Lattice Design in Particle Accelerators

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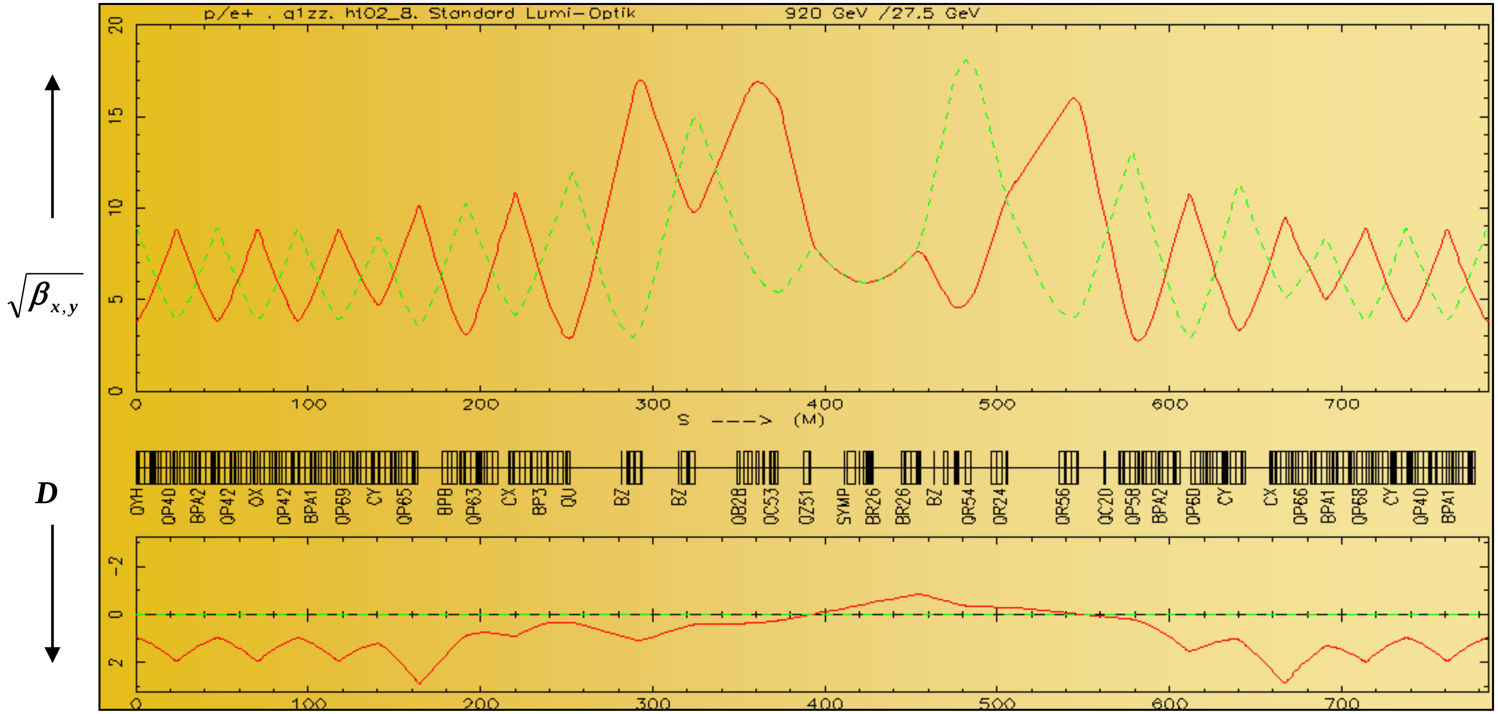


1952: Courant, Livingston, Snyder:

Theory of strong focusing in particle beams

Lattice Design: *„… how to build a storage ring“*

High energy accelerators → circular machines

somewhere in the lattice we need a number of dipole magnets, that are bending the design orbit to a closed ring

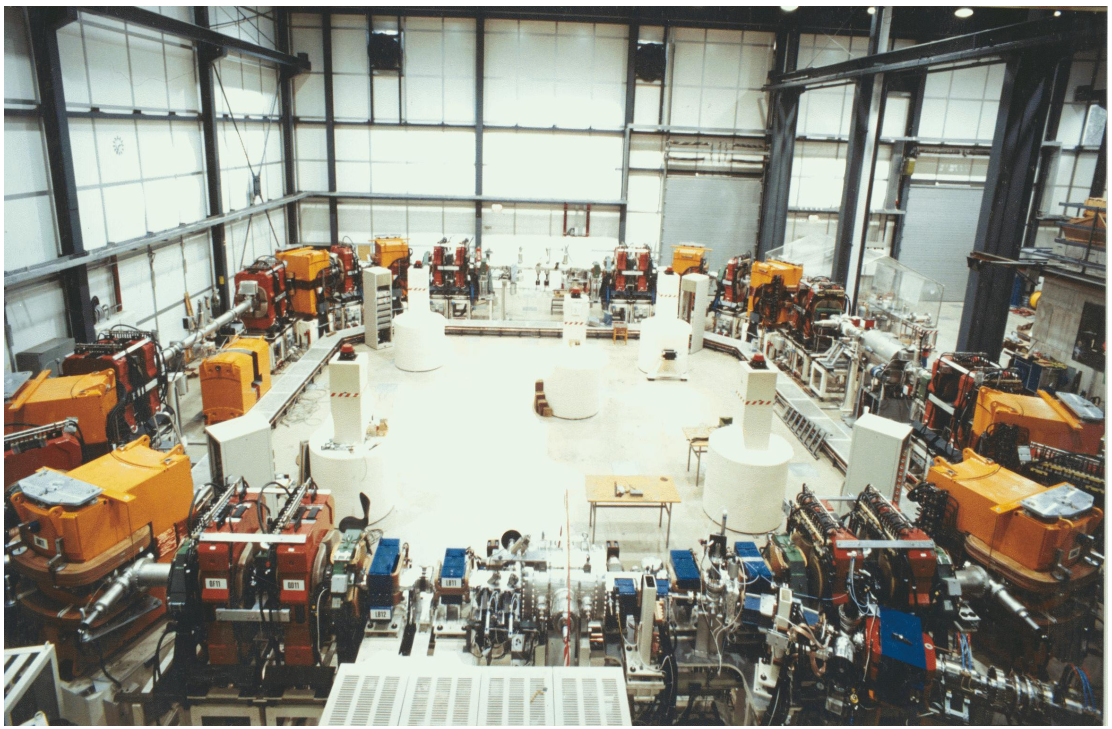
Geometry of the ring:

centrifugal force = Lorentz force

$$e * v * B = \frac{mv^2}{\rho}$$

$$\rightarrow e * B = \frac{mv}{\rho} = p / \rho$$

$$\rightarrow B * \rho = p / e$$



Example: heavy ion storage ring TSR
8 dipole magnets of equal bending strength

p = momentum of the particle,
ρ = curvature radius

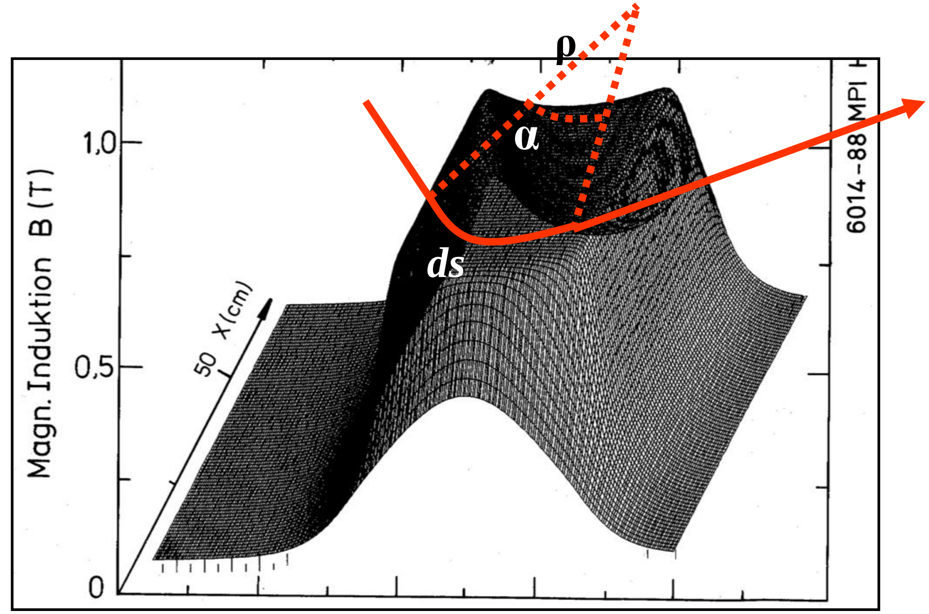
Bρ= beam rigidity

Circular Orbit:

„… defining the geometry"

$$\alpha = \frac{ds}{\rho} \approx \frac{dl}{\rho}$$

$$\alpha = \frac{B*dl}{B*\rho}$$



field map of a storage ring dipole magnet

The angle swept out in one revolution must be 2π, so

$$\alpha = \frac{\int Bdl}{B*\rho} = 2\pi \quad \rightarrow \quad \int Bdl = 2\pi * \frac{p}{q}$$

… for a full circle

Nota bene: $\frac{\Delta B}{B} \approx 10^{-4}$ ***is usually required !!***

Example LHC:

7000 GeV Proton storage ring

dipole magnets N = 1232

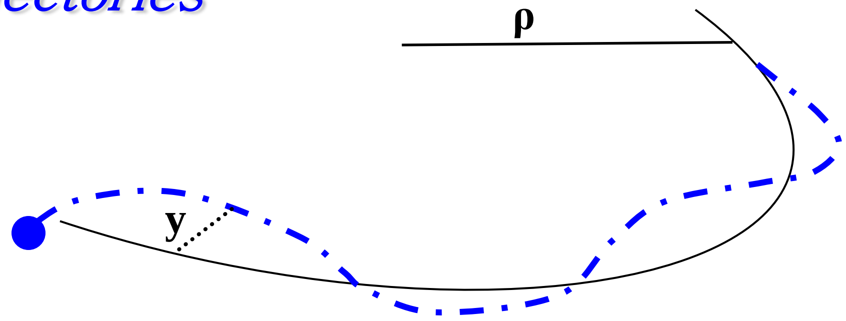
l = 15 m

q = +1 e

$$\int B\,dl \approx N\ l\ B = 2\pi\ p/e$$

$$B \approx \frac{2\pi\ 7000\ 10^{9}\,eV}{1232\ 15\,m\ 3\ 10^{8}\,\frac{m}{s}\ \ e} = 8.3\,Tesla$$

„ Focusing forces ··· single particle trajectories“



$$y'' + K * y = 0$$

$K = -k + 1/\rho^2$ **hor. plane**

$K = k$ **vert. plane**

dipole magnet $\frac{1}{\rho} = \frac{\boldsymbol{B}}{\boldsymbol{p}/\boldsymbol{q}}$

quadrupole magnet $\boldsymbol{k} = \frac{\boldsymbol{g}}{\boldsymbol{p}/\boldsymbol{q}}$

Example: HERA Ring:

Bending radius: ρ = 580 m

Quadrupol Gradient: g = 110 T/m

k = $33.64 * 10^{-3} /m^2$

$1/\rho^2 = 2.97 * 10^{-6} /m^2$

For estimates in large accelerators the weak focusing term $1/\rho^2$ can in general be neglected

Solution for a focusing magnet

$$y(s) = y_0 * \cos(\sqrt{K} * s) + \frac{y'_0}{\sqrt{K}} * \sin(\sqrt{K} * s)$$

$$y'(s) = -y_0 * \sqrt{K} * \sin(\sqrt{K} * s) + y'_0 * \cos(\sqrt{K} * s)$$

Or written more convenient in matrix form:

$$\begin{pmatrix} y \\ y' \end{pmatrix}_s = M * \begin{pmatrix} y \\ y' \end{pmatrix}_0$$

Hor. focusing Quadrupole Magnet

$$M_{QF} = \begin{pmatrix} \cos(\sqrt{K} * l) & \frac{1}{\sqrt{K}} \sin(\sqrt{K} * l) \\ -\sqrt{K} \sin(\sqrt{K} * l) & \cos(\sqrt{K} * l) \end{pmatrix}$$

Hor. defocusing Quadrupole Magnet

$$M_{QD} = \begin{pmatrix} \cosh(\sqrt{K} * l) & \frac{1}{\sqrt{K}} \sinh(\sqrt{K} * l) \\ \sqrt{K} \sinh(\sqrt{K} * l) & \cosh(\sqrt{K} * l) \end{pmatrix}$$

Drift space $$M_{Drift} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$$

$$M_{lattice} = M_{QF1} * M_{D1} * M_{QD} * M_{D1} * M_{QF2} \dots$$

VII.) Transfer Matrix M

which can be expressed ... for convenience ... in matrix form

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M \begin{pmatrix} x \\ x' \end{pmatrix}_0$$

$$M = \begin{pmatrix} \sqrt{\dfrac{\beta_s}{\beta_0}}\left(\cos\psi_s + \alpha_0 \sin\psi_s\right) & \sqrt{\beta_s \beta_0}\,\sin\psi_s \\ \dfrac{(\alpha_0 - \alpha_s)\cos\psi_s - (1 + \alpha_0 \alpha_s)\sin\psi_s}{\sqrt{\beta_s \beta_0}} & \sqrt{\dfrac{\beta_0}{\beta s}}\left(\cos\psi_s - \alpha_s \sin\psi_s\right) \end{pmatrix}$$

**** we can calculate the single particle trajectories between two locations in the ring,***
if we know the α β γ at these positions.

**** and nothing but the α β γ at these positions.***

* ... ***!***

Periodic Lattices

In the case of periodic lattices the transfer matrix can be expressed as a function of a set of periodic parameters α, β, γ

$$M(s) = \begin{pmatrix} \cos\mu + \alpha_S \sin\mu & \beta_S \sin\mu \\ -\gamma_S \sin\mu & \cos(\mu) - \alpha_S \sin\mu \end{pmatrix}$$

$$\mu = \int_s^{s+L} \frac{dt}{\beta(t)}$$

μ = phase advance per period:

For stability of the motion in periodic lattice structures it is required that

$$|trace(M)| < 2$$

In terms of these new periodic parameters the solution of the equation of motion is

$$y(s) = \sqrt{\varepsilon} * \sqrt{\beta(s)} * \cos(\Phi(s) - \delta)$$

$$y'(s) = \frac{-\sqrt{\varepsilon}}{\sqrt{\beta}} * \{\sin(\Phi(s) - \delta) + \alpha\cos(\Phi(s) - \delta)\}$$

VIII.) Transformation of α, β, γ

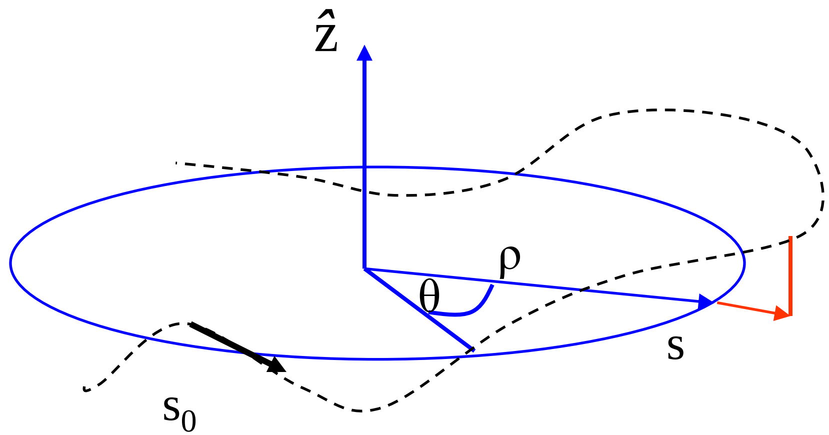
consider two positions in the storage ring: s_0 , s

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M \cdot \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$

since ε = const:

$$\varepsilon = \beta x'^2 + 2\alpha x x' + \gamma x^2$$

$$\varepsilon = \beta_0 x_0'^2 + 2\alpha_0 x_0 x_0' + \gamma_0 x_0^2$$



express x_0, x'_0 as a function of x, x´.

... remember W = CS´-SC´ = 1

$$\left.\begin{aligned} \begin{pmatrix} x \\ x' \end{pmatrix}_0 &= M^{-1} \cdot \begin{pmatrix} x \\ x' \end{pmatrix}_s \\ M^{-1} &= \begin{pmatrix} S' & -S \\ -C' & C \end{pmatrix} \end{aligned}\right\} \Rightarrow \quad \begin{aligned} x_0 &= S'x - Sx' \\ x_0' &= -C'x + Cx' \end{aligned}$$

inserting into ε

$$\varepsilon = \beta x'^2 + 2\alpha x x' + \gamma x^2$$

$$\varepsilon = \beta_0 (Cx' - C'x)^2 + 2\alpha_0 (S'x - Sx')(Cx' - C'x) + \gamma_0 (S'x - Sx')^2$$

sort via x, x´and compare the coefficients to get ....

$$\beta(s) = C^2\beta_0 - 2SC\alpha_0 + S^2\gamma_0$$

$$\alpha(s) = -CC'\beta_0 + (SC' + S'C)\alpha_0 - SS'\gamma_0$$

$$\gamma(s) = C'^2\beta_0 - 2S'C'\alpha_0 + S'^2\gamma_0$$

in matrix notation:

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_s = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC' + CS' & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} \cdot \begin{pmatrix} \beta_0 \\ \alpha_0 \\ \gamma_0 \end{pmatrix}$$

!

1.) this expression is important

2.) given the twiss parameters α, β, γ at any point in the lattice we can transform them and calculate their values at any other point in the ring.

3.) the transfer matrix is given by the focusing properties of the lattice elements, the elements of M are just those that we used to calculate single particle trajectories.

4.) go back to point 1.)

The new parameters α, β, γ can be transformed through the lattice via the matrix elements defined above.

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_S = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC'+S'C & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} * \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_0$$

Question: „ What does that mean ???? "

... and here starts the lattice design !!!

Most simple example: drift space

$$M = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$$

particle coordinates

$$\begin{pmatrix} x \\ x' \end{pmatrix}_l = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} x \\ x' \end{pmatrix}_0 \quad \rightarrow \quad \boxed{\begin{aligned} x(l) &= x_0 + l * x_0' \\ x'(l) &= x_0' \end{aligned}}$$

transformation of twiss parameters:

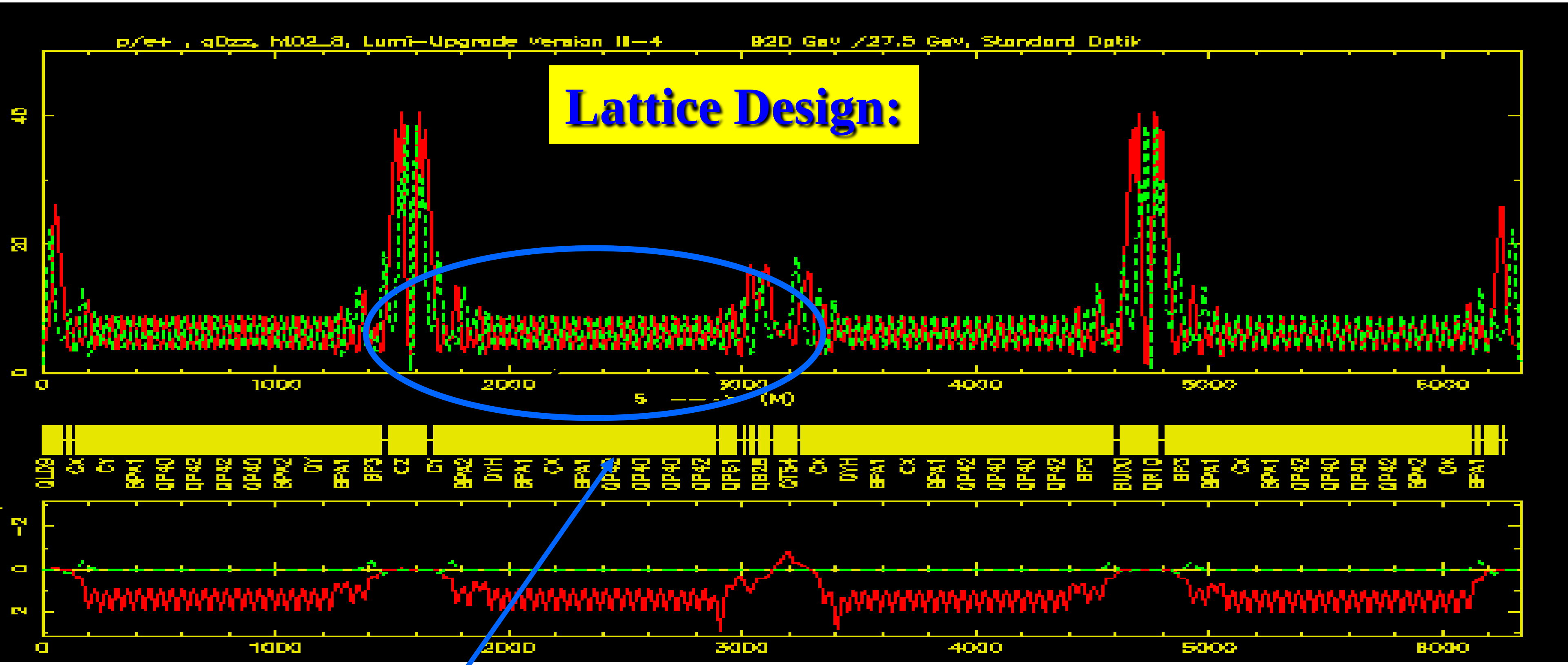
$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_l = \begin{pmatrix} 1 & -2l & l^2 \\ 0 & 1 & -l \\ 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_0$$

$$\boxed{\beta(s) = \beta_0 - 2l * \alpha_0 + l^2 * \gamma_0}$$

Stability ...?

$$trace(M) = 1 + 1 = 2$$

→ ***A periodic solution doesn‘t exist in a lattice built exclusively out of drift spaces.***



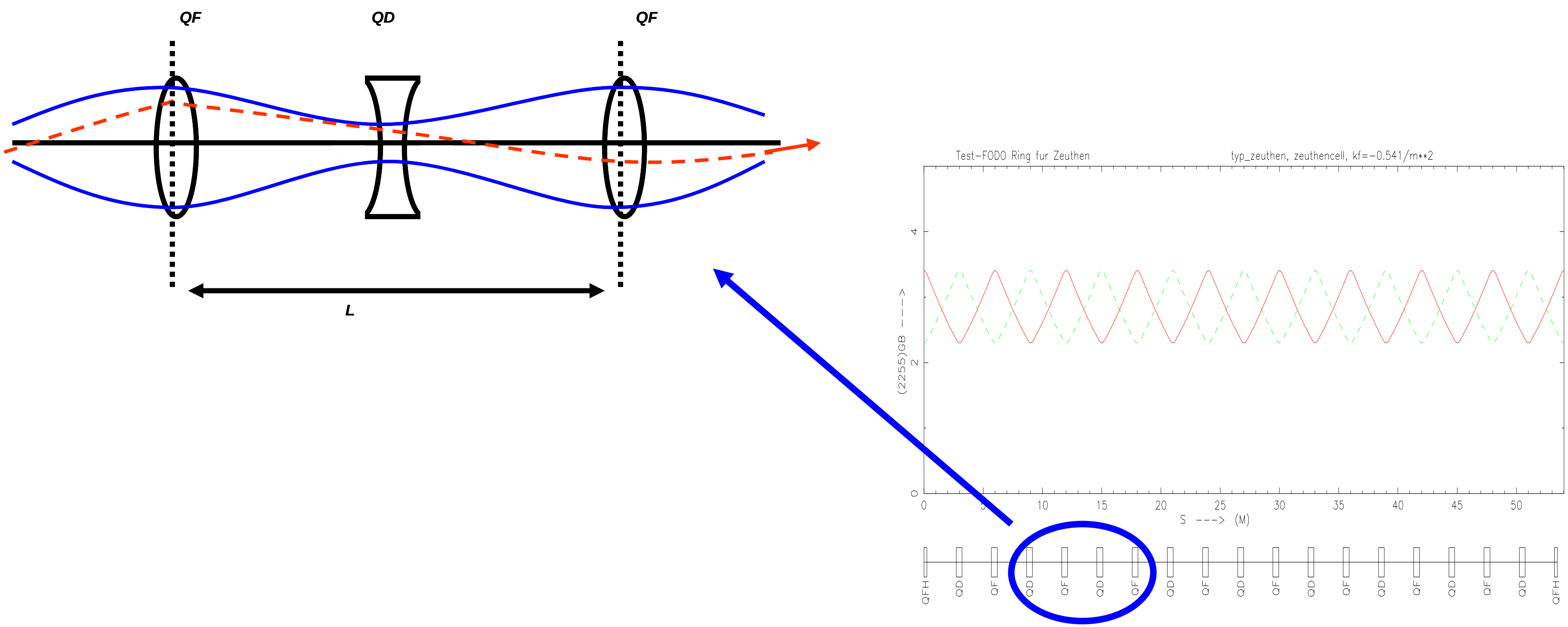
Arc: regular **(periodic) magnet structure:**

bending magnets → define the energy of the ring
main focusing & tune control, chromaticity correction,
multipoles for higher order corrections

Straight sections: **drift spaces for injection, dispersion suppressors,**
low beta insertions, RF cavities, etc....
... and the high energy experiments if they cannot be avoided

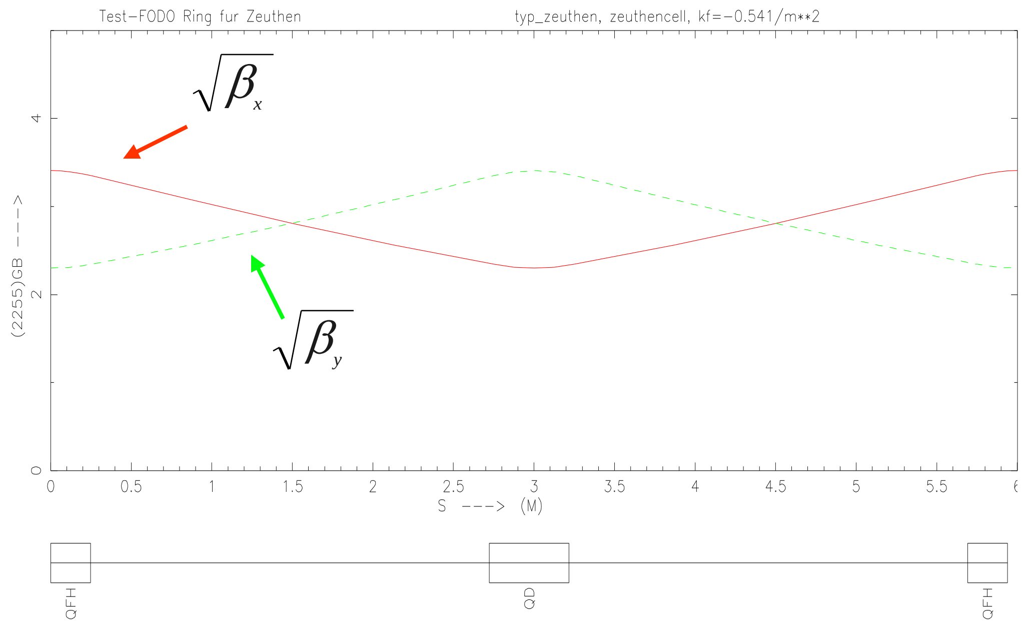
The FoDo-Lattice

A magnet structure consisting of focusing and defocusing quadrupole lenses in alternating order with nothing in between.
(Nothing = elements that can be neglected on first sight: drift, bending magnets, RF structures ... and especially experiments...)



Starting point for the calculation: in the middle of a focusing quadrupole
Phase advance per cell $\mu = 45°$,
→ calculate the twiss parameters for a periodic solution

Periodic solution of a FoDo Cell



Output of the optics program:

Nr	Type	Length	Strength	β_x	α_x	φ_x	β_z	α_z	φ_z
		m	1/m2	m		$1/2\pi$	m		$1/2\pi$
0	IP	0,000	0,000	11,611	0,000	0,000	5,295	0,000	0,000
1	QFH	0,250	-0,541	11,228	1,514	0,004	5,488	-0,781	0,007
2	QD	3,251	0,541	5,488	-0,781	0,070	11,228	1,514	0,066
3	QFH	6,002	-0,541	11,611	0,000	0,125	5,295	0,000	0,125
4	IP	6,002	0,000	11,611	0,000	0,125	5,295	0,000	0,125

QX= 0,125 QZ= 0,125

$$0.125 * 2\pi = 45°$$

Can we understand, what the optics code is doing?

matrices

$$M_{QF} = \begin{pmatrix} \cos(\sqrt{K}*l_q) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}*l_q) \\ -\sqrt{K}\sin(\sqrt{K}*l_q) & \cos(\sqrt{K}*l_q) \end{pmatrix}, \qquad M_{Drift} = \begin{pmatrix} 1 & l \\ 0 & 1_d \end{pmatrix}$$

strength and length of the FoDo elements

***K* = +/- 0.54102 m^{-2}**
***lq* = 0.5 m**
***ld* = 2.5 m**

The matrix for the complete cell is obtained by multiplication of the element matrices

$$M_{FoDo} = M_{qfh} * M_{ld} * M_{qd} * M_{ld} * M_{qfh}$$

Putting the numbers in and multiplying out ...

$$M_{FoDo} = \begin{pmatrix} 0.707 & 8.206 \\ -0.061 & 0.707 \end{pmatrix}$$

The transfer matrix for 1 period gives us all the information that we need !

1.) is the motion stable?

$$trace(M_{FoDo}) = 1.415 \rightarrow \underline{\underline{< 2}}$$

2.) Phase advance per cell

$$M(s) = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos(\mu) - \alpha\sin\mu \end{pmatrix} \rightarrow$$

$$\cos(\mu) = \frac{1}{2} * trace(M) = 0.707$$

$$\mu = arc\cos(\frac{1}{2} * trace(M)) = \underline{\underline{45°}}$$

3.) hor β-function

$$\beta = \frac{M(1,2)}{\sin(\mu)} = \underline{\underline{11.611 \ m}}$$

4.) hor α-function

$$\alpha = \frac{M(1,1) - \cos(\mu)}{\sin(\mu)} = \underline{\underline{0}}$$

Can we do it a little bit easier ?

We can: … the „*thin lens approximation*"

Matrix of a focusing quadrupole magnet:

$$M_{QF} = \begin{pmatrix} \cos(\sqrt{K}*l) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}*l) \\ -\sqrt{K}\sin(\sqrt{K}*l) & \cos(\sqrt{K}*l) \end{pmatrix}$$

If the focal length *f* is much larger than the length of the quadrupole magnet,

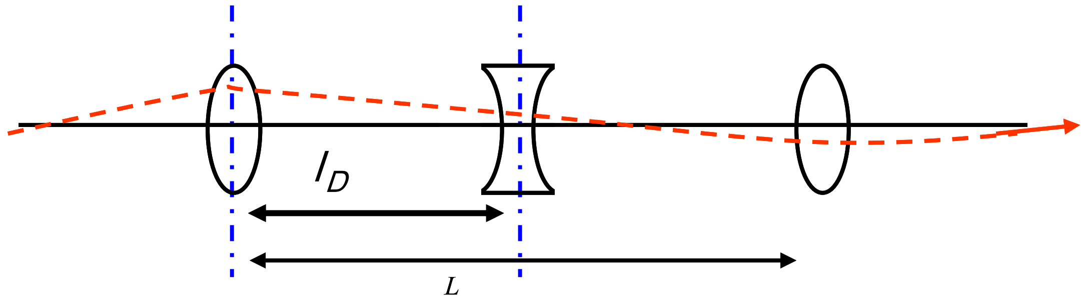
$$f = 1/kl_Q >> l_Q$$

the transfer matrix can be aproximated using

$$kl_q = const, \; l_q \to 0$$

$$M = \begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix}$$

FoDo in thin lens approximation



$$l_D = L/2,$$
$$\tilde{f} = 2f$$

Calculate the matrix for a half cell, starting in the middle of a foc. quadrupole:

$$M_{halfCell} = M_{QD/2} * M_{lD} * M_{QF/2}$$

$$M_{halfCell} = \begin{pmatrix} 1 & 0 \\ 1/\tilde{f} & 1 \end{pmatrix} * \begin{pmatrix} 1 & l_D \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} 1 & 0 \\ -1/\tilde{f} & 1 \end{pmatrix}$$

note: $\tilde{f}$ denotes the focusing strength of half a quadrupole, so $\tilde{f} = 2f$

$$M_{halfCell} = \begin{pmatrix} 1 - l_D/\tilde{f} & l_D \\ -l_D/\tilde{f}^2 & 1 + l_D/\tilde{f} \end{pmatrix}$$

for the second half cell set $f \rightarrow -f$

FoDo in thin lens approximation

Matrix for the complete FoDo cell:

$$M = \begin{pmatrix} 1+{l_D}/{\tilde{f}} & l_D \\ -{l_D}/{\tilde{f}^2} & 1-{l_D}/{\tilde{f}} \end{pmatrix} * \begin{pmatrix} 1-{l_D}/{\tilde{f}} & l_D \\ -{l_D}/{\tilde{f}^2} & 1+{l_D}/{\tilde{f}} \end{pmatrix}$$

$$M = \begin{pmatrix} 1-\frac{2l_D^2}{\tilde{f}^2} & 2l_D(1+\frac{l_D}{\tilde{f}}) \\ 2(\frac{l_D^2}{\tilde{f}^3}-\frac{l_D}{\tilde{f}^2}) & 1-2\frac{l_D^2}{\tilde{f}^2} \end{pmatrix}$$

Now we know, that the phase advance is related to the transfer matrix by

$$\cos\mu = \frac{1}{2} trace \ (M) = {1}/{2} * (2-\frac{4l_D^2}{\tilde{f}^2}) = 1-\frac{2l_D^2}{\tilde{f}^2}$$

After some beer and with a little bit of trigonometric gymnastics

$$\cos(x) = \cos^2({x}/{2}) - \sin^2(x/2) = 1-2\sin^2({x}/{2})$$

we can calculate the phase advance as a function of the FoDo parameter …

$$\cos(\mu) = 1 - 2\sin^2(\mu/2) = 1 - \frac{2l_D^2}{\tilde{f}^2}$$

$$\sin(\mu/2) = l_D / \tilde{f} = \frac{L_{Cell}}{2\tilde{f}}$$

$$\sin(\mu/2) = \frac{L_{Cell}}{4f}$$

Example:

45-degree Cell

$$L_{Cell} = l_{QF} + l_D + l_{QD} + l_D = 0.5m + 2.5m + 0.5m + 2.5m = 6m$$

$$1/f = k * l_Q = 0.5m * 0.541\, m^{-2} = 0.27\, m^{-1}$$

$$\sin(\mu/2) \approx \frac{L_{Cell}}{4f} = 0.405$$

$$\rightarrow \mu \approx 47.8°$$

$$\rightarrow \beta \approx 11.4m$$

Remember:
Exact calculation yields:

$$\mu = 45°$$

$$\beta = 11.6m$$

Stability in a FoDo structure

SPS Lattice

$$M_{FoDo} = \begin{pmatrix} 1 - \frac{2l_D^2}{\tilde{f}^2} & 2l_D(1 + \frac{l_D}{\tilde{f}}) \\ 2(\frac{l_D^2}{\tilde{f}^3} - \frac{l_D}{\tilde{f}^2}) & 1 - 2\frac{l_D^2}{\tilde{f}^2} \end{pmatrix}$$

Stability requires:

$$\left|\boldsymbol{Trace}(\boldsymbol{M})\right| < 2$$

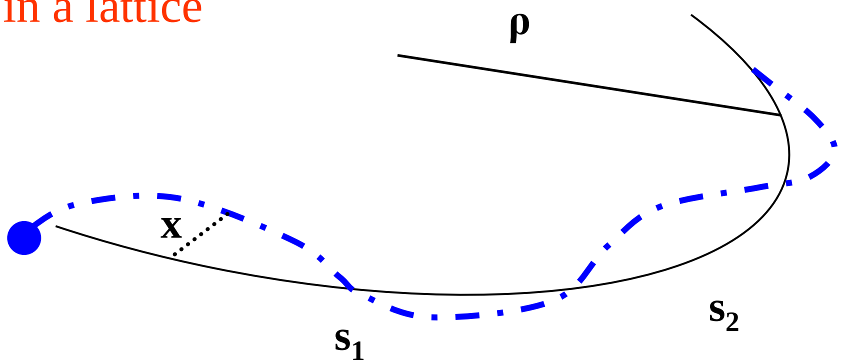
$$\left|\boldsymbol{Trace}(\boldsymbol{M})\right| = \left|2 - \frac{4\boldsymbol{l}_{\boldsymbol{d}}^2}{\tilde{\boldsymbol{f}}^2}\right| < 2$$

$$\rightarrow f > \frac{L_{cell}}{4}$$

For stability the focal length has to be larger than a quarter of the cell length !!

Transformation Matrix in Terms of the Twiss parameters

Transformation of the coordinate vector (x,x´) in a lattice



$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = M_{s1,s2} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

General solution of the equation of motion

$$x(s) = \sqrt{\varepsilon * \beta(s)} * \cos(\psi(s) + \varphi)$$

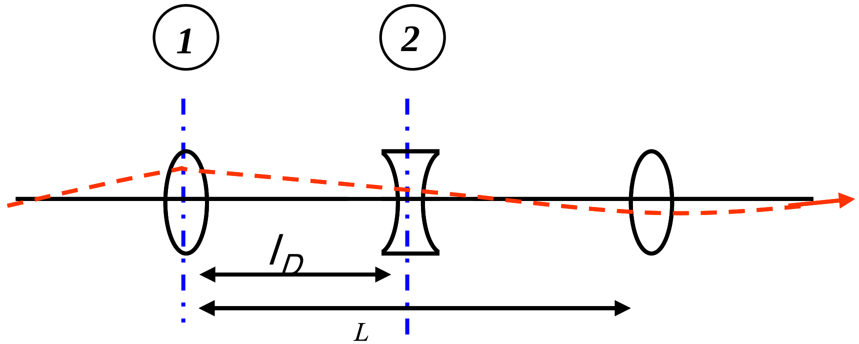
$$x'(s) = \sqrt{\varepsilon / \beta(s)} * \{\alpha(s)\cos(\psi(s) + \varphi) + \sin(\psi(s) + \varphi)\}$$

Transformation of the coordinate vector (x,x´) expressed
as a function of the twiss parameters

$$M_{1\to 2} = \begin{pmatrix} \sqrt{\frac{\beta_2}{\beta_1}}(\cos\psi_{12} + \alpha_1 \sin\psi_{12}) & \sqrt{\beta_1\beta_2}\sin\psi_{12} \\ \frac{(\alpha_1 - \alpha_2)\cos\psi_{12} - (1 + \alpha_1\alpha_2)\sin\psi_{12}}{\sqrt{\beta_1\beta_2}} & \sqrt{\frac{\beta_1}{\beta_2}}(\cos\psi_{12} - \alpha_2 \sin\psi_{12}) \end{pmatrix}$$

Transfer matrix for half a FoDo cell:

$$M_{halfcell} = \begin{pmatrix} 1 - l_D/\tilde{f} & l_D \\ -l_D/\tilde{f}^2 & 1 + l_D/\tilde{f} \end{pmatrix}$$



Compare to the twiss parameter form of M

$$M_{1\to 2} = \begin{pmatrix} \sqrt{\frac{\beta_2}{\beta_1}}(\cos\psi_{12} + \alpha_1 \sin\psi_{12}) & \sqrt{\beta_1\beta_2}\sin\psi_{12} \\ \frac{(\alpha_1 - \alpha_2)\cos\psi_{12} - (1+\alpha_1\alpha_2)\sin\psi_{12}}{\sqrt{\beta_1\beta_2}} & \sqrt{\frac{\beta_1}{\beta_2}}(\cos\psi_{12} - \alpha_2 \sin\psi_{12}) \end{pmatrix}$$

In the middle of a foc (defoc) quadrupole of the FoDo we allways have $\alpha = 0$, and the half cell will lead us from β_{max} to β_{min}

$$M = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{\check{\beta}}{\hat{\beta}}}\cos\frac{\mu}{2} & \sqrt{\check{\beta}\hat{\beta}}\sin\frac{\mu}{2} \\ \frac{-1}{\sqrt{\hat{\beta}\check{\beta}}}\sin\frac{\mu}{2} & \sqrt{\frac{\hat{\beta}}{\check{\beta}}}\cos\frac{\mu}{2} \end{pmatrix}$$

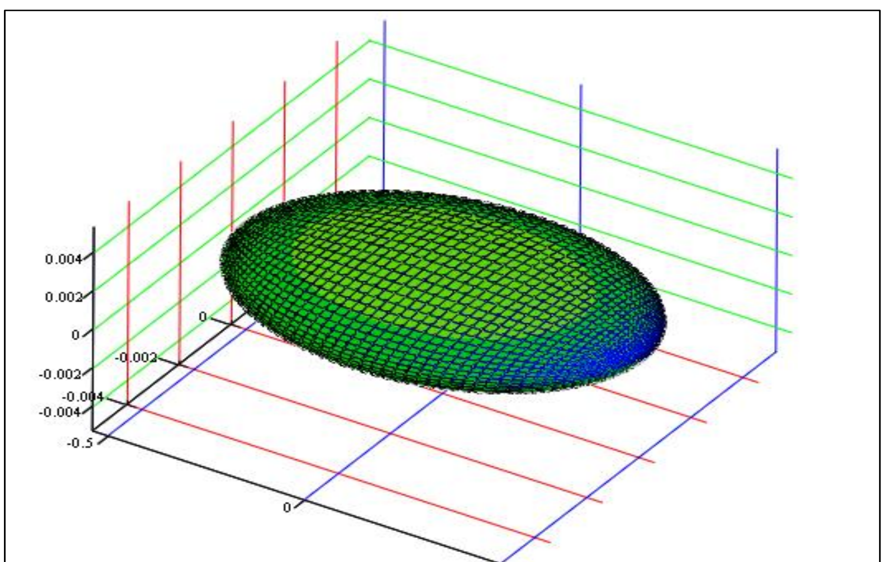
Solving for β_{max} and β_{min} and remembering that …. $\sin\frac{\mu}{2} = \frac{l_D}{\tilde{f}} = \frac{L}{4f}$

$$\frac{S'}{C} = \frac{\hat{\beta}}{\check{\beta}} = \frac{1 + l_D/\tilde{f}}{1 - l_D/\tilde{f}} = \frac{1+\sin\frac{\mu}{2}}{1-\sin\frac{\mu}{2}}$$

$$\frac{S}{C'} = \hat{\beta}\,\check{\beta} = \tilde{f}^{\,2} = \frac{l_D^2}{\sin^2\frac{\mu}{2}}$$

$$\rightarrow \quad \boxed{\hat{\beta} = \frac{(1+\sin\frac{\mu}{2})L}{\sin\mu}} \; !$$

$$\boxed{\check{\beta} = \frac{(1-\sin\frac{\mu}{2})L}{\sin\mu}} \; !$$



(Z, X, Y)

The maximum and minimum values of the β-function are solely determined by the phase advance and the length of the cell.

Longer cells lead to larger β

typical shape of a proton bunch in the HERA FoDo Cell

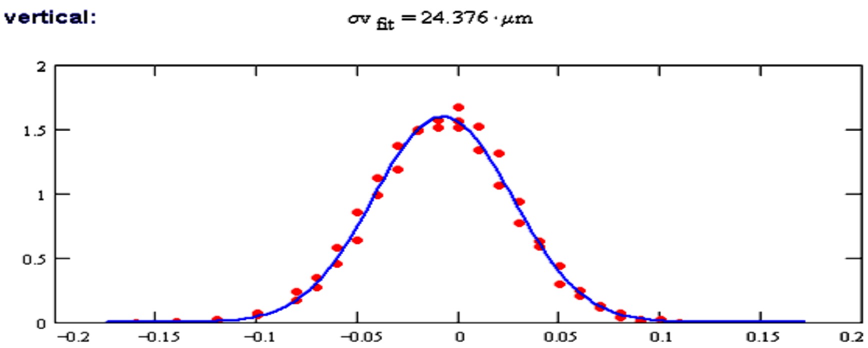
Beam dimension:

Optimisation of the FoDo Phase advance:

In both planes a gaussian particle distribution is assumed, given by the beam emittance ε and the β-function

$$\sigma = \sqrt{\varepsilon\beta}$$

HERA beam size

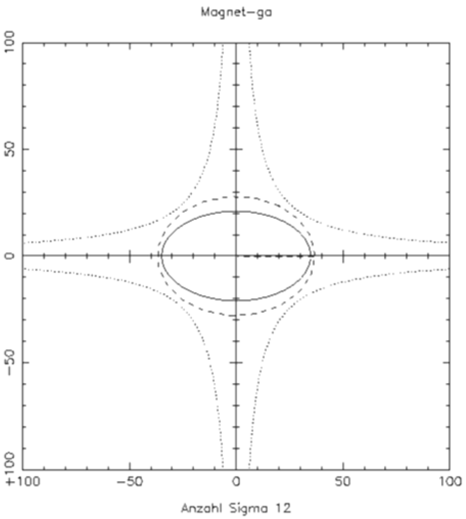


In general proton beams are „*round*" in the sense that

$$\varepsilon_x \approx \varepsilon_y$$

So for highest aperture we have to minimise the β-function in both planes:

$$r^2 = \varepsilon_x\beta_x + \varepsilon_y\beta_y$$



typical beam envelope, vacuum chamber and pole shape in a foc. Quadrupole lens in HERA

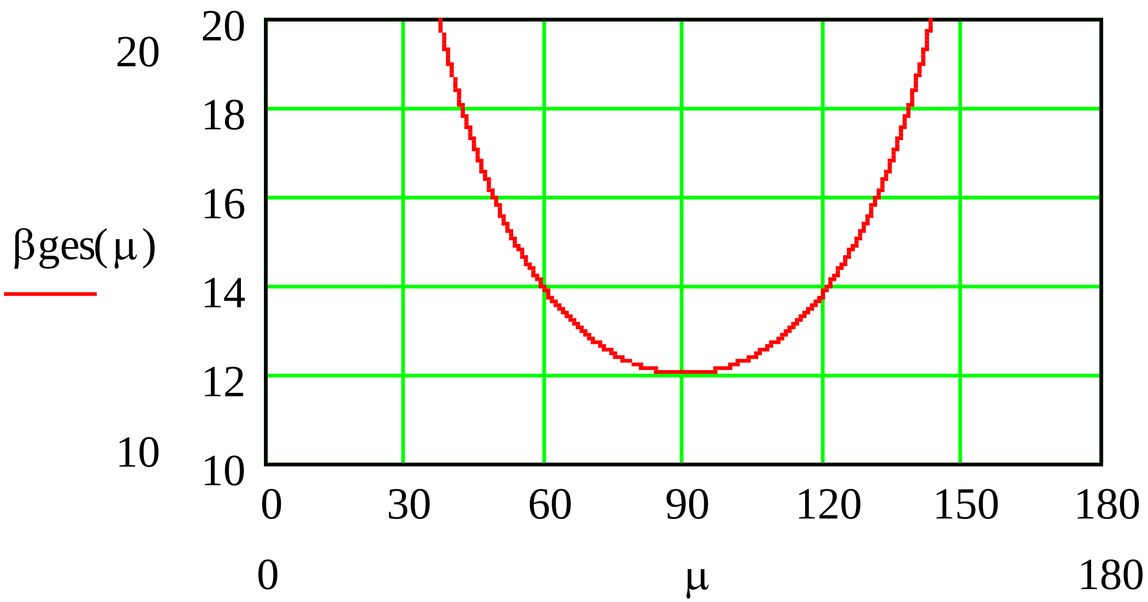
Optimising the FoDo phase advance

$$r^2 = \varepsilon_x \beta_x + \varepsilon_y \beta_y$$

search for the phase advance μ that results in a minimum of the sum of the beta's

$$\hat{\beta} + \check{\beta} = \frac{(1+\sin\frac{\mu}{2})*L}{\sin\mu} + \frac{(1-\sin\frac{\mu}{2})*L}{\sin\mu}$$

$$\hat{\beta} + \check{\beta} = \frac{2L}{\sin\mu} \qquad \frac{d}{d\mu}\left({}^{2L}\!/_{\sin\mu}\right) = 0$$



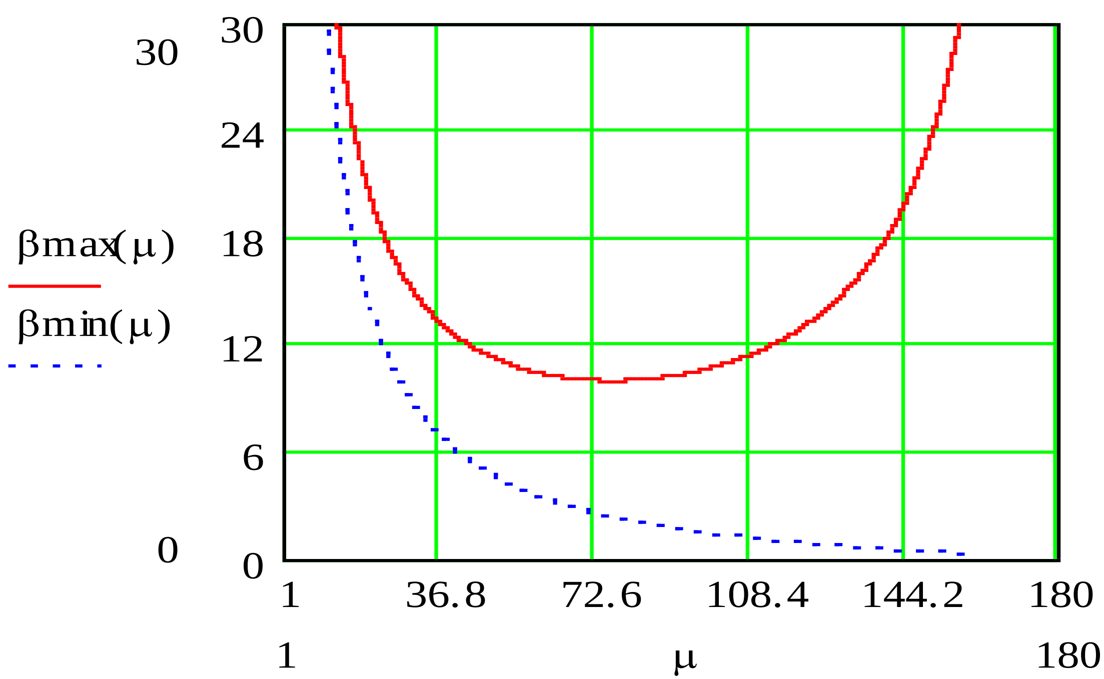
$$\frac{L}{\sin^2\mu} * \cos\mu = 0 \rightarrow \mu = 90^\circ$$

Electrons are different

electron beams are usually flat, $\varepsilon_y \approx 2$ - $10\ \%\ \varepsilon_x$

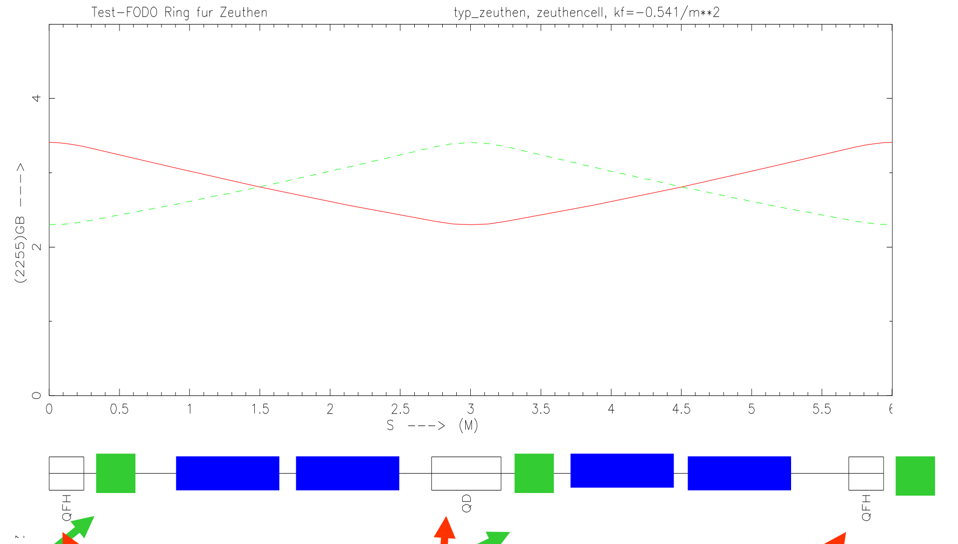
→ *optimise only β_{hor}*

$$\frac{d}{d\mu}(\hat{\beta}) = \frac{d}{d\mu}\frac{L(1+\sin\frac{\mu}{2})}{\sin\mu} = 0 \rightarrow \mu \approx 76°$$



red curve: $\boldsymbol{\beta_{max}}$
blue curve: β_{min}
as a function of the phase advance μ

Orbit distortions in a periodic lattice



field error of a dipole/distorted quadrupole

$$\rightarrow \delta(mrad) = \frac{ds}{\rho} = \frac{\int Bds}{p/e}$$

the particle will follow a new closed trajectory, the distorted orbit:

$$x(s) = \frac{\sqrt{\beta(s)}}{2\sin(\pi Q)} * \oint \sqrt{\beta(\tilde{s})}\,\frac{1}{\rho(\tilde{s})}\cos\left(\left|\psi(\tilde{s}) - \psi(s)\right| - \pi Q\right)d\tilde{s}$$

*** the orbit amplitude will be large if the** **β function at the location of the kick** **is large**
$\beta(\tilde{s})$ indicates the sensitivity of the beam → here orbit correctors should be placed in the lattice

*** the orbit amplitude will be large** **at places where in the lattice β(s)** **is large → here beam position monitors should be installed**

Orbit Correction and Beam Instrumentation in a storage ring

Elsa ring, Bonn

*

Resumé:

1.) Integrated Dipole field: $$\int B ds = N * B_0 * l_{eff} = 2\pi \frac{p}{q}$$

l_{eff} effective magnet length, N number of magnets

2.) Stability condition: $$Trace(M) < 2$$

for periodic structures within the lattice / at least for the transfer matrix of the complete circular machine

3.) Transfer matrix for periodic cell $$M(s) = \begin{pmatrix} \cos\mu + \alpha(s)\sin\mu & \beta(s)\sin\mu \\ -\gamma(s)\sin\mu & \cos(\mu) - \alpha(s)\sin\mu \end{pmatrix}$$

α,β,γ depend on the position s in the ring, μ (phase advance of the period) is independent of s

4.) Thin lens approximation: $$M_{QF} = \begin{pmatrix} 1 & 0 \\ \frac{1}{f_Q} & 1 \end{pmatrix} \qquad f_Q = \frac{1}{k_Q l_Q}$$

focal length of the quadrupole magnet $f_Q = 1/(k_Q l_{Q)} >> l_Q$

5.) Tune (rough estimate):

Phase advance per cell

$$\mu_{cell} = \int_{s}^{s+L_{cell}} \frac{ds}{\beta(s)}$$

Tune = phase advance in units of 2π

$$Q := N * \frac{\mu}{2\pi} = \frac{1}{2\pi} * \oint \frac{ds}{\beta(s)} \approx \frac{1}{2\pi} * \frac{2\pi\bar{R}}{\bar{\beta}} = \bar{R}/\bar{\beta}$$

$\bar{R}, \bar{\beta}$ *average radius and β-function*

$$Q \approx \frac{\bar{R}}{\bar{\beta}}$$

6.) Phase advance per FoDo cell

(thin lens approx)

$$\sin\frac{\mu}{2} = \frac{L_{Cell}}{4 f_Q}$$

L_{Cell} length of the complete FoDo cell, f_Q focal length of the quadrupole, μ phase advance per cell

7.) Stability in a FoDo cell

(thin lens approx)

$$f_Q > \frac{L_{Cell}}{4}$$

8.) Beta functions in a FoDo cell

(thin lens approx)

$$\hat{\beta} = \frac{(1+\sin\frac{\mu}{2})L_{Cell}}{\sin\mu} \qquad \check{\beta} = \frac{(1-\sin\frac{\mu}{2})L_{Cell}}{\sin\mu}$$

L_{Cell} length of the complete FoDo cell, μ phase advance per cell

Conclusion:

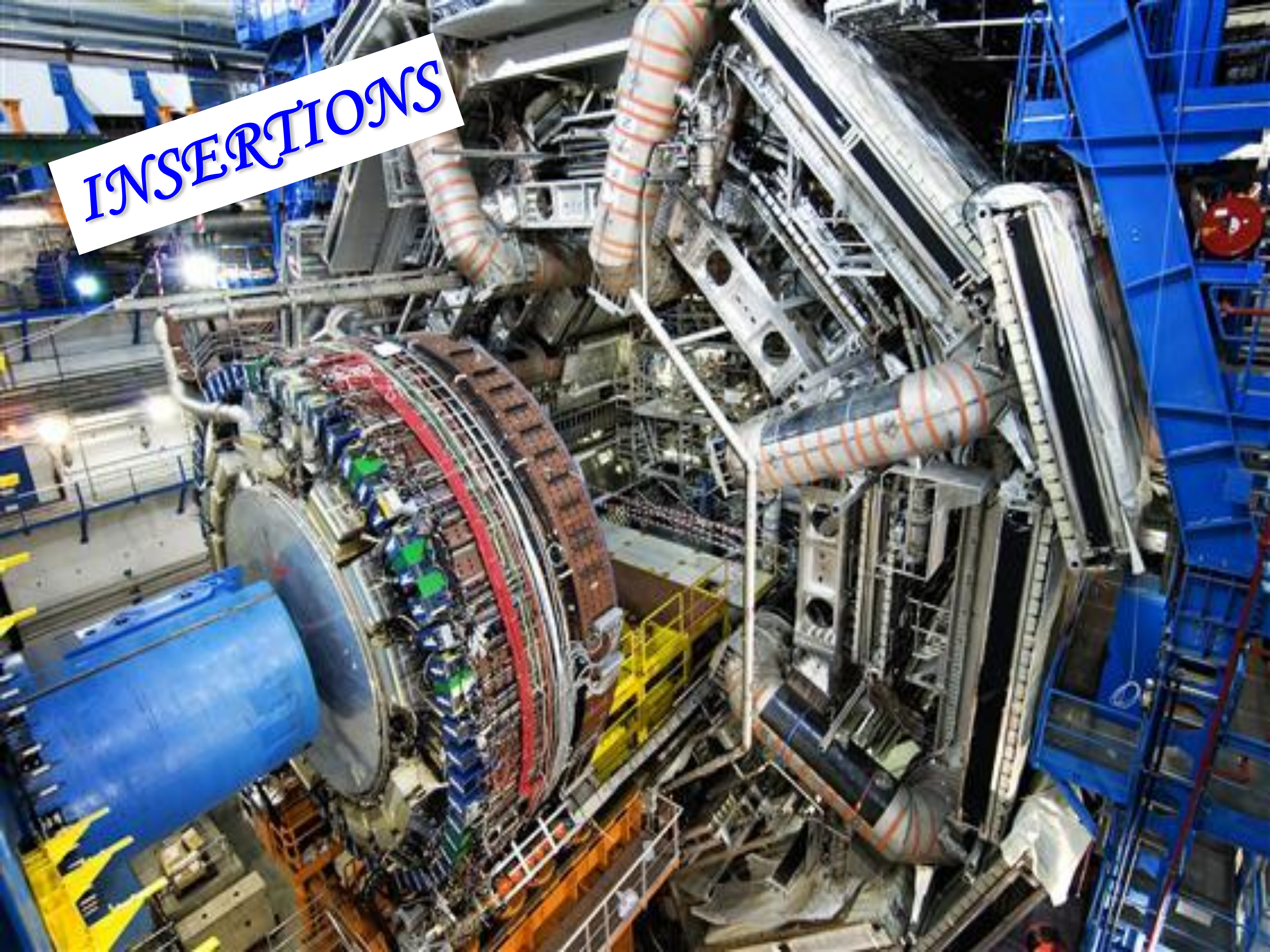
* ***„the arc“ of a storage ring is usually built out of a periodic sequence of single magnet elements eg. FoDo sections***

* ***a first guess of the main parameters of the beam in the arc is obtained by the settings of the quadrupole lenses in this section***

* ***we can get an estimate of the beam parameters using a selection of „rules of thumb“***

Usually the real beam properties will not differ too much from these estimates and we will have a nice storage ring and a beautifull beam and everybody is happy around.

And then someone comes and spoils it all by saying something stupid like installing a tiny little piece of detector in our machine …


INSERTIONS