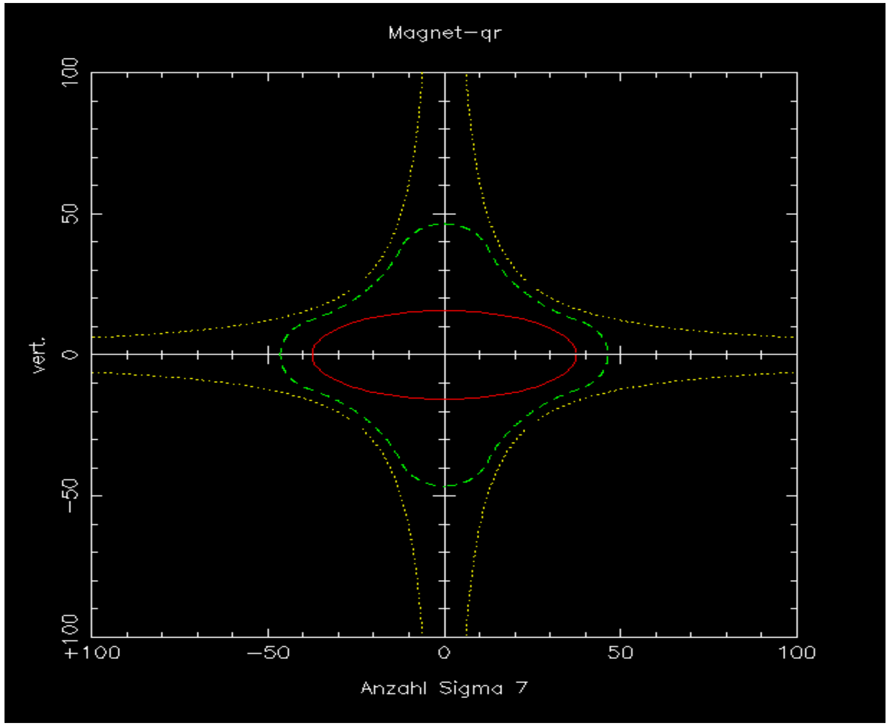


LHC injection optics at 450 GeV

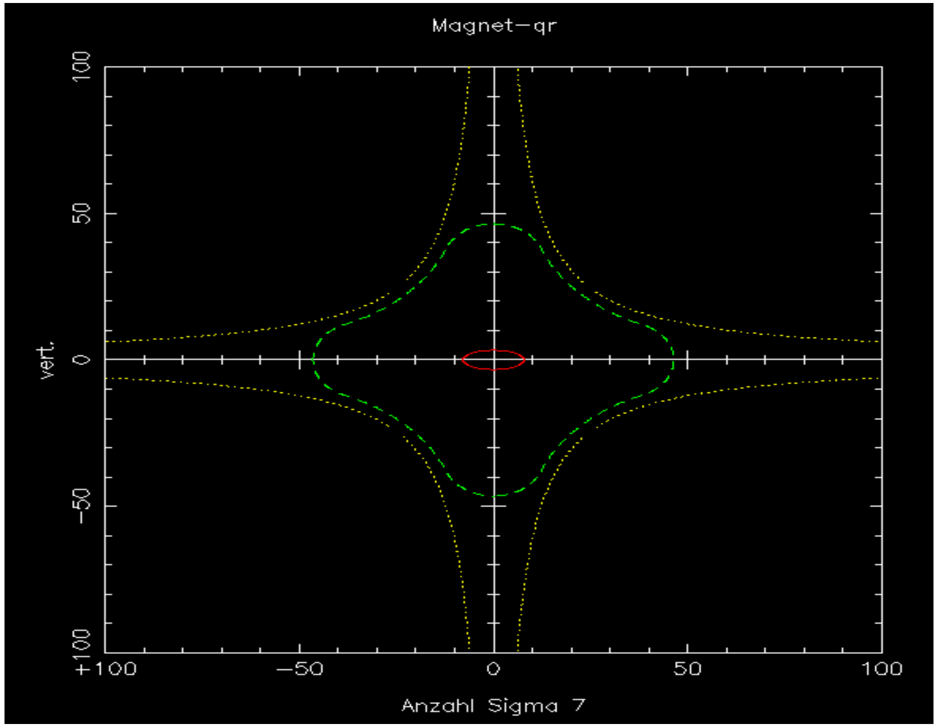
Example: HERA proton ring

injection energy: 40 GeV $\gamma = 43$
flat top energy: 920 GeV $\gamma = 980$

emittance $\varepsilon\,(40GeV) = 1.2 * 10^{-7}$
$\varepsilon\,(920GeV) = 5.1 * 10^{-9}$



7 σ beam envelope at E = 40 GeV



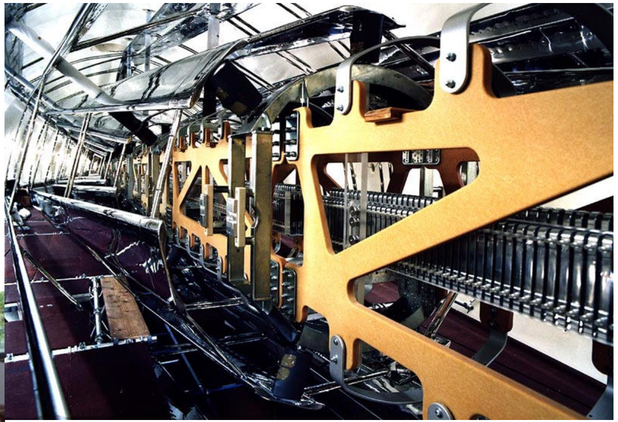
… and at E = 920 GeV

The „ not so ideal world “

15.) The „ Δp / p ≠ 0" Problem

ideal accelerator: *all particles will see the* *same accelerating voltage.*

→ $\Delta p / p = 0$



Vivitron, Straßbourg, inner structure of the acc. section

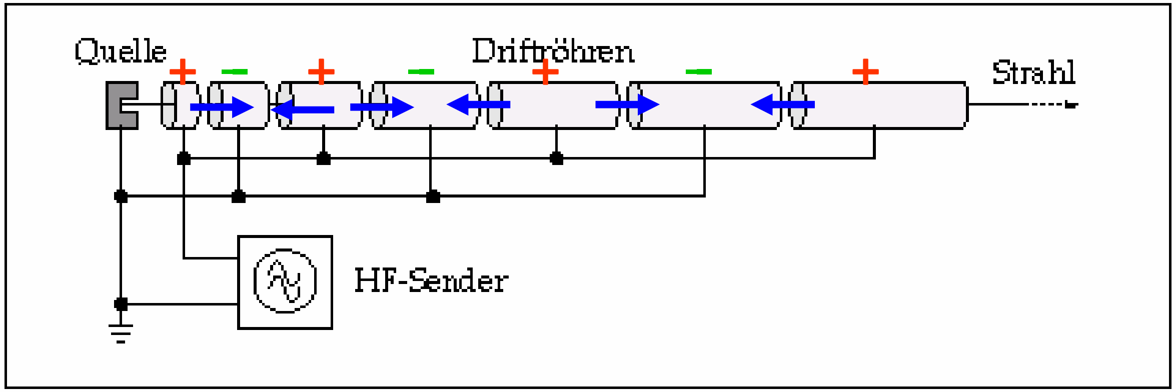
„nearly ideal“ accelerator: *Cockroft Walton or van de Graaf*

$\Delta p / p \approx 10^{-5}$

MP Tandem van de Graaf Accelerator at MPI for Nucl. Phys. Heidelberg

Linear Accelerator

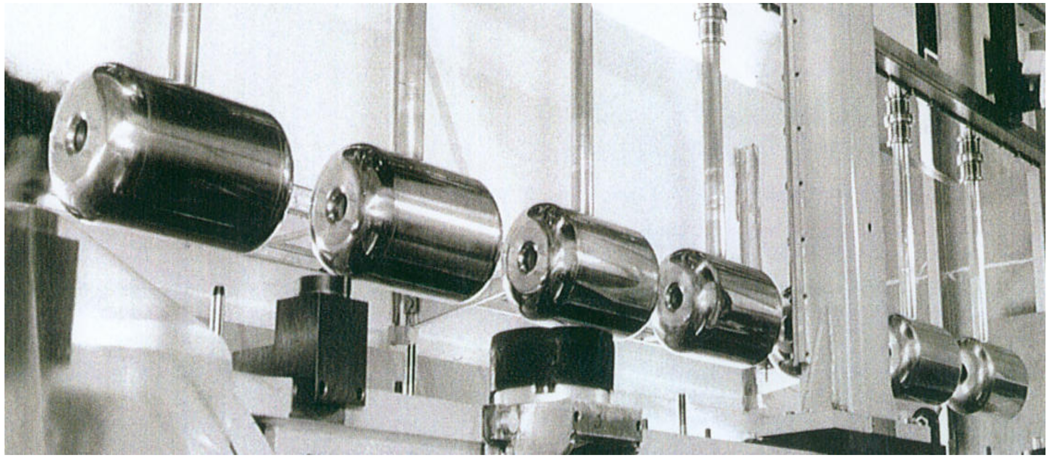
1928, Wideroe *schematic Layout:*



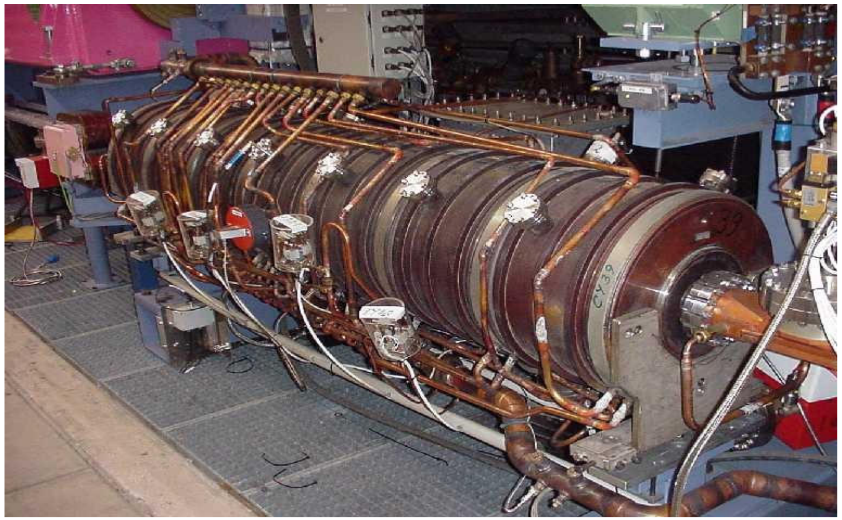
Energy Gain per „Gap":

$$W = q\,U_0 \sin \omega_{RF} t$$

drift tube structure at a proton linac



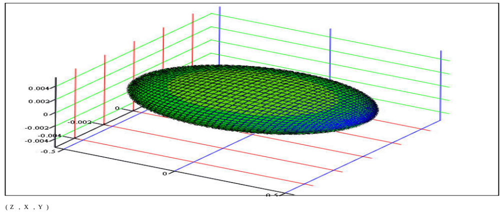
500 MHz cavities in an electron storage ring



* *RF Acceleration:* *multiple application of the same acceleration voltage; brillant idea to gain higher energies*

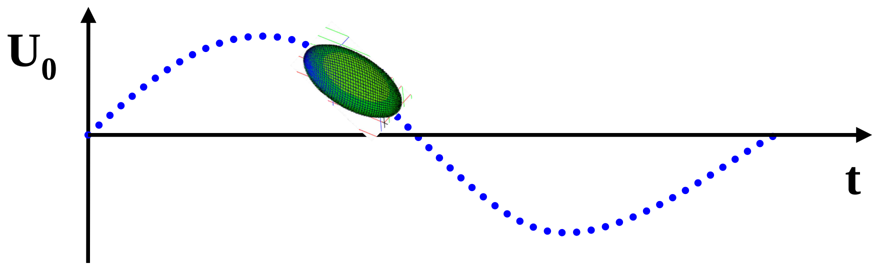
Problem: panta rhei !!!

(Heraklit: 540-480 v. Chr.)



Bunch length of Electrons ≈ 1cm

Example: HERA RF:



$\lambda = 60\ cm$

$$\nu = 500\ \ MHz$$
$$c = \lambda\ \nu$$

$\lambda = 60\ cm$

$$\sin(90^o) = 1$$
$$\sin(84^o) = 0.994$$

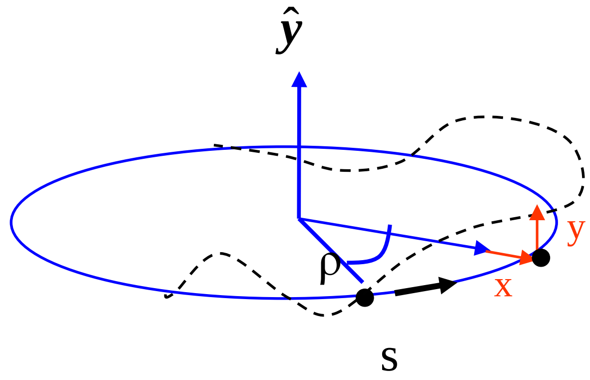
$$\frac{\Delta U}{U} = 6.0\ \ 10^{-3}$$

typical momentum spread of an electron bunch:

$$\frac{\Delta p}{p} \approx 1.0\ \ 10^{-3}$$

16.) Dispersion: trajectories for $\Delta p / p \neq 0$

Question: ***do you remember last session, page 12 ? … sure you do***



***Force** acting on the particle*

$$F = m\frac{d^2}{dt^2}(x+\rho) - \frac{mv^2}{x+\rho} = e\,B_y\,v$$

remember: ***$x \approx mm$, $\rho \approx m$ …*** → ***develop for small x***

$$m\frac{d^2x}{dt^2} - \frac{mv^2}{\rho}(1-\frac{x}{\rho}) = eB_y v$$

consider only linear fields, and change independent variable: $t \rightarrow s$

$$B_y = B_0 + x\frac{\partial B_y}{\partial x}$$

$$x'' - \frac{1}{\rho}(1-\frac{x}{\rho}) = \frac{e\;B_0}{mv} + \frac{e\;x\;g}{mv}$$

$p=p_0+\Delta p$

… but now take a small momentum error into account !!!

Dispersion:

develop for small momentum error

$$\Delta p \ll p_0 \Rightarrow \frac{1}{p_0 + \Delta p} \approx \frac{1}{p_0} - \frac{\Delta p}{p_0^2}$$

$$x'' - \frac{1}{\rho} + \frac{x}{\rho^2} \approx \underbrace{\frac{e\ B_0}{p_0}}_{-\frac{1}{\rho}} - \frac{\Delta p}{p_0^2} eB_0 + \underbrace{\frac{xeg}{p_0}}_{k * x} - \underbrace{xeg \frac{\Delta p}{p_0^2}}_{\approx 0}$$

$$x'' + \frac{x}{\rho^2} \approx \frac{\Delta p}{p_0} * \underbrace{\frac{(-eB_0)}{p_0}}_{\frac{1}{\rho}} + k * x = \frac{\Delta p}{p_0} * \frac{1}{\rho} + k * x$$

$$x'' + \frac{x}{\rho^2} - kx = \frac{\Delta p}{p_0} \frac{1}{\rho} \quad \longrightarrow \quad x'' + x\left(\frac{1}{\rho^2} - k\right) = \frac{\Delta p}{p_0} \frac{1}{\rho}$$

***Momentum spread** of the beam adds a term on the r.h.s. of the equation of motion.*
*→ **inhomogeneous differential equation.***

Dispersion:

$$x'' + x\left(\frac{1}{\rho^2} - k\right) = \frac{\Delta p}{p} \cdot \frac{1}{\rho}$$

general solution:

$$x(s) = x_h(s) + x_i(s)$$

$$\begin{cases} x_h''(s) + K(s) \cdot x_h(s) = 0 \\ x_i''(s) + K(s) \cdot x_i(s) = \frac{1}{\rho} \cdot \frac{\Delta p}{p} \end{cases}$$

Normalise with respect to Δp/p:

$$D(s) = \frac{x_i(s)}{\Delta p / p}$$

Dispersion function D(s)

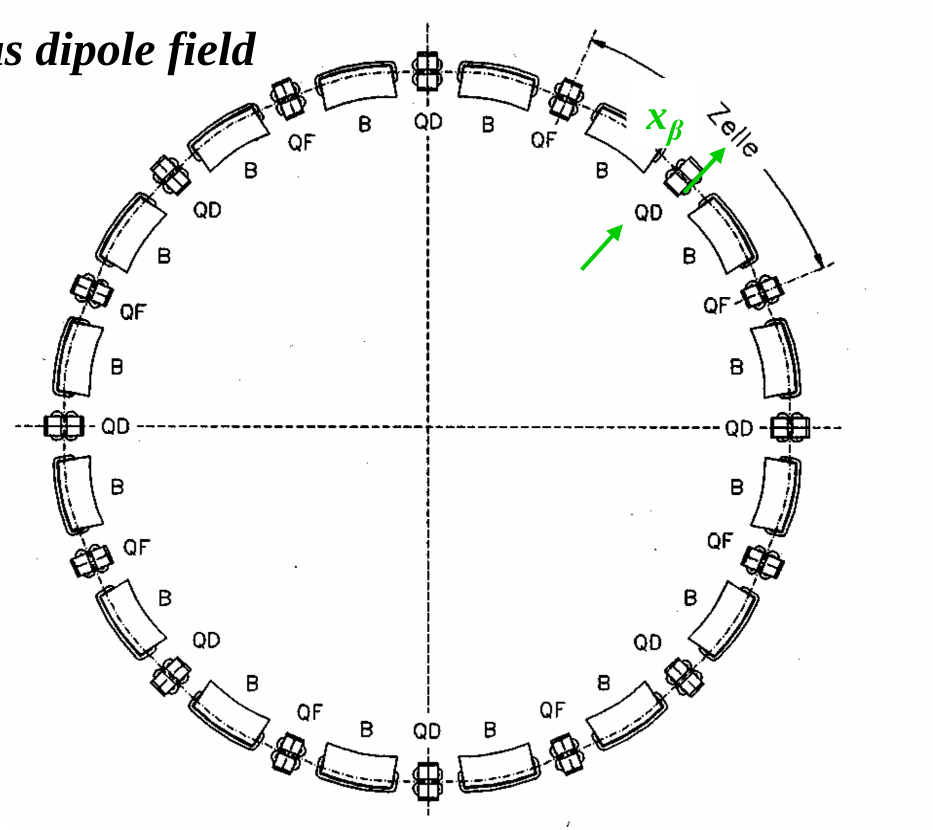
**** is that special orbit, an ideal particle would have for Δp/p = 1***

**** the orbit of any particle is the sum of the well known x_β and the dispersion***

**** as D(s) is just another orbit it will be subject to the focusing properties of the lattice***

Dispersion

Example: homogeneous dipole field



it for $\Delta p/p > 0$

$$: D(s) \cdot \frac{\Delta p}{p}$$

Matrix formalism:

$$x(s) = x_\beta(s) + D(s) \cdot \frac{\Delta p}{p}$$

$$x(s) = C(s) \cdot x_0 + S(s) \cdot x_0' + D(s) \cdot \frac{\Delta p}{p}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_0 + \frac{\Delta p}{p} \begin{pmatrix} D \\ D' \end{pmatrix}$$

Resume´:

beam emittance $$\varepsilon \propto \frac{1}{\beta\gamma}$$

beta function in a drift $$\beta(s) = \beta_0 - 2\alpha_0 s + \gamma_0 s^2$$

… and for α = 0 $$\beta(s) = \beta_0 + \frac{s^2}{\beta_0}$$

particle trajectory for Δp/p ≠ 0
inhomogenious equation

$$x'' + x\left(\frac{1}{\rho^2} - k\right) = \frac{\Delta p}{p_0}\frac{1}{\rho}$$

… and its solution $$x(s) = x_\beta(s) + D(s)\cdot\frac{\Delta p}{p}$$

Appendix:

stability criterion …. proof for the disbelieving collegues !!

Matrix for 1 turn:

$$M = \begin{pmatrix} \cos\psi_{turn} + \alpha_s \sin\psi_{turn} & \beta_s \sin\psi_{turn} \\ -\gamma_s \sin\psi_{turn} & \cos\psi_{turn} - \alpha_s \sin\psi_{turn} \end{pmatrix} = \cos\psi \cdot \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{I} + \sin\psi \underbrace{\begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}}_{J}$$

Matrix for 2 turns:

$$M^2 = (I \cos\psi_1 + J \sin\psi_1)(I \cos\psi_2 + J \sin\psi_2)$$

$$= I^2 \cos\psi_1 \cos\psi_2 + IJ \cos\psi_1 \sin\psi_2 + JI \sin\psi_1 \cos\psi_2 + J^2 \sin\psi_1 \sin\psi_2$$

now …

$$I^2 = I$$

$$I\ J = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}$$

$$J\ I = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} * \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}$$

$$I\ J = J\ I$$

$$J^2 = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} * \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} = \begin{pmatrix} \alpha^2 - \gamma\beta & \alpha\beta - \beta\alpha \\ -\gamma\alpha + \alpha\gamma & \alpha^2 - \gamma\beta \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I$$

$$M^2 = I \cos(\psi_1 + \psi_2) + J \sin(\psi_1 + \psi_2)$$

$$M^2 = I \cos(2\psi) + J \sin(2\psi)$$