TRANSVERSE INSTABILITIES

CAS 2007, Daresbury; Albert Hofmann, September 22, 2007

1) **Introduction**
2) **Transverse impedance**
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4) **Head-tail instability**
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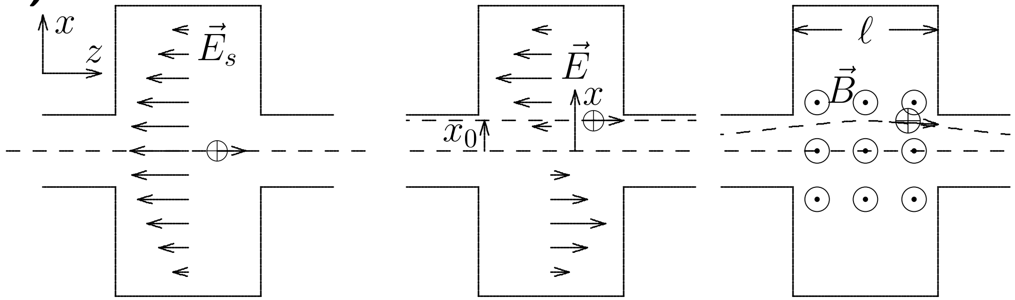
1) INTRODUCTION

The mechanism of transverse instabilities is in many respect similar to the longitudinal case. The transverse motion of a single particle in a storage ring is determined by the external guide fields consisting here mainly of quadrupole magnets, however also the RF-system, initial conditions and synchrotron radiation have an influence. Many particles in a beam may represent a sizable charge and current which act as a source of electromagnetic fields (self fields). They are modified by boundary conditions imposed by the beam surroundings (vacuum chambers, cavities, etc.) and act back on the beam.

However in this case it is the transverse deviation of the beam which represents a dipole moment and excites certain field configuration in cavities which will apply later a transverse force. In case this force increases the original dipole moment we have an instability. Here it is the transverse impedance which describes the relevant properties of the beam surroundings. It is excited by the transverse dipole moment of the beam but is not sensitive to its transverse dimension or higher order moments. The transverse particle distribution has therefore no or very little influence on the instability. However, this impedance has a fast time response and senses a difference in the dipole moment along the bunch, in particular, between the head and tail of the bunch. This can lead to some new effects, called head-tail instabilities.

2) TRANSVERSE IMPEDANCE



longitudinal transverse transverse

In **longitudinal** impedance I induces E_z, force $F_z = eE_z$, voltage $V = -\int E_z ds = -\Delta U/e$

Ohm's law: $V = \propto I,\ I = \hat{I}\mathrm{e}^{j\omega t}$.

Impedance: $Z = |Z|\mathrm{e}^{j\phi} = V/I$

Transverse (dipole) mode, excited by longitudinal bunch motion and transforms into deflection field. Example: mode is excited by bunch with dipole moment Ix_0, has gradient $\partial E_z/\partial x$. After 1/4 oscillation E_z becomes B_y-field with transverse force $\vec{F} = e[\vec{v} \times \vec{B}]$.

Maxwell: $\dot{\vec{B}} = -\mathrm{curl}\vec{E} \ \rightarrow \ \int \dot{\vec{B}} d\vec{a} = -\oint \vec{E} d\vec{s}$

$$\dot{B}_y x \ell = -E_z \ell$$

$$E_z = \frac{\partial \hat{E}}{\partial x} x \cos(\omega t),\ B_y = -\frac{1}{\omega}\frac{\partial \hat{E}}{\partial x}\sin(\omega t)$$

$$E_z = \frac{\partial \hat{E}}{\partial x} x \mathrm{e}^{j\omega t},\ B_y = -\frac{1}{j\omega}\frac{\partial \hat{E}}{\partial x}\mathrm{e}^{j\omega_t} \text{ complex}$$

$$F_x = -evB_y = \dot{p}_x,\ \Delta p_x = F_x \Delta t = F_x \ell / v$$

$$\frac{\Delta(cp_x)}{e} = vB_y\ell = -\frac{v\ell}{j\omega}\frac{\partial E}{\partial x} \propto \frac{1}{j} I x_0$$

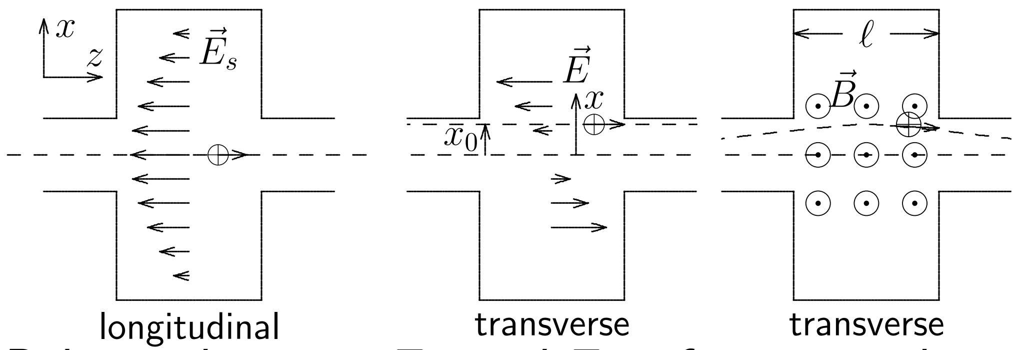
$$Z_T = j\frac{vB_y\ell}{Ix_0} \text{ [Ohm/m]}.$$

On resonance, E_z is in, B_y out of phase of I.

For general deflecting mode, using $x = \hat{x}\mathrm{e}^{j\omega t}$

$$Z_T(\omega) = j\frac{\int \left(\vec{E}(\omega) + [\vec{v} \times \vec{B}(\omega)]\right)_T ds}{Ix(\omega)}$$

$$= -\frac{\omega \int \left(\vec{E}(\omega) + [\vec{v} \times \vec{B}(\omega)]\right)_T ds}{I\dot{x}(\omega)}$$



Relation between Z_L and Z_T of same mode:
A dipole moment Ix_0 induces in longitudinal impedance Z_L a gradient $\partial E_z/\partial x = -kIx_0$

$$E_z(x) = \frac{\partial E_z}{\partial x} x = -kIx_0 x$$

$$E_z(x_0) = -kIx_0^2, \text{ gives long. impedance}$$

$$Z_L(x_0) = -\int E_z(x_0)dz/I = kx_0^2\ell$$

$$d^2Z_L/dx_0^2 = -2k\ell\ , \quad (\ell = \text{cavity length})$$

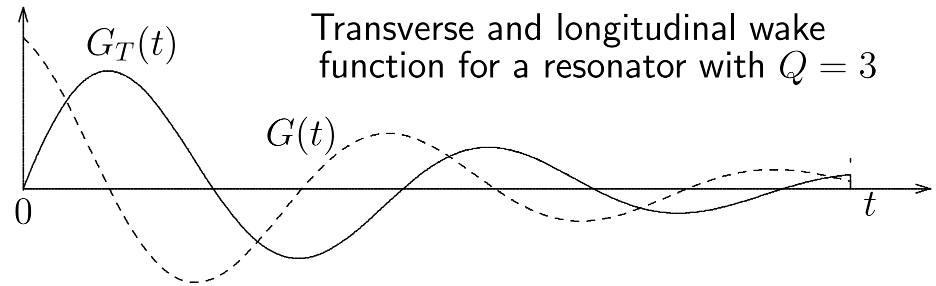
$$B_y = \frac{j}{\omega}\frac{\partial E_z}{\partial x} = \frac{jkIx_0}{\omega}$$

$$Z_T = j\frac{\int[\vec{v}\times\vec{B}]_T ds}{Ix(\omega)} = j\frac{cB_y\ell}{Ix_0} = \frac{ck\ell}{\omega}$$

$$Z_T = \frac{c}{2\omega}\frac{d^2Z_L}{dx^2}\ , \quad \frac{Z_L(\omega)}{\omega/\omega_0} = \frac{\omega_0}{c}Z_T(\omega)x_0^2$$

This gives the ω-symmetry relation for Z_T

$$Z_{Lr}(-\omega) = Z_{Lr}(\omega)\ ,\ Z_{Li}(-\omega) = -Z_{Li}(\omega)$$
$$Z_{Tr}(-\omega) = -Z_{Tr}(\omega),\ Z_{Ti}(-\omega) = Z_{Ti}(\omega)$$



Relation Z_L to Z_T of different modes:
In ring of global and vacuum chamber radii R and b the impedances, averaged for **different** modes and objects, have semi-empirical

$$\text{ratio:} \quad Z_T(\omega) \approx \frac{2R}{b^2}\frac{Z_L(\omega)}{\omega/\omega_0}.$$

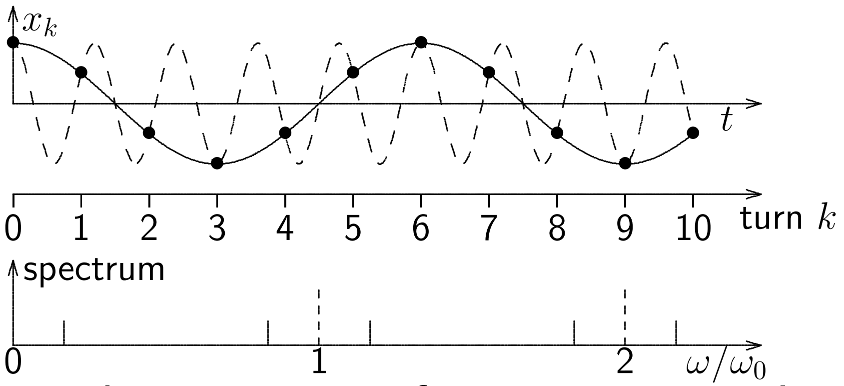
From the area available for the wall current we expect $Z_L \propto 1/b$ and therefore $Z_T \propto 1/b^3$.

3) TRANSVERSE INSTABILITY WITH $Q' = 0$

Transverse dynamics

$$x_k = \hat{x}\cos(2\pi qk)\ ,\ x'_k = -\frac{\hat{x}}{\beta_x}\sin(2\pi qk)$$

$$x_k = \hat{x}\mathrm{e}^{j2\pi qk}\ , \qquad x'_k = \frac{\hat{x}}{\beta_x}\mathrm{e}^{j2\pi qk}$$



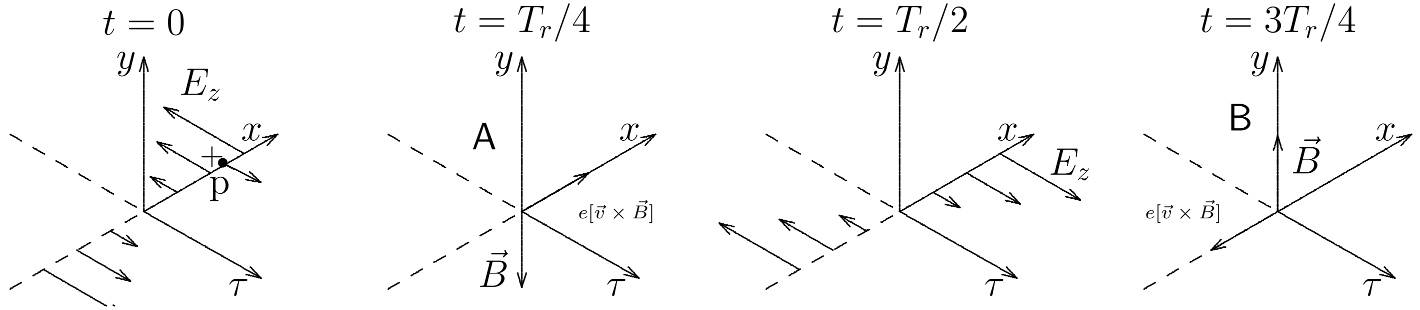
Due to the transverse focusing a particle executes a betatron motion around the orbit. This is an oscillation being locally harmonic but having complicated phase advance around the ring. We approximate by a smooth focusing

$$\ddot{x} + \omega_0^2 Q_x^2 x = 0$$

with revolution frequency ω_0, betatron tune Q_x.

A stationary observer, or impedance, sees the particle position x_k only at one location each turn k and has no information what the particle does in the rest of the ring We observe this motion as function of turn k and make a harmonic fit, i.e. a Fourier analysis. For a single circulating bunch we find no line at the revolution harmonic $p\omega_0$ but an upper and lower sideband at a distance given by the tune $Q_x =$ integer $+q$. The fractional part q is the only part which matters since the integer cannot be observed. For a very short bunch these sidebands will extend to very high frequencies, for longer bunches they level off. A transverse impedance (or a position monitor) is sensitive to the dipole moment Ix of the current and does not see the revolution harmonics.

Multi-traversal instability of a single bunch



A bunch p traverses a cavity with off-set x, excites a field $-E_z$ which converts after $T_r/4$ into field $-B_y$, then into E_z and after into B_y.

The bunch oscillates with tune Q having a fractional part $q = 1/4$ seen as sidebands at $\omega_0(\text{integer} \pm q)$ by a stationary observer.

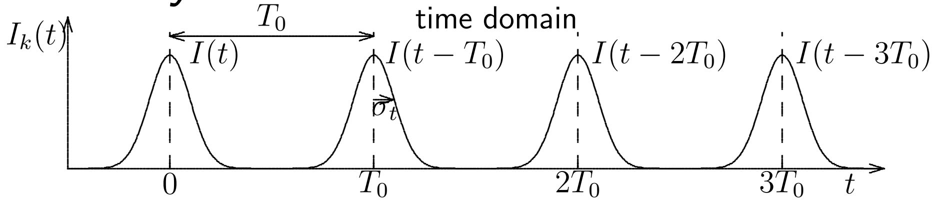
A) A cavity is tuned to upper sideband. Next turn the bunch traverses it in the situation 'A', $t = T_r(k+1/4)$ with a velocity in $-x$-direction and gets by B_y a force in $+x$-direction which damps the oscillation.

B) A cavity is tuned to lower sideband. The bunch traverses it next turn in situation 'B', $t = T_r(k' + 3/4) = T_r(k' + 1 - 1/4)$ with negative velocity and a force in same direction. This increases its velocity and gives instability.

Damping/growth rate $\propto \sum_p \left(I_{p+}^2(Z_{Tr}(\omega_{p+}) - I_{p-}^2 Z_{Tr}(\omega_{p-})\right)$, $I_{p\pm}$ = Fourier lines at side-bands

Transverse instability, single bunch, narrow cavity resonance at $(p+q)\omega_0,\ \beta \approx 1$

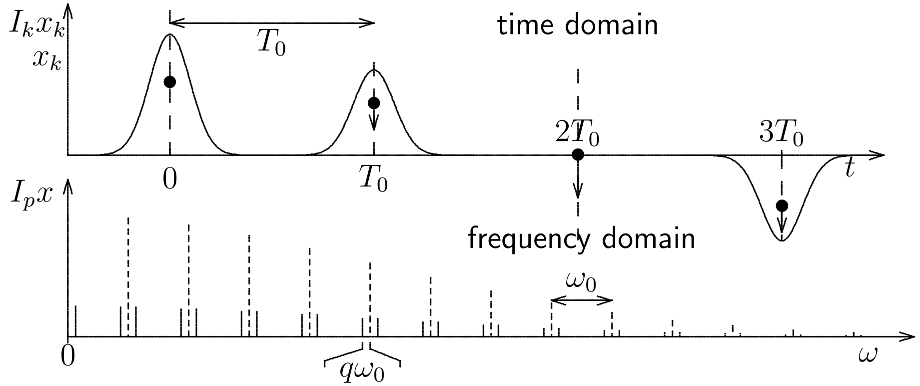
Stationary bunch current



$$I_k(t) = \sum_{-\infty}^{\infty} I(t - kT_0) = I_0 + 2\sum_{1}^{\infty} I_p \cos(p\omega_0 t)$$

$$I_p = \frac{1}{T_0}\int I(t)\cos(p\omega_0 t)dt$$

Dipole moment $I_k x_k$, bunch oscillating $q = \frac{1}{8}$



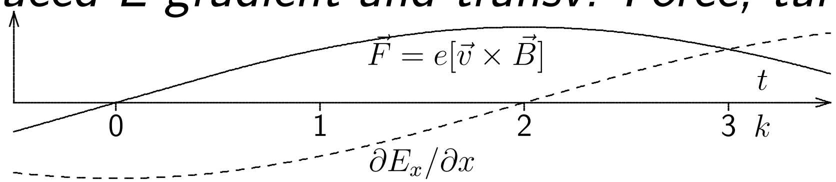
$$I_k x_k = \sum_{-\infty}^{\infty} I(t - kT_0)\hat{x}\cos(2\pi qk)$$

$$= \hat{x} I_{p+} \mathrm{e}^{j\omega_p^+ t},\ \omega_p^+ = \omega_0(p+q),\ p = 0$$

$$I_{p\pm} = \frac{1}{T_0}\int I(t)\mathrm{e}^{-j\omega_p^{\pm} t}dt = \frac{\omega_0}{\sqrt{2\pi}}\tilde{I}(\omega_{p\pm})$$

$$x_k = \hat{x}\mathrm{e}^{j2\pi qk},\ x' = \dot{x}/c = j\frac{\hat{x}}{\beta_x}\mathrm{e}^{j2\pi qk}$$

Induced E-gradient and transv. Force, turn k



$$e\int[\vec{v} \times \vec{B}]_T ds = -jeZ_T I_{p+}\mathrm{e}^{j\omega_p^+ t} x = c\Delta p_i$$

$$\langle \Delta p_i \rangle = \frac{eZ_T I_{p+} x}{jcI_0 T_0}\int I(t)\mathrm{e}^{-j\omega_p^+ t}dt = \frac{eZ_T I_{p+}^2 x}{jcI_0}$$

$$c\langle \Delta p_i \rangle = m_0\gamma c^2 \Delta x' = -\frac{jeZ_T I_{p+}^2}{I_0}x = -\frac{eZ_T I_{p+}^2 \beta_x}{I_0}x'$$

$$\Delta\dot{x} = -\frac{eZ_T I_{p+}^2 \beta_x}{I_0 m_0 \gamma c^2}\dot{x},\ \frac{d\dot{x}}{dt} = \frac{\Delta\dot{x}}{T_0} = -\frac{eZ_T I_{p+}^2 \beta_x \dot{x}}{I_0 T_0 m_0 \gamma c^2}$$

$$\ddot{x}_Z = -2a\dot{x},\ a = \frac{\omega_0 e Z_T \beta_x I_{p+}^2}{4\pi I_0 m_0 \gamma c^2}$$

$$\ddot{x} + 2a\dot{x} + \omega_0^2 Q_x^2 x = 0,\ x = x_0 \exp(-at)$$

Instability due to the resistive impedance

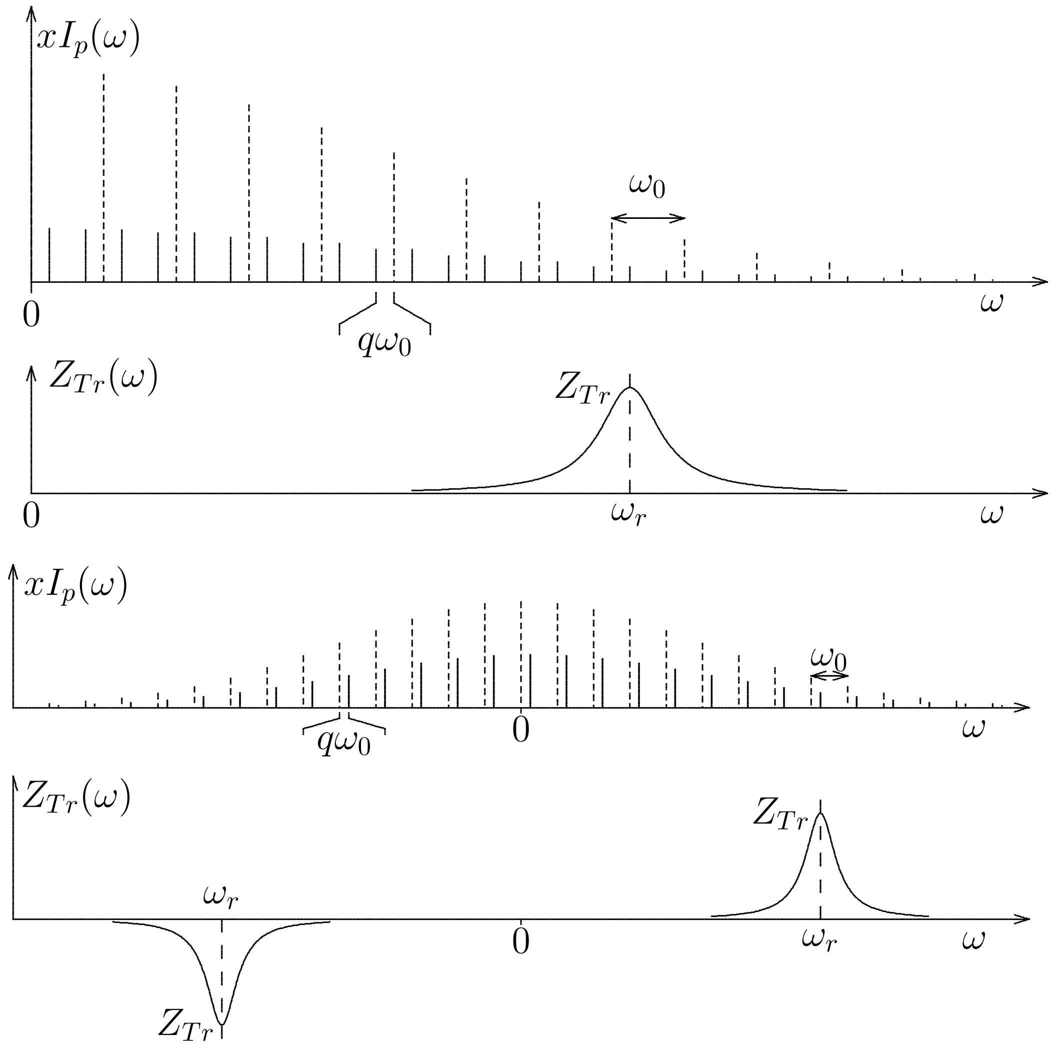
General: impedance is summed over all side-bands, imaginary (reactive) impedance gives frequency shift $\Delta\omega_\beta$ which we neglect.

$$x = x_0 \mathrm{e}^{-at} \cos((Q_x \omega_0 + \Delta\omega_\beta)t + \phi)$$

$$a = \frac{e\omega_0\beta_x}{4\pi m_0 c^2 \gamma I_0} = \sum_{\omega>0} \left(I_{p+}^2 Z_{Tr}(\omega_p^+) - I_{p-}^2 Z_{Tr}(\omega_p^-) \right).$$

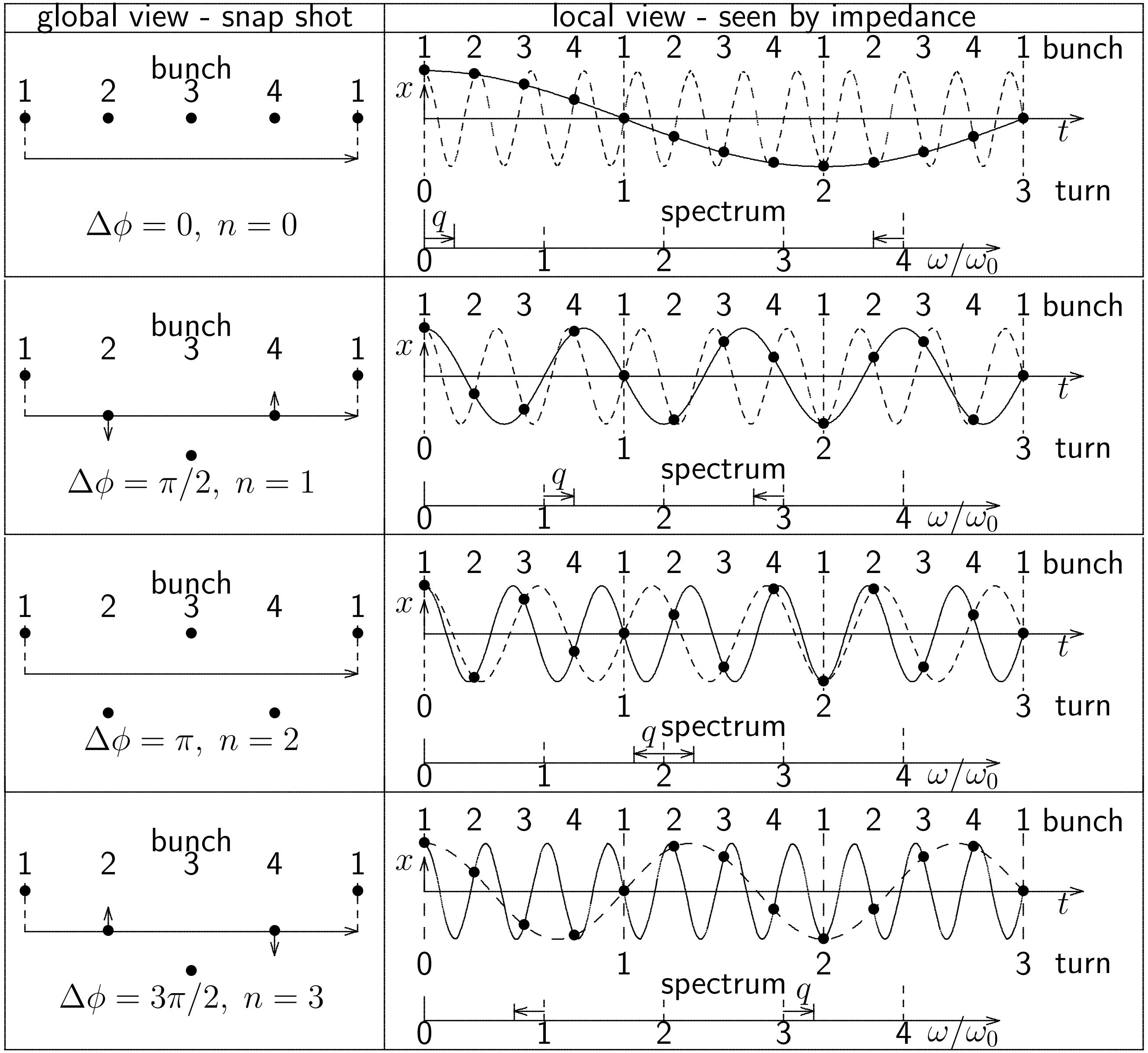
$$a = \frac{e\omega_0\beta_x}{4\pi m_0 c^2 \gamma I_0} \sum_{p=-\infty}^{\infty} I_{p+}^2 Z_{Tr}(\omega_p^+).$$

For a distributed impedance we replace local beta function by average $\beta_x \approx \langle\beta_x\rangle \approx R/Q$ with $R =$ average ring radius. Single strong impedances should be located at a small beta function. this instability is driven by a narrow band impedance with memory.



Transverse instability of many bunches

M circulating bunches can oscillate in M independent modes $n = M\Delta\phi/2\pi$ with phase $\Delta\phi$ between bunches as shown in the global view where all are seen at once. For a local observer the bunches pass by with increasing time delay shown by the bullets which are fitted by an upper (solid line) and lower (dashed line) side-band frequency. Higher frequencies can be fitted and the spectrum repeats every $4\omega_0$.

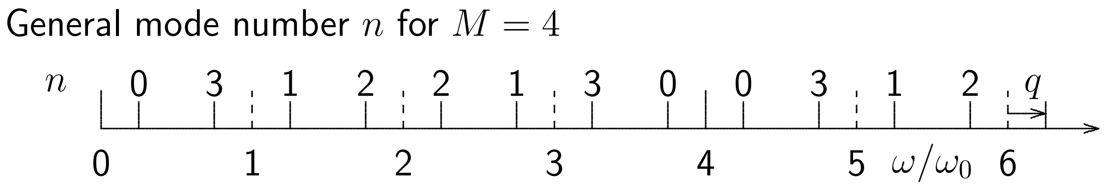


The side-band frequencies of M bunches oscillating in a mode n, fractional tune q are

$$\omega_{p\pm} = \omega_0 \left(pM \pm (n+q)\right)$$

with the running integer p.

Increasing p gives all higher frequency of the spectrum. A picture, shown for $M = 4$ can quickly give the side-band locations for mode n

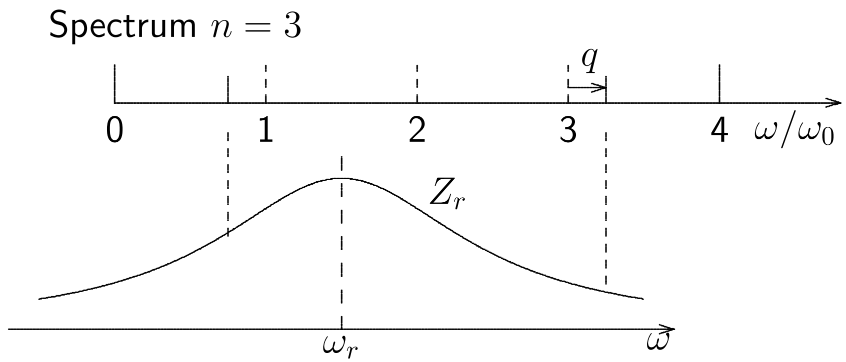


A detailed calculation gives the damping/growth rate $\alpha = 1/\tau$ of coupled bunch oscillations. With $x(t) \propto \mathrm{e}^{-at}$ stable if $a > 0$, that is if $Z_{Tr}(\omega_p^+) > Z_{Tr}(\omega_p^-)$

$$a = \frac{e\omega_0\beta_x}{4\pi m_0 c^2 \gamma I_0} \sum_{\omega>0} \left(I_{p+}^2 Z_{Tr}(\omega_p^+) - I_{P-}^2 Z_{Tr}(\omega_p^-)\right)$$

$$\approx \frac{e\omega_0\beta_x}{4\pi m_0 c^2 \gamma I_0} \sum_{\omega>0} I_p^2 \left(Z_{Tr}(\omega_p^+) - Z_{Tr}(\omega_p^-)\right).$$

We sum over all side-bands of a given mode n.

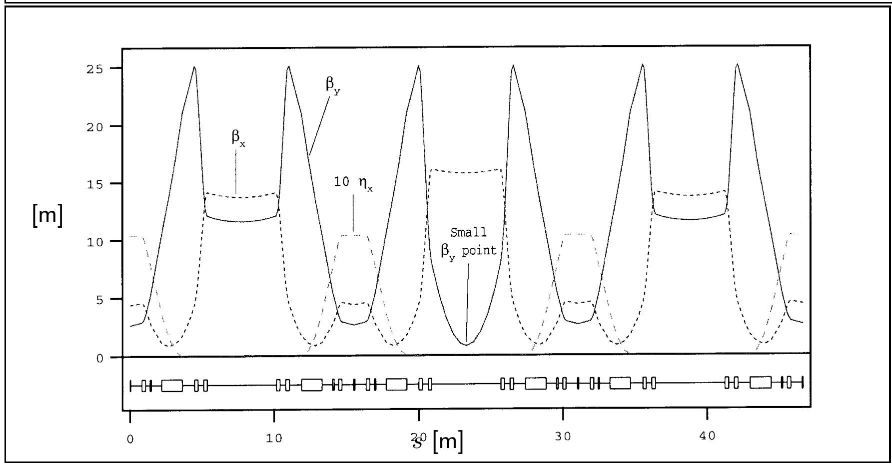


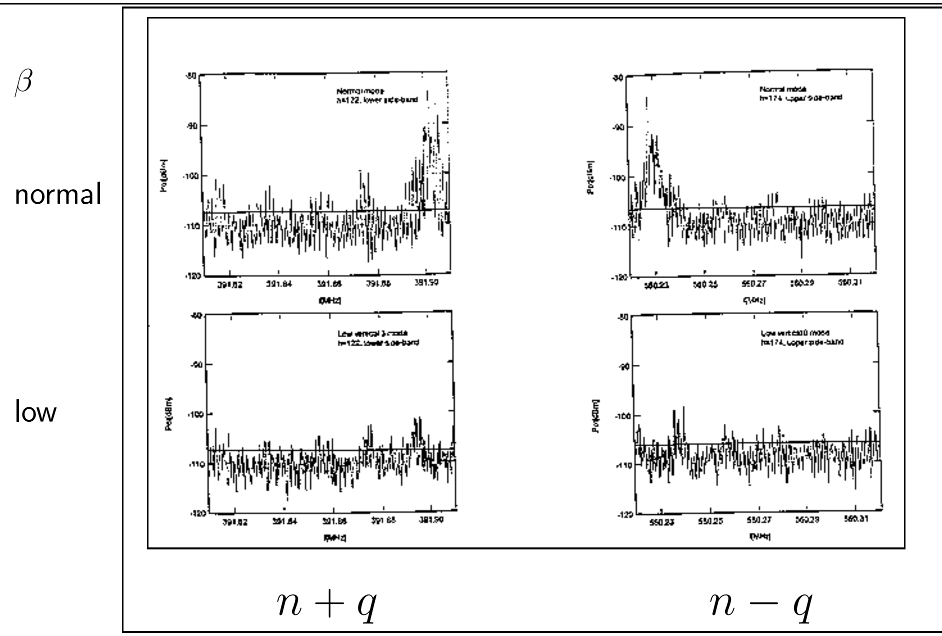
Dependance of the transverse instability on β_y

$$a = \frac{1}{\tau} = \frac{e\omega_0\beta_y}{4\pi m_0 c^2 \gamma I_0} \sum_{\omega>0} \left(I_{p+}^2 Z_{Tr}(\omega_p^+) - I_{p-}^2 Z_{Tr}(\omega_p^-) \right).$$

The transverse instability growth rate is $\propto \beta_y$ since deflection at high β gives larger amplitude. To observe exponential growth we must inject large current or turn a feed-back system off.

In slow accumulation instability current saturates with some unstable betatron lines shown at LNLS (Laboratòrio Nacional de Luz Sínchrotron, Brasil). Reducing β_y in RF section eliminates these lines indicating offending impedance is RF-cavity.



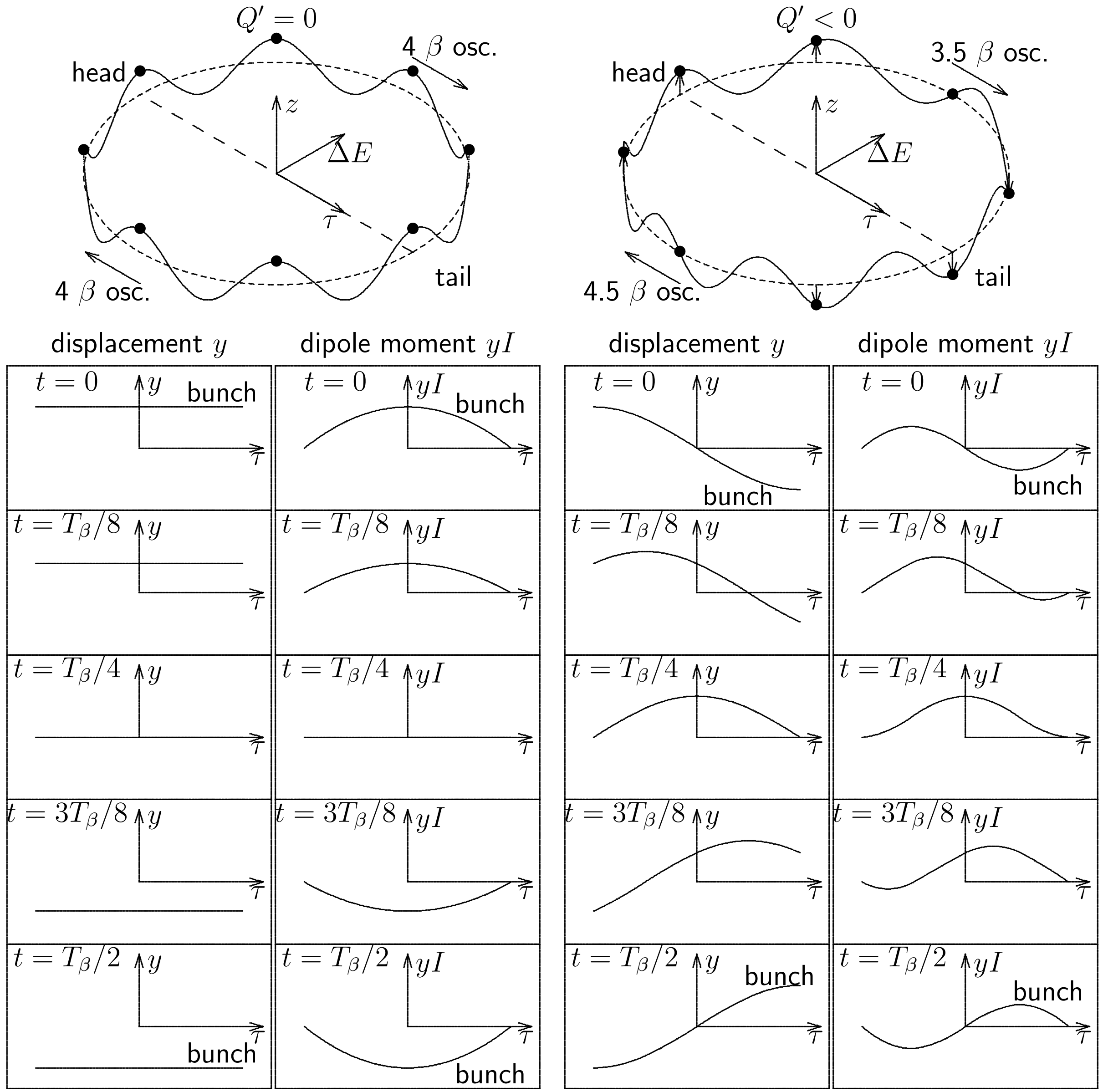


4) HEAD-TAIL INSTABILITY

Head-tail mode oscillation

The synchrotron motion of a particle in energy and time deviation ΔE and τ influences the transverse motion via chromaticity $Q' = dQ/(dp/p)$. For $\gamma > \gamma_T$ it has excess energy while moving from head to tail and a lack moving from tail to head. For $Q' > 0$, the phase advances in the first and lags in the second step; vice versa for $Q' < 0$ or $\gamma < \gamma_T$.

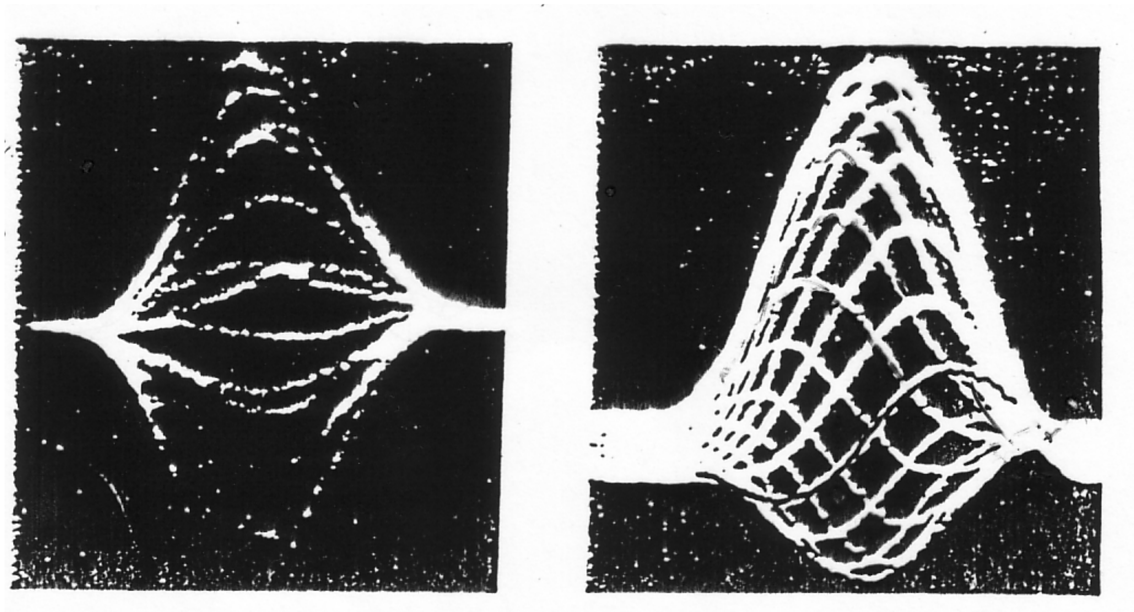
The figure shows the betatron motion in steps of its period $T_\beta = T_0/q$.



Observation of the head-tail mode in the CERN Booster

The head-tail mode oscillation of relatively long bunches can be observed directly with a fast position monitor. The figure shows such a measurement of the head-tail mode at vanishing and finite chromaticity taken by J. Gareyte and F. Sacherer in the CERN Booster.

It shows several traces each corresponding to a turn of the oscillating bunch passing through the transverse position monitor which gives a signal proportional the instantaneous dipole moment $x(t)I(t)$.



$Q' = 0$ $\quad$ $Q' > 0$

Head-tail mode $m = 0$

Head-tail instability

A broad band impedance is excited by oscillating particles A at the bunch head which in turn excite particles B at the tail with a phase shifted by $\Delta\phi$ compared to the head. Half a synchrotron oscillation later particles B are at the head and while particles A are at the tail oscillating with phase $-\Delta\phi$ compared to B (assuming $Q' = 0$). The excitation by the head has the wrong phase to keep oscillation growing unless $Q' \neq 0$ producing a phase shift during a motion from head to tail or vice versa.

The wake field excited by the head affects the tail later which will oscillate with a phase lag. To keep the oscillation growing the head particle must undergo a relative phase delay while moving to the tail and the tail particle a relative phase advance moving to the head. We expect an instability if $Q' < 0$ for $\gamma > \gamma_T$ or if $Q' > 0$ for $\gamma < \gamma_T$.

The 'wiggle' of the head-tail motion is seen by the impedance as an oscillation with chromatic frequency ω_ξ.

$$\frac{\Delta p}{p} = \frac{\hat{\Delta p}}{p}\sin(\omega_s t) \;\; , \;\; \tau = -\hat{\tau}\cos(\omega_s t) \; , \;\; \hat{\tau} = \frac{\omega_s}{\eta_c}\frac{\Delta\hat{p}}{p}$$

The relative betatron phase shift of a particle while executing part of synchrotron oscillation is

$$\Delta\phi_\beta = \omega_0 \int_{t_1}^{t_2} \Delta Q dt = \omega_0 Q' \frac{\Delta\hat{p}}{p} \int_{t_1}^{t_2} \sin(\omega_s t) dt$$

$$= -\omega_0 Q' \frac{\Delta\hat{p}}{p} \left(\cos(\omega_s t_2) - \cos(\omega_s t_2)\right) = \frac{\omega_0 Q'}{\eta_c}(\tau_2 - \tau_1)$$

giving the chromatic frequency

$$\omega_\xi = \frac{\Delta\phi_\beta}{\Delta\tau} = \frac{\omega_0 Q'}{\eta_c}.$$

Model of head-tail instability

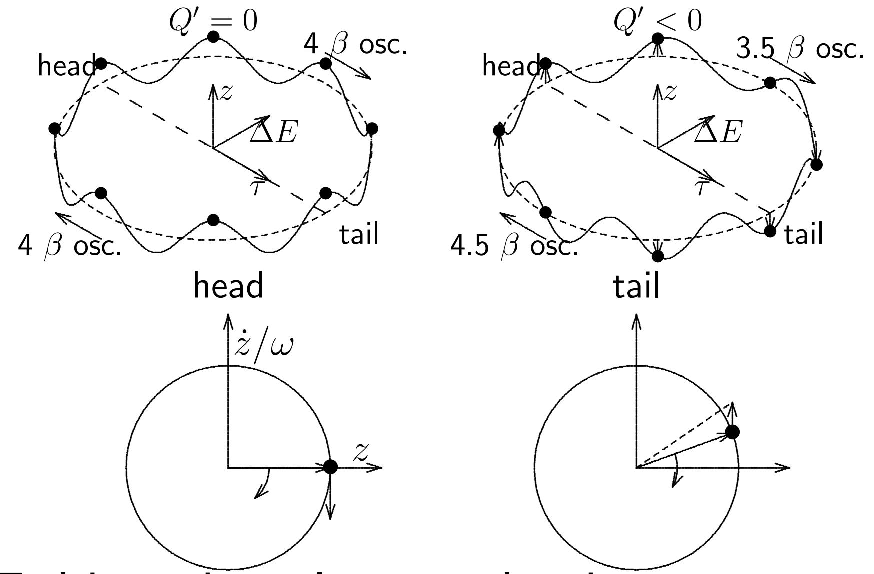
Above transition energy:

$Q' = 0$: Going from head to tail or from tail to head has same phase change. Phase lag and advance between head an tail interchange, neither damping nor growth.

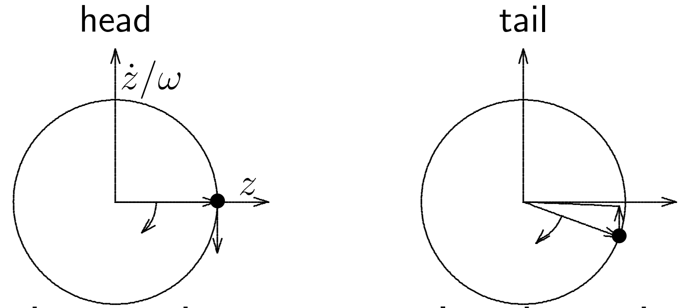
$Q' < 0$: Going from head to tail there is a loss in phase, going from tail to head a gain (picture), giving a systematic phase advance between head and tail and in average growth.

$Q' > 0$: Going from head to tail there is a gain in phase, going from tail to head a loss, giving a systematic phase lag between head and tail and in average damping.

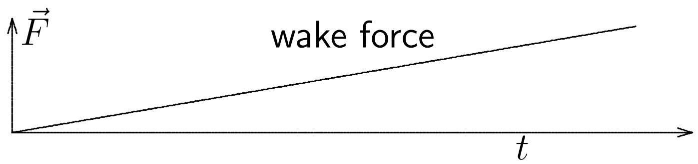
Below transition this situation is reversed.



Tail has phase lag, amplitude is increased



Tail has phase advance, amplitude is decreased

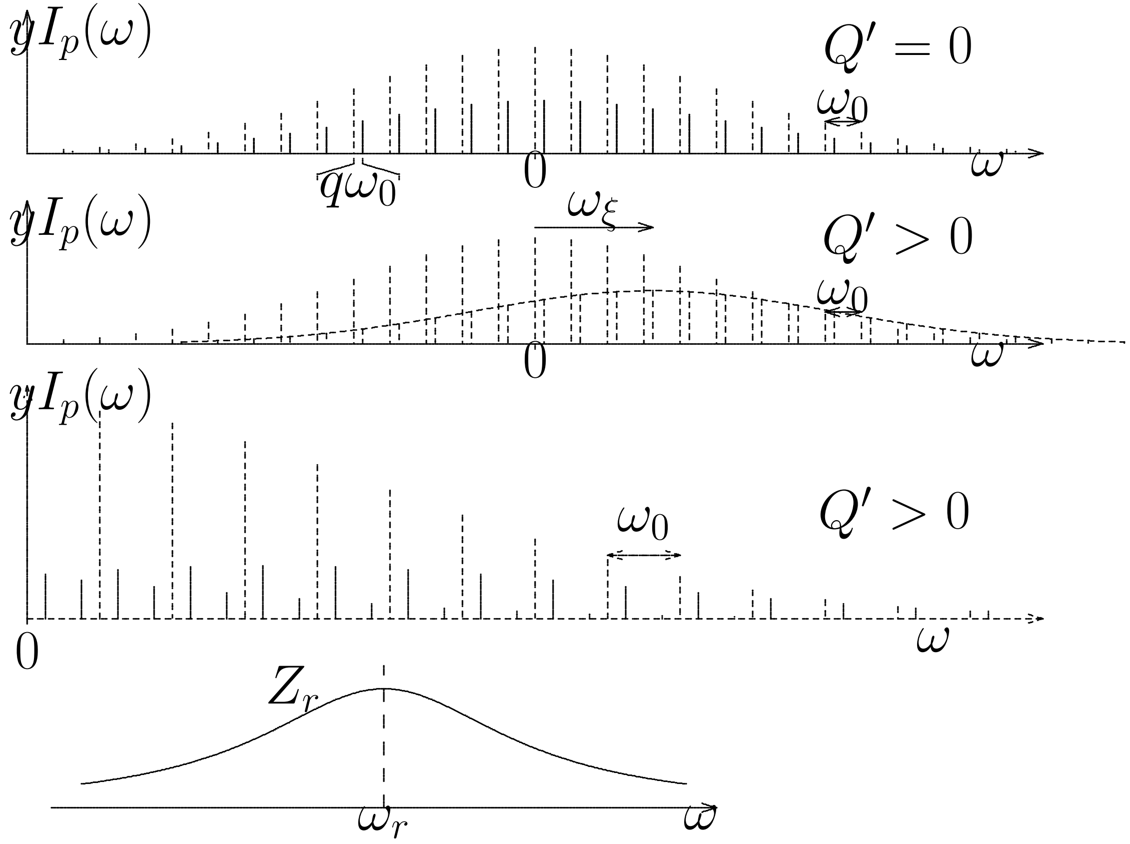


'Wiggle' of head-tail mode shifts envelope of sidebands by chromatic frequency ω_ξ which can be different for adjacent sidebands. Broad band impedance gives instability with growth $(a < 0)$, damping $(a > 0)$ rate

$$I_{p\xi\pm} = \frac{\omega_0}{\sqrt{2\pi}} \tilde{I}(\omega_{p\pm} + \omega_\xi)$$

$$\omega_{p\pm} = \omega_0 \left(pM \pm (n+q)\right)$$

$$a = \frac{e\omega_0\beta_x}{4\pi m_0 c^2 \gamma I_0} \sum_{\omega>0} \left(I_{p\xi+}^2 Z_{Tr}(\omega_p^+) - I_{p\xi-}^2 Z_{Tr}(\omega_p^-)\right).$$

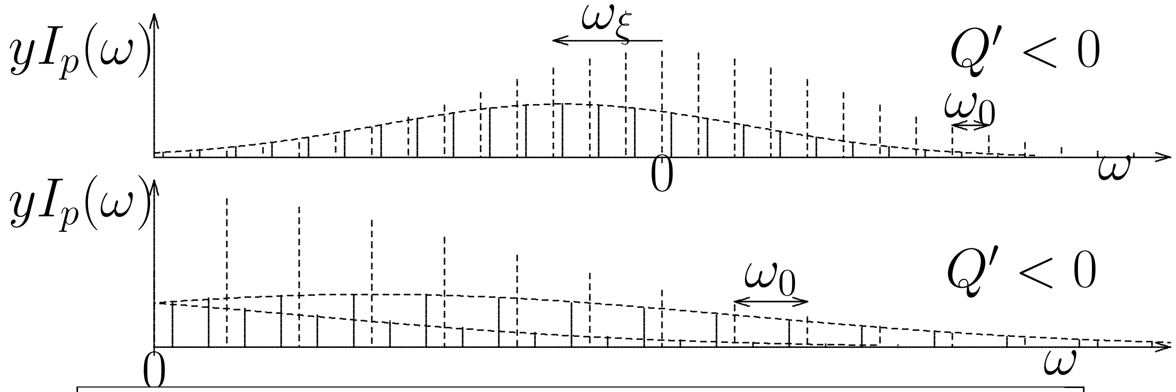


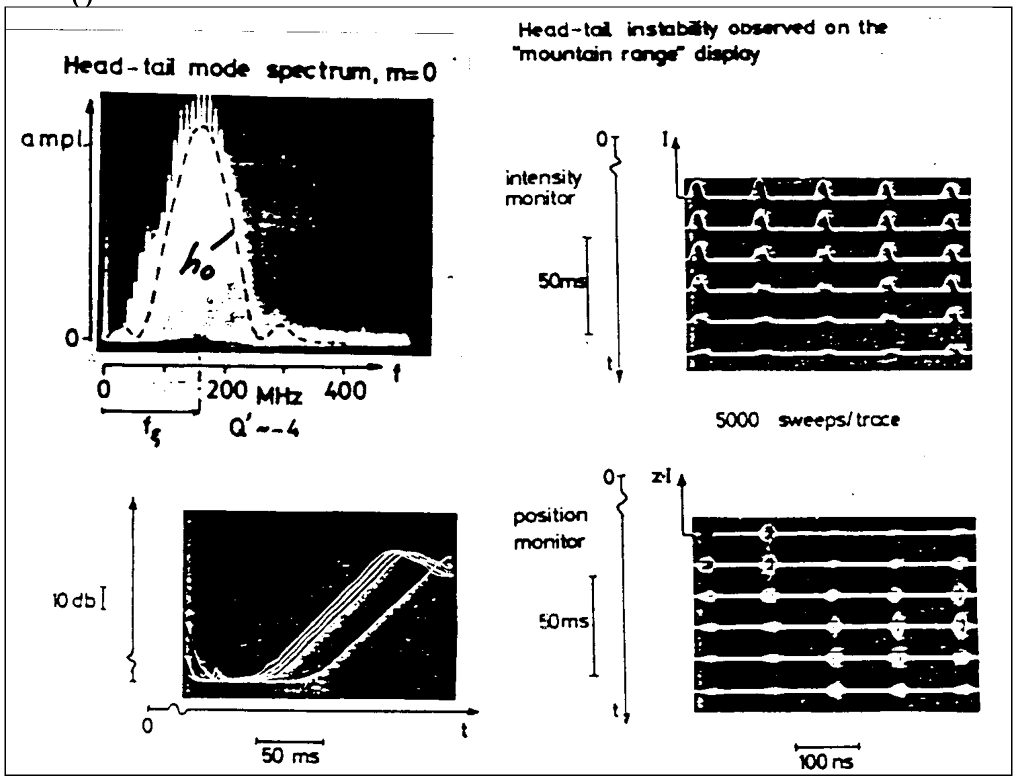
Head-tail instability seen in the CERN ISR

We inject proton bunches with $Q' < 0$ and observe with intensity (u.r.) and position (l.r.) monitors. The first shows decaying bunches, the second growing head-tail oscillations. The filtered betatron signal (l.l.) increases linearly on a logarithmic scale indicating exponential growth up to beam loss. A transverse spectrum 'snap shot' (u.l) during growth shows the envelope of betatron lines shifted by ω_ξ. The current components and frequencies are

$$I_{p\xi\pm} = \frac{\omega_0}{\sqrt{2\pi}} \tilde{I}(\omega_{p\pm} + \omega_\xi)$$

$$\omega_{p\pm} = \omega_0\,(pM \pm (n+q))\,,\ \frac{\omega_\xi}{\omega_0} = \frac{Q'}{\eta_c}$$



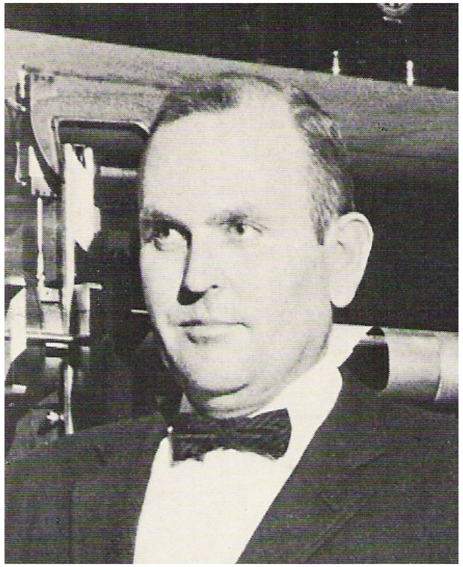


5) SUMMARY

The instability treatment used here was invented by K. Robinson. This was gerneralized to nearly all longitudinal and transverse bunched beam instabilities by Frank Sacherer.

This demands for resistive impedance at upper, Z^+, and lower, Z^-, side-band to fulfill a **stability condition**

	above transition	below transition
longitudinal, stability	$Z_r^+ < Z_r^-$	$Z_r^+ > Z_r^-$
transverse $Q' = 0$, stability	$Z_{Tr}^+ > Z_{Tr}^-$	$Z_{Tr}^+ > Z_{Tr}^-$
transverse head-tail, stability	$Q' < 0$	$Q' > 0$


Ken Robinson


Frank Sacherer