LANDAU DAMPING

CAS 2009, Darmstadt; Albert Hofmann, 06-07.10.09

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1) Introduction

Mechanism of Landau damping

A single, undamped oscillator with resonant frequency ω_r reacts to a pulse excitation with a free oscillation. For a set of oscillators with resonant frequencies ω_{rj}, of distribution $f(\omega_r)$, a pulse excitation gives each oscillator the same initial velocity $\dot{x}(0)$ followed by a free oscillation with ω_{rj}. For the interaction of the beam with its surroundings (impedance) and for observing the beam only the **center-of-mass motion** is relevant. By the different ω_{ri} the oscillating particles change phase relative to each other and the center-of-mass motion is reduced while the **incoherent** motion of the particles continues. This reduction is faster the larger the frequency spread. It differs from other damping mechanisms as the decay is usually not exponential.

In a beam instability the field created by particles in the beam induce a currents current in the impedance of the beam surroundings. Since it is at a relatively large distance from the beam only the average field determined by the center-of-mass and not the one of the individual particles. The induced current produces a field on its own which acts back on the beam. Depending on its phase it can increase an initial beam motion and lead to an instability. If the decay of the coherent center-of-mass motion by the frequency spread is sufficiently fast it can interrupt the interaction with the wall at infinitesimally small amplitude level and avoid the instability.

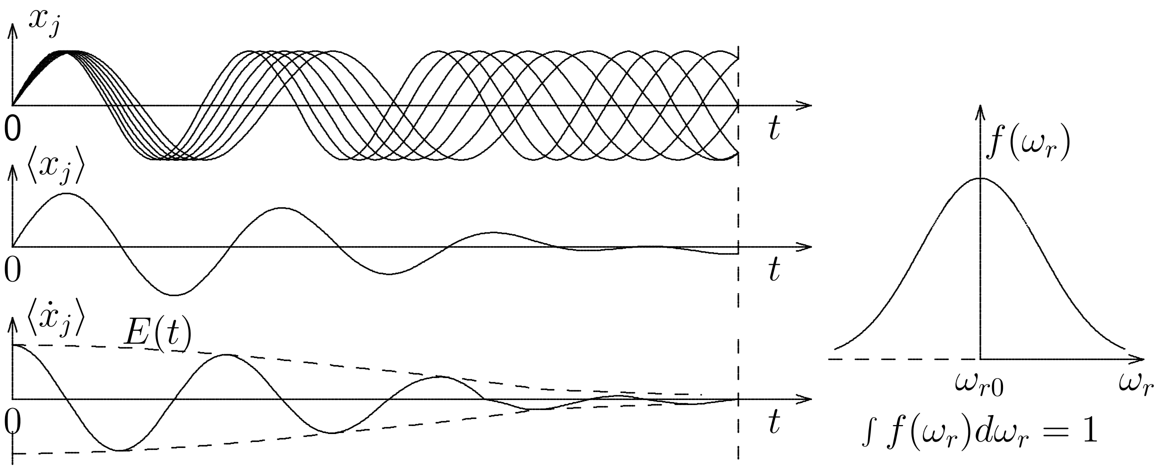
Treatment of Landau damping

Landau damping can be understood from different points of view. We treat it here in a manner which is close to beam observation and experiment and relate it to the beam response to a harmonic excitation, called transfer function.

The fields induced by the center-of-mass motion are modified by the beam surroundings (impedance) and act back on the beam. This can lead to an instability with a threshold determined by the beam response. Below this threshold the frequency spread eliminates any coherent motion at **infinitesimal small amplitudes**. Above, the voltage induced in the resistive part of the impedance leads to an increase of the initial coherent motion and we have an instability.

The amount of Landau damping depends on the frequency distribution $f(\omega_r)$ or its derivative at the frequency ω of the instability. It can be enhanced for coasting beams by increasing the tune dependence on momentum (chromaticity) and for bunches by introducing nonlinearities which lead to a frequency dependence on oscillation amplitude. The reactive part of the wall impedance due to many resonances often leads to a frequency difference between the coherent (center-of-mass) motion and the incoherent oscillations of the individual particles. This makes Landau damping becomes ineffective.

2) Response of an oscillator-set to excitation
Response to a pulse excitation



Oscillators j with ω_{rj} get at $t = 0$ a kick with $\dot{x}_j(0^+) = \dot{x}_0$ and oscillate freely with different ω_{rj} and fixed amplitude $\hat{x}_j = \dot{x}_0/\omega_{rj}$

$$\dot{x}_j(t) = \dot{x}_0 \cos(\omega_{rj} t)$$
$$x_j = \hat{x}_j \sin(\omega_{rj} t)$$

Response of single particle and center-of-mass

$\dot{x}_j(t)\rangle = \dot{x}_0 \cos(\omega_r t),\ x_j = (\dot{x}_0/\omega_{rj}) \sin(\omega_{rj} t)$

$\langle \dot{x}(t) \rangle = \dot{x}_0 \int f(\omega_r) \cos(\omega_r t) d\omega_r \propto$ inv. FT $= \mathcal{F}^{-1}(f)$

narrow distribution $\Delta\omega_r = \omega_r - \omega_{r0} \ll \omega_{r0}$

$\langle \dot{x}(t) \rangle = \dot{x}_0 \int f(\omega_{r0} + \Delta\omega_r) \cos((\omega_{r0} + \Delta\omega_r) t) d\omega_r$

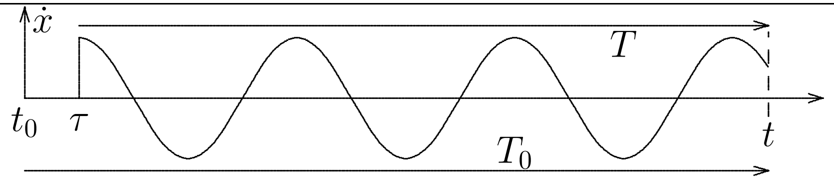
$$g(t) = \langle \dot{x}(t) \rangle / \dot{x}_0 = \cos(\omega_{r0} t) I_1(t) + \sin(\omega_{r0}) I_2(t)$$
$$= \cos(\omega_{r0} t - \phi) E(t), \quad (E(t) = \text{ envelope})$$

with inverse Fourier integrals

$$I_1(t) = \int f(\Delta\omega_r) \cos(\Delta\omega_r t) d\omega_r$$
$$I_2(t = -\int f(\Delta\omega_r) \sin(\Delta\omega_r t) d\omega_r$$
$$g(t) \propto \mathcal{F}^{-1}_{\cos}(f(\Delta\omega_r)) \cos(\omega_{r0} t) - \mathcal{F}^{-1}_{\sin}(f(\Delta\omega_r)) \sin(\omega_{r0} t)$$
$$E(t) = \sqrt{I_1^2(t) + I_2^2(t)}$$

The center-of-mass velocity response $g(t)$ of a frequency distribution $f(\Delta\omega_r)$ to a pulse excitation is proportional to its inverse Fourier transform times an oscillation at ω_{r0}.

Single oscillator response to harmonic excitation



Velocity response of single oscillator with ω_r to a pulse excitation at a time t_0 is

$$\dot{x}(t) = \dot{x}_0 \cos(\omega_r(t-\tau)) = \dot{x}_0 \cos(\omega_r T).$$

Harmonic excitation at ω starting at t_0 and observing at t. with $(\omega_r - \omega)/\omega_r \ll 1$.

Done by small kicks with harmonic modulation

$$d\dot{x}_\tau = \frac{d\hat{\dot{x}}}{d\tau} \cos(\omega\tau) d\tau = G d\tau = \hat{G} \cos(\omega\tau) d\tau$$

with acceleration $G(t)$. Velocity at time t is

$$\dot{x}(t) = \hat{G} \int_{t_0}^{t} \cos(\omega\tau) \cos(\omega_r(t-\tau)) d\tau.$$

$T = t - \tau,\ T_0 = t - t_0$, develop $\cos(\omega(t-T))$, single oscillator response to harmonic excitation.

$$\frac{\dot{x}(t)}{\hat{G}} = -\int_{T_0}^{0} \cos(\omega(t-T)) \cos(\omega_r T) dT = \int_0^{T_0} (\cos(\omega t)\cos(\omega T) + \sin(\omega t)\sin(\omega T)) \cos(\omega_r T) dT,$$

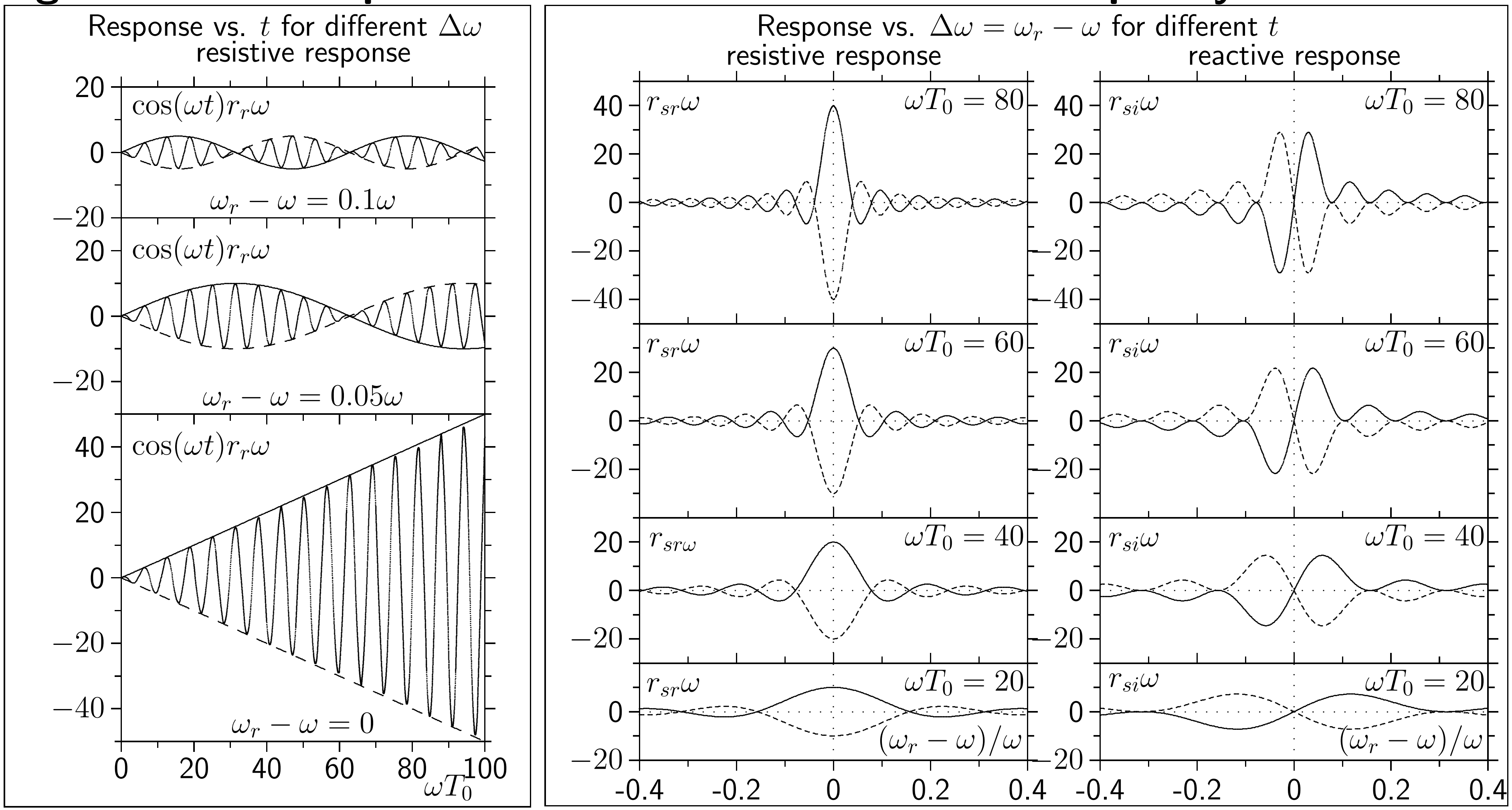
$$\approx \frac{1}{2}\left[\cos(\omega t)\frac{\sin((\omega_r-\omega)T_0)}{\omega_r-\omega} - \sin(\omega t)\frac{1-\cos((\omega_r-\omega)T_0)}{\omega_r-\omega}\right] = \cos(\omega t) r_{sr} + \sin(\omega t) r_{si}.$$

Large T_0 : $r_{sr} = \frac{1}{2}\frac{\sin((\omega_r-\omega)T_0)}{\omega_r-\omega} \approx \begin{cases} \infty & \text{if } \omega = \omega_r \\ 0 & \text{if } \omega \neq \omega_r \end{cases} \approx \frac{\pi}{2}\delta(\omega_r-\omega)$ since $\int \frac{\sin(ax)}{x} dx = \pi$,

$$r_{si} = -\frac{1-\cos((\omega_r-\omega)T_0)}{2(\omega_r-\omega)} \approx \frac{1}{2(\omega_r-\omega)}_{\omega\neq\omega_r}$$

local averaging gives envelope

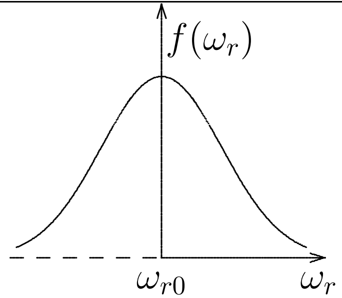
Single oscillator response as a function of excitation frequency and time



Velocity response of a single oscillator after a long harmonic excitation time

$$\dot{x} = \hat{G}\cos(\omega t) r_{sr}(\omega_r) + \sin(\omega t) r_{si}(\omega_r) \approx \frac{\hat{G}}{2}\left[\cos(\omega t)\pi\delta(\omega_r - \omega) - \sin(\omega t)\left(\frac{1}{\omega_r - \omega}\right)_{\omega \neq \omega_r}\right]$$

Response of an oscillator set to a harmonic excitation



$$f(\omega_r) = \frac{1}{N}\frac{dN}{d\omega_r}$$

$$\int_0^\infty f(\omega_r)d\omega_r = 1$$

Oscillator set with resonant frequency distribution $f(\omega_r)$ gets harmonic excitation at ω. Center-of-mass velocity response $\langle \dot{x}(t) \rangle$ is convolution of single oscillator velocity response $\dot{x}(t)$ and distribution $f(\omega_r)$.

$$\dot{x}(t) \approx \frac{\hat{G}}{2}\left[\cos(\omega t)\pi\delta(\omega_r - \omega) - \sin(\omega t)\left(\frac{1}{\omega_r - \omega}\right)_{\omega \neq \omega_r}\right] \;,\quad \langle \dot{x}(t) \rangle = \int_0^\infty \dot{x}(t,\omega_r) f(\omega_r) d\omega_r$$

$$\frac{\langle \dot{x}(t) \rangle}{\hat{G}} = \frac{1}{2}\left(\cos(\omega t)\pi f(\omega) - \sin(\omega t)\text{PV}\int_0^\infty \frac{f(\omega_r)d\omega_r}{\omega_r - \omega}\right) = \cos(\omega t) r_r(\omega) + \sin(\omega t) r_i(\omega)$$

$$\text{PV}\int \frac{f(\omega_r)}{\omega - \omega_r} d\omega_r = \lim_{\epsilon \to 0}\left[\int_0^{\omega-\epsilon} \frac{f(\omega_r)}{\omega_r - \omega} + \int_{\omega+\epsilon}^\infty \frac{f(\omega_r)}{\omega_r - \omega}\right] d\omega_r \quad = \text{ 'principle value integral'}$$

$$r_r(\omega) = \frac{\pi}{2} f(\omega) \;,\quad r_i(\omega) = \frac{1}{2}\text{PV}\int_0^\infty \frac{f(\omega_r)d\omega_r}{\omega_r - \omega} \quad = \text{ normalized response components}$$

$$\frac{\langle x(t) \rangle}{\hat{G}} = \frac{1}{\hat{G}}\int \langle \dot{x}(t) \rangle dt = \frac{1}{\omega}\left[\sin(\omega t) r_r(\omega) - \cos(\omega t) r_i(\omega)\right] \quad = \text{ spatial response}$$

Response, **transfer function**, has resistive part absorbing energy from coherent (center-of-mass) motion converting it into incoherent one (thermal energy), gives damping. Is proportional to distribution at ω, vanishes outside. Reactive part is proportional to 'principle value' integral.

Short derivation using complex notation

$$\cos(\omega t) = \frac{e^{j\omega t} + e^{-j\omega t}}{2} \;\to\; \frac{e^{j\omega t}}{2},\; -\infty \le \omega \le \infty$$

$$\ddot{x} + \omega_r^2 x = \frac{\hat{G}}{2} e^{j\omega t}\;,\; (-\omega^2 + \omega_r^2)x = \frac{\hat{G}}{2} e^{j\omega t}$$

$$x = \frac{\hat{G} e^{j\omega t}}{2(\omega_r^2 - \omega^2)} = \frac{\hat{G} e^{j\omega t}}{4\omega}\left(\frac{1}{\omega_r - \omega} - \frac{1}{\omega_r + \omega}\right)$$

$$\frac{\langle x \rangle_+}{\hat{G}} = \frac{e^{j\omega t}}{4\omega} \int_0^\infty \frac{f(\omega_r)}{\omega_r - \omega} d\omega_r\;, \quad \frac{\langle \dot{x} \rangle_+}{\hat{G}} = j\omega \frac{\langle x \rangle_+}{\hat{G}}$$

$$\int_{-\infty}^{\infty} \frac{f(\omega_r) d\omega_r}{\omega_r - \omega} = \pm j\pi f(\omega) + \text{PV}\int \frac{f(\omega_r) d\omega_r}{\omega_r - \omega}$$

Real notation, $\omega > 0$, $\to$ complex one with positive and negative frequencies and solution $x = \hat{x} e^{j\omega t}$

Displacement response, single oscillator. For $\omega > 0$ only first and for $\omega < 0$ only second term is large. Taking the first and integrating over $f(\omega_r)$

Integration over pole gives PV (principle value) integral plus imaginary residue. Resolve sign ambiguity by $\dot{x}(-\infty) = 0$

$$\text{for } \omega > 0 \quad \frac{\langle \dot{x} \rangle_+}{\hat{G}} = \frac{e^{j\omega t}}{4}\left[\pi f(\omega) + j\text{PV} \int_0^\infty \frac{f(\omega_r) d\omega_r}{\omega_r - \omega}\right] = \frac{e^{j\omega t}}{2}\; [r_r(\omega) + j r_i(\omega)]$$

$$\text{for } \omega < 0 \quad \frac{\langle \dot{x} \rangle_-}{\hat{G}} = \frac{e^{-j\omega t}}{4}\left[\pi f(\omega) - j\text{PV} \int_{-\infty}^0 \frac{f(\omega_r)}{\omega_r + \omega} d\omega_r\right] = \frac{e^{-j\omega t}}{2}\, [r_r(\omega) - j r_i(\omega)]$$

$$\frac{\langle \dot{x} \rangle_+}{\hat{G}} + \frac{\langle \dot{x} \rangle_-}{\hat{G}} = \frac{1}{2}\left[\cos(\omega t)\pi f(\omega) - \sin(\omega t)\text{PV} \int_{-\infty}^\infty \frac{f(\omega_r)}{\omega_r - \omega} d\omega_r\right],\; \text{(agrees with previous)}$$

Response for a Gaussian frequency distribution

$$f(\omega_r) = \frac{1}{\sqrt{2\pi}\sigma_\omega} e^{-\frac{\Delta\omega_r^2}{2\sigma_\omega^2}}, \quad \int_{-\infty}^{\infty} f(\Delta\omega_r) d\omega_r = 1$$

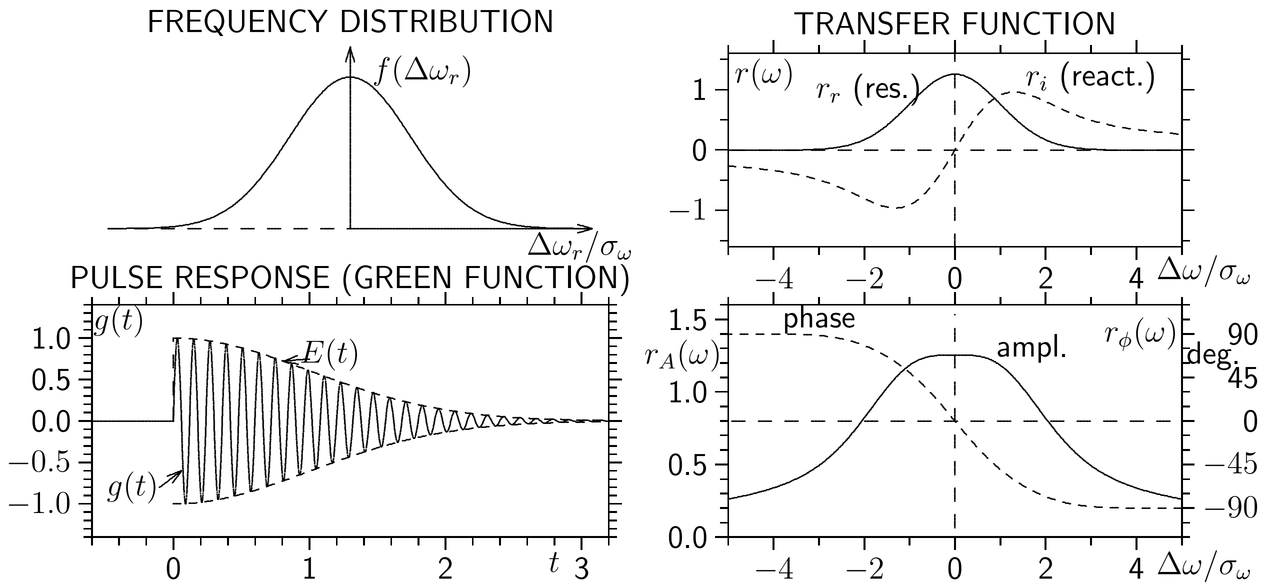
$$g(t) = \frac{\cos(\omega_r t)}{\sqrt{2\pi}\sigma_\omega} \int_{-\infty}^{\infty} e^{-\frac{\Delta\omega_r^2}{2\sigma_\omega^2}} \cos(\Delta\omega_r t) d\omega_r = e^{-\frac{\sigma_\omega^2 t^2}{2}} \cos(\omega_{r0} t)$$

$$r_r(\omega) = \frac{\pi}{2\sqrt{2\pi}\sigma_\omega} e^{-\Delta\omega^2/2\sigma_\omega^2}$$

$$r_i(\omega) = \frac{1}{\sqrt{2}\sigma_\omega} e^{-\frac{\Delta\omega^2}{2\sigma_\omega^2}} \int_0^{\Delta\omega/(\sqrt{2}\sigma_\omega)} e^{t'^2} dt'$$

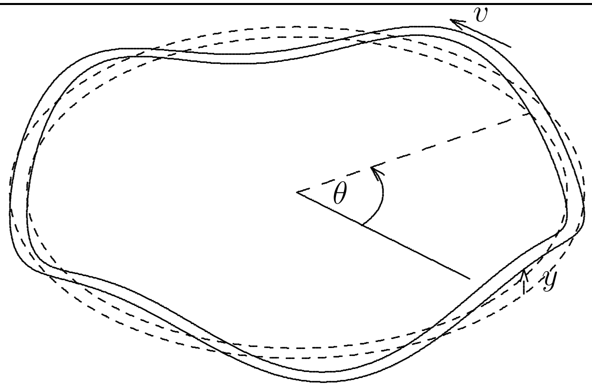
ω_r-distribution around ω_{rc}
$\Delta\omega_r = \omega_r - \omega_{rc}$
$\Delta\omega = \omega - \omega_{rc}$
pulse response
$g(t) = E(t)\cos(\omega_{r0} t)$

transfer function, from FT of $E(t)$, Dawson integral



The response of this frequency distribution has similarities to the one of a harmonic oscillator. The transfer function phase changes by 180^0. However, the pulse response decays like a Gaussian and not exponentially.

3) Transverse coasting beam instability
Oscillation modes



Coasting beam of N particles circulates with ω_0, current $I = eN\omega_0/(2\pi)$ in ring of uniform focusing. Particle rotates with ω_0 and makes betatron oscillation with $Q\omega_0$

$$\theta_i = \theta_{0i} + \omega_0 t, \; y_i(t) = \hat{y}\cos(Q\omega_0 t - \phi_i)).$$

Phase difference $d\phi$ between adjacent particles gives oscillation mode seen in rotating frame $\theta' = \theta - \omega_0 t$ with motion $\hat{y}\cos(Q\omega_0 t - \phi(\theta'))$ representing a wave moving in $+\theta$-direction if $d\phi/d\theta' > 0$ and vice versa; called forward/backward waves.

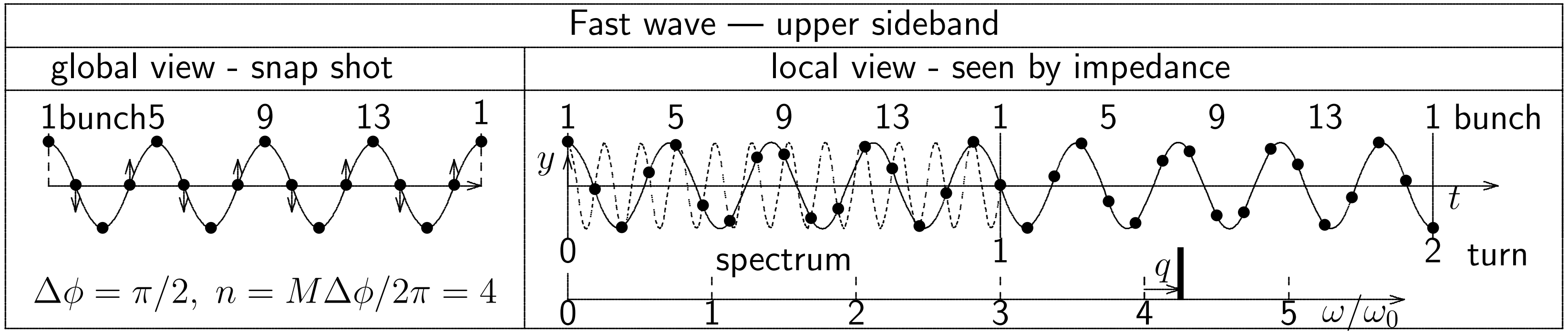
We chose set of closed modes $\phi = \pm n\theta'$ seen as forward and backward waves for $+$ or - sign in rotating frame or as fast and slow waves by stationary observer with frequencies $\omega_{\beta f}$, $\omega_{\beta s} > 0$

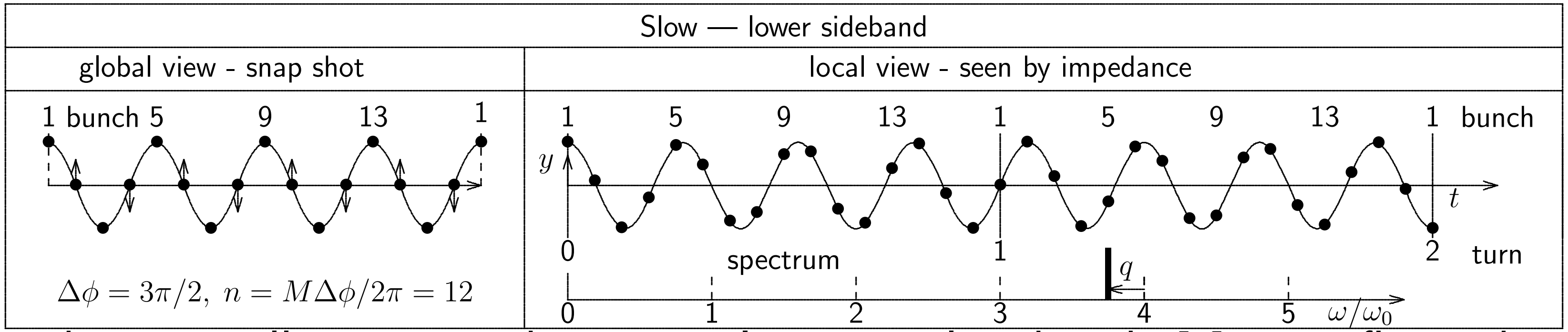
$$y = \hat{y}\cos(Q\omega_0 t \pm n\theta') = \hat{y}\cos(n\theta - (n \pm Q)\omega_0 t)$$

$$\omega_{\beta f} = (n + Q)\omega_0, \; , n > -Q$$
$$\omega_{\beta s} = (n - Q)\omega_0, \; n > Q$$

with conditions on n for positive frequencies. For a fixed $t = 0$ we get the form of the frozen oscillation $\cos(n\theta)$ and $\sin(n\theta)$ forming a base of orthogonal modes which describe any closed distortion of the beam. Small n result in relatively small observed frequencies which are more important since the wall impedance becomes smaller at very high frequencies and only a limited number of modes n have to be considered.

Coasting beam and coupled bunch modes





Coasting beam oscillates in mode $n = 4$. It is completed with $M = 16$ fictive bunches having relative phase between $\Delta\phi = \pi/2$ for fast and $\Delta\phi = 3\pi/2$, or $-\pi/2$, for slow wave. Coupled bunch modes $n = M\Delta\phi/(2\pi) = 4$, for first and $n = 12$, or $n = -4$, for second case.
Global view shows all bunches at the **same time**, in local view bunches are seen in **successive** times and advance in phase. Through bunches higher frequencies fits are possible giving higher sidebands in spectrum, absent in coasting beam where each mode $\pm n$ has only one sideband.
Merry-go-round with horses moving up and down illustrates coupled bunch modes.

Effect of momentum spread

Betatron frequencies for a stationary observer

$$\omega_{\beta f} = (n_f + Q)\omega_0 \quad , \quad \omega_{\beta s} = (n_s - Q)\omega_0$$
$$\omega_{\beta f} = (n + q)\omega_0 \quad , \quad \omega_{\beta s} = (n + 1 - Q)\omega_0,$$

depend on momentum deviation Δp through the slip factor η_c and the chromaticity Q'

$$\Delta\omega_0/\omega_0 = -\eta_c \Delta p/p \; , \quad \Delta Q = Q' \Delta p/p$$
$$\Delta\omega_{\beta f} = (Q' - \eta_c(n_f + Q))\omega_0 \Delta p/p$$
$$\Delta\omega_{\beta s} = (Q' - \eta_c(n_s - Q))\omega_0 \Delta p/p.$$

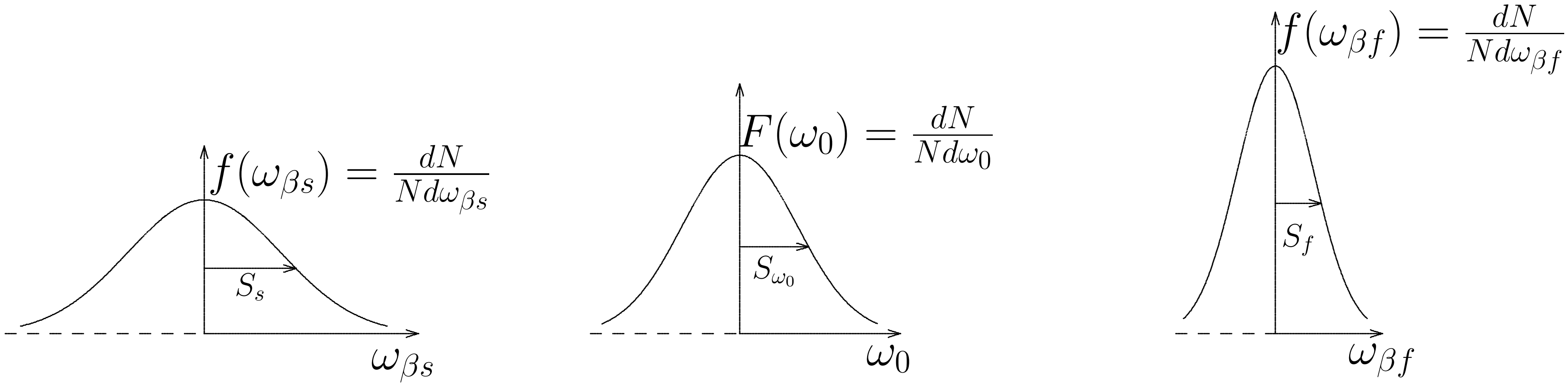
This gives two distributions $f(\omega_{\beta f}),\ f(\omega_{\beta s})$. The HWHH spread δp in momentum gives widths in revolution and betatron frequencies

$$S_{\omega_0} = \eta_c \omega_0 \delta p/p$$
$$S_{\beta f} = (Q' - \eta_c(n_f + Q))\omega_0 \delta p/p$$
$$S_{\beta s} = (Q' - \eta_c(n_s - Q))\omega_0 \delta p/p$$

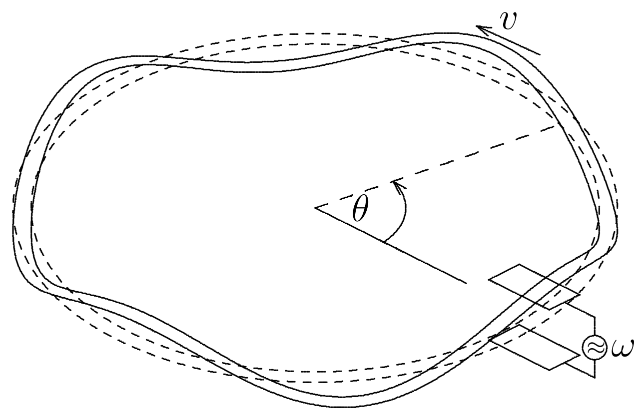
normalization $x_{\beta f} = \omega_{\beta f}/S_{\beta f},\ x = \omega/S_{\beta f}$

$g(x_{\beta f}) = dN/N dx_{\beta f} = f(\omega_{\beta f})/S_{\beta f}$ and for s correspondingly for slow wave

Distributions at revolution and betatron frequencies



Response of narrow particle string



Excite a ring of monoenergetic particles

$$\ddot{y} + \omega_0 Q^2 y = \hat{G}\cos(\omega t)$$

Seek solution $y(t) = \hat{y}\cos(n\theta - \omega)$, drive particles not at $\omega_0 Q$ but close to either the fast or slow wave $\omega_{\beta f} = (n_f + Q)\omega_0$ or $\omega_{\beta s} = (n_s - Q)\omega_0$.

$$\left(-(n\omega_0 - \omega)^2 + Q^2\omega_0^2\right)\hat{y}\cos(n\theta - \omega t) = \hat{G}\cos(\omega t)$$

$$\hat{y} = \frac{\hat{G}}{\omega_0^2 Q^2 - (n\omega_0 - \omega)^2} = \frac{-\hat{G}}{(\omega - \omega_0(n_f + Q)(\omega - \omega_0(n_s - Q)}$$

$$= \frac{-\hat{G}}{2\omega_0 Q}\left[\frac{1}{\omega - \omega_0(n_f + Q)} - \frac{1}{\omega - \omega_0(n_s - Q)}\right]$$

$$= \frac{-\hat{G}}{(\omega_{\beta f - \omega})(\omega_{\beta s - \omega})} = \frac{-\hat{G}}{2\omega_0 Q}\left(\frac{1}{\omega_{\beta s} - \omega} - \frac{1}{\omega_{\beta f} - \omega}\right)$$

$$\left(\frac{\hat{y}}{\hat{G}}\right)_f \approx \frac{1}{2\omega_0 Q}\left(\frac{1}{\omega_{\beta f} - \omega}\right), \quad \left(\frac{\hat{y}}{\hat{G}}\right)_s \approx \frac{-1}{2\omega_0 Q}\left(\frac{1}{\omega_{\beta s} - \omega}\right).$$

substitute into diff. eqn.

excite, observe at $\theta = 0$

gives response,
excite fast wave $\omega \approx \omega_{\beta f}$, first term much smaller than second and vice versa.
Note, the two responses have opposite sign, to be discussed later.

Response of the whole beam

Whole beam with frequency distributions $f(\omega_{\beta f})$, $f(\omega_{\beta s})$ is excited by $G(t) = \hat{G}\mathrm{e}^{j\omega t}$. Center-of-mass velocity response is:

$$\langle\hat{\dot{y}}\rangle_f = j\omega\langle y\rangle = j\frac{\hat{G}\omega}{2Q\omega_0}\int\frac{f(\omega_{\beta f})d\omega_{\beta f}}{\omega_{\beta f}-\omega}$$

$$= \frac{\hat{G}\omega}{2Q\omega_0}\left(\pi f(\omega) + jPV\int\frac{f(\omega_{\beta f})d\omega_{\beta f}}{\omega_{\beta f}-\omega}\right)$$

$$= j\frac{\hat{G}\omega}{2S_{\beta f}Q\omega_0}\int\frac{g(x_{\beta f})dx_{\beta f}}{x_{\beta f}-x} \text{ normalized}$$

$$= \frac{\hat{G}\omega}{2S_{\beta f}Q\omega_0}\left(\pi g(x) + jPV\int\frac{g(x_{\beta f})dx_{\beta f}}{x_{\beta f}-x}\right)$$

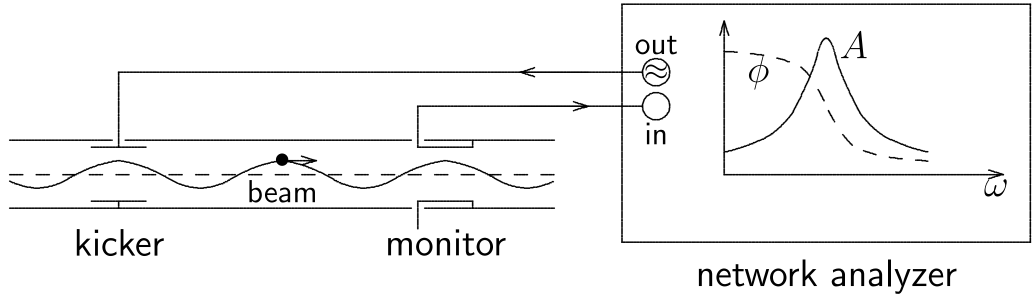
$$\langle\hat{\dot{y}}\rangle_s = -j\frac{\hat{G}\omega}{2Q\omega_0}\int\frac{f(\omega_{\beta s})d\omega_{\beta s}}{\omega_{\beta s}-\omega}$$

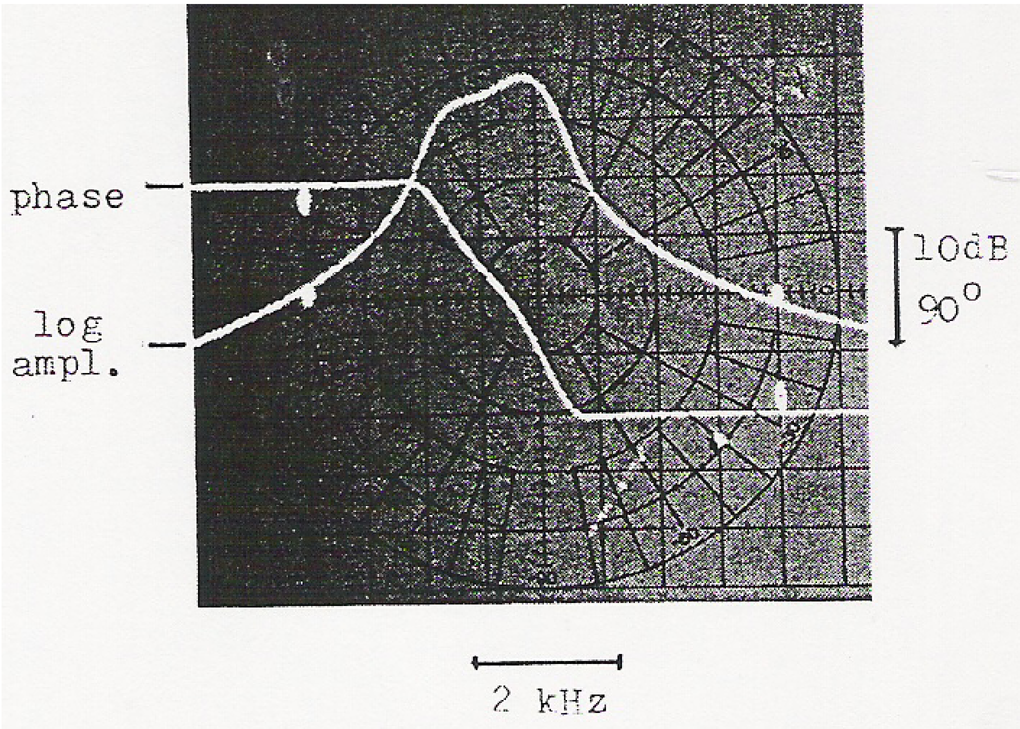
$$= -\frac{\hat{G}\omega}{2Q\omega_0}\left(\pi f(\omega) + jPV\int\frac{f(\omega_{\beta s})d\omega_{\beta s}}{\omega_{\beta s}-\omega}\right)$$

The term $\pi f(\omega)$ is real, exciting acceleration and responding velocity are in phase resulting in an absorption of energy and damping, called Landau damping. It is only present if the excitation frequency ω is within the frequency distribution of the individual particles. The second term is imaginary and gives the out-of-phase response being of less interest.

The spread in betatron frequencies is given by the momentum spread and the dependence of revolution frequency ω_0 and betatron tune Q on momentum deviation $\Delta p/p$. It is therefore determined by an **external parameter** which is not affected by the excitation of betatron oscillations.

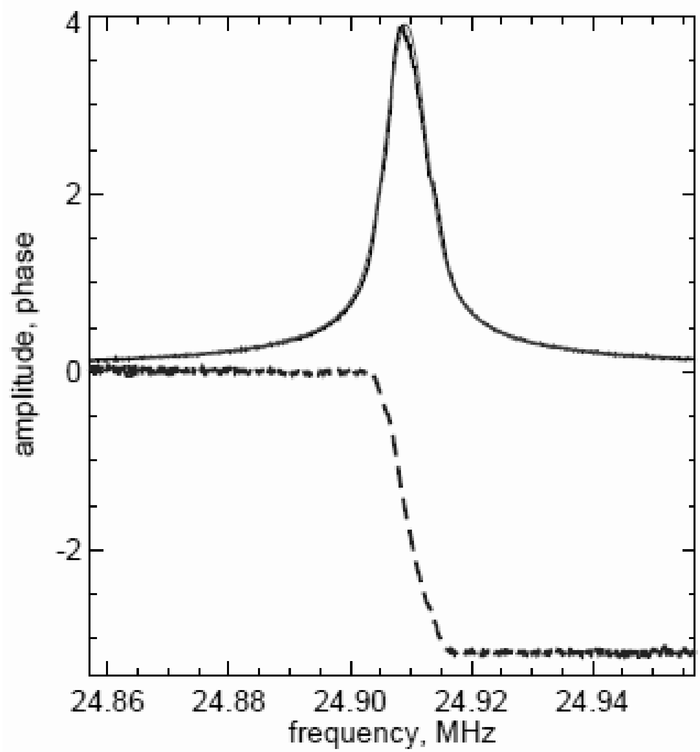
Measuring the beam response





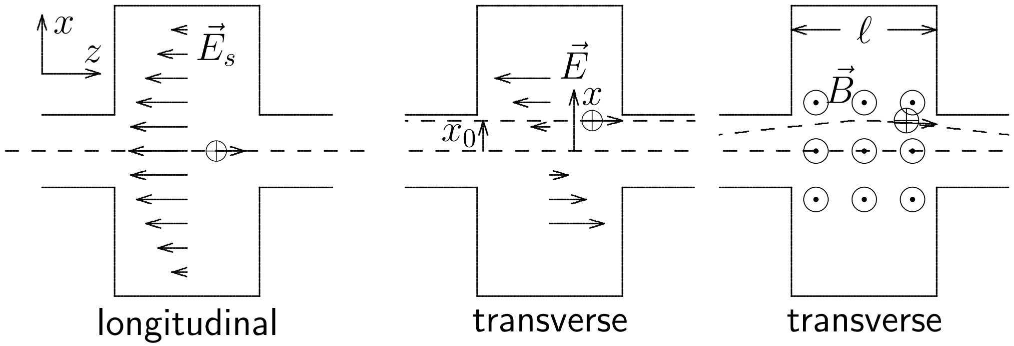
Vertical BTF of an unbunched beam in ISR

Network analyzer gives amplitude and phase of center-of-mass response to harmonic excitation — beam transfer function, BTF.



Vertical BTF of a coasting beam in the SIS 18 synchrotron at GSI, V. Kornilov et al.[6)]

Transverse impedance



$$Z_T(\omega) = j\frac{\int\left(\vec{E}(\omega)+[\vec{v}\times\vec{B}(\omega)]\right)_T ds}{I x(\omega)} = -\frac{\omega\int\left(\vec{E}(\omega)+[\vec{v}\times\vec{B}(\omega)]\right)_T ds}{I\dot{x}(\omega)}$$

Response to applied G of fast and slow wave

$$\langle\dot{x}\rangle_f = \frac{G\omega}{2Q\omega_0}\left(\pi f(\omega) + jPV\int\frac{f(\Delta\omega_r)}{\omega_r-\omega}d\omega_r\right)$$

$$\langle\dot{x}\rangle_s = -\frac{G\omega}{2Q\omega_0}\left(\pi f(\omega) + jPV\int\frac{f(\Delta\omega_r)}{\omega_r-\omega}d\omega_r\right)$$

induced fields in Z_T give G_s, averaged over $2\pi R$

$$G_s = \frac{e\oint\left(\vec{E}(\omega)+[\vec{\beta}\times\vec{B}(\omega)]\right)_T ds}{2\pi R m_0\gamma} = \frac{-eZ_T I\langle\dot{x}\rangle}{2\pi\gamma m_0 R\omega}$$

The external acceleration G excites a coherent oscillation $\langle\dot{x}\rangle$ which itself induces fields in the impedance leading to an additional self acceleration G_s.
It is added to the external one resulting in a modification of the beam response.
If the induced acceleration is just as large as the external one, $G_s = G$, we can turn the latter off and the oscillation continues at fixed amplitude — stability limit.

Stability limit

$$\langle \dot{x} \rangle_s = -\frac{G\omega}{2Q\omega_0}\left(\pi f(\Delta\omega) + jPV\int \frac{f(\Delta\omega_r)}{\omega_r - \omega} d\omega_r\right)$$

$$= -\frac{G\omega}{2Q\omega_0}\frac{\pi}{\sqrt{2\pi}\sigma_\omega} e^{-\frac{\Delta\omega^2}{2\sigma_\omega^2}}\left[1 + j\frac{2}{\sqrt{\pi}}\int_0^{\frac{\Delta\omega}{\sqrt{2}\sigma_\omega)}} e^{t'^2} dt'\right]$$

$$Z_T = -\frac{\omega}{I\langle \dot{x} \rangle(\omega)}\oint \left(\vec{E}(\omega) + [\vec{\beta} \times \vec{B}(\omega)]\right)_T ds$$

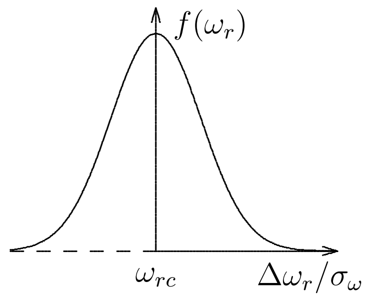
$$\hat{G}_s = -\frac{eZ_T I\langle \dot{x} \rangle_s}{\gamma m_0 2\pi R\omega}, \quad \hat{G}_f = -\frac{eZ_T I\langle \dot{x} \rangle_f}{\gamma m_0 2\pi R\omega}$$

response, Gaussian dist. center ω_{rc}

$$f(\omega_r) = \frac{e^{-\Delta\omega_r^2/2\sigma_\omega^2}}{\sqrt{2\pi}\sigma_\omega}$$

$$\Delta\omega_r = \omega_r - \omega_{rc}$$

$$\Delta\omega = \omega - \omega_{rc}$$



voltage induced in Z_T gives acceleration G_s If $G_s = G$, self sustained oscillation without drive, inst.threshold.

$$\langle \dot{x} \rangle_s = -\frac{G_s\omega}{2Q\omega_0}\frac{\pi}{\sqrt{2\pi}\sigma_\omega} = -\frac{ecZ_{Tr}I\langle \dot{x} \rangle_s}{4\pi Q m_0 c^2\gamma}$$

$$1 = \frac{ecZ_{Tr}I}{4\pi Q m_0 c^2\gamma}\frac{\sqrt{\pi}}{\sqrt{2}\sigma_\omega}, \quad Z_{Trs} \le \frac{4\sqrt{2\pi}\sigma_\omega Q E}{ecI}$$

$$Z_{Trs} \le \frac{4\sqrt{2\pi}R\sigma_\omega E}{ecI\beta_x}$$ slow wave stability condition

assume $\omega = \omega_{rc}$, excitation at center of distribution, get only real response

Using $E = m_0c^2\gamma$ and $\beta_x \approx R/Q$ with ring radius R.

instability by large positive Z_{Tr}

$$Z_{Trf} \ge -\frac{4\sqrt{2\pi}R\sigma_\omega E}{ecI\beta_x}$$ fast wave stability condition

instability by large negative Z_{Tr}

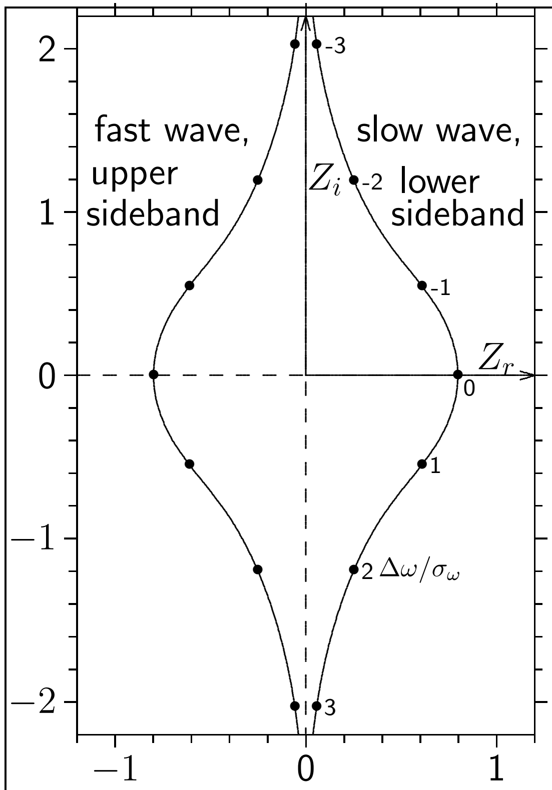
Stability diagram

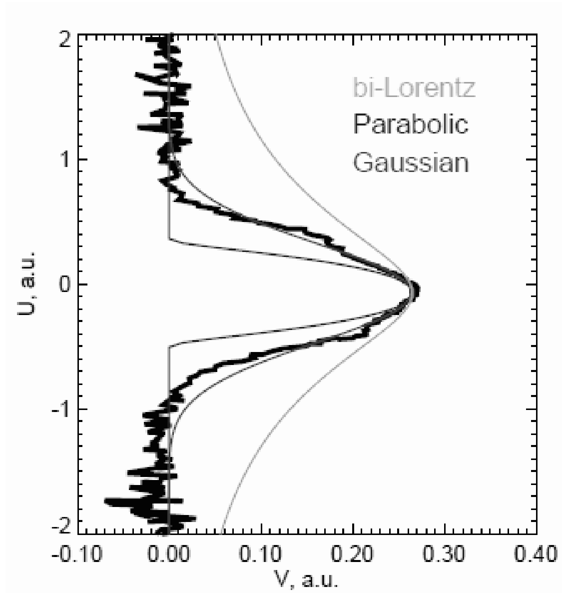
For the general complex response and impedance the stability condition can be expressed as a diagram by relating the beam parameter against the inverse response of the beam, i.e. inverse amplitude plotted against the negative phase, inverse Nyquist diagram.

slow wave: $$\frac{jecIZ_T(\omega)}{4\pi QE} \leq \frac{1}{\int \frac{f(\omega_{\beta s})}{\omega_{\beta s}-\omega} d\omega_{\beta s}}$$

fast wave: $$\frac{jecIZ_T(\omega)}{4\pi QE} \geq \frac{-1}{\int \frac{f(\omega_{\beta f})}{\omega_{\beta f}-\omega} d\omega_{\beta f}}.$$

Relation between complex impedance and complex beam response to excitation. If impedances are inside central curve we have stability, outside an instability. Curve itself represents threshold.





Left: Transverse stability diagrams calculated for at the upper and lower side-band of a coasting beam with Gaussian distribution.

Right: Calculated and measured stability diagram based on a coasting beam BTF measured at the SS18 synchrotron at GSI, Kornilov et al.[6)]

Response in the presence of an impedance

Beam response to external acceleration

$$\langle \hat{\dot{y}} \rangle_s = \frac{\hat{G}\omega}{2Q\omega_0} \int \frac{f(\omega_{\beta s})}{\omega_{\beta s} - \omega} d\omega_{\beta s}$$

Oscillating beam induces a voltage in a transverse impedance giving a self acceleration G_Z

$$Z_T(\omega) = -\frac{\omega}{I\dot{y}(\omega)} \oint \left(\vec{E}(\omega) + [\vec{\beta} \times \vec{B}(\omega)]\right)_T ds$$

$$\hat{G}_Z = -\frac{eZ_T I \langle \dot{y} \rangle}{\gamma m_0 2\pi R\omega}$$

This self excitation ads to the external one. Inverse response (stability diagram) due to both

$$\frac{(\hat{G} + \hat{G}_Z)}{\langle \hat{\dot{y}} \rangle_s} = \frac{\omega}{2Q\omega_0 \int \frac{f(\omega_{\beta s})}{\omega_{\beta s} - \omega} d\omega_{\beta s}}$$

Since only the external excitation is known, express the response with respect to it.

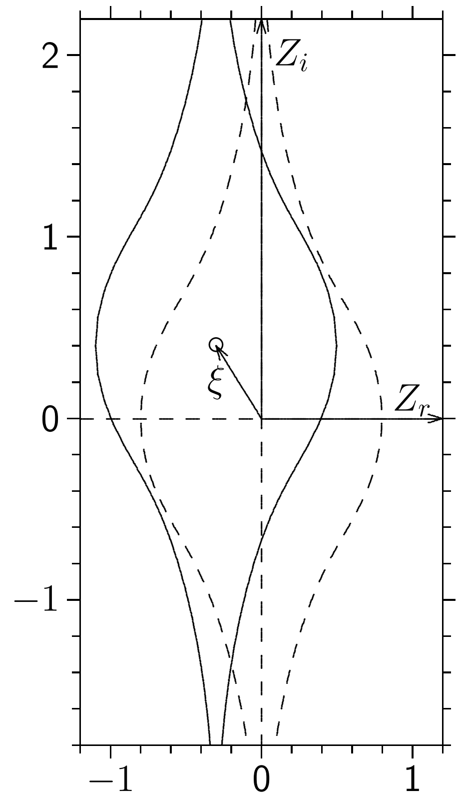
$$\frac{\hat{G}}{\langle \hat{\dot{y}} \rangle_s} = \frac{\omega}{2Q\omega_0 \int \frac{f(\omega_{\beta s})}{\omega_{\beta s} - \omega} d\omega_{\beta s}} - \frac{\hat{G}_Z}{\langle \hat{\dot{y}} \rangle_s}$$

$$= \frac{\omega}{2Q\omega_0 \int \frac{f(\omega_{\beta s})}{\omega_{\beta s} - \omega} d\omega_{\beta s}} - \frac{eZ_T I}{\gamma m_0 2\pi R\omega}$$

Impedance shifts stability diagram by vector $\xi \propto Z_T$

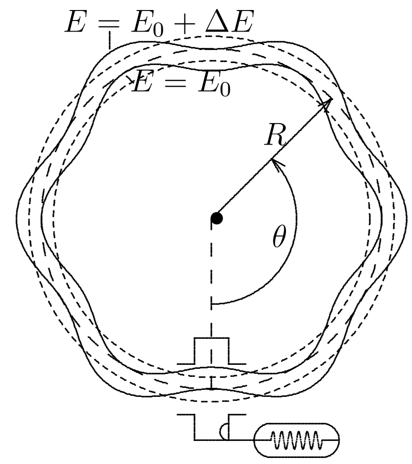
$$\xi = \frac{-eZ_T I}{2\pi R\gamma m_0 \omega}.$$

Can be used to measure Z_T.



4) Longitudinal coasting beam instability

Dynamics: Distribution $f(\Delta E)$, or $F(\Delta\omega_0)$ around ω_c



$$\frac{\Delta\omega_0}{\omega_0} = -\frac{\eta_c}{\beta^2}\frac{\Delta E}{E},\ F_0(\omega_0) = \frac{dN}{N d\omega_0}$$

$$I = (Ne\omega_0/2\pi)\int F_0(\Delta\omega_0)d\omega_0.$$

$t = 0$: pulse excite harmonic E or ω_0 modulation of n periods: $\delta E = \delta\hat{E}\cos(n\theta) = -\frac{\beta^2 E}{\eta_c\omega_c}\delta\hat{\omega}\cos(n\theta)$

$$F_{0+} = F_0(\Delta\omega_0 + \delta\omega_0) \approx F_0(\Delta\omega_0) + \frac{dF_0}{d\omega_0}\delta\hat{\omega}\cos(n\theta)$$

$$F(t) = F_0 + \frac{dF_0}{d\omega_0}\delta\hat{\omega}\cos(n\theta - \omega_0 t),$$ use $\theta = 0, n = 1$

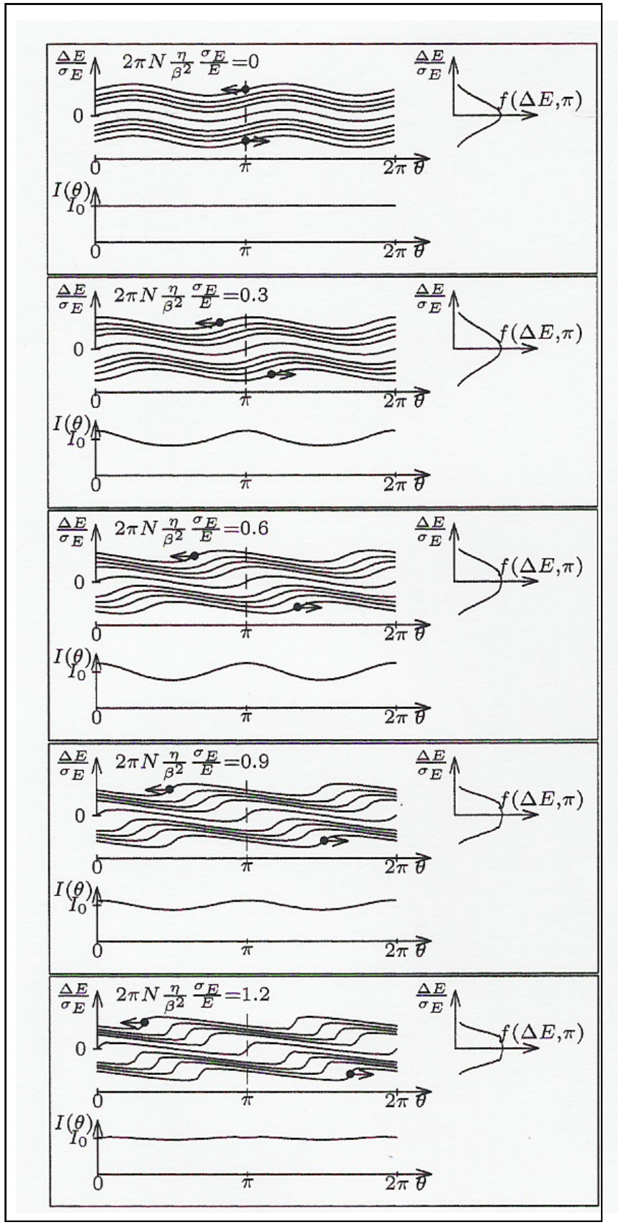
$$\omega_0 = \omega_c + \Delta\omega_0 = \omega_c - \omega_0\eta_c\Delta E/\beta^2 E$$

$$F(t) = F_0(\omega_0) + (dF_0/d\omega_0)\delta\hat{\omega}\cos(\omega_c t + \Delta\omega_0)$$

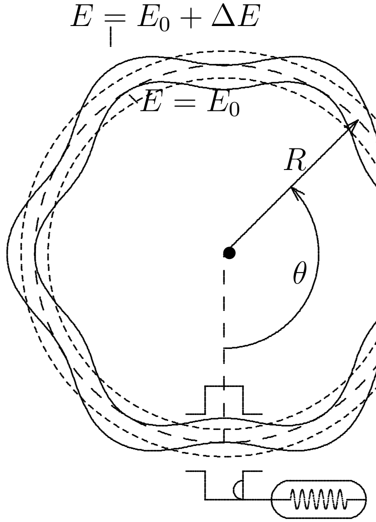
$$I(t) = Ne\omega_0/(2\pi)\int F(\Delta\omega)d\omega = I_0 + I_1(t)$$ current

$$I_1 = \frac{Ne\omega_0}{2\pi}\delta\hat{\omega}\left[\cos(\omega_c t)\int(dF_0/d\omega_0)\cos(\Delta\omega_0 t)d\omega_0 + \sin(\omega_c t)\int(dF_0/d\omega_0)\sin(\Delta\omega_0 t)d\omega_0\right]$$

$$\propto \left[\cos(\omega_c t)\mathcal{F}^{-1}_{\cos}(dF_0/d\omega_0) + \sin(\omega_c t)\mathcal{F}^{-1}_{\sin}(dF_0/d\omega_0)\right]$$



Response to a harmonic excitation



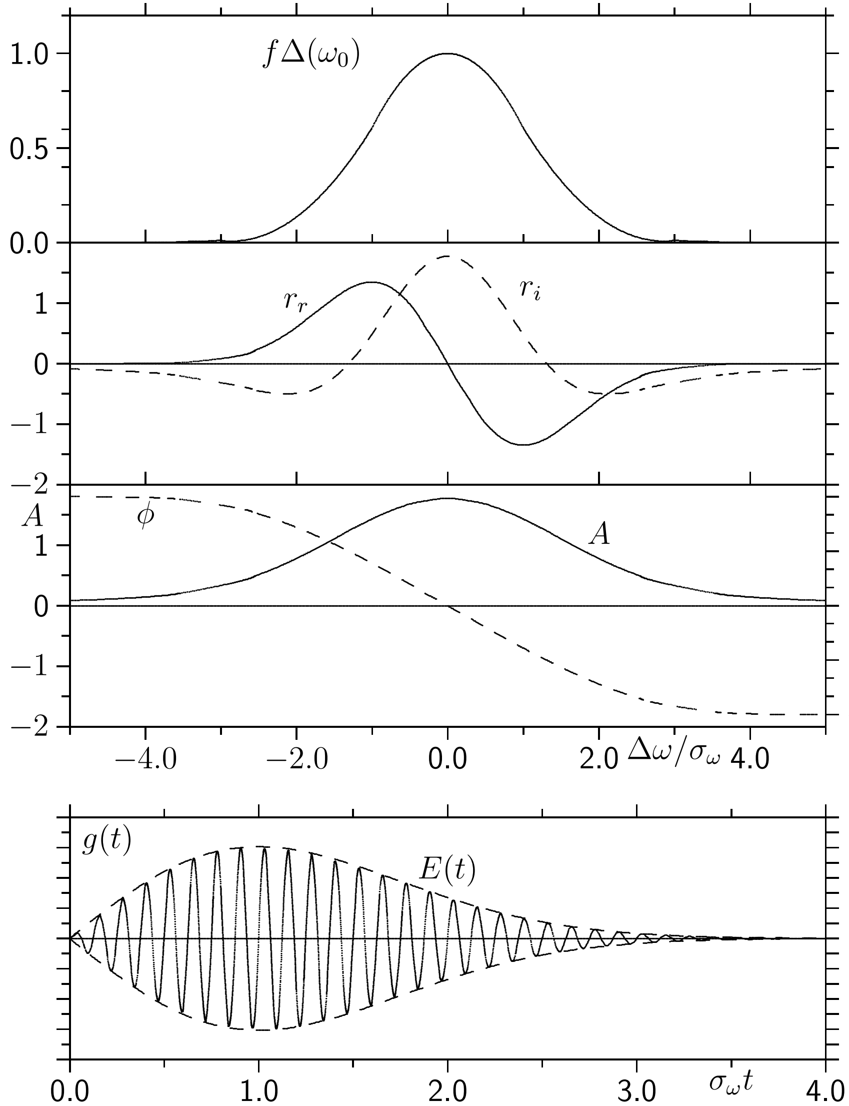
$$\frac{\Delta E}{E} = -\frac{1}{\eta_c}\frac{\Delta\omega_0}{\omega_0}\ ,\quad \Delta\omega_0 = \omega_r - \omega_0$$

$$I(t) = Ne\omega_0 \int f(\Delta E) dE = I_0 + I_1(t)$$

I_1 response to excitation U, give only result

$$I_1(t) \propto \mathrm{e}^{i\omega t} = U(t)(r_r(\omega) + jr_i(\omega))$$

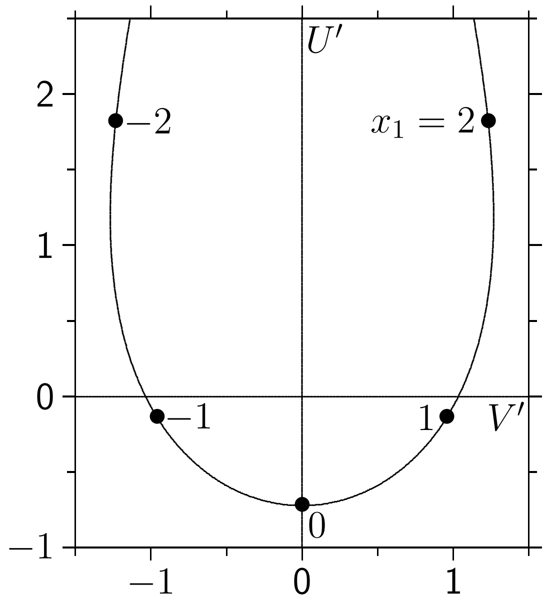
$$I_1(t) = \frac{-jNe^2\omega_0^3 U(t)}{2\pi\beta^2 E}\int \frac{dF_0/d\omega_0}{\omega - n\omega_0} d\omega_0$$



Longitudinal stability limit

$$I_1 = \frac{Ne^2\omega_0^3 U(t)}{2\pi\beta^2 E}\left(\pi\frac{dF_0}{d\omega_0}(\omega) - jPV\int\frac{dF_0(\omega_0)/d\omega_0}{\omega - n\omega_0}d\omega_0\right)$$
$$1 = \frac{N\mathrm{e}^2\omega_0^3 \eta Z(\omega)}{2\pi\beta^2 E}\left(\frac{\pi dF_0}{d\omega_0}(\omega) - PVj\int\frac{dF_0(\omega_0)/d\omega_0}{\omega - n\omega_0}d\omega_0\right)$$
]

Response is perturbed current $I_1(t) = U(t)(r_r(\omega) + jr_i(\omega))$
Stability limit if $I_1(t)$ induces in $Z(\omega)$ just voltage used to excite beam $U(t) = I_1(t)Z(\omega)$

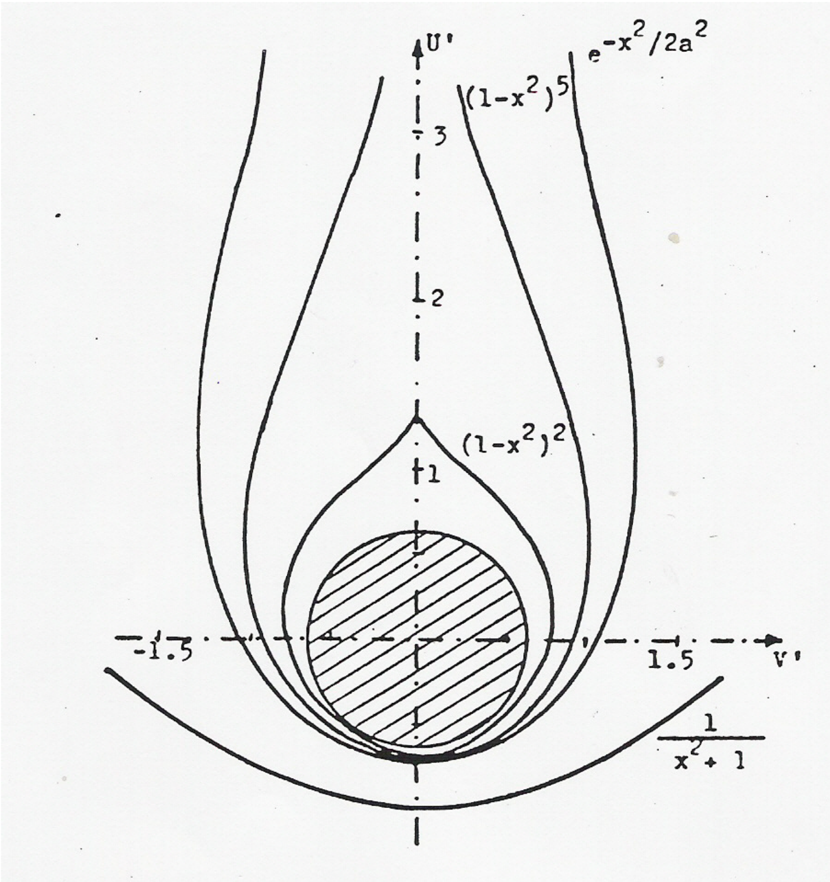


Mapping between complex impedance and complex response presented as stability diagram.
Separate beam and distribution form parameters.

$$I_0 = \frac{Ne\omega_0}{2\pi}, \qquad \delta p = \text{half width half height}$$
$$S = \eta\omega_0\frac{\delta p}{p}\ \text{spread},\ x = \frac{\omega_r - \omega_0}{S},\quad x_1 = \frac{\omega - n\omega_0}{nS}$$
$$g_0(x) = \frac{2\pi S F_0(\omega_r)}{N},\ \int g_0(x)dx = 1.$$

$$1 = -[V' + iU']\left[\pi\frac{dg_0}{dx}(x_1) - iPV\int\frac{dg_0/dx}{x - x_1}dx\right],\ [V' + iU'] = \frac{eI_0 Z(\omega)/n}{2\pi\beta^2 E_0\eta(\Delta p/p)^2}$$

Longitudinal stability criterion (Keil-Schnell)[2)]



Stability diagrams, (A. Ruggiero, V. Vaccaro[1)])

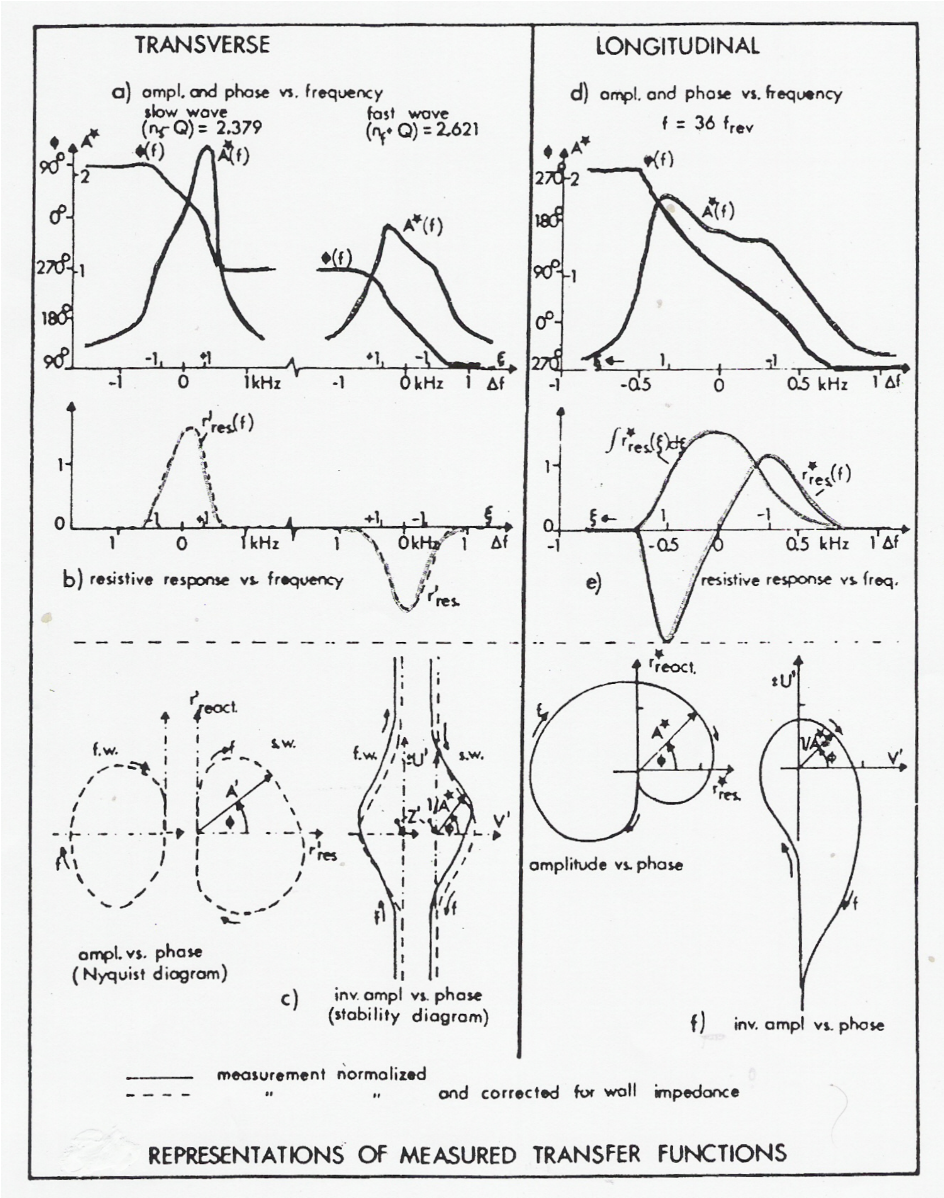
We separated effects due the distribution form and the ones due to beam and accelerator parameters and got the normalized stability diagram

$$V' + iU' = \frac{eI_0 Z(\omega)/n}{2\pi\beta^2 E\eta_c(\delta p/p)^2}$$
$$= -\left[\pi \frac{dg_0}{dx}(x_1) - iPV \int \frac{dg_0/dx}{x - x_1} dx\right]^{-1}$$

Approximate these diagrams by circle of radius 0.6, get Keil-Schnell stability criterion

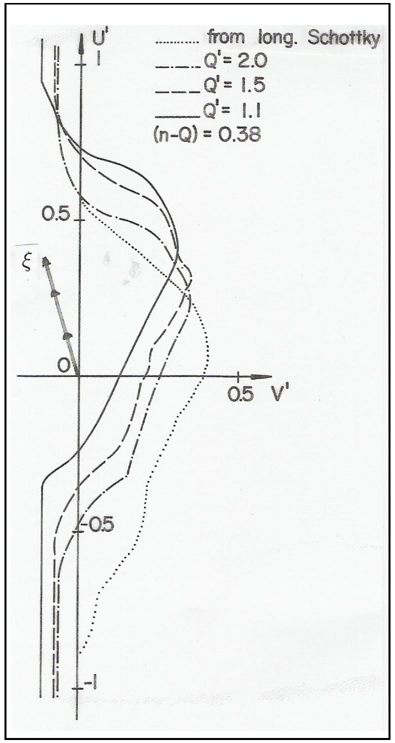
$$\left|\frac{Z}{n}\right| \leq \frac{2\pi\beta^2 E\eta_c(\delta p/p)^2}{eI_0}.$$

Measured coasting beam responses at ISR[3)]

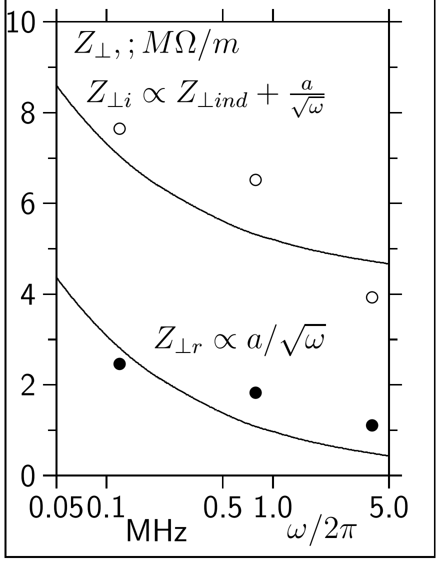


Transv.: side-band changes phase by π. Resistive response positive for slow,negative for fast wave.

Longitudinal: Each revolution harmonics gives 2π phase change.



measured impedance $Z_r + jZ_i$

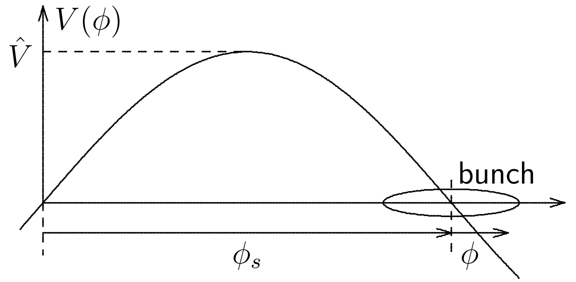


stability diagram shifted by impedance.

5) Bunched beams

In bunched beams spread in frequencies caused by non-linearities making them dependent on amplitude. Calculations are complicated but mainly amount of spread is of interest. To get Landau damping coherent frequency ω must be inside the spread.

Example: synchr. oscil., energy E_0, phase deviation ϕ, syn. phase $\phi_s \approx \pi$, rev. freq. ω_0



$$\omega_0^2 = \frac{\eta_c h e \hat{V}}{2\pi E}$$

$$\ddot{\phi} + \omega_{s0}^2 \sin\phi \approx \ddot{\phi} + \omega_{s0}^2 \phi = 0\ ,\ \ \phi = \hat{\phi}\cos(\omega_{s0} t)$$

$$H' = \frac{\dot{\phi}^2}{2} + \frac{\omega_{s0}^2 \phi^2}{2} = \frac{\omega_{s0}^2 \hat{\phi}^2}{2} = \text{ Hamiltonian}$$

$$\text{Gaussian } \psi(\dot{\phi}, \phi) \propto \mathrm{e}^{H'/\langle H' \rangle},\ \ \langle H' \rangle = \frac{\omega_{s0}^2 \sigma_\phi^2}{2}$$

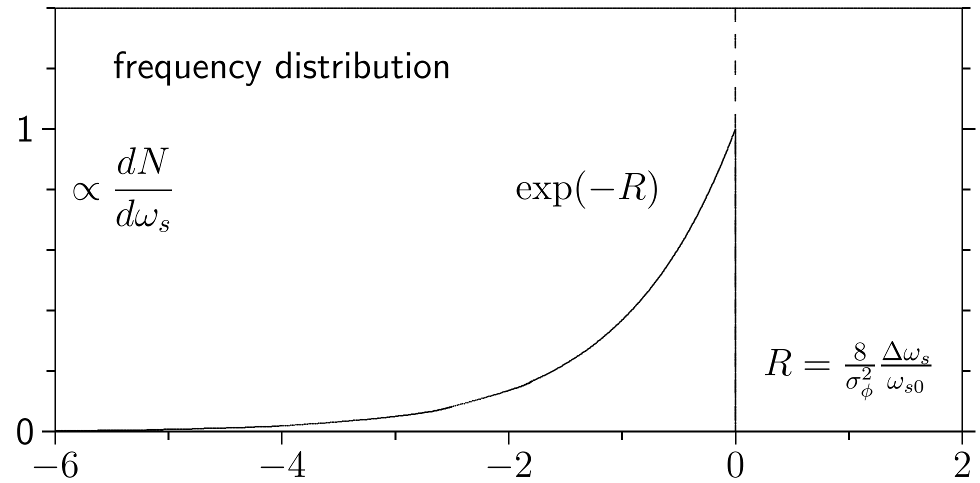
next approximation, $\sin\phi \approx \phi - \phi^3/6$, seeking solution of form $\phi = \hat{\phi}\cos(\omega_s t)$

$$(-\omega_s^2 + \omega_{s0}^2)\hat{\phi}\cos(\omega_s t) - \hat{\phi}^3 \omega_{s0}^2 \cos^3(\omega_s t)/6 = 0$$

using $\cos^3 x = (3\cos x + \cos(3x))/4$, neglecting higher harmonics gives

$$\frac{\omega_s}{\omega_{s0}} = \sqrt{1 - \frac{1}{8}\hat{\phi}^2} \approx 1 - \frac{1}{16}\hat{\phi}^2,\ \ \frac{\Delta\omega_s}{\omega_{s0}} = -\frac{\hat{\phi}^2}{16}$$

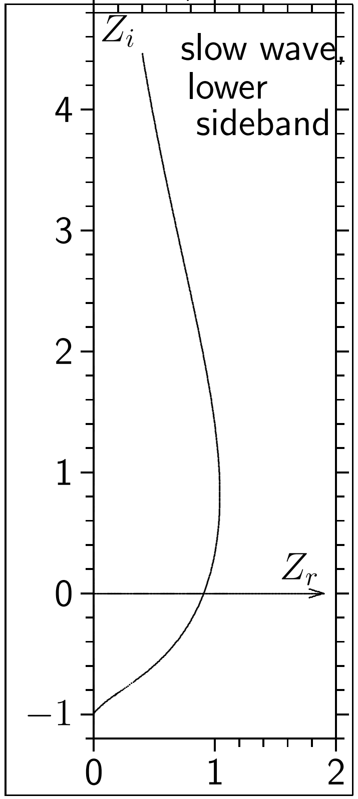
$$\frac{\Delta\omega_s}{\omega_{s0}} = -\frac{H'}{8\omega_{s0}^2} = -\frac{\langle H' \rangle}{8\omega_{s0}^2} R,\ \ R = \frac{H'}{\langle H' \rangle}.$$

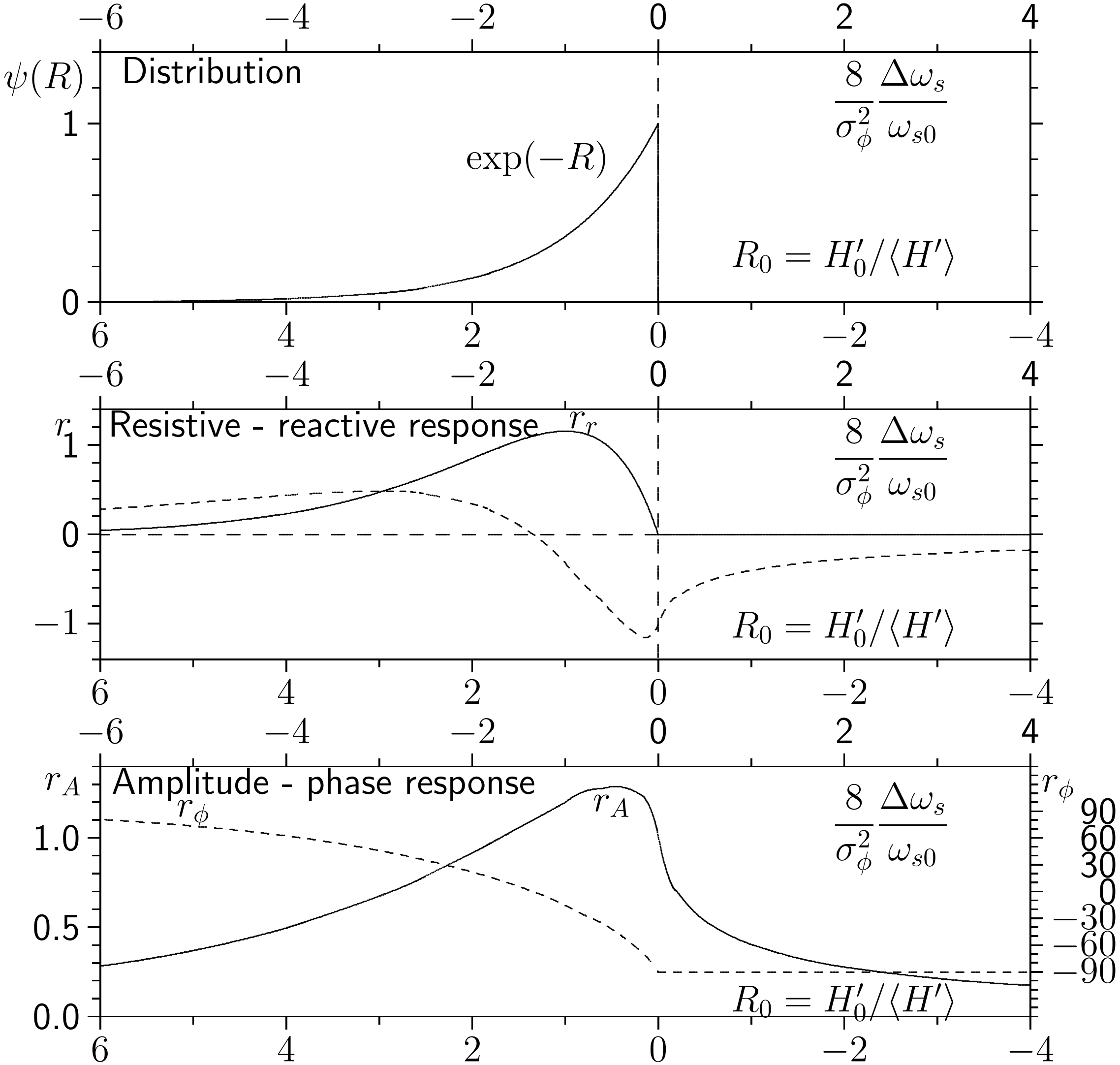


Harmonic excitation

Apply harmonic excitation to bunch $G = \hat{G}\cos(\omega t_0)$ with ω corresponding to Hamiltonian H_0', get dipole moment response D/G.

Stability diagram, arbitrary units

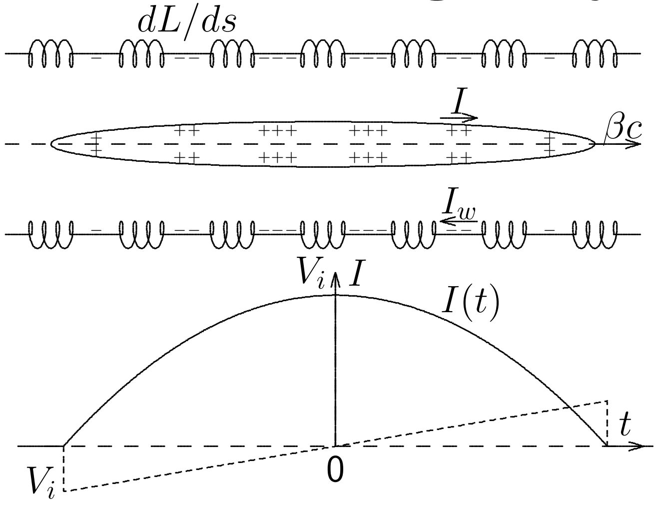




$$\frac{D}{C} \propto \frac{8\pi\alpha_c}{\omega_{s0}^2\sigma_\phi^2}\left[\pi e^{-R_0} R_0 \cos(\omega t) - PV\int_0^\infty \frac{e^{-R} R dR}{(R_0 - R)}\sin(\omega t)\right],$$

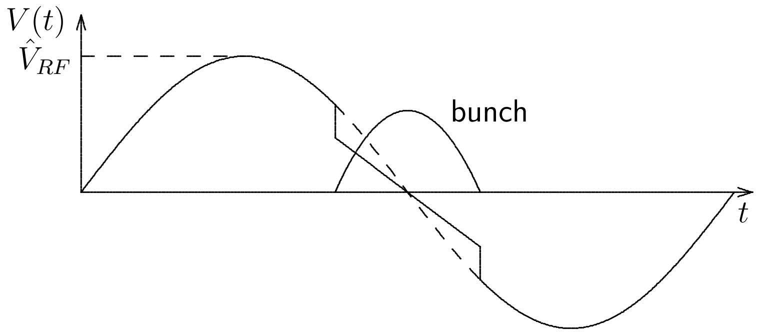
Boni et al. LNF- RM-23, 1981[4)]

Inductive wall — gives synchrotron frequency shift in long bunches



$$E_z = -\frac{dL}{dz}\frac{dI_w}{dt} = \frac{dL}{dz}\frac{dI_b}{dt}$$

$$V = -\int E_z dz = -L\frac{dI_b}{dz}$$



We take a parabolic bunch form

$$I_b(\tau) = \hat{I}\left(1-\frac{\tau^2}{\hat{\tau}^2}\right) = \frac{3\pi I_0}{2\omega_0\hat{\tau}}\left(1-\frac{\tau^2}{\hat{\tau}^2}\right)$$

$$\frac{dI_b}{d\tau} = -\frac{3\pi I_0 \tau}{\omega_0\hat{\tau}^3},\ I_0 = \langle I_b\rangle,$$

$$V = \hat{V}(\sin\phi_s + h\omega_0\cos\phi_s\tau) + \frac{3\pi I_0 L\tau}{\omega_0\hat{\tau}^3},\ L\omega_0 = \left|\frac{Z}{n}\right|$$

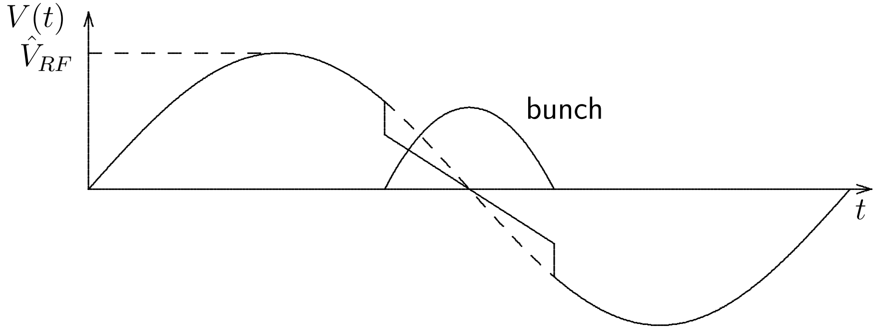
$$V = \hat{V}\left[\sin\phi_s + \cos\phi_s h\omega_0\left(1+\frac{3\pi|Z/n|_0 I_0}{h\hat{V}\cos\phi_s(\omega_0\hat{\tau})^3}\right)\tau\right]$$

$$\omega_{s0}^2 = -\frac{\omega_0^2 h\eta_c e\hat{V}\cos\phi_s}{2\pi E}$$

$$\omega_s^2 = \omega_{s0}^2\left[1+\frac{3\pi|Z/n|_0 I_0}{h\hat{V}_{RF}\cos\phi_s(\omega_0\hat{\tau})^3}\right]$$

$$\frac{\Delta\omega_s}{\omega_0} = \frac{\omega_s-\omega_{s0}}{\omega_{s0}} \approx \frac{3\pi|Z/n|_0 I_0}{2h\hat{V}_{RF}\cos\phi_s(\omega_0\hat{\tau}_0)^3}$$

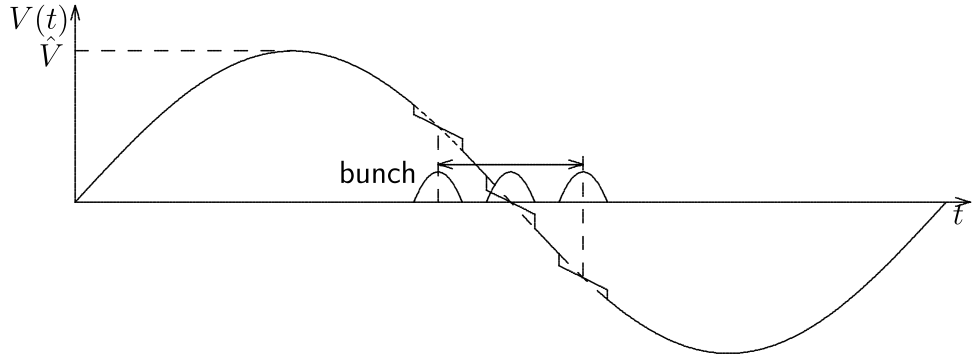
Coherent and incoherent frequency

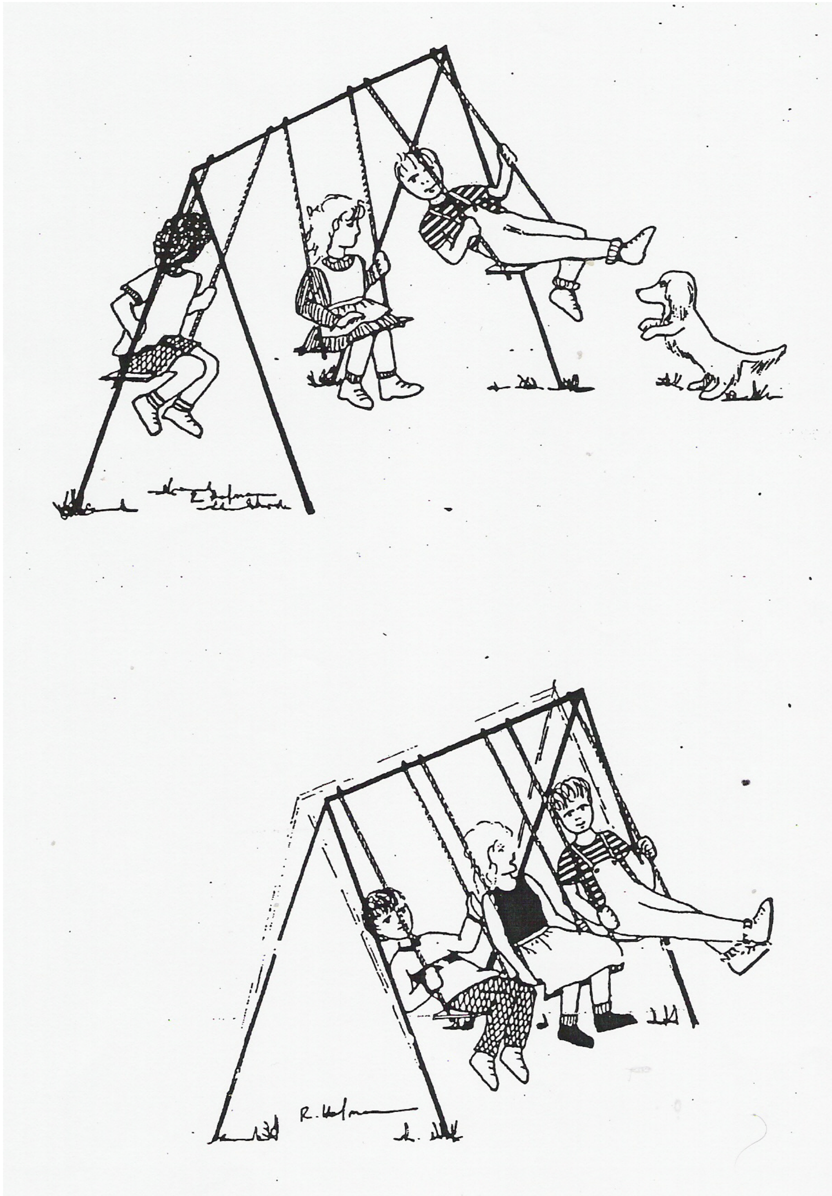


$$\frac{\omega_s^2}{\omega_{s0}^2} = 1 + \frac{3\pi|Z/n|_0 I_0}{h\hat{V}_{RF}\cos\phi_s(\omega_0\hat{\tau})^3}$$

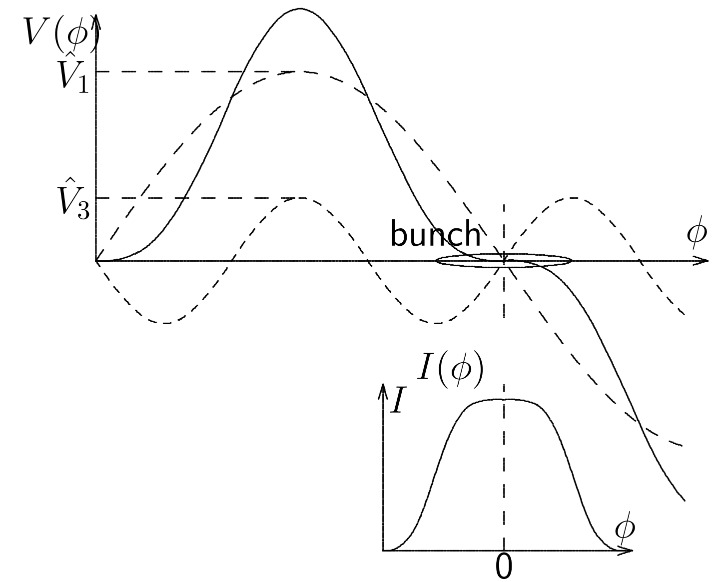
$$\frac{\omega_s - \omega_{s0}}{\omega_{s0}} = \frac{\Delta\omega_s}{\omega_s} \approx \frac{3\pi|Z/n|_0 I_0}{2h\hat{V}\cos\phi_s(\omega_0\hat{\tau}_0)^3}$$

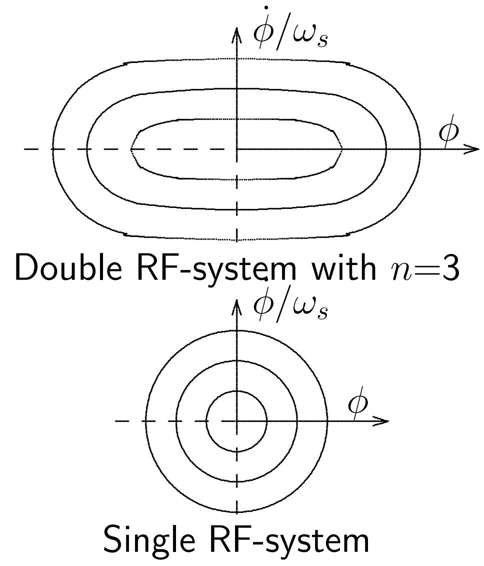
Incoherent frequency of single particles is changed (reduced for $\gamma > \gamma_T$ and vice versa), but coherent dipole (rigid bunch) mode. This separation reduces Landau damping.

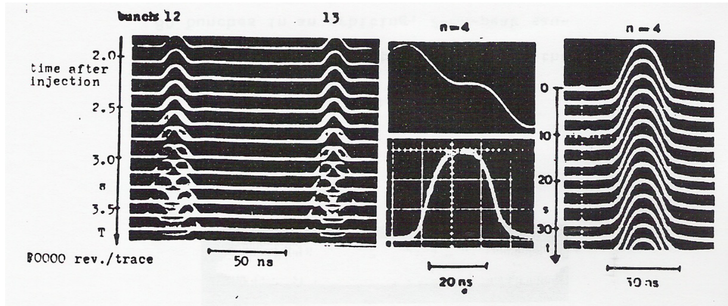




7Increase Landau damping - double RF-system







Stability by double RF-system; left: single RF, middle: RF and bunch form, right: stable bunch

Two RF-systems ω_{RF} and $n\omega_{RF}$, $\phi_s \approx \pi$,

$$V(\phi) \;=\; -\hat{V}_1 \sin\phi + \hat{V}_n \sin(n\phi) \text{ with } \hat{V}_n/ = \hat{V}_1/n$$

$$V(\phi) \;\approx\; \hat{V}_1 \left(-\left(\phi - \frac{\phi^3}{6}\right) + \frac{1}{n}\left(n\phi - \frac{n^3\phi^3}{6}\right)\right)$$

$$= \;-\hat{V}_1 \frac{n^2-1}{6}\phi^3$$

$$\ddot{\phi} + \omega_{s0}^2 \frac{n^2-1}{6}\phi^3 \;=\; 0\;,\;\; \omega_s \propto \hat{\phi}$$

$$\frac{\dot{\phi}^2}{2} + \omega_{s0}^2 \frac{n^2-1}{24}\phi^4 \;=\; H = \text{const.}$$

$$I(\phi) = \hat{I}\exp\left[-\omega_{s0}^2 \frac{n^2-1}{24\sigma_{\dot{\phi}}^2}\phi^4\right]$$

ω_{s0} and ω_s are synchrotron frequencies of basic and double RF-system. The strong amplitude dependence of ω_s gives spread and Landau damping. Flat voltage form leads to long bunch.

Bibliography

Different treatments of Landau are possible. This approach is close to observation and experiments using pulsed or harmonic excitation as described in theory of filters complex functions.

Some BTF papers used are found below general Landau damping on next page.

1) Ruggiero, Vaccaro; CERN ISR-TH-RF/69-48.

2) Keil, Schnell; CERN ISR-TH/68-33.

3) Borer, Guignard, Hofmann, Peschardt, Sacherer, Zotter; IEEE, NS-26, No 3 1979

4) Boni, Guiducci, Serio, Tazzioli, Wang; Frascati LNF-RM23, 1981.

5) J. Byrd; 1998 IEEE.

6) Kronilov, Boine-Frankenheim, Kaufmann, Moritz; GSI Acc-Note-2006-12-001.

7) O. Chorniy; thesis T.U. Darmstadt, 2008.

H.G. Hereward, "The elementary theory of Landau damping"; CERN 1965 - 20, PS-Division.

H.G. Hereward, "Landau damping by non-linearity"; CERN/MPS/DL 1969 - 11.

D. Sagan, "On the physics of Landau damping"; Amer. J. Physics, 1984.

R.D. Kohaupt, "Landau damping"; CAS 1989 Uppsala, CERN 90-04.

A. Hofmann, "Landau Damping", CAS 2003 Zeuthen, CERN 2006-002.

J. Rees, A. Chao, "Landau Damping Revisited", SLAC-PUB-13469.