Non–Linear

Imperfections

Intermediate Level CAS

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Non–Linear Imperfections

- equation of motion
  - → Hills equation
  - → sine and cosine like solutions + one turn map
- Poincare section → normalized coordinates
- smooth approximation
- resonances → tune diagram and fixed points
- non–linear resonances
  - → driving terms and magnetic multipole expansion
- perturbation treatment of non–linear maps
  - → amplitude growth and detuning guadrupole
  - → fixed points and slow extraction sextupole
  - → resonance islands octupole
- pendulum model equation of motion and phase space
- Hills equations in Cylindrical coordinates
- examples → resonance islands
- higher order perturbation treatment

Equations of Motion I

Lorentz Force:

$$\frac{d\vec{p}}{dt} = q \cdot (\vec{E} + \vec{v} \times \vec{B})$$

path length as free parameter:

replace time 't' by path length 's': $x' = \frac{d}{ds} x$

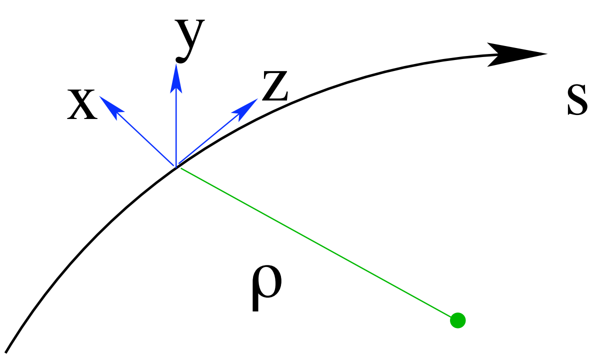
$$\frac{d}{dt} = \underbrace{\frac{ds}{dt}}_{v} \cdot \frac{d}{ds} \longrightarrow x' = \frac{p_x}{p_0}$$

Equation of motion:

$$\frac{d^2x}{ds^2} = \frac{F}{v \cdot p_0}$$

Equations of Motion II

Variables in rotating coordinate system:



geometrical focusing

Hills equation:

$$\frac{d^2x}{d\,s^2} + K(s)\cdot\, x = 0 \qquad K(s) = K(s+L);$$

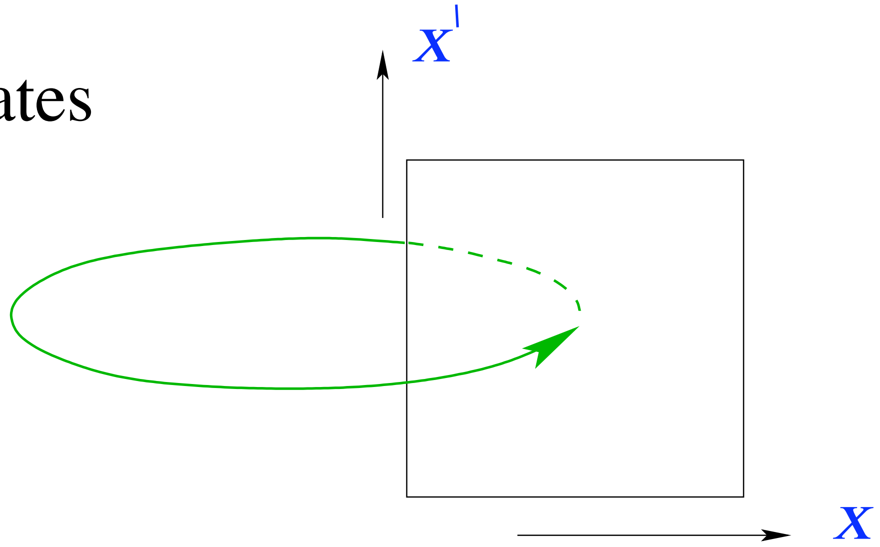
$$K(s) = \begin{cases} 0 & \text{drift} \\ 1/\rho^2 & \text{dipole} \\ 0.3\bullet\dfrac{B[T/m]}{p[GeV/c]} & \text{quadrupole} \end{cases}$$

Non–linear equation of motion:

$$\frac{d^2x}{d\,s^2} + K(s)\cdot\, x = \frac{F_x}{v\bullet p}$$

Poincare Section I

- Display coordinates after each turn:



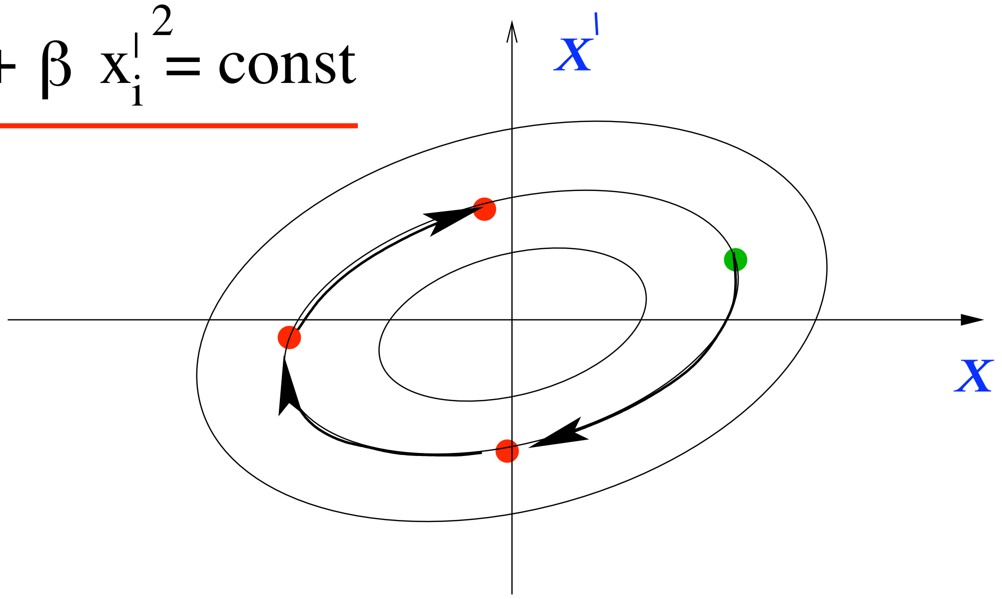
- Linear β – motion:

$$x_i = \sqrt{R} \cdot \sqrt{\beta(s)} \cdot \sin(2\pi Q\, i + \phi_0)$$

$$x'_i = \sqrt{R} \cdot [\cos(2\pi Q\, i + \phi_0) + \alpha(s) \cdot \sin(2\pi Q\, i + \phi_0)] / \sqrt{\beta(s)}$$

$\rightarrow$ $\gamma\, x_i^2 + 2\,\alpha\, x_i x'_i + \beta\, x'^2_i = \mathrm{const}$

$\rightarrow$ ***ellipse***



- the ellipse orientation and the half axis length vary along the machine

Poincare Section II

- for the sake of simplicity assume $\alpha = 0$

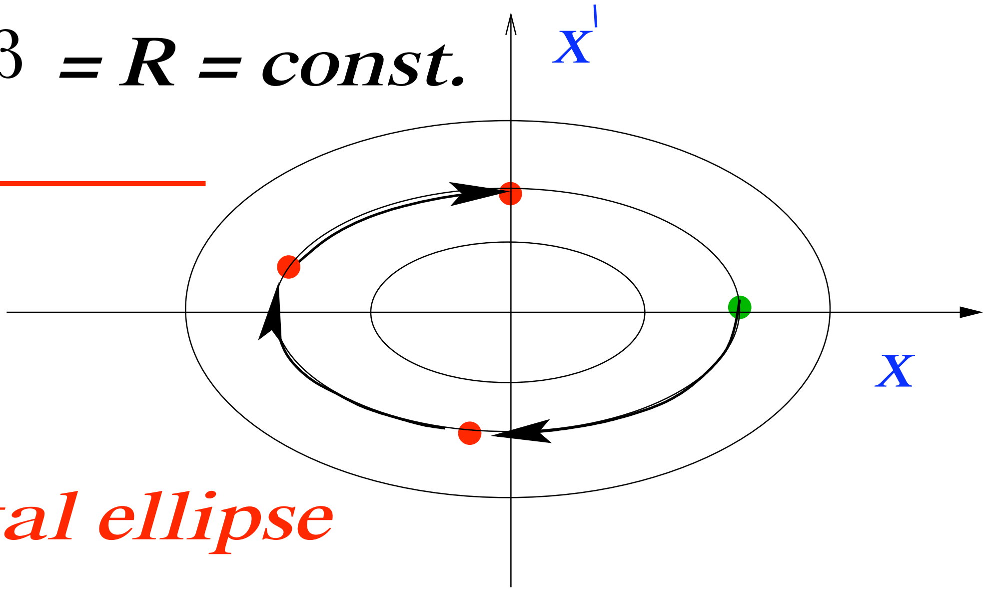
  at the location of the Poincare Section

→

$$x = \sqrt{\beta} \cdot \sqrt{R} \cdot \cos(2\pi\, Q\, i + \phi_0)$$

$$x' = \sqrt{R} \cdot \sin(2\pi\, Q\, i + \phi_0)/\sqrt{\beta}$$

$$\frac{x^2}{\beta} + x'^2 \cdot \beta = R = const.$$



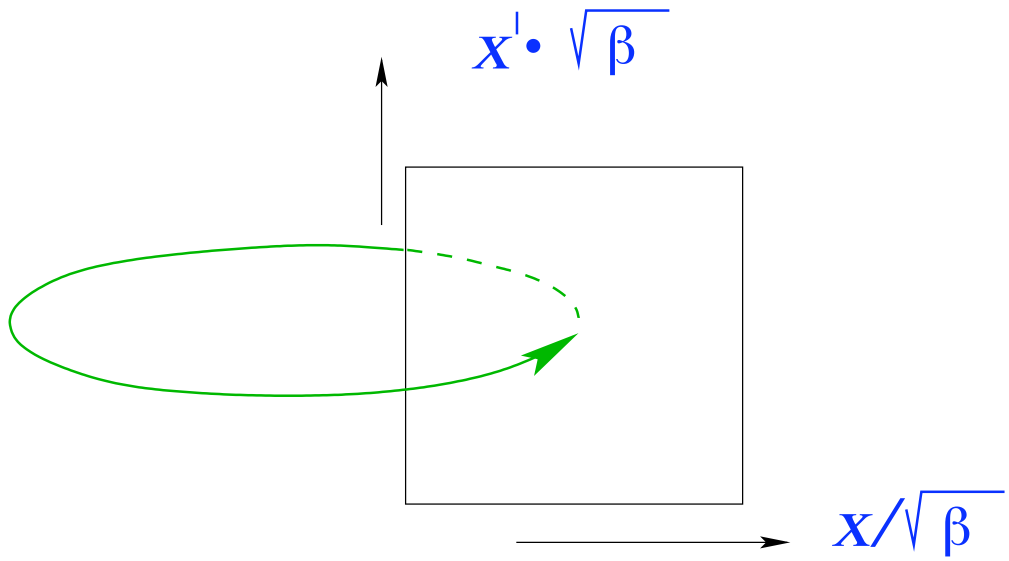
→ *horizontal ellipse*

- for $\alpha \neq 0$

  one can define a new set of coordinates via linear combination of x and x' such that one axis of the ellipse is parallel to x–axis

Poincare Section III

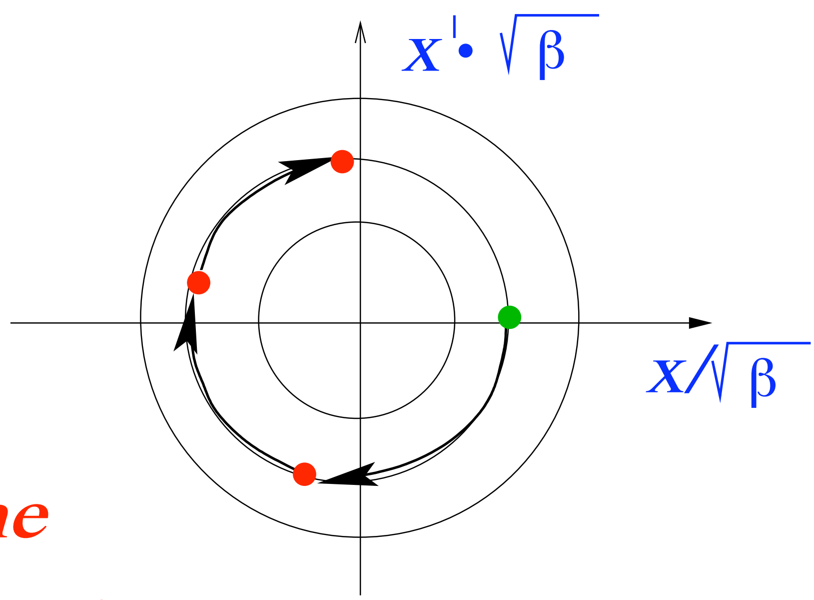
- **Display normalized coordinates:**



- **normalized coordinates:**

$$x/\sqrt{\beta} = \sqrt{R} \cdot \cos(2\pi\, Q\, i + \phi_0)$$

$$\sqrt{\beta} \cdot x' = -\sqrt{R} \cdot \sin(2\pi\, Q\, i + \phi_0)$$



→ circles in the Poincare Section

Smooth Approximation

assume: $\beta = \text{constant}$

$$\longrightarrow \quad x = A \cdot \cos[\phi(s)] \qquad \text{with:} \qquad \phi(s) = \int_{s0}^{s} \frac{1}{\beta}\, dt$$

$$\longrightarrow \quad \frac{d\phi}{ds} = \boxed{\frac{1}{\beta} = \omega = \frac{2\pi Q}{L}}$$

Linear β – motion: $\beta = const \longrightarrow \alpha = 0$

$$x_i = \sqrt{R} \cdot \sqrt{\beta(s)} \cdot \sin(2\pi Q\, i + \phi_0)$$

$$x'_i = \sqrt{R} \cdot \cos(2\pi\, Q\, i + \phi_0)/\sqrt{\beta(s)}$$

Linear equation of motion:

$$\frac{d^2x}{ds^2} + \left(\frac{2\pi}{L} \cdot Q\right)^2 \cdot x = 0 \longrightarrow \text{Harmonic Oscillator}$$

Resonances I

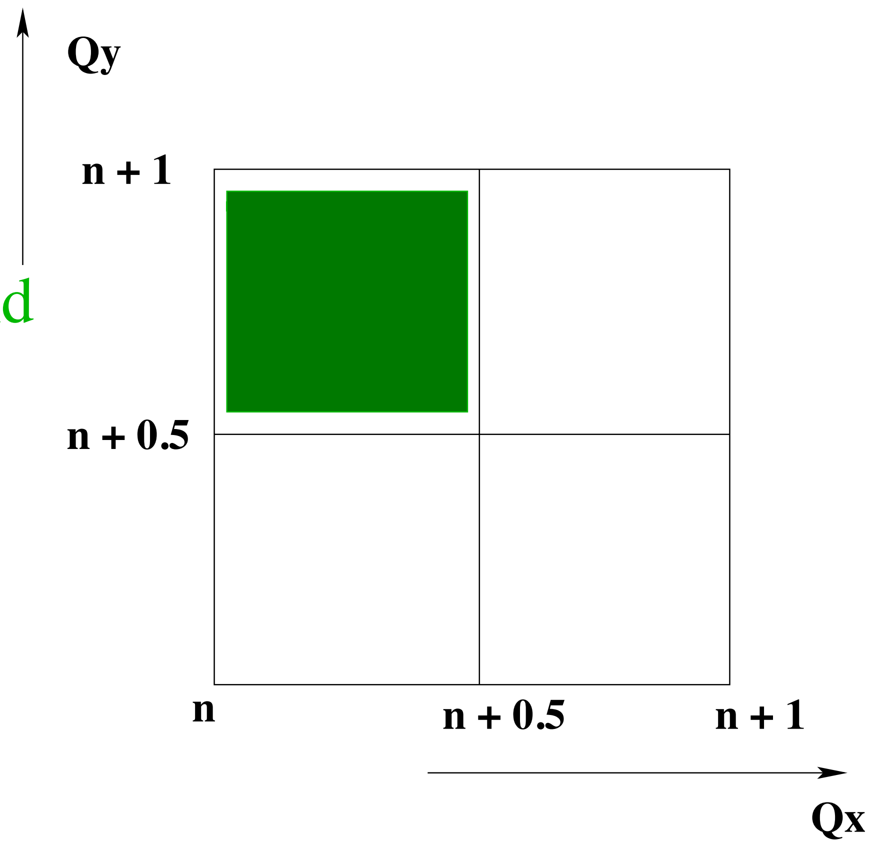
tune diagram with linear resonances:

stability:

avoid integer and

half integer

resonances!



higher order resonances:

$$n\,Q_x + m\,Q_y = r$$

the rational numbers

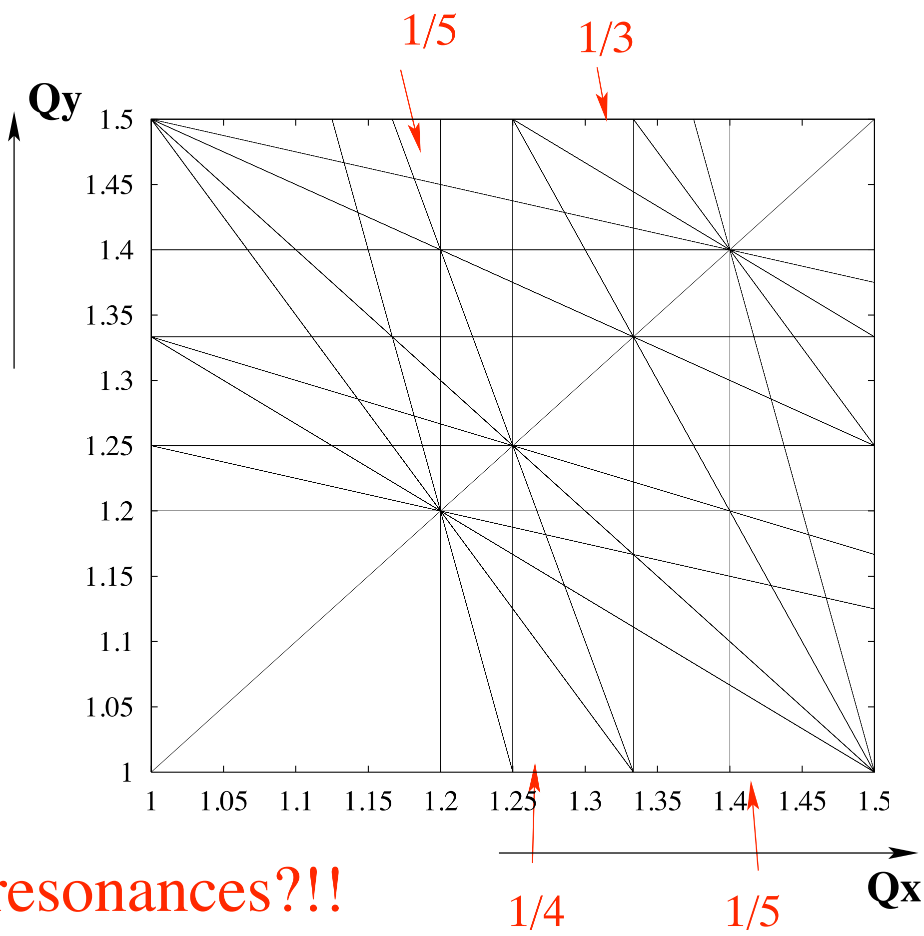
lie 'dense' in the

real numbers

there are resonances

everywhere

stability of low order resonances?!!

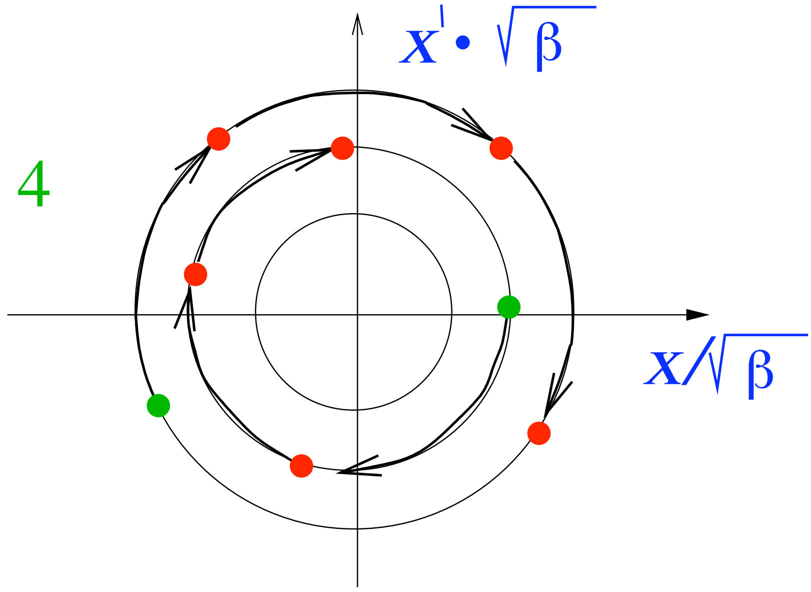


Resonances II

fixed points in the Poincare section:

$Q = N + 1 / n$

example: n = 4



→ every point is mapped on itself after n turns!

→ every point is a 'fixed point'

→ motion remains stable if the resonances are not driven

→ sources for resonance driving terms?

Non–Linear Resonances I

- Sextupoles + octupoles

- Magnet errors:

  pole face accuracy

  geometry errors

  eddy currents

  edge effects

- Vacuum chamber:

  LEP I welding

- Beam–beam interaction

$\longrightarrow$ careful analysis of all components

Non–Linear Resonances II

Taylor expansion for upright multipoles:

$$B_y + i \cdot B_x = \sum_{n=0} \frac{1}{n!} \cdot f_n \cdot (x + i\,y)^n$$

with: $$f_n = \frac{\partial^n B_y}{\partial x^n}$$

multipole	order	B_x	B_y
dipole	0	0	B_0
quadrupole	1	$f_1 \cdot y$	$f_1 \cdot x$
sextupole	2	$f_2 \cdot x \cdot y$	$\frac{1}{2} f_2 \cdot (x^2 - y^2)$
octupole	3	$\frac{1}{6} f_3 \cdot (3y\,x^2 - y^3)$	$\frac{1}{6} f_3 \cdot (x^3 - 3x\,y^2)$

convergence:

the Taylor series is normally not convergent for

$|x + i\,y| > 1$ $\longrightarrow$ define ´normalized´ coefficients

$$b_n = \frac{f_{n-1}}{(n-1)! \cdot B_0} \cdot R_{ref}^{\,n-1}$$

Non–Linear Resonances III

- **normalized multipole expansion:**

$$B_y + i \cdot B_x = B_{main} \cdot \sum_{n=1} b_n \cdot \left(\frac{x + i\,y}{R_{ref}}\right)^{n-1}$$

b_n is the relative field contribution of the n–th multipole at the reference radius

b_1 = dipole; b_2 = quadrupole; b_3 = sextupole; etc

- **skew multipoles:**

rotation of the magnetic field by 1/2 of the azimuthal magnet symmetry: 90^o for dipole

45^o for quadrupole

30^o for sextupole; etc

- **general multipole expansion:**

$$B_y + i \cdot B_x = B_{main} \cdot \sum_{n=1} (b_n - i\,a_n) \cdot \left(\frac{x + i\,y}{R_{ref}}\right)^{n-1}$$

Perturbation I

- perturbed equation of motion:

$$\frac{d^2x}{ds^2} + \left(\frac{2\pi}{L} \cdot Q_x\right)^2 \cdot x = \frac{F_x(x,y)}{v \cdot p}$$

$$\frac{d^2y}{ds^2} + \left(\frac{2\pi}{L} \cdot Q_y\right)^2 \cdot y = \frac{F_y(x,y)}{v \cdot p}$$

- assume motion in one degree only:

$y \equiv 0$ is a solution of the vertical equation of motion

$$\longrightarrow \quad B_x \equiv 0; \qquad B_y = \frac{1}{n!} \cdot f_n \cdot x^n \qquad F_x = -v_s \cdot B_y$$

- perturbed horizontal equation of motion:

$$\frac{d^2x}{ds^2} + \left(\frac{2\pi}{L} \cdot Q_x\right)^2 \cdot x = \frac{-1}{n!} \cdot k_n(s) \cdot x^n$$

- normalized strength:

$$k_n = 0.3 \cdot \frac{f_n\,[\mathrm{T/m}^n]}{p\,[\mathrm{GeV/c}]}\,; \quad [k_n] = 1\,/\,\mathrm{m}^{n+1}$$

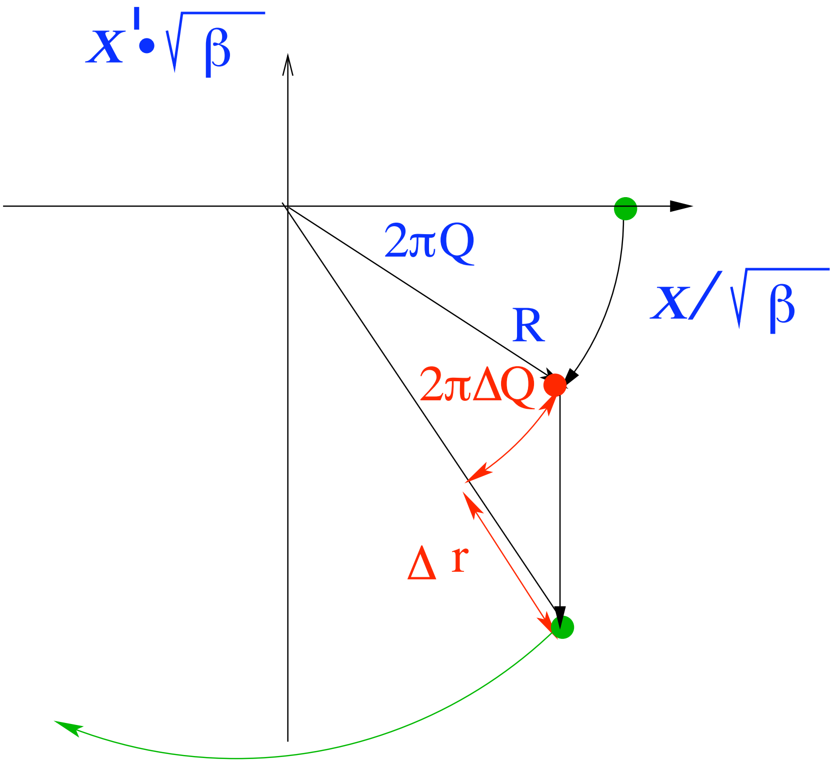
Perturbation II

- perturbation just infront of Poincare Section:

$$\Delta x' = \int \frac{F_y}{v \cdot p}\, ds \longrightarrow = \frac{-l}{n!} \cdot k_n \cdot x^n$$

where 'l' is the length of the perturbation

- perturbed Poincare Map:



$k_n > 0$ ⟶

increase in 'R'

increase in 'Q'

- stability of particle motion over many turns?

Perturbation III

- coordinates after 'i' itteration and before kick:

(1) $x_i / \sqrt{\beta} = r \cdot \cos(\phi_i)$ $\quad x'_i \cdot \sqrt{\beta} = -r \cdot \sin(\phi_i)$

(2) with: $\phi_i = \phi_{i-1} + 2\pi Q$ and: $\boxed{r = \sqrt{R}}$

- coordinates after the perturbation kick:

(3) $$x_{i+kick} / \sqrt{\beta} = x_i / \sqrt{\beta}$$

(4) $$x'_{i+kick} \cdot \sqrt{\beta} = x'_i \cdot \sqrt{\beta} - \frac{1}{n!} \cdot k_n \cdot x_i^n \cdot \sqrt{\beta}$$

- write new coordinates in circular coordinates

(5) $$x_{i+kick} / \sqrt{\beta} = (r + \Delta r_i) \cdot \cos(\phi_i + \Delta\phi_i)$$

(6) $$x'_{i+kick} \cdot \sqrt{\beta} = (r + \Delta r_i) \cdot \sin(\phi_i + \Delta\phi_i)$$

Perturbation IV

- solve for 'Δr_i' and '$\Delta\phi_i$':

  → substitute (1) and (2) into (3) and (4)

  → set new expression equal to (5) and (6)

  → use: $\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$

  $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$

  and: $\sin(\Delta\phi) = \Delta\phi$; $\cos(\Delta\phi) = 1$

→ solve for 'Δr_i' and '$\Delta\phi_i$':

→ $$\Delta r_i = -\Delta x_i' \cdot \sqrt{\beta} \cdot \sin(\phi_i)$$

$$\Delta\phi_i = \frac{-\Delta x_i' \cdot \sqrt{\beta} \cdot \cos(\phi_i)}{[r + \Delta x_i' \cdot \sqrt{\beta} \cdot \sin(\phi_i)]}$$

- substitute the kick expression:

(7) $$\Delta r_i = \frac{l}{n!} \cdot k_n \cdot x_i^n \cdot \sqrt{\beta} \cdot \sin(\phi_i)$$

(8) $$\Delta\phi_i = \frac{\frac{l}{n!} \cdot k_n \cdot x_i^n \cdot \sqrt{\beta} \cdot \cos(\phi_i)}{[r + \Delta r_i]}$$

Perturbation V

- quadrupole perturbation:

$$\Delta r_i = l \cdot k_l \cdot x_i \cdot \sqrt{\beta} \cdot \sin(\phi_i)$$

with: $x_i = \sqrt{\beta} \cdot r \cdot \cos(\phi_i)$

$$\Delta r_i = l \cdot k_l \cdot r \cdot \beta \cdot \sin(2\phi_i)$$

sum over many turns with: $\phi_i = 2\pi Q \cdot i$

$$\longrightarrow \boxed{\sum_i \Delta r_i = 0 \quad \text{unless:} \quad Q = p/2}$$

(half integer resonance)

- tune change (first order in the perturbation):

$$\Delta\phi_i = l \cdot k_l \cdot \beta \cdot [1+\cos(2\phi_i)]/2$$

average change per turn: $\phi_i = 2\pi Q \cdot i$

$$\boxed{\langle \Delta Q \rangle = l \cdot k_l \cdot \beta / 4\pi} \longrightarrow Q = Q_0 + \langle \Delta Q \rangle$$

Perturbation VI

- resonance stop band: $Q \neq p/2$

the map perturbation generates a tune oscillation

$$\delta Q_i = l \cdot k_1 \cdot \beta \cdot \cos(4\pi \cdot Q \cdot i + 2\phi_o)/4\pi$$

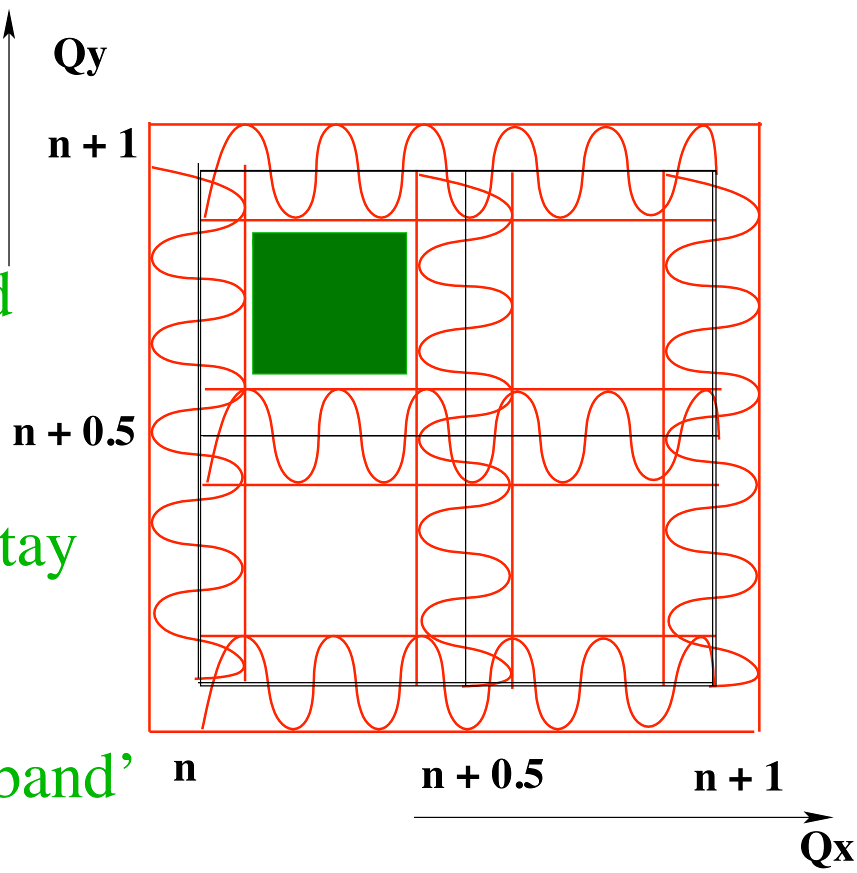
$$= \langle \Delta Q \rangle \cdot \cos(4\pi Q i + 2\phi_o)/4\pi$$

→ particles will experience the half integer resonance if their tune satisfies:

$$\boxed{(p/2 - \langle \Delta Q \rangle) < Q < (p/2 + \langle \Delta Q \rangle)}$$

- tune diagram:

avoid integer and half integer resonances and stay away from the resonance 'stop band'



Perturbation VII

- sextupole perturbation:

$$\Delta r_i = \boldsymbol{l} \cdot k_2 \cdot x_i^2 \cdot \sqrt{\beta} \cdot \sin(\phi_i)/2$$

with: $x_i = \sqrt{\beta} \cdot r \cdot \cos(\phi_i)$

$$\Delta r_i = \boldsymbol{l} \cdot k_2 \cdot r_i^2 \cdot \beta^{3/2} \cdot [\sin(\phi_i) + \sin(3\phi_i)]/8$$

sum over many turns: $\phi_i = 2\pi Q \cdot i$

→ $\boxed{r = 0 \quad \text{unless:} \quad Q = p \text{ or } Q = p/3}$

- tune change (first order in the perturbation):

$$2\pi \, \Delta Q_i = \boldsymbol{l} \cdot k_2 \cdot r_i \cdot \beta^{3/2} \cdot [3 \cos(2\pi Q i + \phi_0) + \cos(6\pi Q i + 3\phi_0)]/8$$

sum over many turns:
(unless: $Q = p$ or $Q = p/3$) $\boxed{\langle \Delta Q \rangle = 0}$

→ stop band increases with amplitude!

Perturbation VIII

what happens for Q = p; p/3 ?

$$\Delta r_i = l \cdot k_2 \cdot r_i^2 \cdot \beta^{3/2} \cdot \boxed{[\sin(2\pi Q i + \phi_o) + \sin(6\pi Q i + 3\phi_o)]/8}$$

constant for each kick

$$2\pi \Delta Q_i = l \cdot k_2 \cdot r_i \cdot \beta^{3/2} \cdot \boxed{[3\cos(2\pi Q i + \phi_o) + \cos(6\pi Q i + 3\phi_o)]/8}$$

amplitude 'r' increases every turn → instability

→ dephasing and tune change

→ motion moves off resonance

→ stop of the instability

→ what happens in the long run?

Perturbation IX

- let us assume: Q = p/3

$$\Delta r_i = \boldsymbol{l} \cdot k_2 \cdot r_i^2 \cdot \beta^{3/2} \cdot [\boxed{\sin(\phi_i)} + \sin(3\phi_i)] / 8$$

$$\Delta\phi_i = \boldsymbol{l} \cdot k_2 \cdot r_i \cdot \beta^{3/2} \cdot [\boxed{3\cos(\phi_i)} + \cos(3\phi_i)] / 8$$

$$+ 2\pi Q$$

the first terms change rapidly for each turn

→ the contribution of these terms are small
and we omit these terms in the following
(method of averaging)

→ $$\Delta r_i = \boldsymbol{l} \cdot k_2 \cdot r_i^2 \cdot \beta^{3/2} \cdot \sin(3\phi_i) / 8$$

$$\Delta\phi_i = \boldsymbol{l} \cdot k_2 \cdot r_i \cdot \beta^{3/2} \cdot \cos(3\phi_i) / 8 + 2\pi Q$$

Perturbation X

fixed point conditions: $Q_0 \gtrsim p/3;\ k_2 > 0$

$\Delta r / \text{turn} = 0$ and $\Delta\phi / \text{turn} = 2\pi p / 3$

with: $$\Delta r_i = \boldsymbol{l} \cdot k_2 \cdot r_i^2 \cdot \beta^{3/2} \cdot \sin(3\,\phi_{\boldsymbol{i}}) / 8$$

$$\Delta\phi_i = 2\pi Q_0 + \boldsymbol{l} \cdot k_2 \cdot r_i \cdot \beta^{3/2} \cdot \cos(3\,\phi_{\boldsymbol{i}}) / 8$$

$\longrightarrow$ $$\phi_{\text{fixed point}} = \pi/3;\ \pi;\ 5\pi/3;$$

$$r_{\text{fixed point}} = \frac{16\pi\,(Q_0 - p/3)}{\boldsymbol{l}\ k_2\ \beta^{3/2}}$$

$\longrightarrow$ $r = 0$ also provides a fixed point in the

$x;\ x'$ plane

(infinit set in the r, ϕ plane)

Perturbation XI

fixed point stability:

linearize the equation of motion around the fixed points:

Poincare map:

$$r_{i+1} = r_i + f(r_i, \phi_i)$$

$$\phi_{i+1} = \phi_i + g(r_i, \phi_i)$$

single sextupole kick:

→ $$f = \boldsymbol{l} \cdot k_2 \cdot r_i^2 \cdot \beta^{3/2} \sin(3\phi_i) / 8$$

$$g = \boldsymbol{l} \cdot k_2 \cdot r_i \cdot \beta^{3/2} \cos(3\phi_i) / 8$$

→ linearized map around fixed points:

$$\begin{pmatrix} r_{i+1} \\ \phi_{i+1} \end{pmatrix} = \left.\begin{pmatrix} \frac{\partial r_{i+1}}{\partial r_i} & \frac{\partial r_{i+1}}{\partial \phi_i} \\ \frac{\partial \phi_{i+1}}{\partial r_i} & \frac{\partial \phi_{i+1}}{\partial \phi_i} \end{pmatrix}\right|_{\text{fixed point}} \circ \begin{pmatrix} r_i \\ \phi_i \end{pmatrix}$$

Perturbation XII

- Jacobin matrix for single sextupole kick:

Jacobian matrix

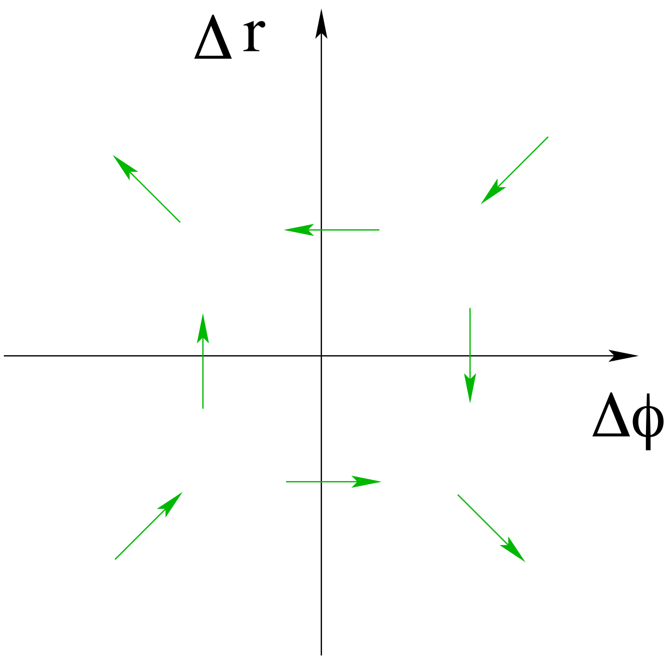
$$\frac{\partial r_{i+1}}{\partial r_i} = 1; \qquad \frac{\partial r_{i+1}}{\partial \phi_i} = -3\boldsymbol{l} \cdot k_2 \cdot \beta^{3/2} \cdot r^2_{\text{fixed point}} / 8$$

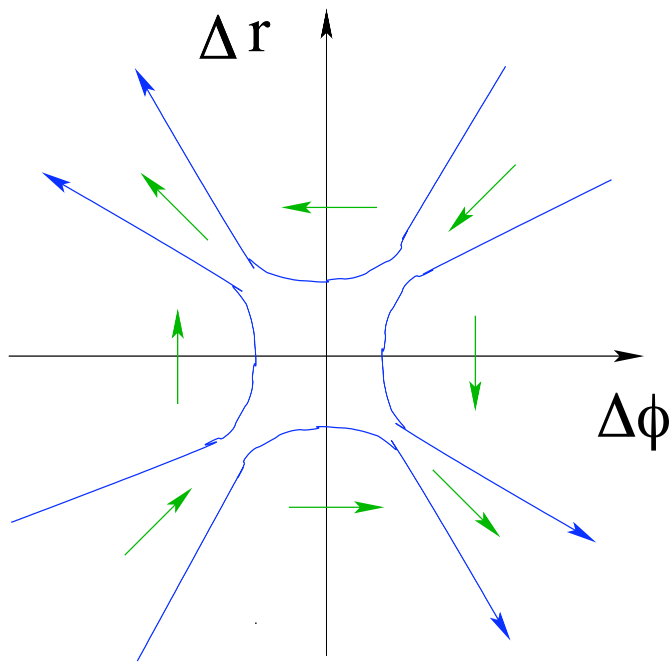
$$\frac{\partial \phi_{i+1}}{\partial r_i} = -\boldsymbol{l} \cdot k_2 \cdot \beta^{3/2} / 8; \qquad \frac{\partial \phi_{i+1}}{\partial \phi_i} = 1$$

$\phi_{\text{fixed point}} = \pi/3;\ \pi;\ 5\pi/3;$ and $r_{\text{fixed point}} \neq 0$

$\longrightarrow$ $\Delta r_{i+1} = -3\boldsymbol{l} \cdot k_2 \cdot \beta^{3/2} \cdot r^2_{\text{fixed point}} / 8 \cdot \Delta\phi_i$

$\Delta\phi_{i+1} = -\boldsymbol{l} \cdot k_2 \cdot \beta^{3/2} / 8 \cdot \Delta r_i$ stability?

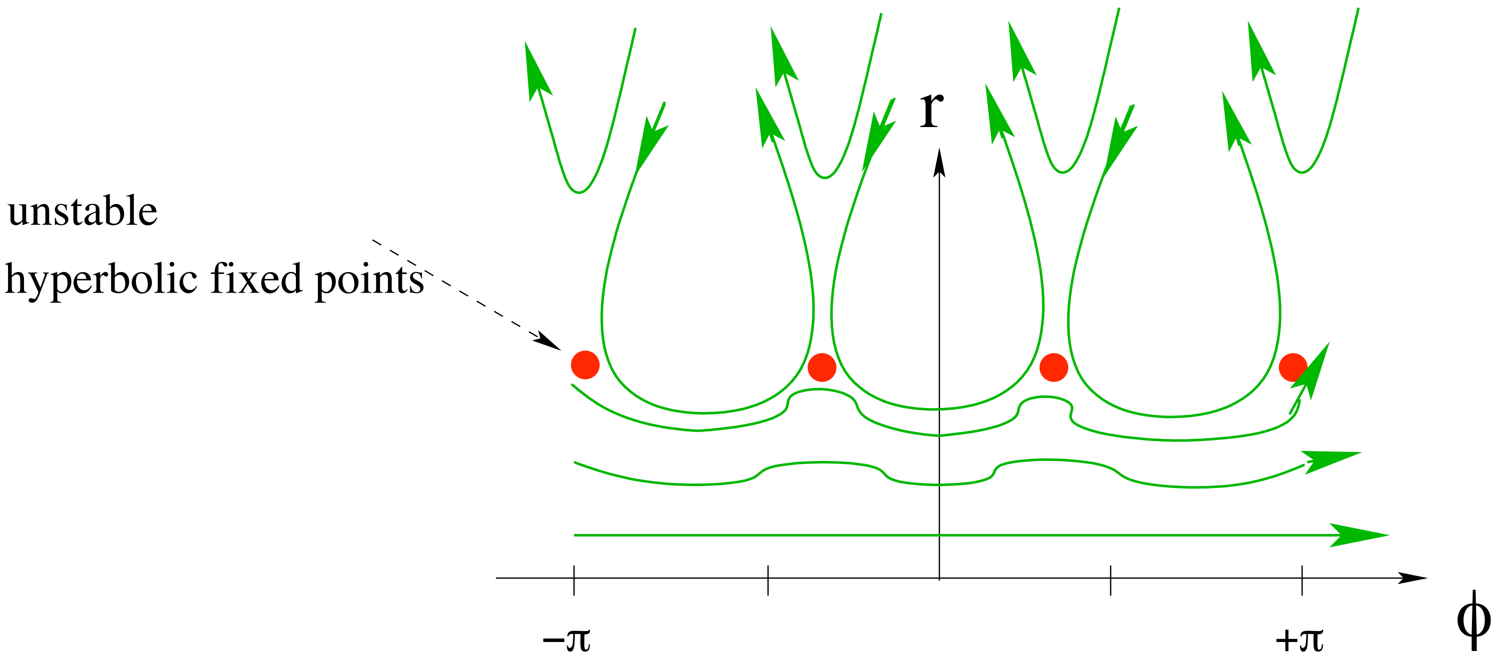




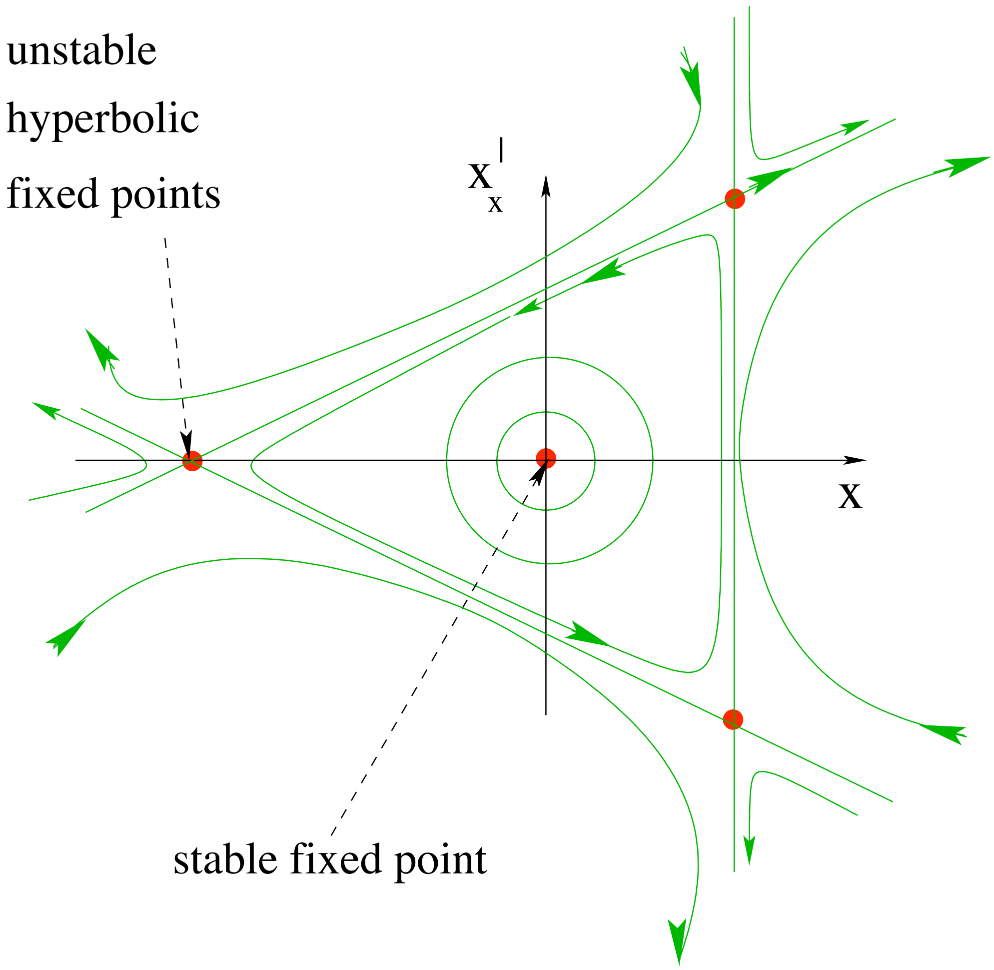
hyperbolic fixed point

Perturbation XIII

- Poincare Section for 'r' and ϕ ':

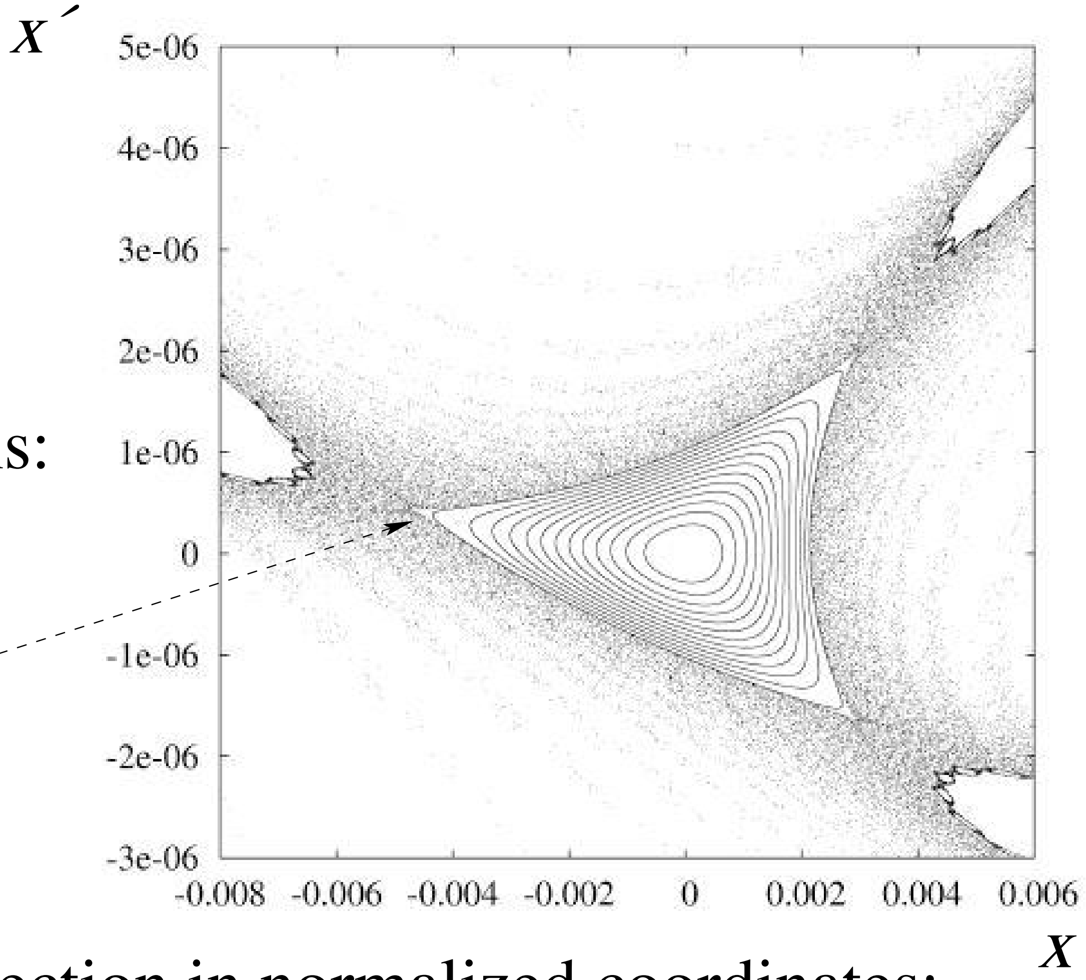


- Poincare section in normalized coordinates:



Perturbation XIV

- Sextupole Poincare Section from simulations:



unstable

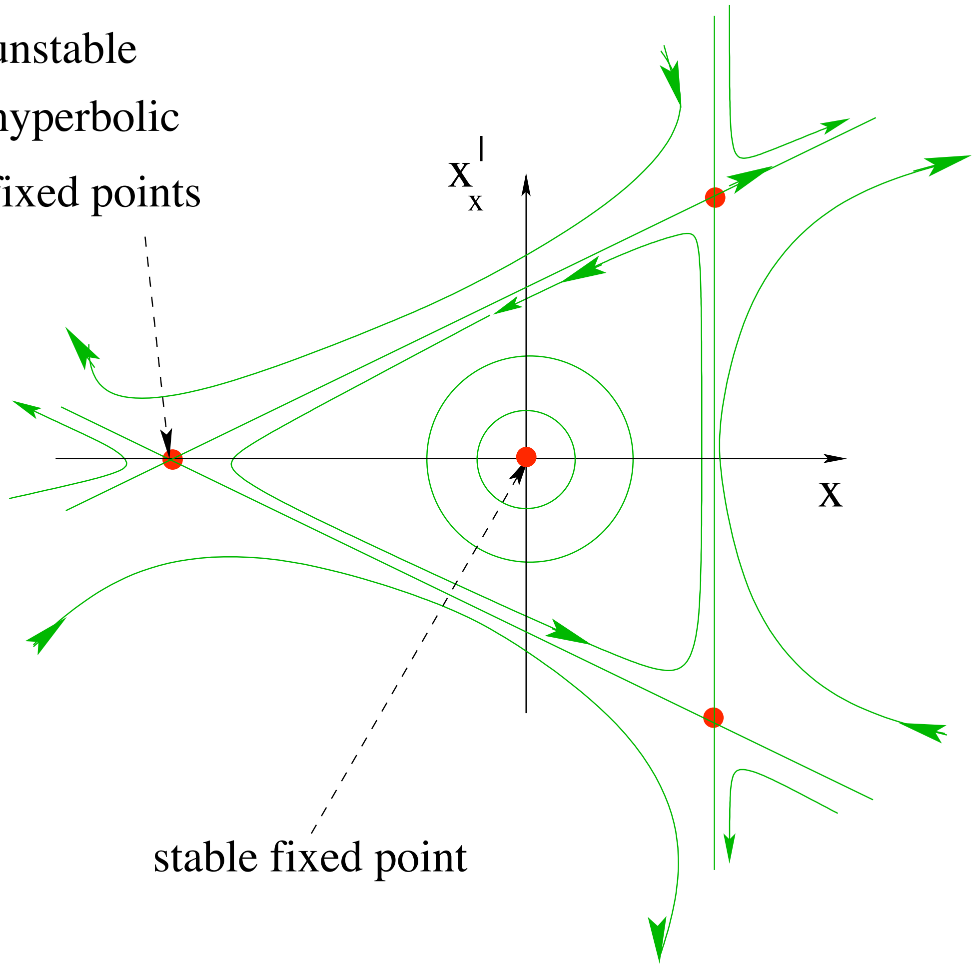
hyperbolic fixed point

- Poincare section in normalized coordinates:

unstable

hyperbolic

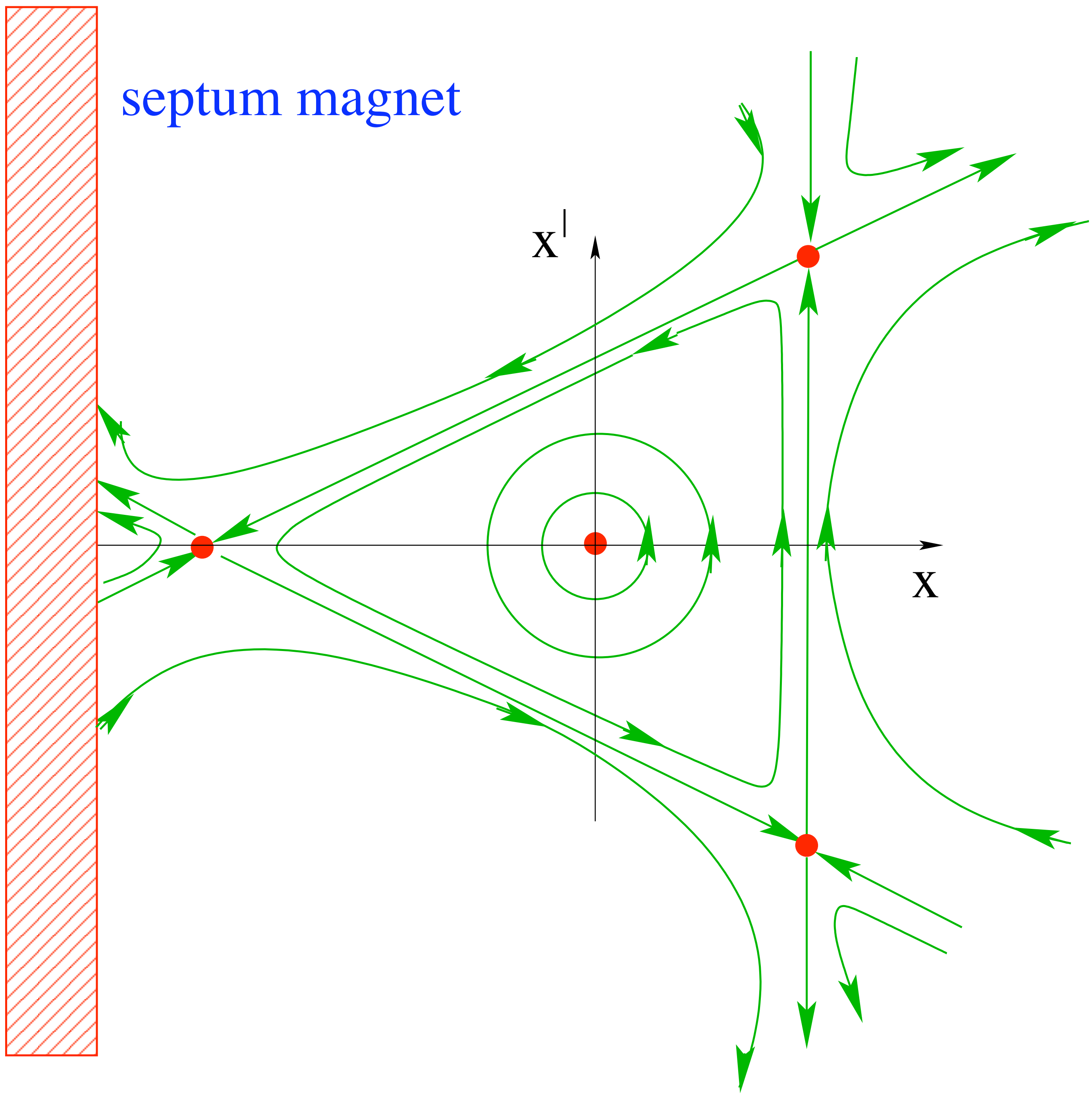
fixed points



stable fixed point

Perturbation XVI

slow extraction:



fixed point position:

$$r_{\text{fixed point}} = \frac{16\pi\left(Q - \frac{p}{3}\right)}{l \cdot k_2 \cdot \beta^{3/2}} \longrightarrow$$

changing the tune during extraction!

Perturbation XVII

- octupole perturbation:

$$\Delta r_i = l \cdot k_3 \cdot x_i^3 \cdot \sqrt{\beta} \cdot \sin(\phi_i)/6$$

with: $x_i = \sqrt{\beta} \cdot r \cdot \cos(\phi_i)$

$$\Delta r_i = l \cdot k_3 \cdot r_i^3 \cdot \beta^2 \cdot [2 \sin(2\phi_i) + \sin(4\phi_i)] / 48$$

sum over many turns: $\phi_i = 2\pi Q \cdot i + \phi_0$

$\rightarrow$ $\boxed{r = 0 \quad \text{unless:} \quad Q = p, p/2, p/4}$

- tune change (first order in the perturbation):

$$2\pi \Delta Q_i = l \cdot k_3 \cdot r_i^2 \cdot \beta^2 \cdot [4 \cos(4\pi Q i + 2\phi_0) + 3 + \cos(8\pi Q i + 4\phi_0)]/48$$

sum over many turns (unless: $Q = p$ or $Q = p/4$):

$\rightarrow$ $\boxed{\langle \Delta Q \rangle = l \cdot k_3 \cdot r^2 \cdot \beta^2 / 16 / 2\pi}$

Perturbation XVIII

- detuning with amplitude:

  particle tune depends on particle amplitude

  → tune spread for particle distribution

  → stabilization of collective instabilities

  → install octupoles in the storage ring

  → distribution covers more resonances in the tune diagram

  → avoid octupoles in the storage ring

  → requires a delicate compromise

- Poincare section topology:

  $Q = p/4$ and apply method of averaging

→ $$\Delta r_i = l \cdot k_3 \cdot r_i^3 \cdot \beta^2 \cdot \sin(4\phi_i) / 48$$

$$\Delta\phi_i = l \cdot k_3 \cdot r_i^2 \cdot \beta^2 \cdot [3 + \cos(4\phi_i)] / 48 + 2\pi Q$$

Perturbation XIX

- fixed point conditions: $Q_0 \lesssim p/4;\ k_3 > 0$

Δr / turn = 0 and $\Delta\phi$ / turn = $2\pi p / 4$

with: $$\Delta r_i = \boldsymbol{l} \bullet k_3 \bullet r_i^3 \bullet \beta^2 \bullet \sin(4\phi_i) / 48$$

$$\Delta\phi_i = 2\pi Q_0 + \boldsymbol{l} \bullet k_3 \bullet r_i^2 \bullet \beta^2 \bullet [3 + \cos(4\phi_i)] / 48$$

→ $$\phi_{\text{fixed point}} = \pi/2;\ \pi;\ 3\pi/2;\ 2\pi$$

$$r_{\text{fixed point}} = \sqrt{\frac{96\pi\,(p/4 - Q_0)}{\boldsymbol{l}\ k_3\ \beta^2\ (3+1)}}$$

→ $$\phi_{\text{fixed point}} = \pi/4;\ 3\pi/4;\ 5\pi/4;\ 7\pi/4$$

$$r_{\text{fixed point}} = \sqrt{\frac{96\pi\,(p/4 - Q_0)}{\boldsymbol{l}\ k_3\ \beta^2\ (3-1)}}$$

Perturbation XX

- fixed point stability for single octupole kick:

Jacobian matrix

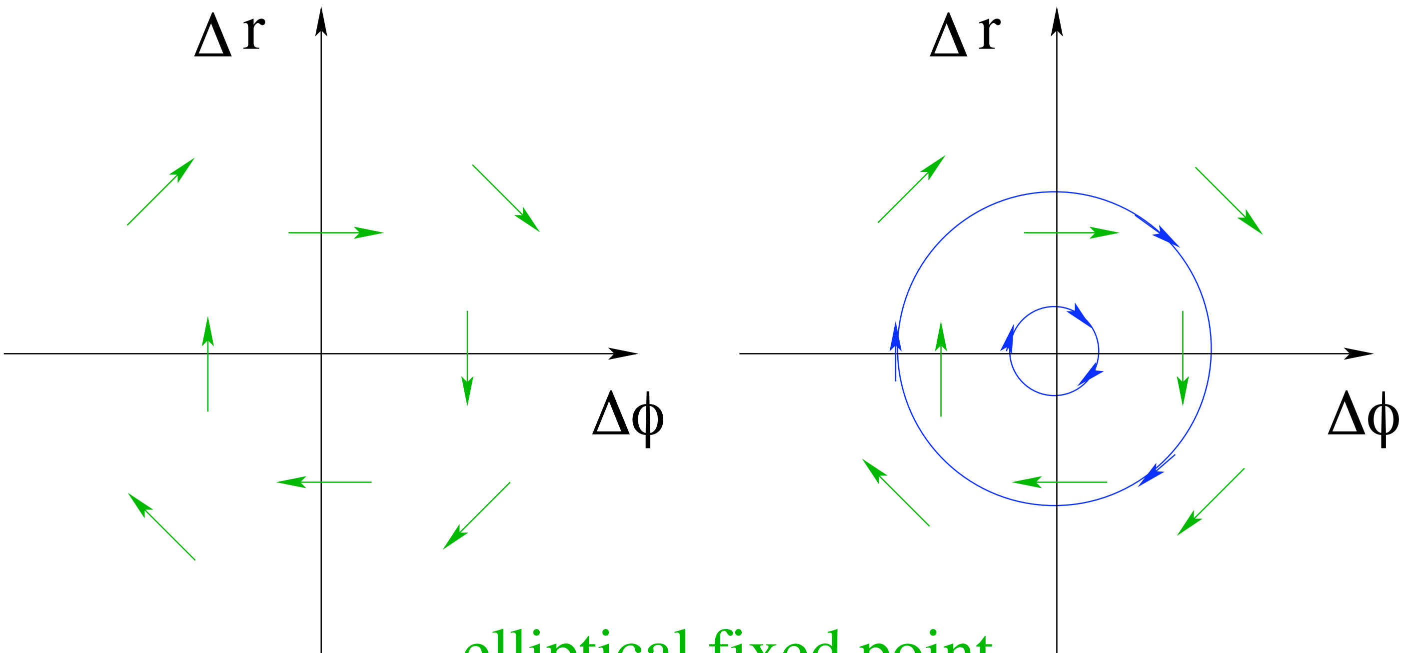
$$\frac{\partial r_{i+1}}{\partial r_i} = 1; \qquad \frac{\partial r_{i+1}}{\partial \phi_i} = \pm 4\, l \cdot k_3 \cdot \beta^2 \cdot r^3_{\text{fixed point}} / 48$$

$$\frac{\partial \phi_{i+1}}{\partial r_i} = + l \cdot k_3 \cdot \beta^2 \cdot r\ (3 \pm 1) / 24; \qquad \frac{\partial \phi_{i+1}}{\partial \phi_i} = 1$$

$$\longrightarrow \quad \Delta r_{i+1} = \pm 4\, l \cdot k_3 \cdot \beta^2 \cdot r^3_{\text{fixed point}} / 48 \cdot \Delta\phi_i$$

$$\Delta\phi_{i+1} = l \cdot k_3 \cdot \beta^2\ (3 \pm 1) / 24 \cdot \Delta r_i$$

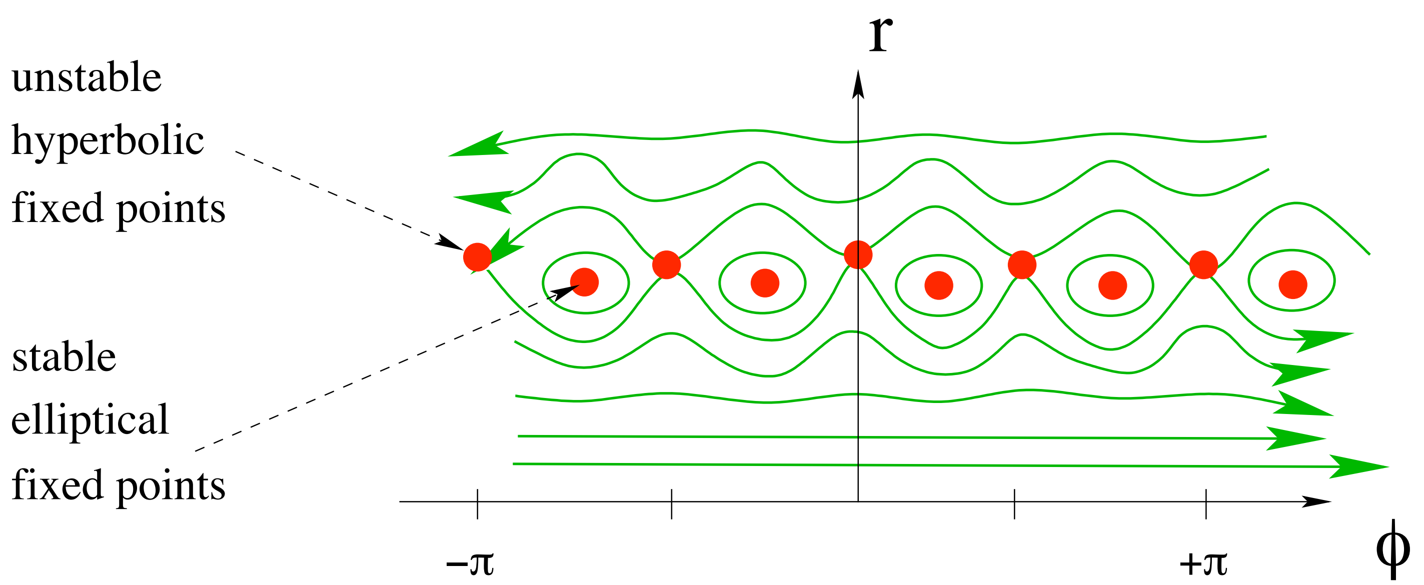
Stability for '$-$' sign and $k_3 > 0$?



elliptical fixed point

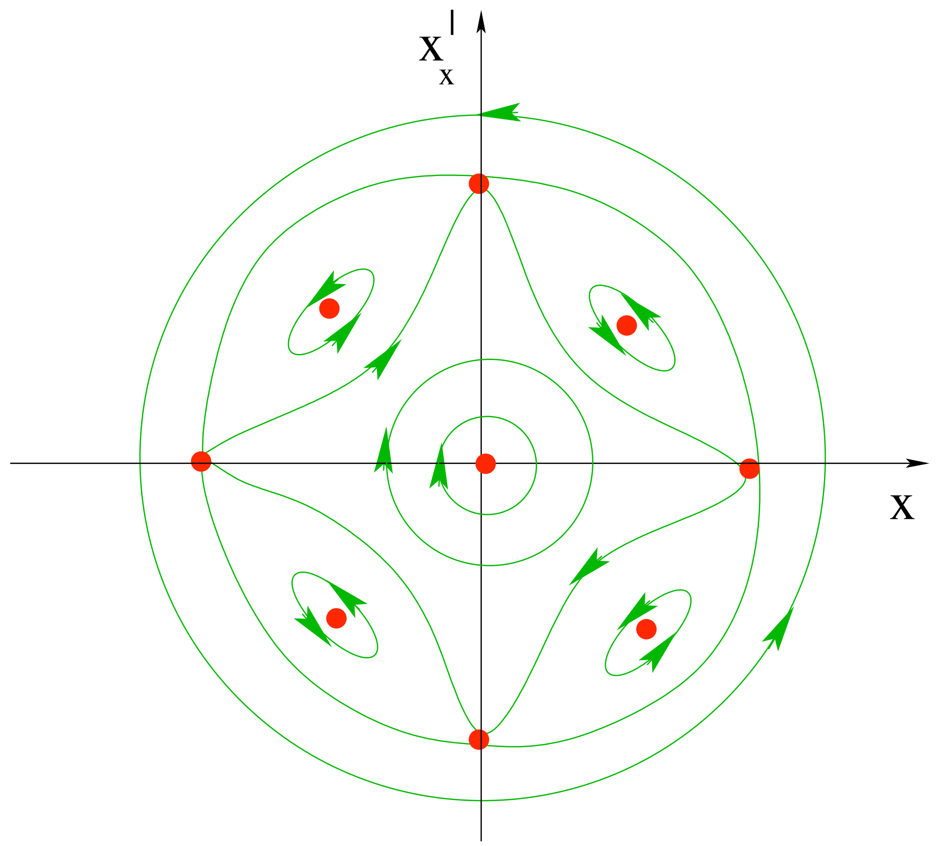
Perturbation XXI

- Poincare Section for 'r' and ϕ ':



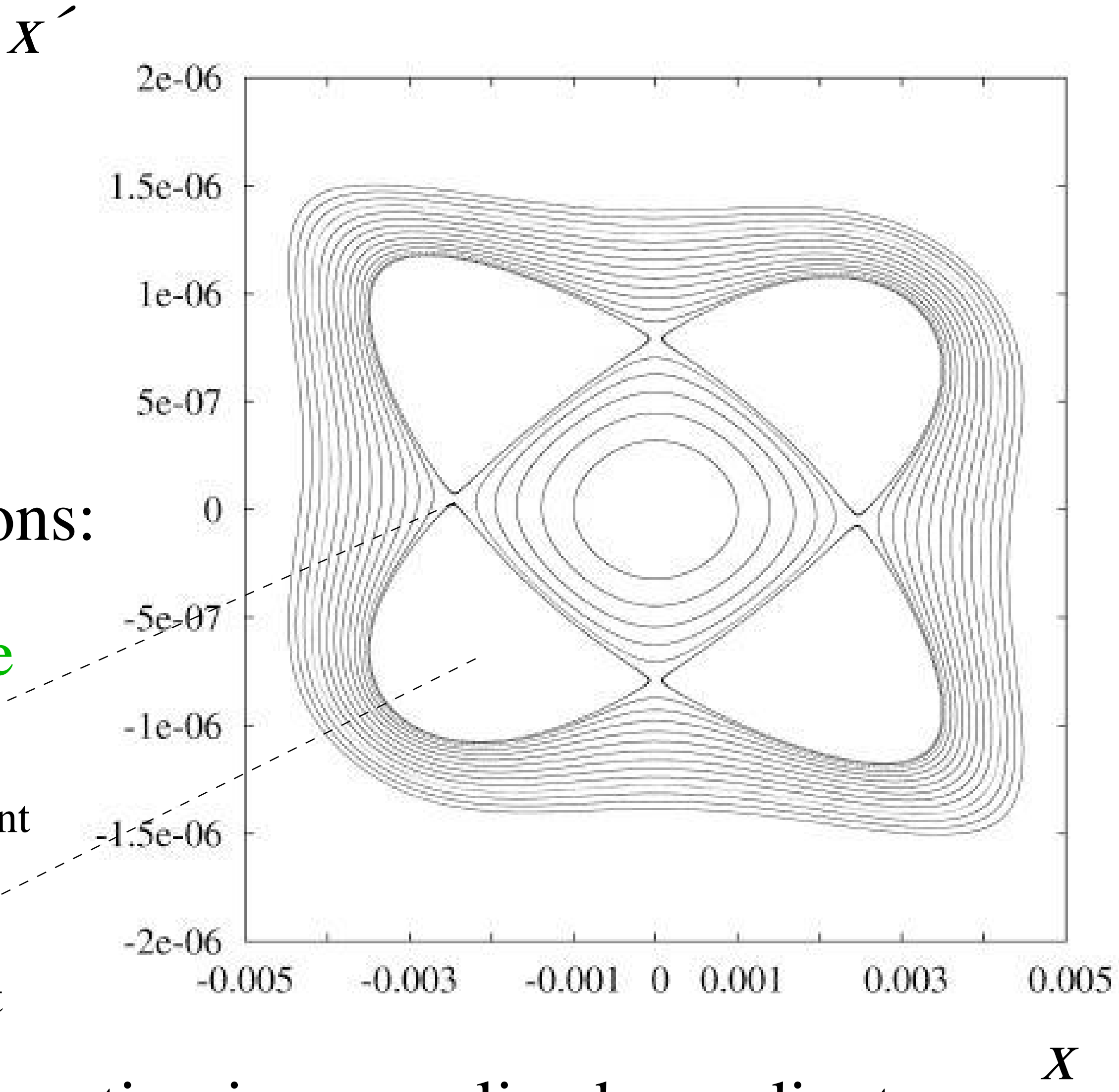
island structure

- Poincare section in normalized coordinates:



Perturbation XXII

- Octupole Poincare Section from Simulations:

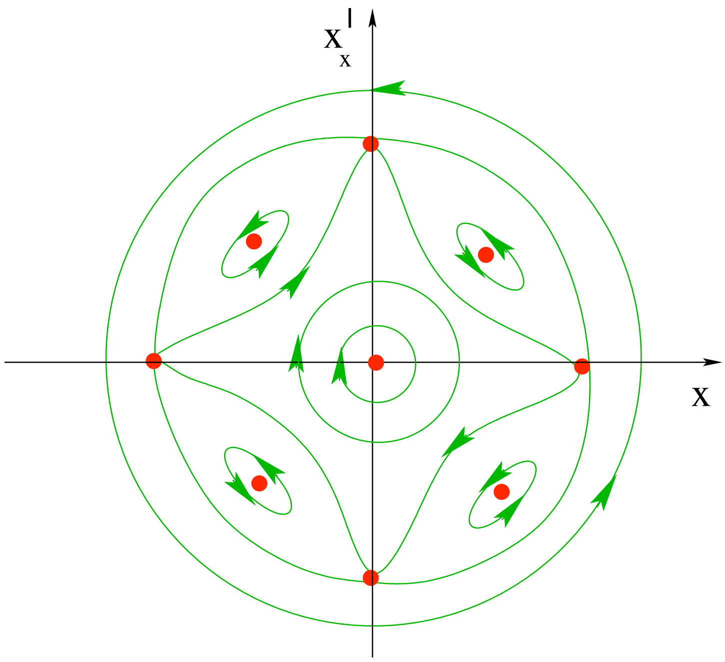


island structure

unstable hyperbolic fixed point

stable elliptical fixed point

- Poincare section in normalized coordinates:



- generic signature of non–linear resonances:

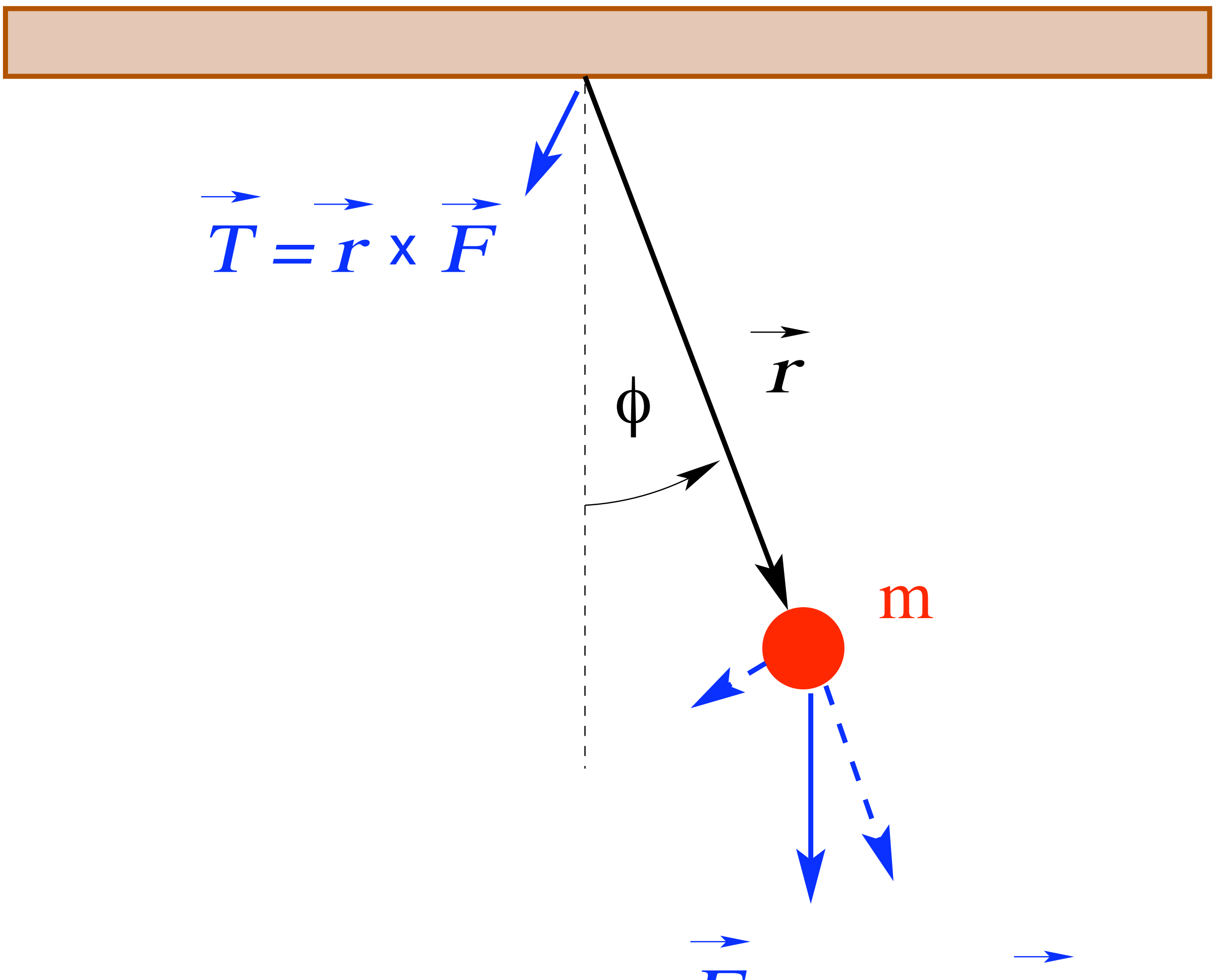
→ chain of resonance islands

Pendulum Dynamics I

- generic signature of non–linear resonances:

  → chain of resonance islands

- pendulum dynamics:



- pendulum coordinates:

  angle variable: ϕ

  angular momentum: $L = m \bullet r \bullet v$

$$v = \frac{ds}{dt} = r \bullet \frac{d\phi}{dt} \longrightarrow L = m \bullet r^2 \bullet \frac{d\phi}{dt}$$

Pendulum Dynamics II

- equations of motion:

$$\frac{d\phi}{dt} = \frac{1}{m \cdot r^2} \cdot L \qquad \frac{dL}{dt} = -r \cdot g \cdot m \cdot \sin(\phi)$$

- generic form:

$$\boxed{\frac{d\phi}{dt} = G \cdot p \qquad \frac{dp}{dt} = -F \cdot \sin(\phi)}$$

- constant of motion:

$$E_{tot} = E_{kin} + U_{pot}$$

$$\rightarrow \quad E_{kin} = \frac{1}{2} G \cdot p^2 \qquad U_{pot} = -F \cdot \cos(\phi)$$

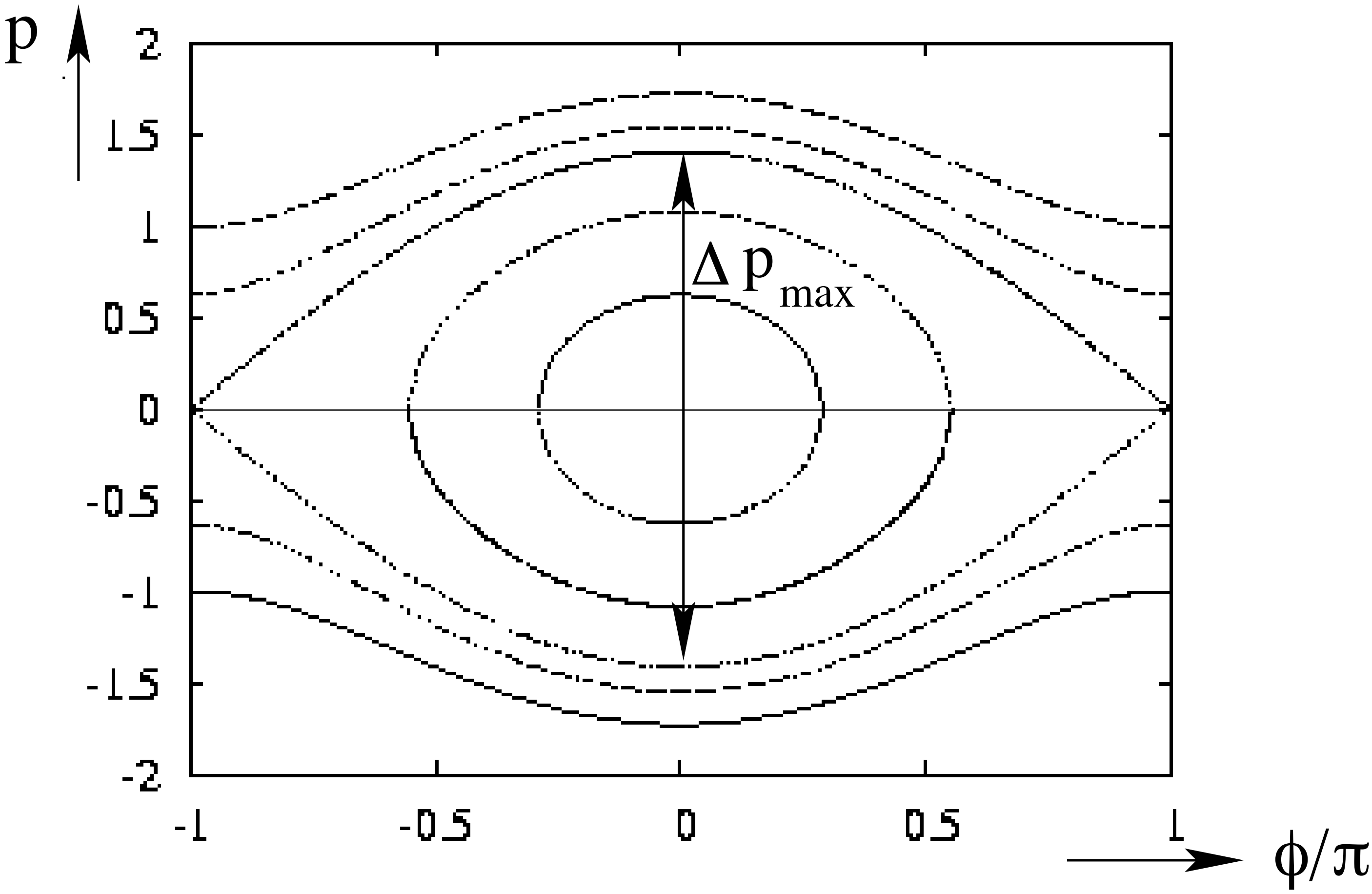
- solution:

$$\frac{d\phi}{dt} = G \cdot p \qquad p = \sqrt{[E + F \cdot \cos(\phi)]} \cdot \sqrt{\frac{2}{G}}$$

$$\rightarrow \quad \boxed{t - t_0 = \sqrt{\frac{1}{2G}} \int \frac{d\phi}{\sqrt{[E + F \cdot \cos(\phi)]}}}$$

Pendulum Dynamics III

- phase space:



$\longrightarrow$ island width: $\Delta p_{max} = 4\sqrt{F/G}$

$E_{tot} = F$ and $\phi = 0$

island oscillation frequency: $\omega_{island} = \sqrt{F \cdot G}$

- pendulum motion:

libration: oscillation around stable fixed point

rotation: continuous increase of phase variable

separatrix: separation between the two types

Cylindrical Coordinates I

- linear solution:

$$x = \sqrt{\beta} \cdot \sqrt{R} \cdot \cos(\phi) \qquad x' = -\sqrt{R} \cdot \sin(\phi) / \sqrt{\beta}$$

with: $$\frac{d\phi}{ds} = \omega = \frac{2\pi Q}{L} = \frac{1}{\beta}$$

- perturbed Hill´s equation:

$$\frac{d^2 x}{d s^2} + \omega^2 \cdot x = \frac{F_x(x,y)}{v \cdot p}$$

$$\longrightarrow \qquad x'' = \frac{-1}{n!} \cdot k_n(s) \cdot x^n - \omega^2 \cdot x$$

- equation of motion in cylindrical coordinates:

$$\frac{d\phi}{ds} = \frac{d\phi}{dx} \cdot x' + \frac{d\phi}{dx'} \cdot x''$$

$$\frac{dR}{ds} = \frac{dR}{dx} \cdot x' + \frac{dR}{dx'} \cdot x''$$

Cylindrical Coordinates II

- radial coordinate: $$R = \frac{x^2}{\beta} + x'^2 \cdot \beta$$

$\longrightarrow$

$$\frac{dR}{ds} = \cancel{\frac{2\,x\,x'}{\beta}} \cancel{-2\,\beta\,\omega^2}\; x\,x' + 2\,x'\beta \cdot \frac{F_x(s,r,\phi)}{v \cdot p}$$

$$\boxed{\frac{dR}{ds} = \frac{-2}{n!} \cdot k_n(s) \cdot \left(R \cdot \beta\right)^{(n+1)/2} \cdot \sin(\phi) \cdot \cos^n(\phi)}$$

- angular coordinate: $$\phi = atan\left(\frac{-x' \cdot \beta}{x}\right)$$

with: $$\frac{d}{ds}\, atan(f[s]) = \frac{1}{f^2(s) + 1} \cdot \frac{df}{ds}$$

$$\left(\frac{1}{\beta} = \omega\right) \longrightarrow \frac{d\phi}{ds} = \omega - \frac{x}{R} \cdot \frac{F_x(s,r,\phi)}{v \cdot p}$$

$$\boxed{\frac{d\phi}{ds} = \omega + \frac{1}{n!} \cdot k_n(s) \cdot R^{(n-1)/2} \cdot \beta^{(n+1)/2} \cdot \cos^{n+1}(\phi)}$$

Examples for Equation of Motion I

- quadrupole: n = 1

$$\frac{dR}{ds} = -k_1(s) \cdot R \cdot \beta \cdot \sin(2\phi)$$

$$\frac{d\phi}{ds} = \omega + k_1(s) \cdot \beta \cdot \left(1 + \cos(2\phi)\right)/2$$

→ similar expressions as with the map approach
but we can now treat distributed perturbations!

- sextupole: n = 2

$$\frac{dR}{ds} = \frac{-1}{4} \cdot k_2(s) \cdot \left(R \cdot \beta\right)^{3/2} \cdot \left(\sin(\phi) + \sin(3\phi)\right)$$

$$\frac{d\phi}{ds} = \omega + \frac{1}{8} \cdot k_2(s) \cdot R^{1/2} \cdot \beta^{3/2} \cdot \left(3\cos(\phi) + \cos(3\phi)\right)$$

→ similar expressions as with the map approach

Examples for Equation of Motion II

octupole: $n = 3$

$$\frac{dR}{ds} = \frac{-1}{24} \cdot k_3(s) \cdot R^2 \cdot \beta^2 \cdot \left(2\sin(\phi) + \sin(4\phi)\right)$$

$$\frac{d\phi}{ds} = \omega + \frac{1}{48} \cdot k_3(s) \cdot R \cdot \beta^2 \cdot \left(3 + 4\cos(2\phi) + \cos(4\phi)\right)$$

one single kick at one location:

$$\longrightarrow \quad \frac{F(s)}{v \cdot p} = 1\, k_n(s) \cdot \delta_L(s - s_0)$$

$$\text{with: } \delta = \begin{cases} 1 & \text{for } s = s + n \cdot L \\ 0 & \text{else} \end{cases}$$

$\longrightarrow$ Fourier series of δ –function:

$$\frac{F(s)}{v \cdot p} = 1\, k_n(s) \cdot \frac{1}{L} \cdot \sum_{n=-\infty}^{+\infty} \cos(n \cdot 2\pi \cdot s/L)$$

Examples for Equation of Motion III

- single octupole magnet at s_0 : n = 3

$$\frac{dR}{ds} = \frac{-1}{24 \cdot L} \cdot lk_3(s) \cdot R^2 \cdot \beta^2 \cdot \sum_{n=0}^{+\infty} \Big(2\sin(\phi + n \cdot 2\pi \cdot s/L) + \sin(4\phi + n \cdot 2\pi \cdot s/L) \Big)$$

$$\frac{d\phi}{ds} = \frac{2\pi Q}{L} + \frac{1}{48 \cdot L} \cdot lk_3(s) \cdot R \cdot \beta^2 \cdot \sum_{n=0}^{+\infty} \Big(3 + 2\cos(\phi + n \cdot 2\pi \cdot s/L) + \cos(4\phi + n \cdot 2\pi \cdot s/L) \Big)$$

- resonance: $\phi = \frac{2\pi \overline{Q}}{L} \cdot s + \phi_0$

with $\overline{Q} = N + 1/n$

→ all but one term change rapidly with s!

→ method of averaging!

Examples for Equation of Motion IV

- 1/4 resonance : $p = 4$

$$\frac{dR}{ds} = \frac{-1}{24 \cdot L} \cdot l k_3 \cdot R^2 \cdot \beta^2 \cdot \sin(4\phi_0)$$

$$\frac{d\phi}{ds} = \frac{2\pi Q}{L} + \frac{1}{48 \cdot L} \cdot l k_3 \cdot R \cdot \beta^2 \cdot \left(3 + \cos(4\phi_0)\right)$$

- fixed point conditions: $Q_0 \lessgtr p/4;\ k_3 > 0$

$\Delta R / \text{turn} = 0$ and $\Delta\phi / \text{turn} = 2\pi p / 4$

$\rightarrow$ $\phi_{\text{fixed point}} = \pi/2;\ \pi;\ 3\pi/2;\ 2\pi$

$$R_{\text{fixed point}} = \frac{96\pi\,(p/4 - Q_0)}{l\ k_3\ \beta^2\ (3+1)}$$

$\rightarrow$ $\phi_{\text{fixed point}} = \pi/4;\ 3\pi/4;\ 5\pi/4;\ 7\pi/4$

$$R_{\text{fixed point}} = \frac{96\pi\,(p/4 - Q_0)}{l\ k_3\ \beta^2\ (3-1)}$$

Example Octupole

$lk_3 = 4\ \mathrm{m}^{-3}$

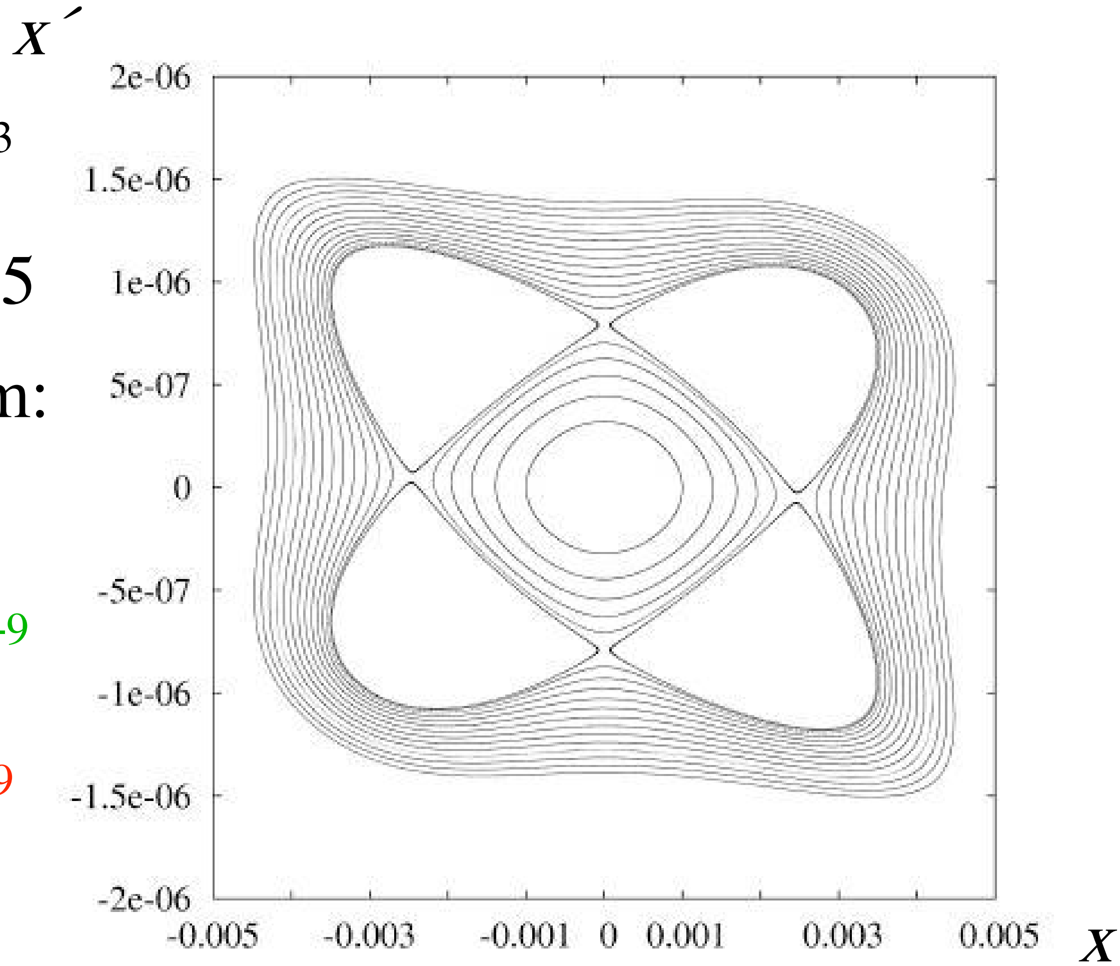
$Q = 0.2495$

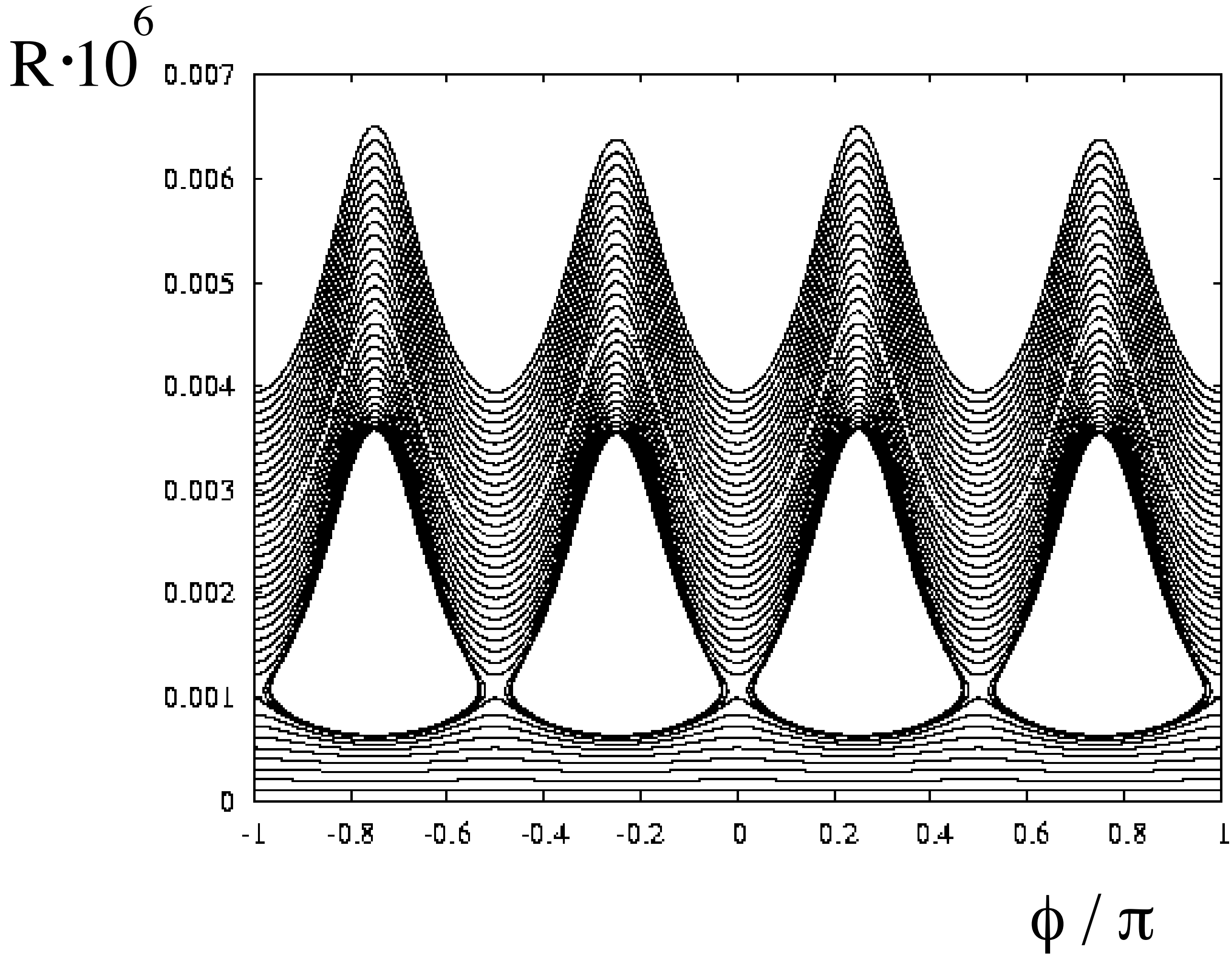
$L = 4.7$ km:

$\beta = 3000$

$R = 2 \cdot 10^{-9}$

$R = 1 \cdot 10^{-9}$





Examples for Equation of Motion V

- expand motion around stabel fixed point:

$$\phi = \frac{2\pi Q}{L} s + \phi_{fix} + \Delta\phi$$

$$R = R_{fix} + \Delta R$$ and keep only first order in ΔR

$$\frac{d\Delta R}{ds} = \frac{-1}{24 \cdot L} \cdot lk_3 \cdot R_{fix}^2 \cdot \beta^2 \cdot \sin(4\Delta\phi)$$

$$\frac{d\phi}{ds} = \frac{2\pi Q_0}{L} + \frac{1}{48 \cdot L} \cdot lk_3 \cdot R_{fix} \cdot \beta^2 \cdot \left(3 - \overset{1}{\cancel{\cos(4\Delta\phi)}}\right)$$

$$+ \frac{1}{48 \cdot L} \cdot lk_3 \cdot \Delta R \cdot \beta^2 \cdot \left(3 - \overset{1}{\cancel{\cos(4\Delta\phi)}}\right)$$

- change to new angular variable:

$$\varphi = 4\phi - 8\pi Q \cdot s / L \qquad r = 4 \cdot \Delta R$$

with $$Q = Q_0 + \frac{1}{48 \cdot \pi} \cdot lk_3 \cdot R_{fix} \cdot \beta^2$$

Examples for Equation of Motion VI

- pendulum approximation:

$$\frac{dr}{ds} = -F \cdot \sin(\varphi)$$

with $\boxed{F = \frac{4}{24 \cdot L} \cdot lk_3 \cdot \beta^2 \cdot R_{fix}^2}$

$$\frac{d\varphi}{ds} = G \cdot r$$

and $\boxed{G = \frac{1}{24 \cdot L} \cdot lk_3 \cdot \beta^2}$

- resonance width:

$$\Delta r_{max} = 4\sqrt{F/G} = 8 \cdot \Delta R_{fix}$$

$$\longrightarrow \quad \Delta R_{max} = 2 \cdot \Delta R_{fix}$$

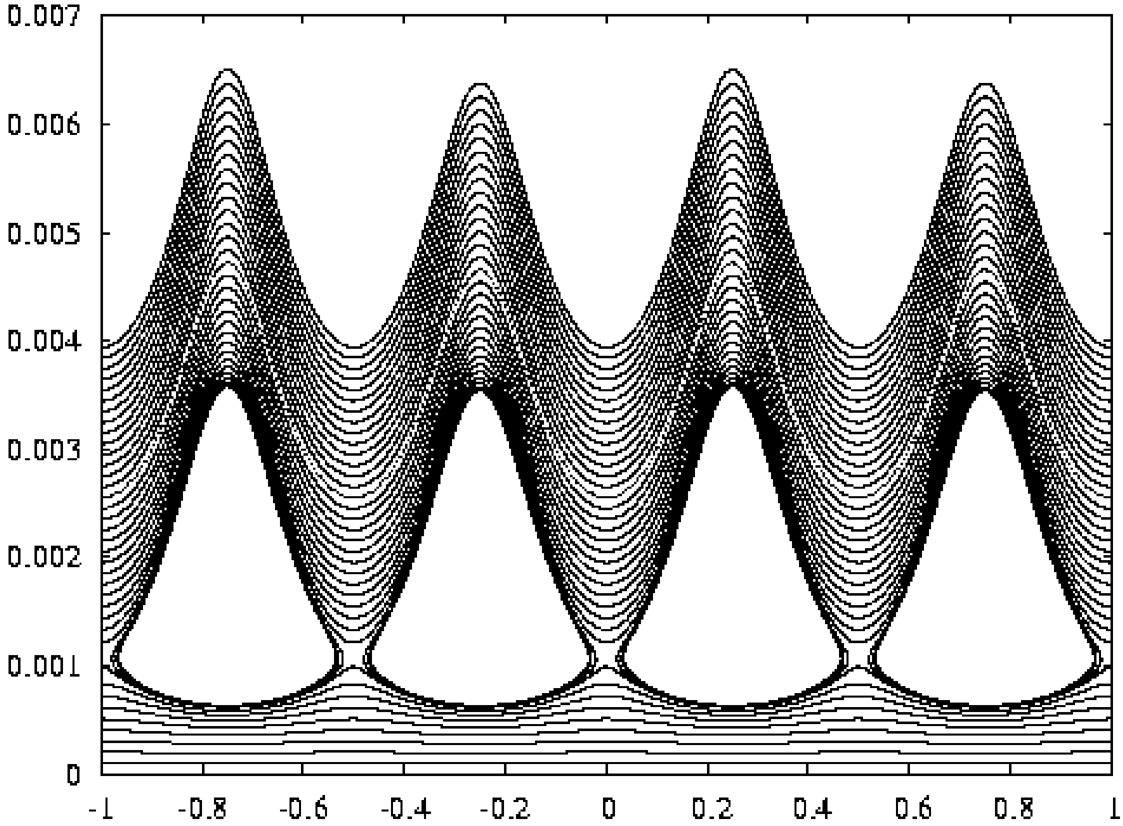
resonance width equals twice the stable fixed point

resonance width increases with decreasing k_3 !

Example Octupole

$R \cdot 10^6$

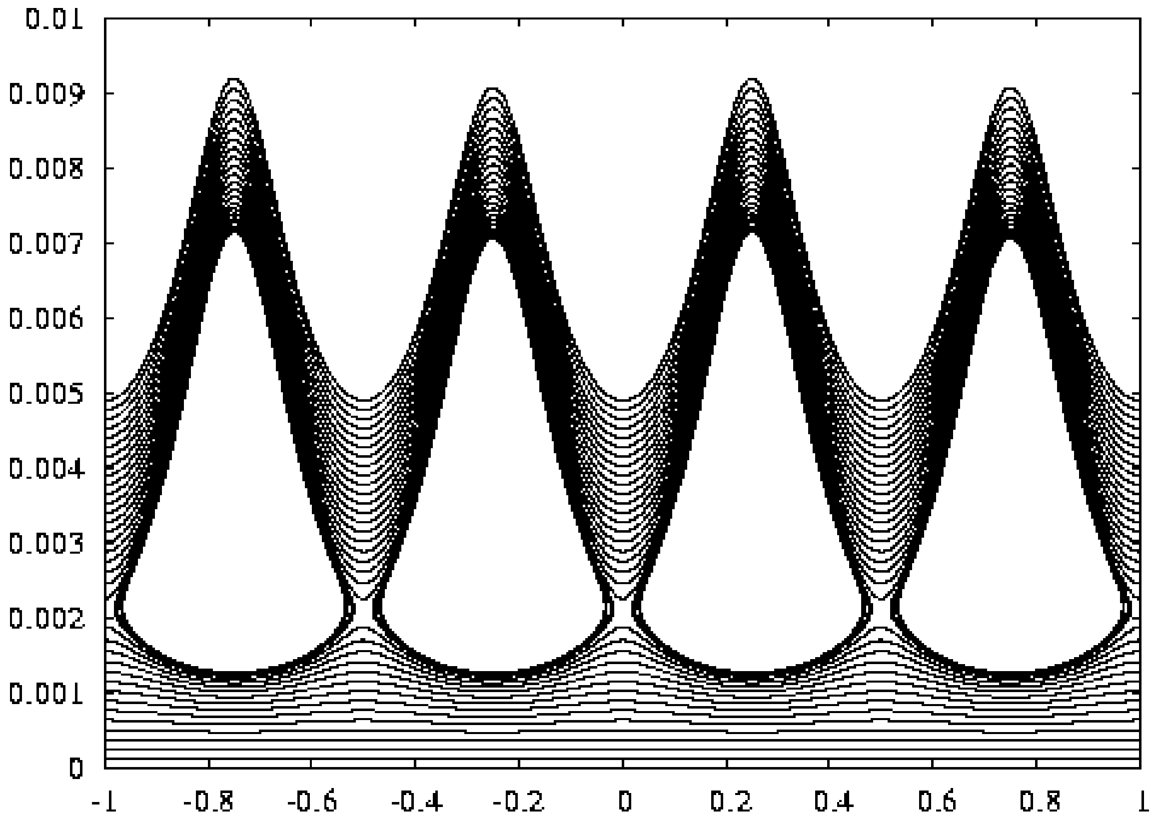
$lk_3 = 4\ m^{-3}$



ϕ / π

$R \cdot 10^6$

$lk_3 = 2\ m^{-3}$



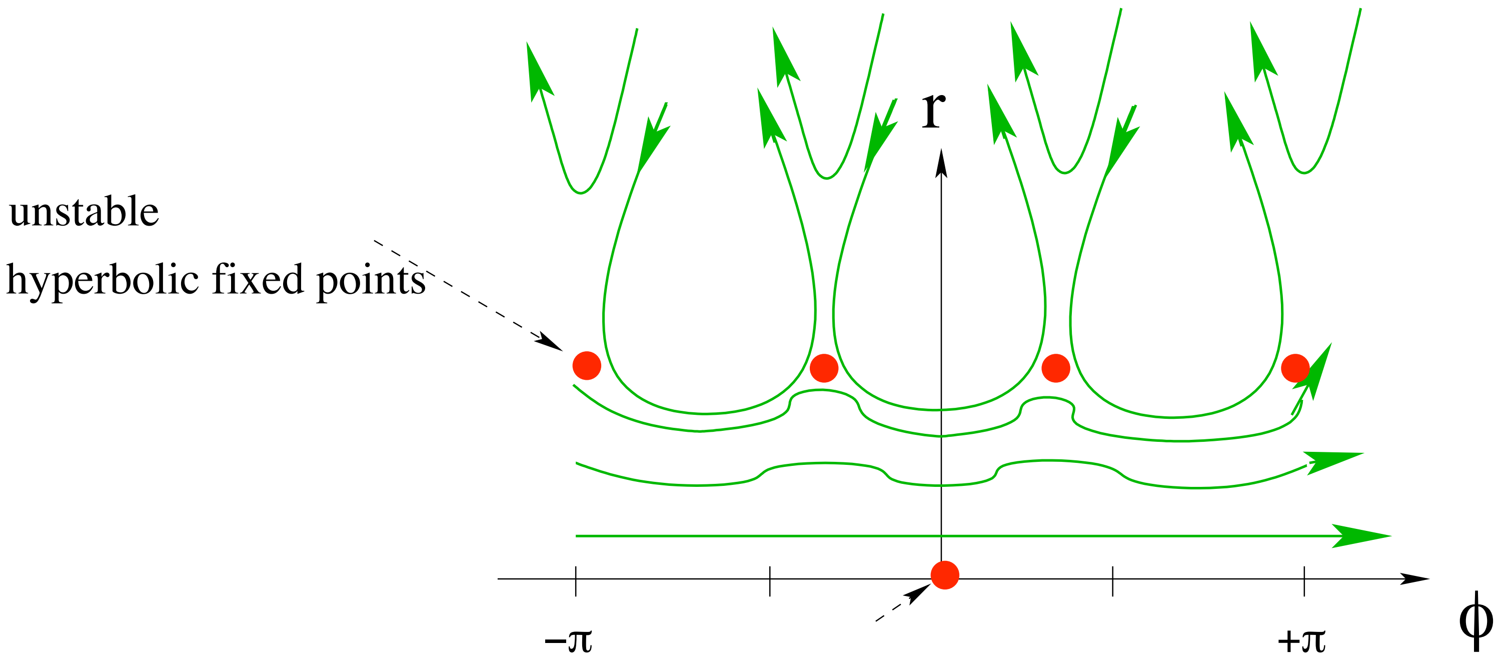
ϕ / π

Example Sextupole

- why did we not find islands for a sextupole?

  ➤ the pendulum approximation requires an amplitude dependent tune!

  ➤ $$\frac{d\phi}{ds} = G \cdot r$$

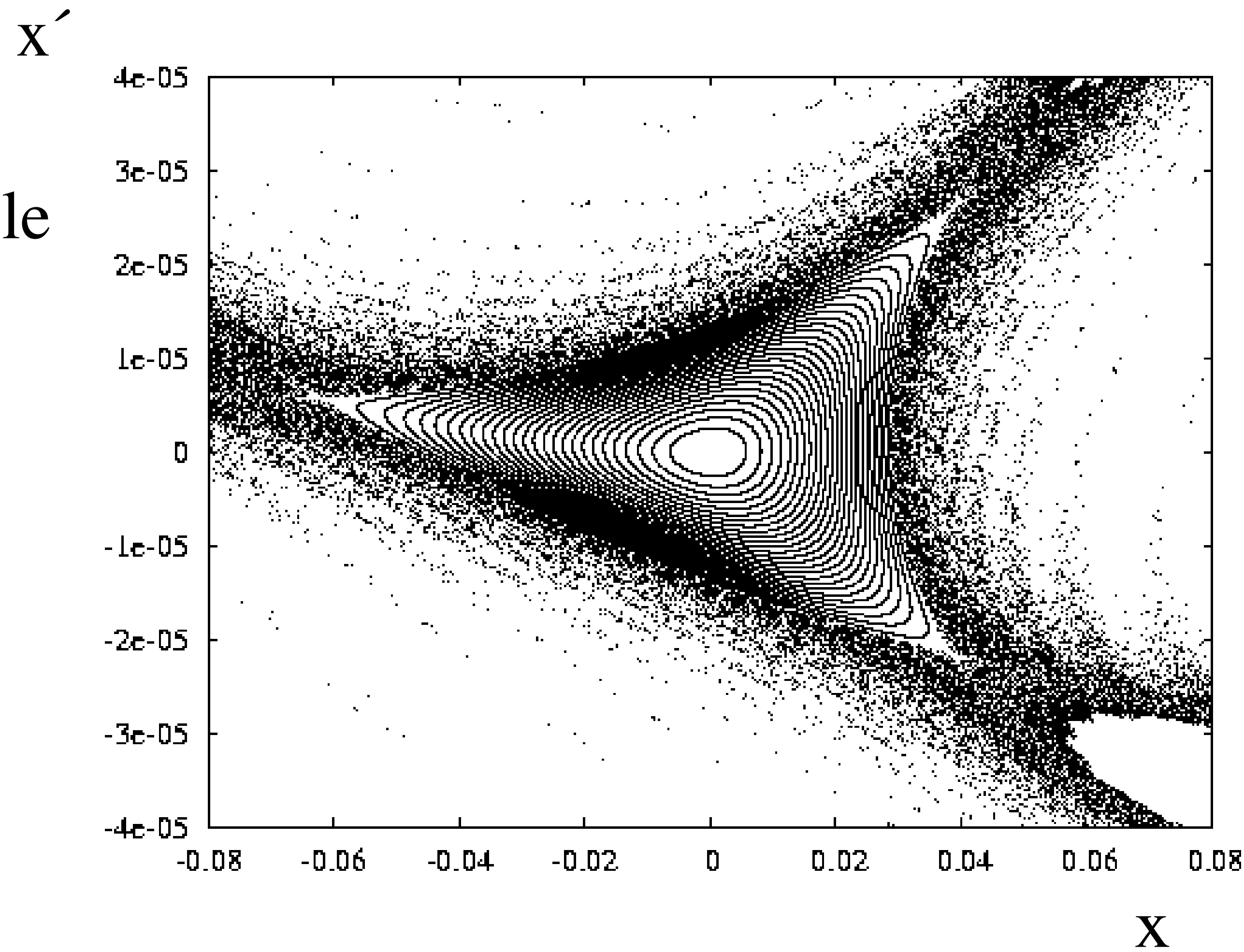


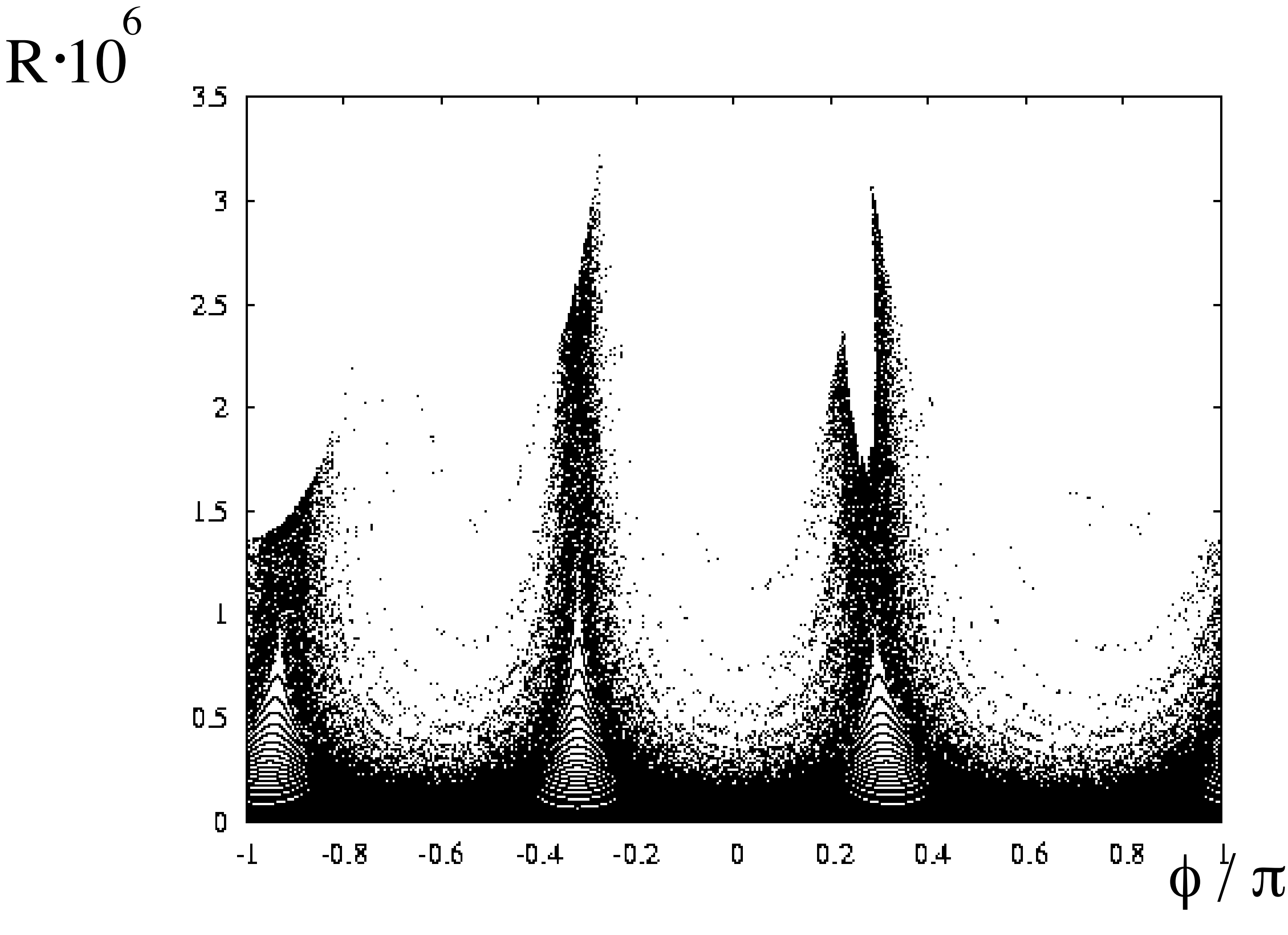
- the sextupole perturbation has not amplitude dependent tune (to first order)

  ➤ stabilization by an octupole term?

Example Sextupole

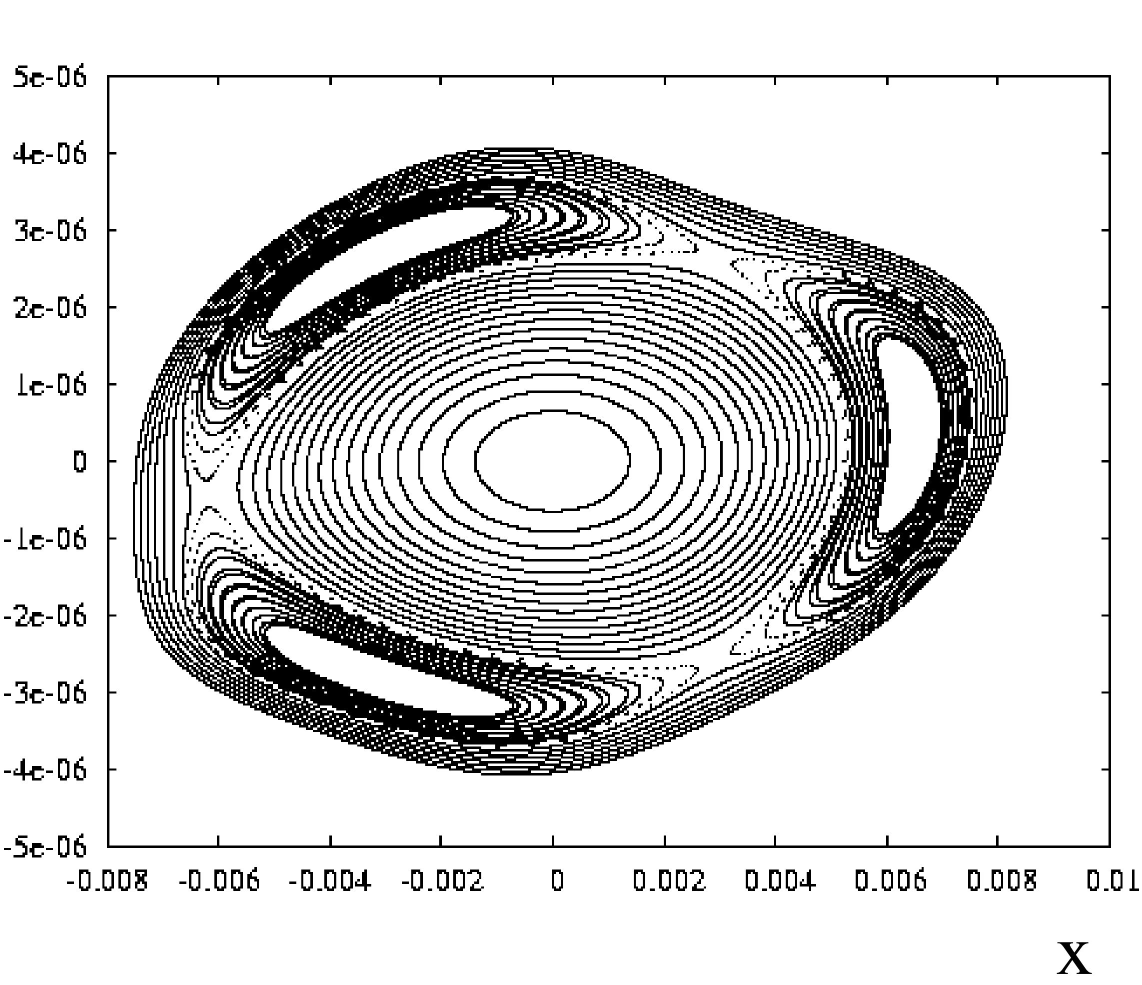
sextupole only

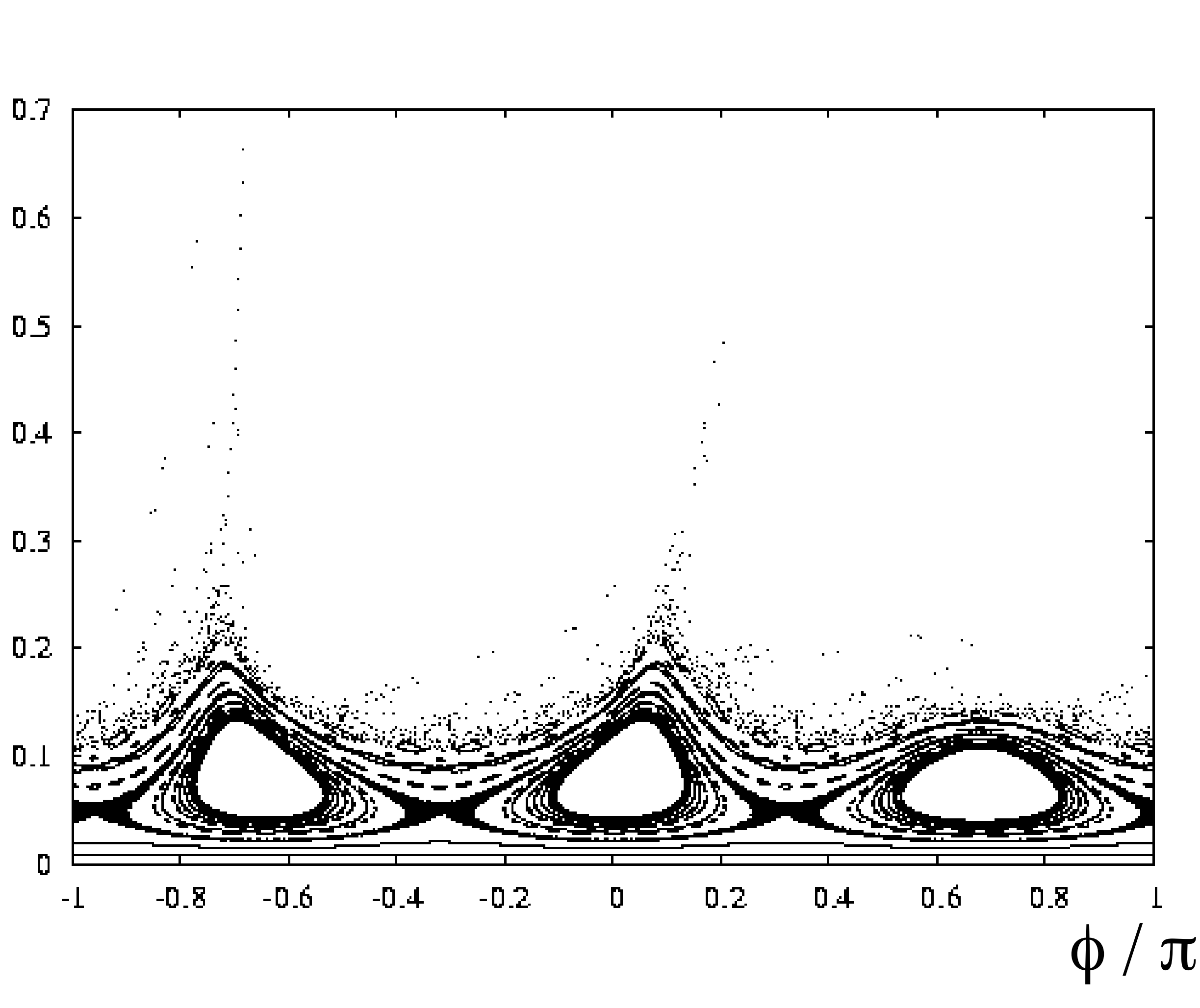




Example Sextupole + Octupole

sextupole plus octupole





Higher Order

- so far we assumed on the right–hand side:

$$\phi = 2\pi Q_0 \cdot s/L + \phi_{fix} + \Delta\phi$$

$$R = R_{fix} + \Delta R$$

and kept only first order terms in Δ R

- higher order perturbation treatment:

$$R(s) = R_0(s) + \varepsilon R_1(s) + \varepsilon^2 R_2(s) + O(\varepsilon^3)$$

$$\phi(s) = \phi_0(s) + \varepsilon \phi_1(s) + \varepsilon^2 \phi_2(s) + O(\varepsilon^3)$$

with: $\varepsilon = (\beta \cdot R_{fix})^{(n+1)/2} \cdot lk_n / L$

- match powers of ε :

match powers of ´ε´

solve lowest order without perturbation

substitute solution in next higher order equations

solve next order etc

Higher Order II

- expand equation of motion into a Taylor series around zero order solution

$$\frac{dr}{ds} = F(r,\phi) \qquad \frac{d\phi}{ds} = G(r,\phi)$$

→ single sextupole kick:

$$F = f(R) \cdot [\sin(3\phi) + 3\sin(\phi)]$$

$$G = g(R) \cdot [\cos(3\phi) + 3\cos(\phi)] + \frac{2\pi Q}{L}$$

$$\rightarrow \quad \frac{dR}{ds} = \varepsilon \cdot f + \left[\frac{\partial f}{\partial r} \cdot r_1 + \frac{\partial F}{\partial \phi} \cdot \phi_1\right] \cdot \varepsilon^2 + O(\varepsilon^3)$$

$$\frac{d\phi}{ds} = \frac{2\pi Q}{L} + \varepsilon \cdot g + \left[\frac{\partial g}{\partial r} \cdot r_1 + \frac{\partial G}{\partial \phi} \cdot \phi_1\right] \cdot \varepsilon^2 + O(\varepsilon^3)$$

Higher Order III

- match powers of ε and solve equation of motion in ascending order of ε^n:

- zero order: $\phi_0(s) = \frac{2\pi Q}{L} \cdot s + \phi_0$

$$R_0(s) = R_0 \qquad (Q = p + \nu)$$

→ substitute into equation of motion and solve for $\phi_1(s)$ and $r_1(s)$

- first order:

$$\phi_1(s) \propto \Big[\sin\!\big(\frac{6\pi Q}{L} \cdot s + 3\phi_0\big)/3 + 3 \cdot \sin\!\big(\frac{2\pi Q}{L} \cdot s + \phi_0\big)\Big]$$

$$R_1(s) \propto \Big[\cos\!\big(\frac{6\pi Q}{L} \cdot s + 3\phi_0\big)/3 + 3 \cdot \cos\!\big(\frac{3\pi Q}{L} \cdot s + \phi_0\big)\Big]$$

Perturbation IV

- second order:

  → substitute $\phi_1(s)$ and $r_1(s)$ into equation

  of motion and order powers of ε^2

you get terms of the form: $\frac{dr_2}{ds} = \left[\frac{\partial f}{\partial r} \cdot r_1 + \frac{\partial f}{\partial \phi} \cdot \phi_1\right]$

$$\frac{d\phi}{ds} = \left[\frac{\partial g}{\partial r} \cdot r_1 + \frac{\partial g}{\partial \phi} \cdot \phi_1\right]$$

→

$\sin(3\phi) \cdot \cos(3\phi)$; $\sin(3\phi) \cdot \cos(\phi)$; $\sin(\phi) \cdot \cos(\phi)$

$\cos(3\phi) \cdot \cos(3\phi)$; $\cos(3\phi) \cdot \cos(\phi)$; $\cos(\phi) \cdot \cos(\phi)$

→ $\frac{d\phi}{ds} \propto \cos(6\phi)$; $\cos(4\phi)$; $\cos(2\phi)$; $\boxed{1}$

→ $\frac{dr}{ds} \propto \sin(6\phi)$; $\sin(4\phi)$; $\sin(2\phi)$

- higher order resonances: ε^n

  a single perturbation generates ALL resonances

  driving term strength and resonance width

  decrease with increasing order!

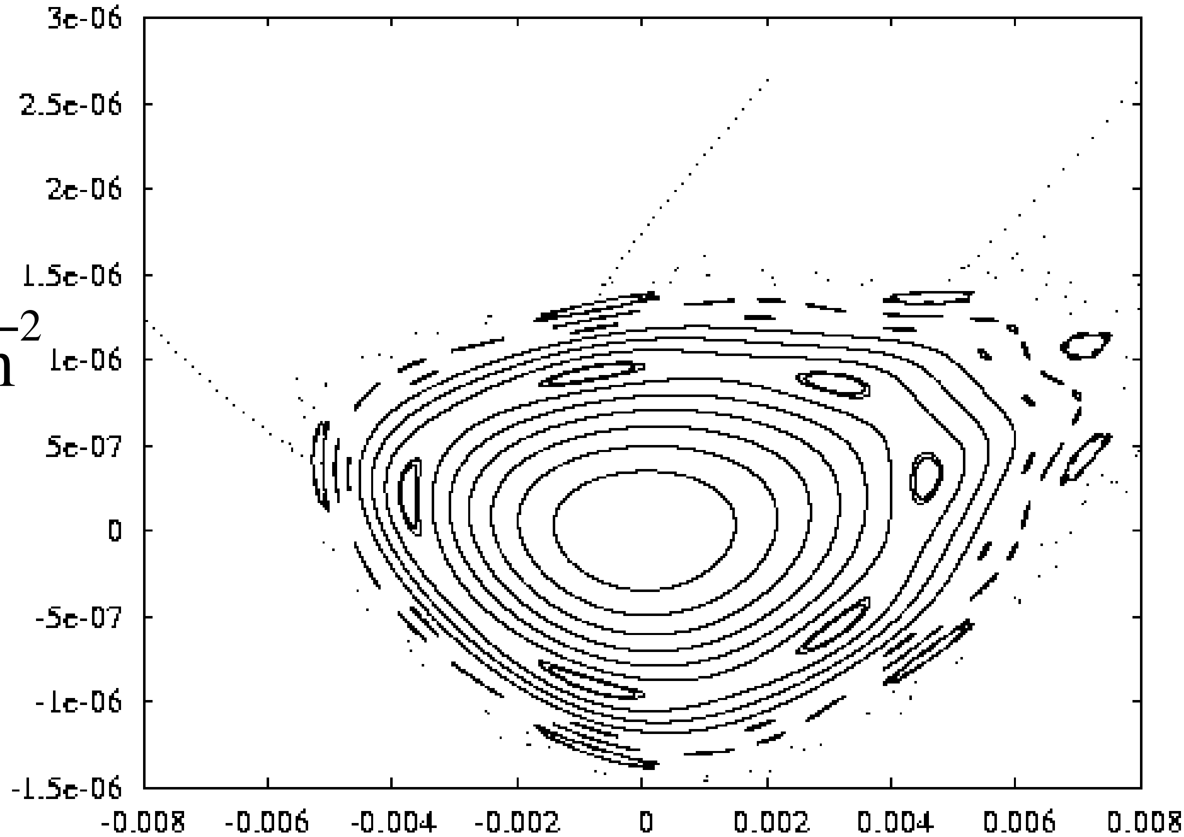
Perturbation V

x´

sextupole only

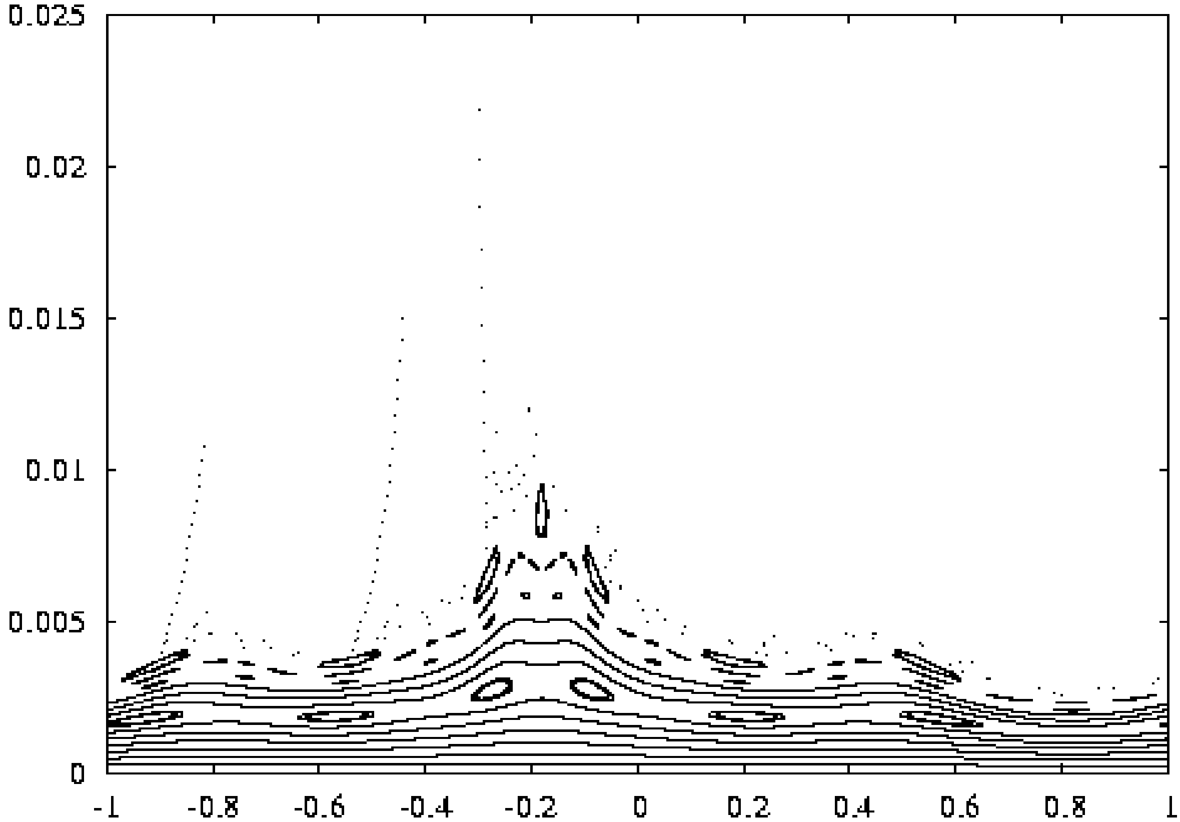
$k_2 = -0.06 m^{-2}$

$Q = 0.18$



x

$R \cdot 10^6$



ϕ / π

Integrable Systems

- trajectories in phase space do not intersect

  deterministic system

- integrable systems:

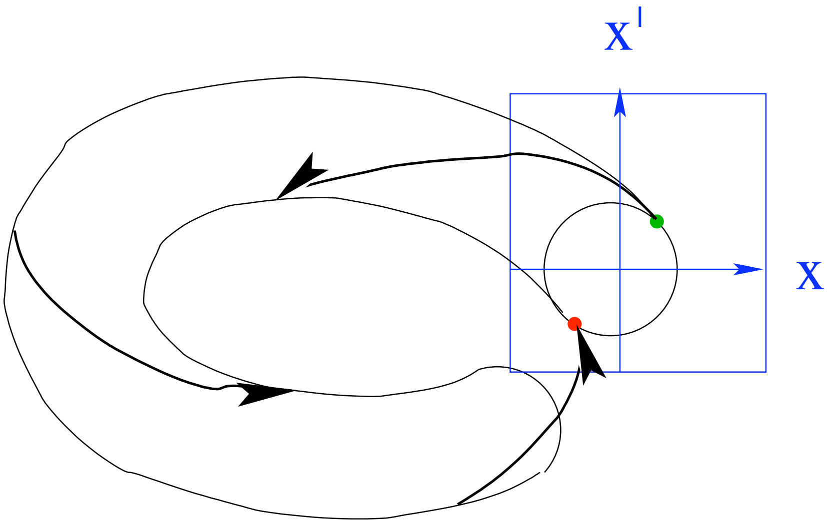
  all trajectories lie on invariant surfaces

  n degrees of freedom

  $\longrightarrow$ n dimensional surfaces

- two degrees of freedom:

  x, s $\longrightarrow$ motion lies on a torus



- Poincare section for two degrees of freedom:

  $\longrightarrow$ motion lies on closed curves

  $\longrightarrow$ indication of integrability

Non–Integrable Systems

'chaos' and non–integrability:

so far we removed all but one resonance

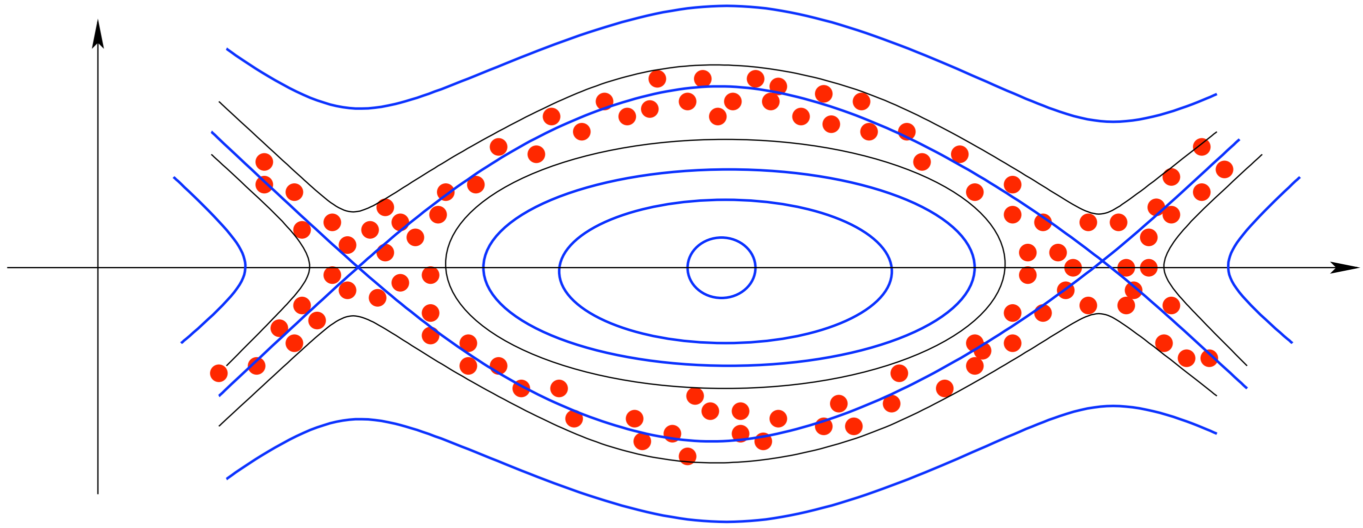
(method of averaging)

⟶ dynamics is integrable and therefore predictable

re–introduction of the other resonances 'perturbs' the separatrix motion

➤ motion can 'change' from libration to rotation

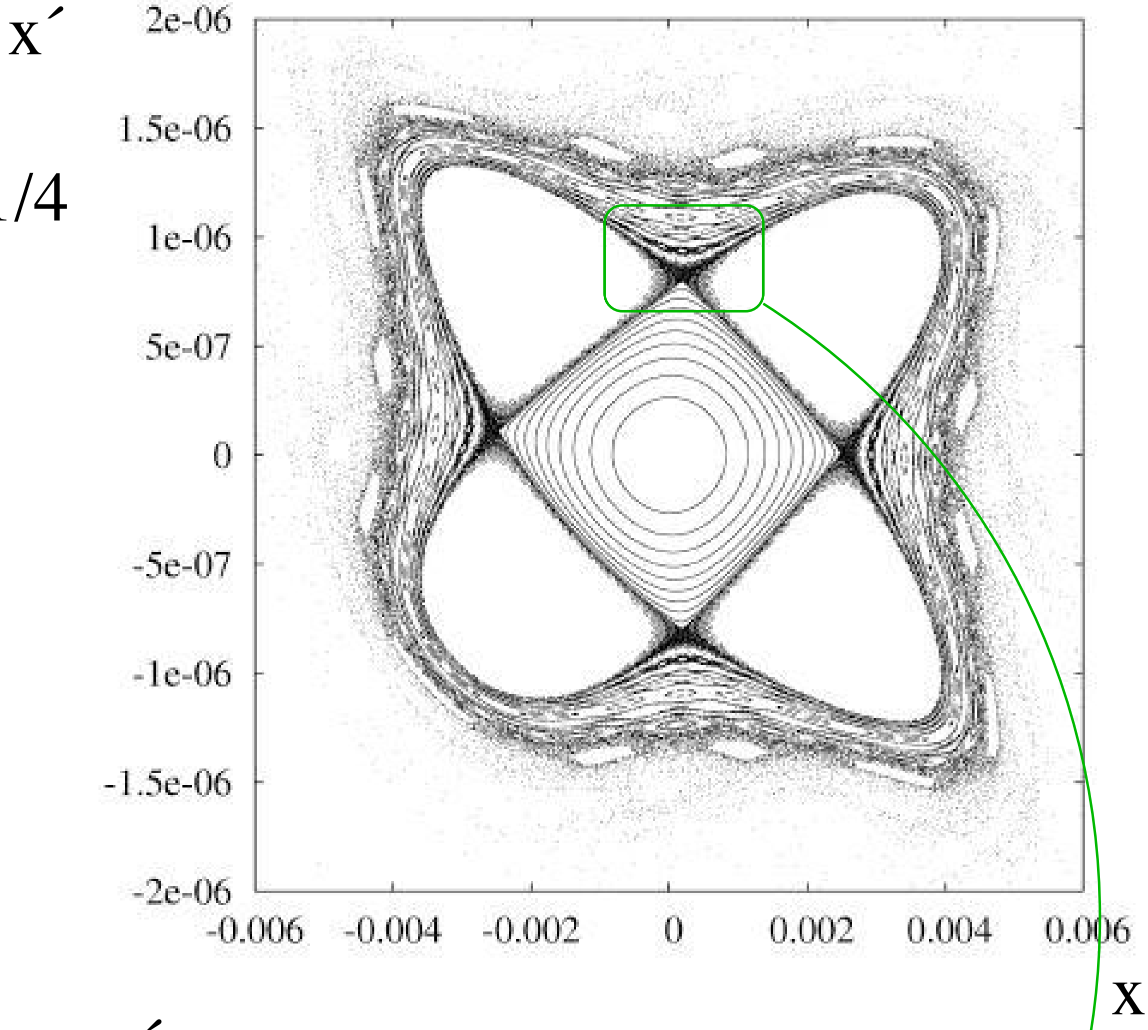
➤ generation of a layer of 'chaotic motion'



no hope for exact deterministic solution in this area!

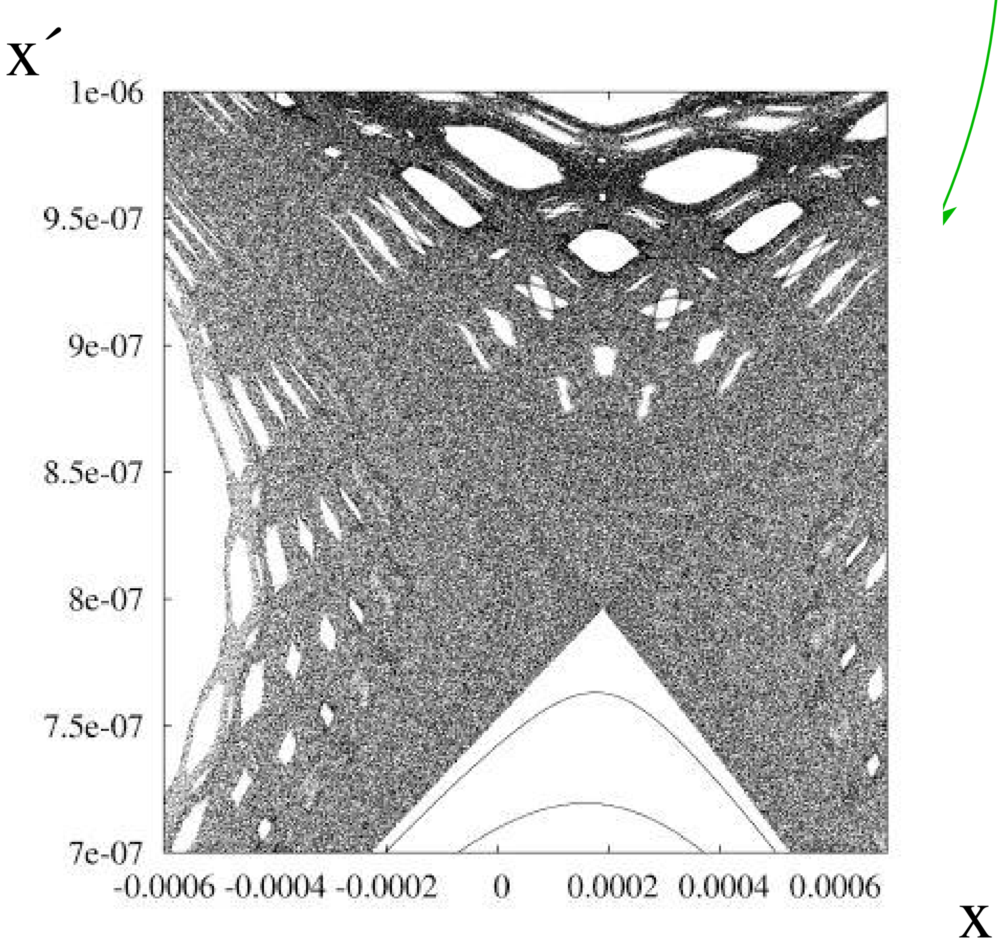
Sextupole + Octupole

motion near 1/4 resonance:



pendulum island sctructure appears on all scales!

→ renormalization theory



Non–Integrable Systems

- slow particle loss:

  particles can stream along the 'stochastic layer'

  for 1 degree of freedom (plus 's' dependence) the particle amplitude is bound by neighboring integrable lines

  not true for more than one degree of freedom

- global 'chaos' and fast particle losses:

  if more than one resonance are present their resonance islands can overlap

  → the particle motion can jump from one resonance to the other

  → 'global chaos'

  → fast particle losses and dynamic aperture

Summary

Non–linear Perturbation:

- amplitude growth
- detuning with amplitude
- coupling

→ **Complex dynamics:**

3 degrees of freedom

\+ 1 invariant of the motion

\+ non–linear dynamics

→ **no global analytical solution!**

→ analytical analysis relies on perturbation theory