Introduction to Transverse Beam Optics II

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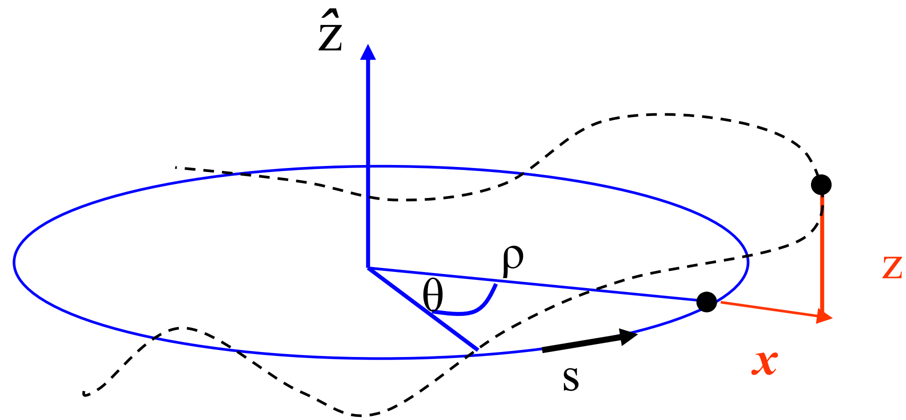
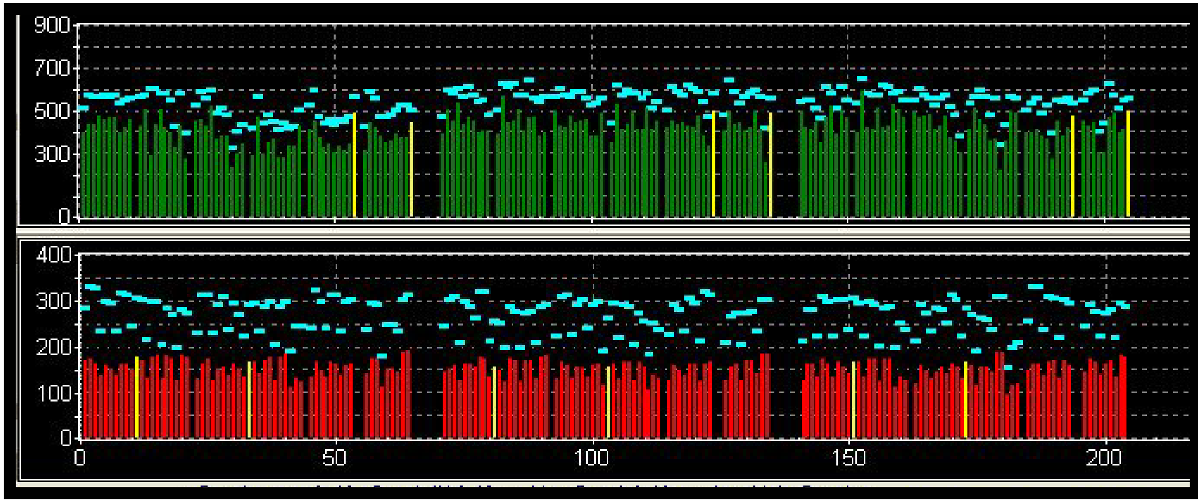
I.) Reminder: the ideal world

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M \cdot \begin{pmatrix} x \\ x' \end{pmatrix}_0$$



$$M_{foc} = \begin{pmatrix} \cos(\sqrt{|K|}s) & \frac{1}{\sqrt{|K|}}\sin(\sqrt{|K|}s \\ -\sqrt{|K|}\sin(\sqrt{|K|}s) & \cos(\sqrt{|K|}s) \end{pmatrix}_0$$

The Beta Function



Beam parameters of a typical high energy ring: ***Ip = 100 mA***

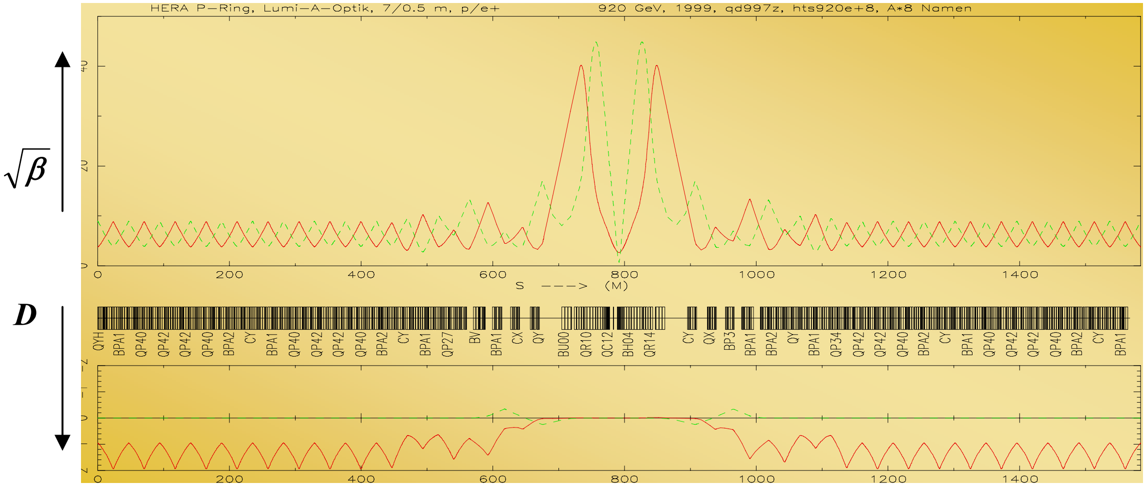
particles per bunch: ***$N \approx 10^{11}$***

Example: HERA Bunch pattern

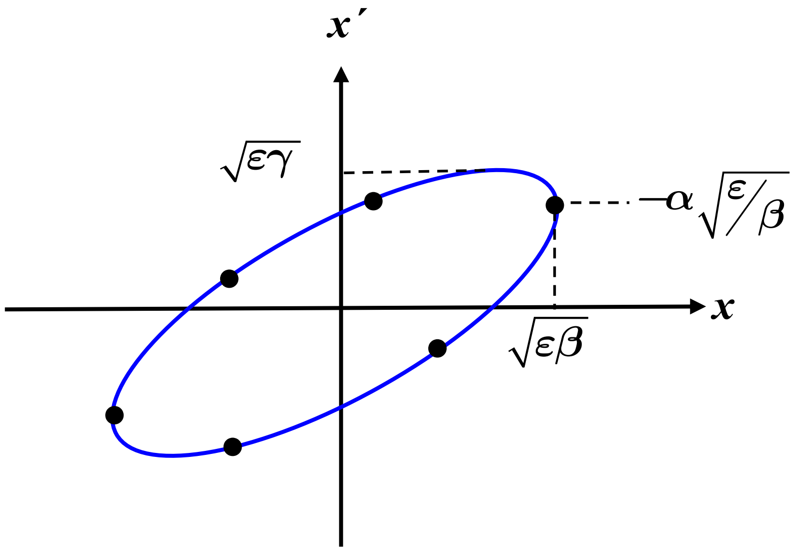
... question: do we really have to calculate some 10^{11} single particle trajectories ?

$$x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)} \cdot \cos(\psi(s) + \phi)$$

$$\psi(s) = \int_0^s \frac{ds}{\beta(s)}$$

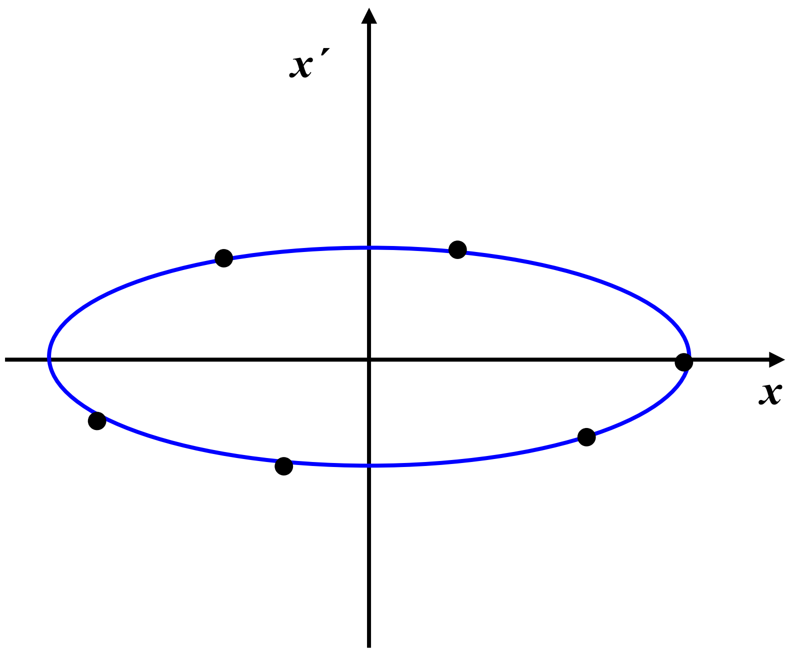


Beam Emittance and Phase Space Ellipse



$$\varepsilon = \gamma(s) * x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)x'(s)^2$$

Usually we get in a quadrupole $\alpha(s) = 0$



Inside foc quadrupoles β reaches maximum
→ largest aperture needed

⋯ *the not so ideal world*

II.) Emittance ... so sorry $\varepsilon \neq const.$

According to Hamiltonian mechanics:
phase space diagram relates the variables q and p

$q = position = x$
$p_x = momentum = mc\gamma\beta_x$

Liouvilles Theorem: $\int p\,dq = const$

for convenience (i.e. because we are lazy bones) we use in accelerator theory:

$$x' = \frac{dx}{ds} = \frac{dx}{dt}\frac{dt}{ds} = \frac{\beta_x}{\beta}$$

where $\beta = v/c$

$$\int p\,dq = const = mc\int \gamma\beta_x\,dx = mc\gamma\beta\int x'dx$$

$$\Rightarrow \quad \varepsilon = \int x'dx \propto \frac{1}{\beta\gamma}$$

the beam emittance shrinks
during acceleration $\varepsilon \sim 1/\gamma$

III.) Dispersion

Momentum error: $\Delta p / p \neq 0$

***Question:* do you remember yesterday on page 11 … sure you do:**

Force acting on the particles

$$F = m\frac{d^2}{dt^2}(x+\rho) - \frac{mv^2}{x+\rho} = eB_z v$$

$$x'' - \frac{1}{\rho}(1-\frac{x}{\rho}) = \frac{eB_0}{mv} + \frac{exg}{mv}$$

$$\frac{1}{mv} = \frac{1}{p_0 + \Delta p} \approx \frac{1}{p_0}(1 - \frac{\Delta p}{p_0})$$

neglecting higher order terms …

$$\boxed{x'' + x(\frac{1}{\rho^2} - k) = \frac{\Delta p}{p} \cdot \frac{1}{\rho}}$$

***Momentum spread* of the beam adds a term on the r.h.s. of the equation of motion.**
→ *inhomogeneous differential equation.*

$$x'' + x\left(\frac{1}{\rho^2} - k\right) = \frac{\Delta p}{p} \cdot \frac{1}{\rho}$$

general solution:

$$x(s) = x_h(s) + x_i(s)$$

$$\begin{cases} x_h''(s) + K(s) \cdot x_h(s) = 0 \\ x_i''(s) + K(s) \cdot x_i(s) = \dfrac{1}{\rho} \cdot \dfrac{\Delta p}{p} \end{cases}$$

Normalise with respect to Δp/p:

$$D(s) = \frac{x_i(s)}{\Delta p / p}$$

Dispersion function D(s)

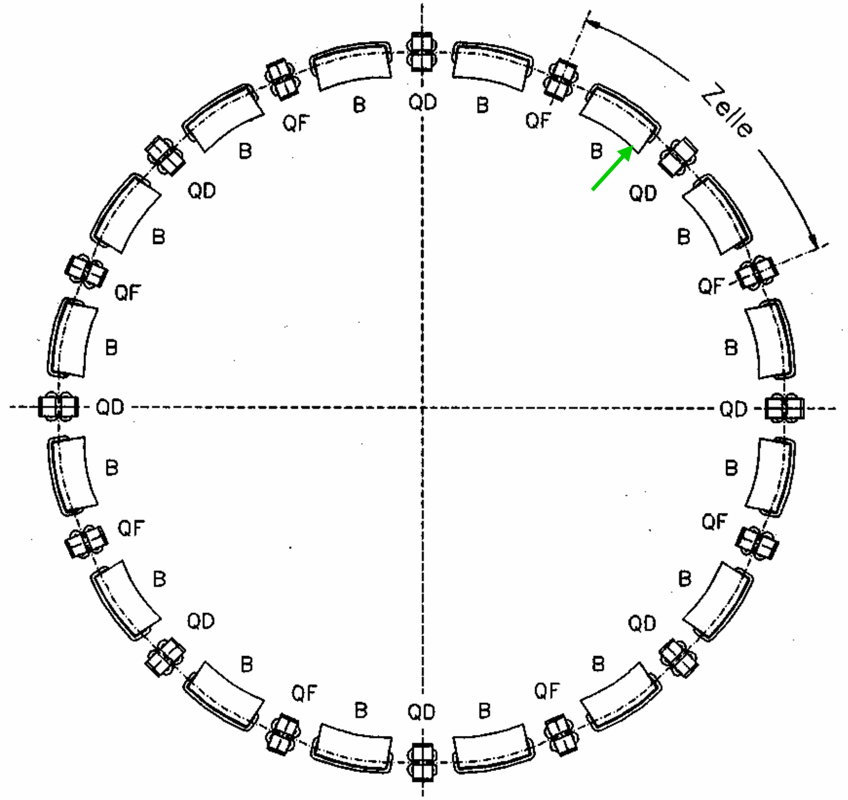
**** is that special orbit, an ideal particle would have for Δp/p = 1***

**** the orbit of any particle is the sum of the well known x_β and the dispersion***

**** as D(s) is just another orbit it will be subject to the focusing properties of the lattice***

Dispersion

Example: homogenous dipole field



bit for Δp/p > 0

$$= D(s) \cdot \frac{\Delta p}{p}$$

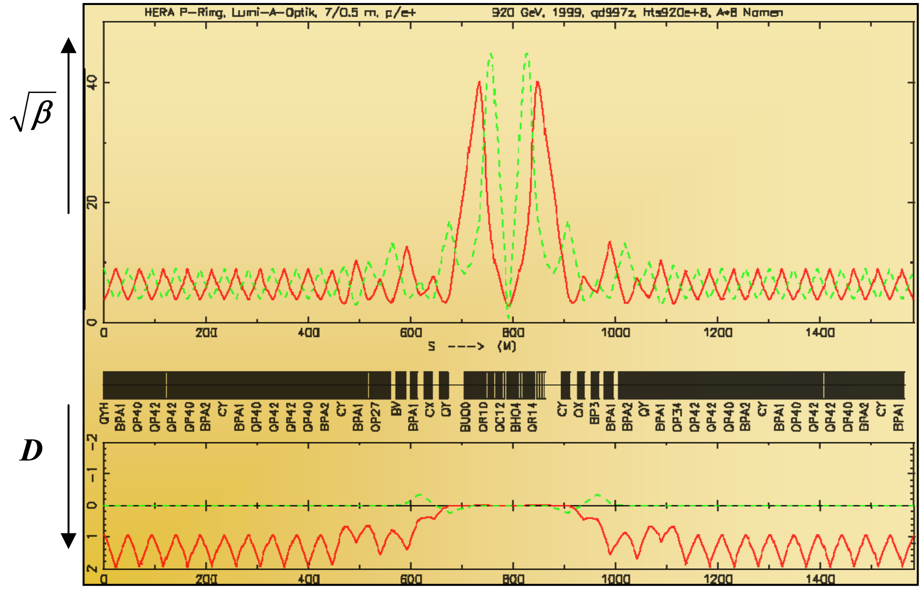
Matrix formalism:

$$x(s) = x_\beta(s) + D(s) \cdot \frac{\Delta p}{p}$$

$$x(s) = C(s) \cdot x_0 + S(s) \cdot x'_0 + D(s) \cdot \frac{\Delta p}{p}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_0 + \frac{\Delta p}{p} \begin{pmatrix} D \\ D' \end{pmatrix}$$

$$\begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_s = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_0$$



Example HERA

$$\left. \begin{aligned} x_\beta &= 1 \ldots 2\ mm \\ D(s) &\approx 1 \ldots 2\ m \\ \Delta p/p &\approx 1 \cdot 10^{-3} \end{aligned} \right\}$$

Amplitude of Orbit oscillation

contribution due to Dispersion ≈ beam size

Calculate D, D´

$$D(s) = S(s) \int_{s0}^{s1} \frac{1}{\rho} C(\tilde{s})\, d\tilde{s} - C(s) \int_{s0}^{s1} \frac{1}{\rho} S(\tilde{s})\, d\tilde{s}$$

Example: Drift

$$M_{Drift} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$$

$$D(s) = S(s)\underbrace{\int_{s0}^{s1} \frac{1}{\rho} C(\tilde{s})d\tilde{s}}_{=0} - C(s)\underbrace{\int_{s0}^{s1} \frac{1}{\rho} S(\tilde{s})d\tilde{s}}_{=0}$$

$$M_{Drift} = \begin{pmatrix} 1 & l & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Example: Dipole

$$M_{Dipole} = \begin{pmatrix} \cos\frac{l}{\rho} & \rho\sin\frac{l}{\rho} \\ -\frac{1}{\rho}\sin\frac{l}{\rho} & \cos\frac{l}{\rho} \end{pmatrix} \quad \rightarrow$$

$$D(s) = \rho\cdot(1 - \cos\frac{l}{\rho})$$

$$D'(s) = \sin\frac{l}{\rho}$$

IV.) Momentum Compaction Factor:

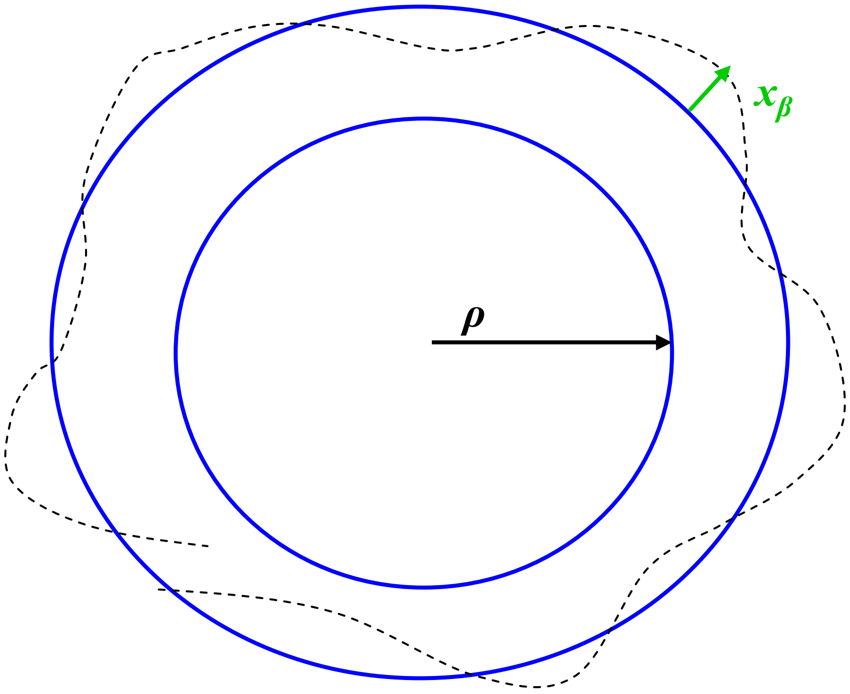
The dispersion function relates the momentum error of a particle to the horizontal orbit coordinate.

inhomogeneous differential equation

$$x'' + K(s) * x = \frac{1}{\rho} \frac{\Delta p}{p}$$

general solution

$$x(s) = x_\beta(s) + D(s) \frac{\Delta p}{p}$$

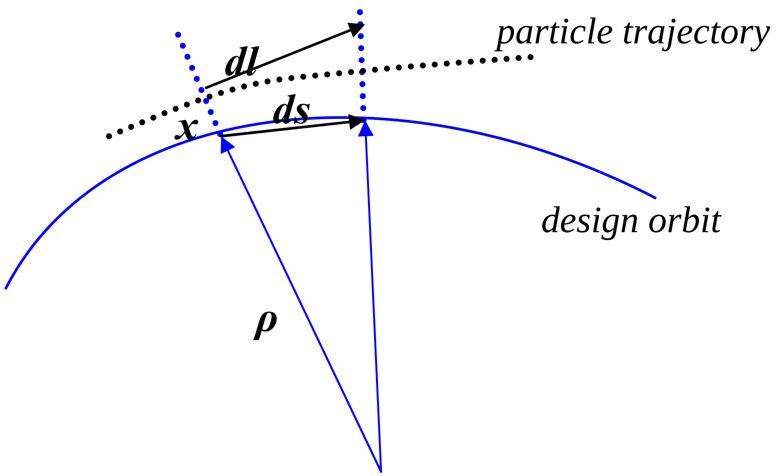


But it does much more:

it changes the length of the off - energy - orbit !!

particle with a displacement x to the design orbit
→ path length dl ...

$$\frac{dl}{ds} = \frac{\rho + x}{\rho}$$



$$\rightarrow \quad dl = \left(1 + \frac{x}{\rho(s)}\right) ds$$

circumference of an off-energy closed orbit

$$l_{\Delta E} = \oint dl = \oint \left(1 + \frac{x_{\Delta E}}{\rho(s)}\right) ds$$

remember:

$$x_{\Delta E}(s) = D(s)\,\frac{\Delta p}{p}$$

$$\delta l_{\Delta E} \approx \frac{\Delta p}{p} \int \left(\frac{D(s)}{\rho(s)}\right) ds$$

*** The lengthening of the orbit for off-momentum particles is given by the dispersion function and the bending radius.***

Definition:

$$\frac{\delta l_\varepsilon}{L} = \alpha_{cp} \frac{\Delta p}{p}$$

$$\rightarrow \quad \boxed{\alpha_{cp} = \frac{1}{L} \oint \left(\frac{D(s)}{\rho(s)} \right) ds}$$

For first estimates assume:

$$\frac{1}{\rho} = const$$

$$\int_{dipoles} \boldsymbol{D(s) ds} = \Sigma\, (\boldsymbol{l}_{dipoles}) * \langle \boldsymbol{D} \rangle_{dipole}$$

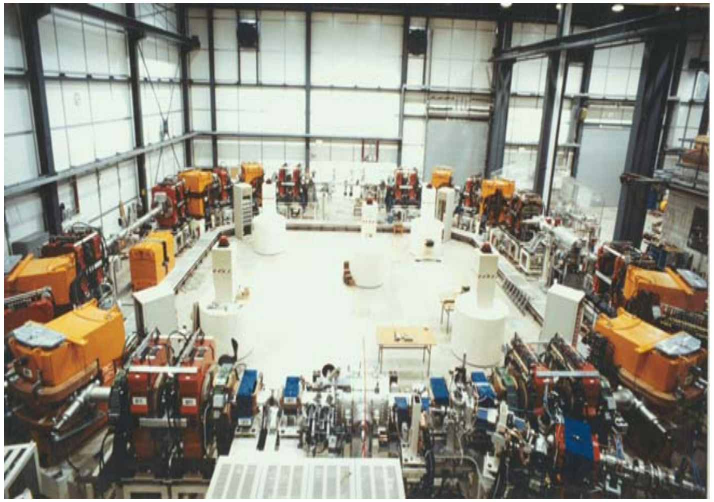
$$\alpha_{cp} = \frac{1}{L} l_{dipoles} \langle D \rangle \frac{1}{\rho} = \frac{1}{L} 2\pi\rho \langle D \rangle \frac{1}{\rho} \quad \rightarrow \quad \boxed{\alpha_{cp} \approx \frac{2\pi}{L} \langle D \rangle \approx \frac{\langle D \rangle}{R}}$$

Assume: $v \approx c$

$$\boxed{\rightarrow \frac{\delta T}{T} = \frac{\delta l_\varepsilon}{L} = \alpha_{cp} \frac{\Delta p}{p}}$$

α_{cp} combines via the dispersion function the momentum spread with the longitudinal motion of the particle.

V.) Tune and Quadrupoles



Question: ***what will happen, if you do not make too many mistakes and your particle performs one complete turn ?***

Transfer Matrix from point „0“ in the lattice to point „s“:

$$M = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}\left(\cos\psi_s + \alpha_0 \sin\psi_s\right) & \sqrt{\beta_s\beta_0}\,\sin\psi_s \\ \frac{(\alpha_0 - \alpha_s)\cos\psi_s - (1+\alpha_0\alpha_s)\sin\psi_s}{\sqrt{\beta_s\beta_0}} & \sqrt{\frac{\beta_0}{\beta s}}\left(\cos\psi_s - \alpha_s \sin\psi_s\right) \end{pmatrix}$$

Matrix for one complete turn

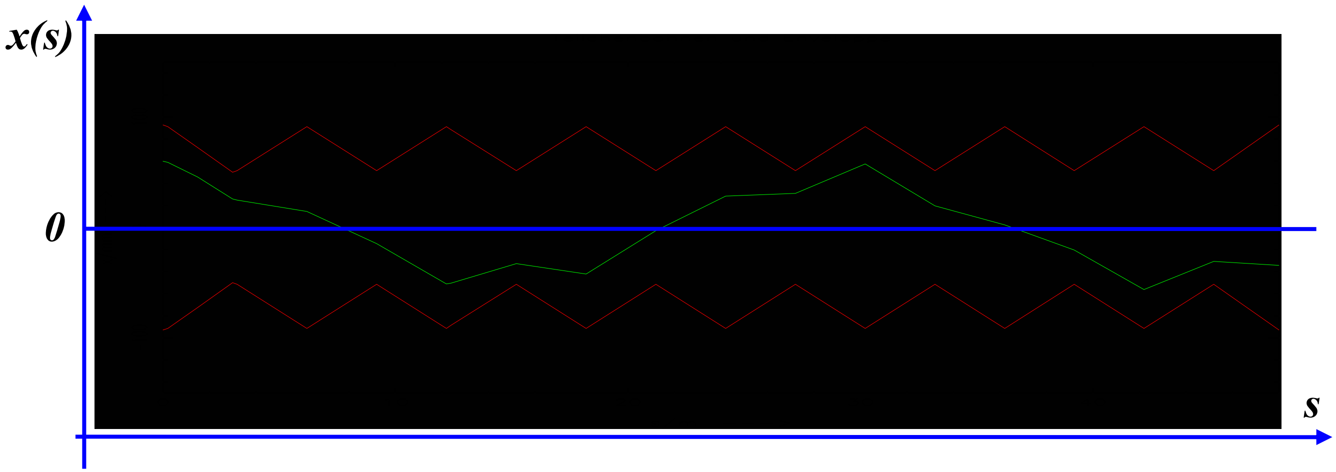
the Twiss parameters are periodic in L:

$$\beta(s+L) = \beta(s)$$
$$\alpha(s+L) = \alpha(s)$$
$$\gamma(s+L) = \gamma(s)$$

$$M_{turn} = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} \cos\psi_{turn} + \alpha\sin\psi_{turn} & \beta\sin\psi_{turn} \\ -\gamma\sin\psi_{turn} & \cos\psi_{turn} - \alpha\sin\psi_{turn} \end{pmatrix}$$

Definition: ***phase advance of the particle oscillation per revolution in units of 2π is called tune***

$$Q = \frac{\Delta\psi_{turn}}{2\pi} = \frac{\mu}{2\pi}$$



Quadrupole Error in the Lattice

optic perturbation described by thin lens quadrupole

$$\boldsymbol{M}_{dist} = \boldsymbol{M}_{\Delta k} * \boldsymbol{M}_0 = \underbrace{\begin{pmatrix} 1 & 0 \\ \Delta \boldsymbol{K} \boldsymbol{ds} & 1 \end{pmatrix}}_{\text{quad error}} * \underbrace{\begin{pmatrix} \cos\psi_{turn} + \alpha \sin\psi_{turn} & \beta \sin\psi_{turn} \\ -\gamma \sin\psi_{turn} & \cos\psi_{turn} - \alpha \sin\psi_{turn} \end{pmatrix}}_{\text{ideal storage ring}}$$

$$\boldsymbol{M}_{dist} = \begin{pmatrix} \cos\psi_{turn} + \alpha \sin\psi_{turn} & \beta \sin\psi_{turn} \\ \Delta \boldsymbol{K} \boldsymbol{ds} (\cos\psi_{turn} + \alpha \sin\psi_{turn}) - \gamma \sin\psi_{turn} & \Delta \boldsymbol{K} \boldsymbol{ds} * \beta \sin\psi_{turn} + \cos\psi_{turn} - \alpha \sin\psi_{turn} \end{pmatrix}$$

rule for getting the tune

$$\boldsymbol{Trace}(\boldsymbol{M}) = 2\cos\psi = 2\cos\psi_0 + \Delta \boldsymbol{K} \boldsymbol{ds} \beta \sin\psi_0$$

$$\psi = \psi_0 + \Delta\psi$$ ***Quadrupole error → Tune Shift***

$$\cos(\psi_0 + \Delta\psi) = \cos\psi_0 + \frac{\Delta K ds \beta \sin\psi_0}{2}$$

remember the old fashioned trigonometric stuff and assume that the error is small !!!

$$\cos\psi_0 * \underbrace{\cos\Delta\psi}_{\approx 1} - \sin\psi_0 * \underbrace{\sin\Delta\psi}_{\approx \Delta\psi} = \cos\psi_0 + \frac{\Delta K ds \beta \sin\psi_0}{2}$$

$$\Delta\psi = \frac{\Delta K ds \beta}{2}$$

and referring to Q instead of ψ: $\psi = 2\pi Q$

$$\Delta Q = \int_{s0}^{s0+L} \frac{\Delta K(s)\beta(s)ds}{4\pi}$$

a quadrupol error leads to a shift of the tune:

$$\Delta Q = \int_{s0}^{s0+L} \frac{\Delta K(s)\beta(s)ds}{4\pi} \approx \frac{\Delta K l_{quad}\overline{\beta}}{4\pi}$$

! the tune shift is proportional to the β-function at the quadrupole

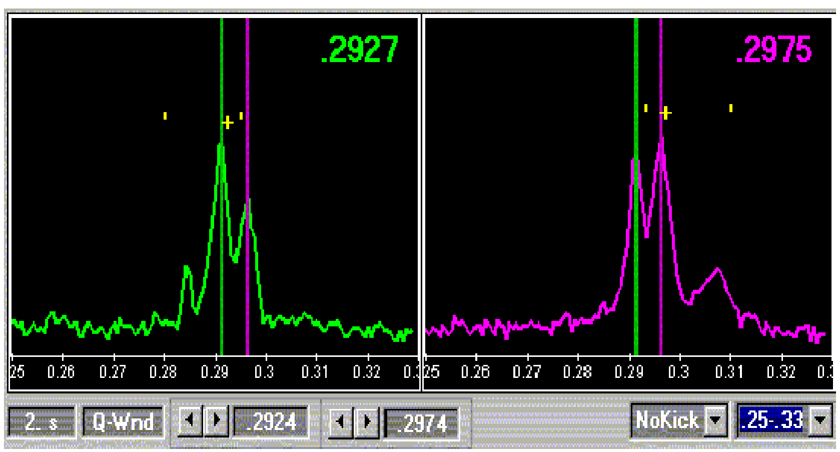
!! field quality, power supply tolerances etc are much tighter at places where β is large

!!! mini beta quads: β ≈ 1900

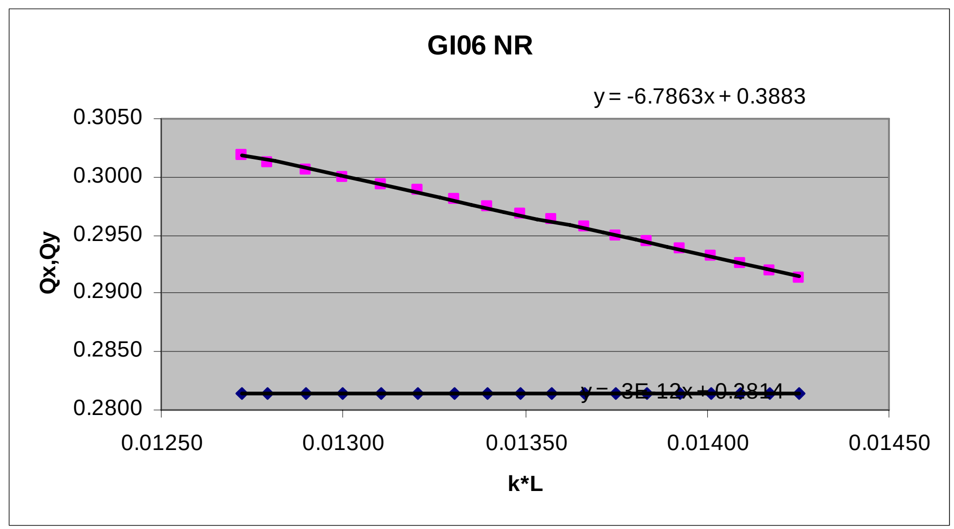
arc quads: β ≈ 80

!!!! β is a measure for the sensitivity of the beam

Example: measurement of β in a storage ring:



tune spectrum ...



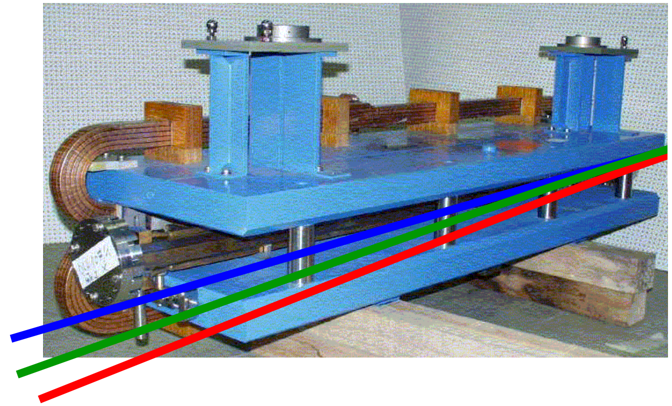
tune shift as a function of a gadient change

VI.) Chromaticity: ξ

Influence of external fields on the beam: ***prop. to magn. field & prop. zu 1/p***

dipole magnet

$$\alpha = \frac{\int B * dl}{p/e}$$



$$x_d(s) = D(s) * \frac{\Delta p}{p}$$

focusing lens

$$k = \frac{g}{p/e}$$

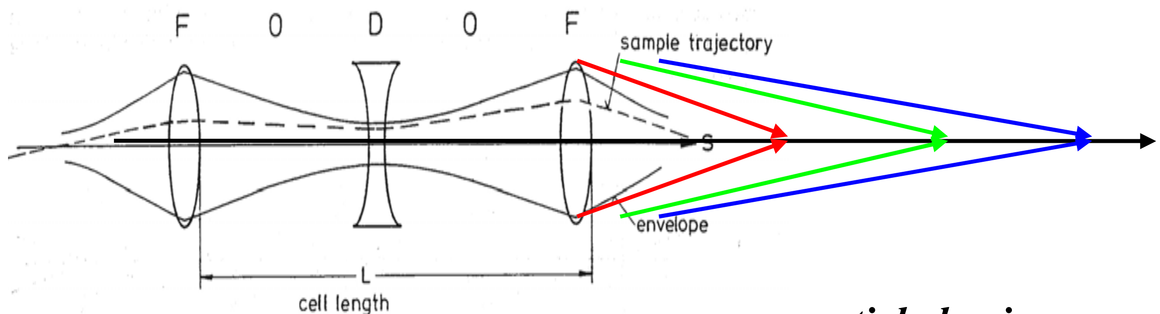


Figure 29: FODO cell

particle having ...

to high energy

to low energy

ideal energy

Chromaticity: ξ

$$k = \frac{g}{p/e} \qquad\qquad p = p_0 + \Delta p$$

in case of a momentum spread:

$$k = \frac{eg}{p_0 + \Delta p} \approx \frac{e}{p_0}(1 - \frac{\Delta p}{p_0})g = k_0 + \Delta k$$

$$\Delta k = -\frac{\Delta p}{p_0} k_0$$

… which acts like a quadrupole error in the machine and leads to a tune spread:

$$dQ = -\frac{\Delta p}{p_0}\frac{1}{4\pi} k_0 \beta(s) ds$$

definition of chromaticity:

$$\boxed{\Delta Q = \xi \frac{\Delta p}{p_0}}$$

Problem: chromaticity is generated by the lattice itself !!

ξ is a number indicating the size of the tune spot in the working diagram,
ξ is always created if the beam is focussed
→ it is determined by the focusing strength k of all quadrupoles

$$\boxed{\xi = \frac{-1}{4\pi} * \oint K(s)\beta(s)\, ds}$$

k = quadrupole strength
β = betafunction indicates the beam size … and even more the sensitivity of the beam to external fields

Example: HERA

HERA-p: ξ ***= -70 … -80***
Δ p/p = 0.5 *10^{-3}
Q = 0.257 … 0.337

→Some particles get very close to resonances and are lost

Correction of ξ:

1.) sort the particles acording to their momentum

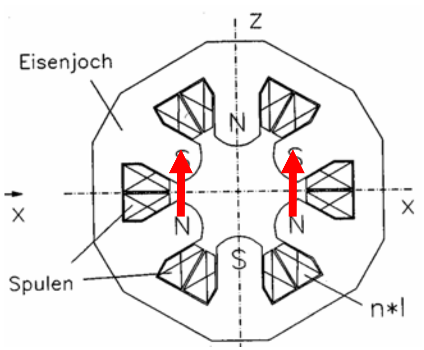
$$x_D(s) = D(s)\frac{\Delta p}{p}$$

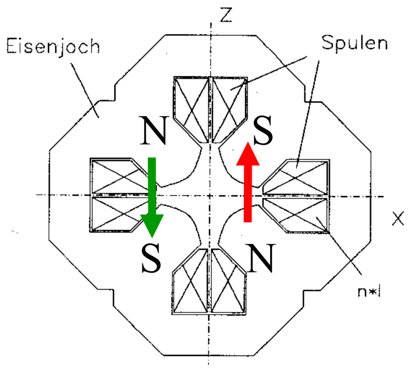
2.) apply a magnetic field that rises quadratically with x (sextupole field)

$$\left.\begin{aligned} B_x &= \tilde{g}xz \\ B_z &= \frac{1}{2}\tilde{g}(x^2 - z^2) \end{aligned}\right\} \qquad \frac{\partial B_x}{\partial z} = \frac{\partial B_z}{\partial x} = \tilde{g}x$$

linear rising „gradient“:

Sextupole Magnets:





normalised quadrupole strength:

$$k_{sext} = \frac{\tilde{g}x}{p/e} = m_{sext.}x$$

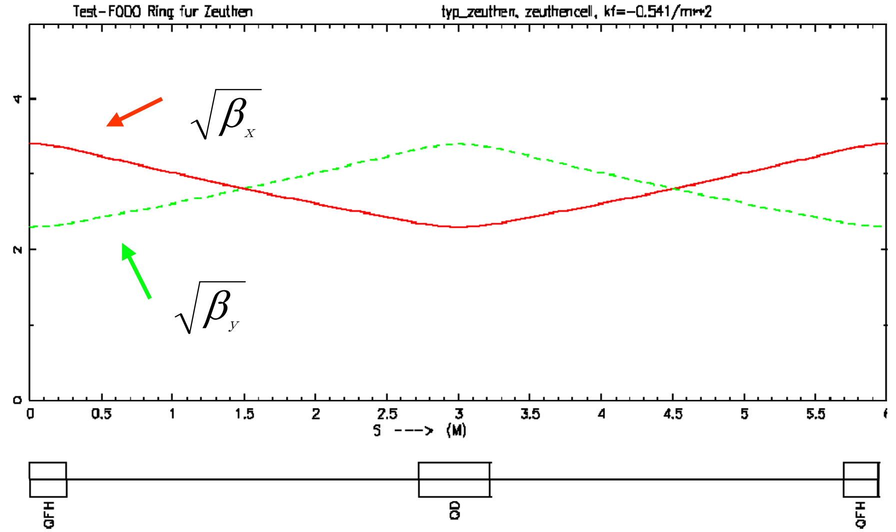
$$k_{sext} = m_{sext.}D\frac{\Delta p}{p}$$

corrected chromaticity:

$$\xi = -\frac{1}{4\pi}\oint\{K(s) - mD(s)\}\beta(s)ds$$

Chromaticity in the FoDo Lattice

$$\xi = -\frac{1}{4\pi}\int \beta(s) * k(s) ds$$



β-Function in a FoDo

$$\hat{\beta} = \frac{(1+\sin\frac{\mu}{2})L}{\sin\mu} \qquad \check{\beta} = \frac{(1-\sin\frac{\mu}{2})L}{\sin\mu}$$

$$\xi \approx -\frac{1}{4\pi} N * \frac{\hat{\beta} - \check{\beta}}{f_Q}$$

$$\xi = -\frac{1}{4\pi} N * \frac{1}{f_Q} * \left\{ \frac{L(1+\sin\frac{\mu}{2}) - L(1-\sin\frac{\mu}{2})}{\sin\mu} \right\}$$

using some TLC transformations ... ξ can be expressed in a very simple form:

$$\xi = -\frac{1}{4\pi} N \frac{1}{f_Q} \frac{2L \sin\frac{\mu}{2}}{\sin\mu}$$

$$\xi = -\frac{1}{4\pi} N \frac{1}{f_Q} \frac{L \sin\frac{\mu}{2}}{\sin\frac{\mu}{2} \cos\frac{\mu}{2}}$$

remember ...

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$$

$$\xi_{Cell} = -\frac{1}{4\pi f_Q} * \frac{L \tan\frac{\mu}{2}}{\sin\frac{\mu}{2}}$$

putting ...

$$\sin\frac{\mu}{2} = \frac{L}{4 f_Q}$$

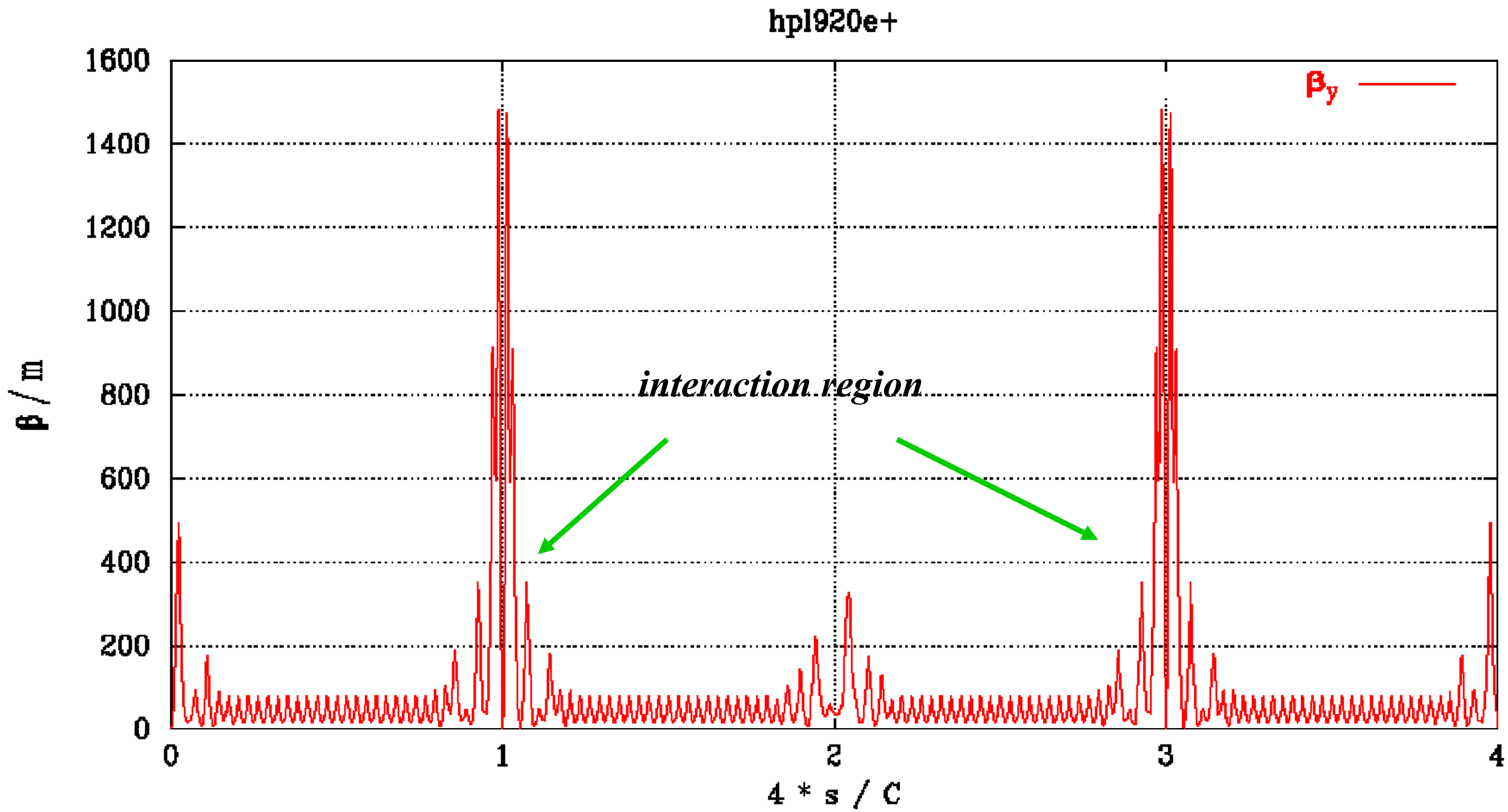
$$\boxed{\xi_{Cell} = -\frac{1}{\pi} * \tan\frac{\mu}{2}}$$

contribution of one FoDo Cell to the chromaticity of the ring:

Chromaticity

$$\xi = \frac{-1}{4\pi} \oint k(s)\beta(s)ds$$

question: main contribution to ξ in a lattice … ?



Resume:

beam emittance

$$\varepsilon \propto \frac{1}{\beta\gamma}$$

dispersion orbit

$$x(s) = x_{\beta}(s) + D(s)\frac{\Delta p}{p}$$

momentum compaction

$$\frac{\delta l_{\varepsilon}}{L} = \alpha_{cp}\frac{\Delta p}{p}$$

$$\alpha_{cp} \approx \frac{2\pi}{L}\langle D\rangle \approx \frac{\langle D\rangle}{R}$$

quadrupole error

$$\Delta Q = \int_{s0}^{s0+l} \frac{\Delta K(s)\beta(s)ds}{4\pi}$$

chromaticity

$$\xi = -\frac{1}{4\pi}\oint K(s)\beta(s)ds$$