LONGITUDINAL INSTABILITIES

CAS 2007, Daresbury; A. Hofmann

1) Introduction
2) Impedances and wake functions
3) Longitudinal dynamics
4) Robinson instability
5) Potential well bunch lengthening

1) INTRODUCTION

Overview

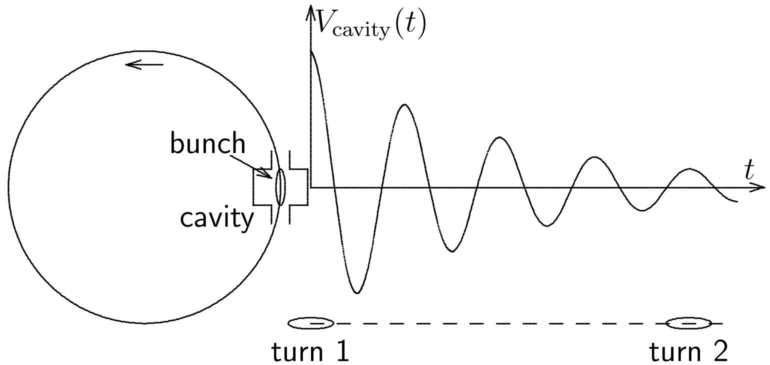
The single particle motion is given by external guide fields (dipoles, quadrupoles, RF), initial conditions and synchrotron radiation.

Beam with many particles induces currents in vacuum chamber **impedance** and creates **self fields** acting back on it. This **collective** action by many particles can: give **synchrotron frequency shift** due to modified focusing; increase initial disturbance, **instability**; **change particle distribution**, (bunch lengthening).

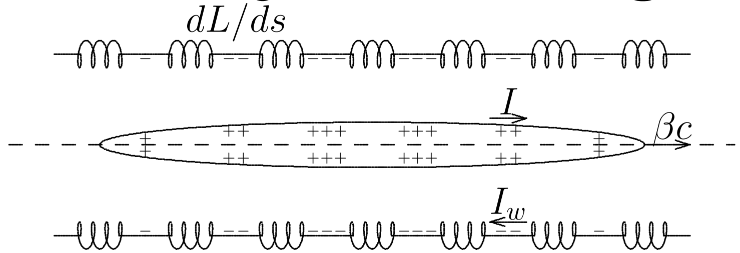
Multi-turn effects driven by narrow-band cavity with memory build up instability in many turns with small self-fields treated as **perturbation**.

Start a small disturbance from a stationary beam, calculate fields it produces in impedance, check if they increase/decrease the initial amplitude, give growth/damping rate. Check this for orthogonal (independent) modes of disturbances.

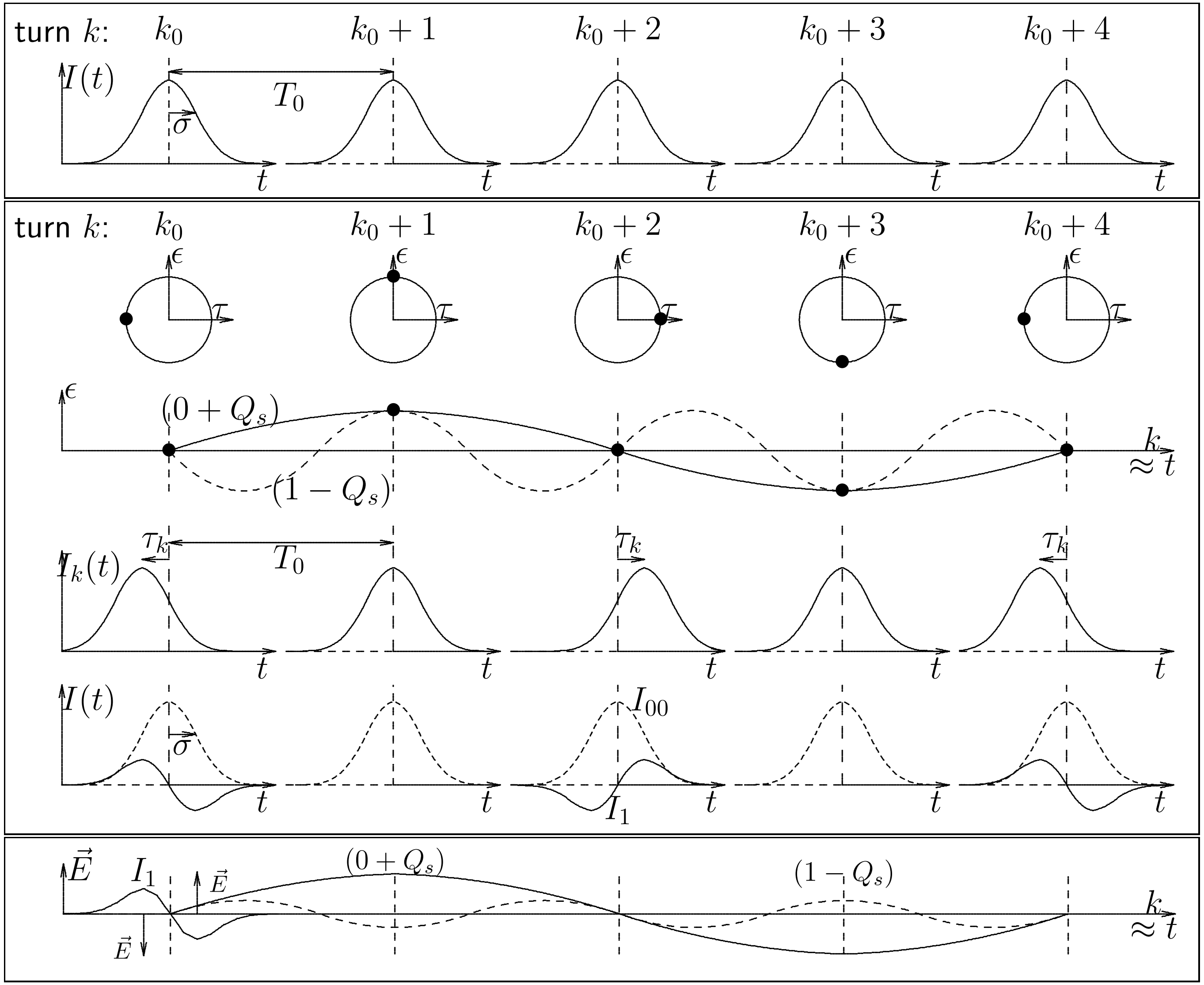
Bunch induces fields in passive cavity, they oscillate and act back next turn, in- or decrease initial disturbance depending on phase.



Single traversal effects driven by strong self-fields from broad impedances change distribution, modify oscillation modes and can couple them. Self consistent solutions are difficult to get, **bunch lengthening**.



Mechanism of single bunch, multi-turn instability



stationary bunch

$I_k = I_0 + 2\sum I_p \cos(p\omega_0 t)$

$I_p = \frac{1}{T_0}\int_0^{T_0} I(t)\cos(p\omega_0 t)dt$

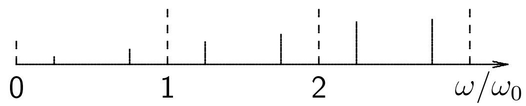
oscillating bunch, $Q_s = \frac{1}{4}$

$\epsilon_k = \hat{\epsilon}\sin(2\pi Q_s k)$ phase-

$\tau_k = \hat{\tau}\cos(2\pi Q_s k)$ space

$$I = I_0 + \sum I_p \left[\cos(p\omega_0 t) + \frac{p\omega_0\hat{\tau}}{2}\left(\sin(\omega_p^+ t) + \sin(\omega_p^- t)\right)\right]$$

$\omega_p^\pm = (p \pm Q_s)\omega_0.$



field $\vec{E}$ induced in $Z_r(\omega_p^\pm)$ acts on energy deviation ϵ

4) IMPEDANCE, WAKE FUNCTION

Resonator

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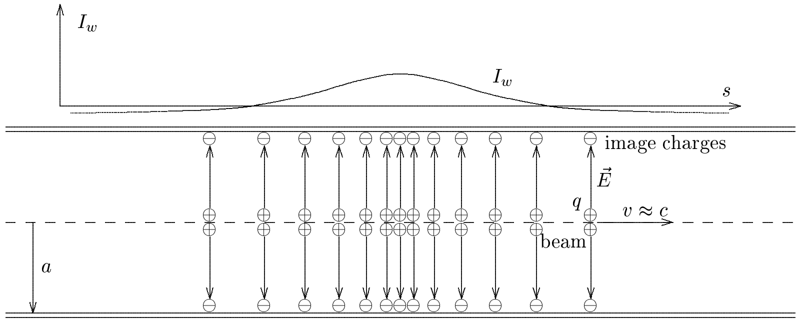
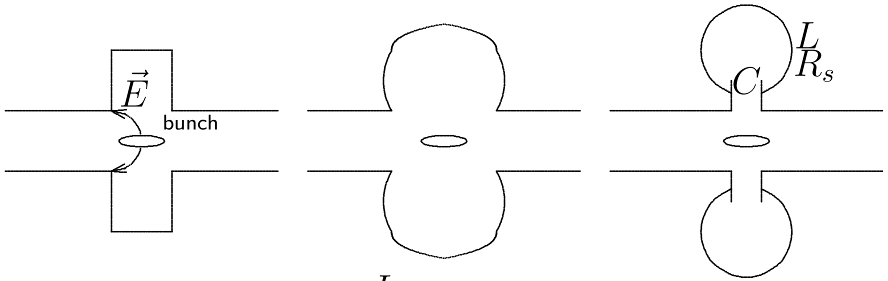
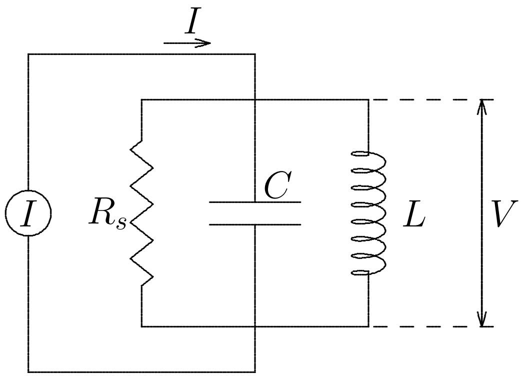


Beam induces wall current $I_w = -(I_b - \langle I_b \rangle)$



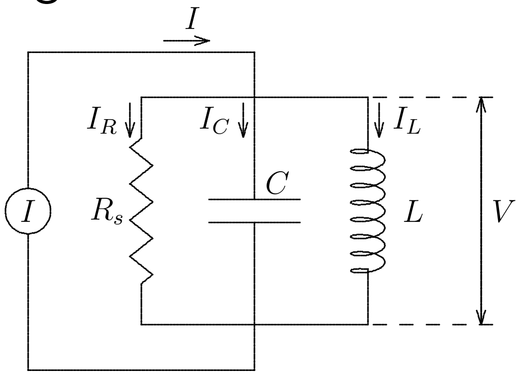


Cavities have narrow band oscillation modes which can drive coupled bunch instabilities. Each resembles an **RCL - circuit** and can, in good approximation, be treated as such. This circuit has a shunt impedance R_s, an inductance L and a capacity C. In a real cavity these parameters cannot easily be separated and we use others which can be measured directly: The **resonance frequency** ω_r, the **quality factor** Q and the **damping rate** α:

$$\omega_r = \frac{1}{\sqrt{LC}}\,, \quad Q = R_s\sqrt{\frac{C}{L}} = \frac{R_s}{L\omega_r} = R_s C \omega_r$$

$$\alpha = \frac{\omega_r}{2Q}\,, \quad L = \frac{R_s}{Q\omega_r}\,, \quad C = \frac{Q}{\omega_r R_s}.$$

Driving this circuit with a current I gives the voltages and currents across the elements



$$\begin{aligned} V_R &= I_R R_s \\ V_C &= \frac{1}{C}\int I_C dt \\ V_L &= L\frac{dI_L}{dt} \end{aligned}$$

$$V_R = V_C = V_L = V$$

$$I_R + I_C + I_L = I$$

Differentiating with respect to t gives

$$\dot{I} = \dot{I}_R + \dot{I}_C + \dot{I}_L = \frac{\dot{V}}{R_s} + C\ddot{V} + \frac{V}{L}.$$

Using $L = R_s/(\omega_r Q)$, $C = Q/(\omega_r R_s)$ and $\alpha = \omega_r/(2Q)$, $\omega_r = 1/\sqrt{LC}$ gives diff. eq.

$$\ddot{V} + \frac{\omega_r}{Q}\dot{V} + \omega_r^2 V = \frac{\omega_r R_s}{Q}\dot{I}$$

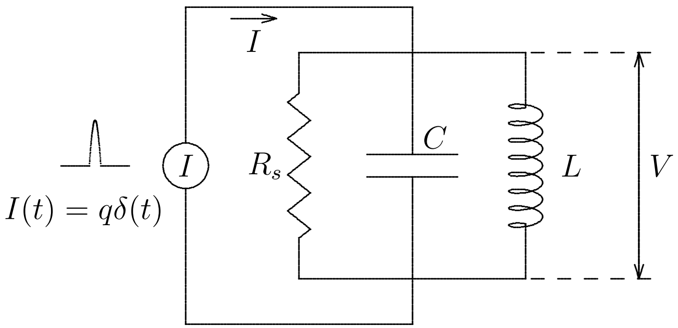
The solution of the homogeneous equation represents a damped oscillation

$$V(t) = \hat{V}\mathrm{e}^{-\alpha t}\cos\left(\omega_r\sqrt{1-\frac{1}{4Q^2}}t+\phi\right)$$

$$\begin{aligned} V(t) = \mathrm{e}^{-\alpha t}\Bigg(&\mathrm{A}\cos\left(\omega_r\sqrt{1-\frac{1}{4Q^2}}t\right) \\ &+B\sin\left(\omega_r\sqrt{1-\frac{1}{4Q^2}}t\right)\Bigg) \end{aligned}$$

Wake function – Green function

Response of RCL circuit to a delta pulse



$$\omega_r = \frac{1}{\sqrt{LC}}$$
$$Q = R_s\sqrt{\frac{C}{L}}$$
$$\alpha = \frac{\omega_r}{2Q}$$

$$\ddot{V} + \frac{\omega_r}{Q}\dot{V} + \omega_r^2 V = \frac{\omega_r R_s}{Q}\dot{I}$$

Charge q brings the capacity to a voltage

$$V(0^+) = \frac{q}{C} = \frac{\omega_r R_s}{Q} q \text{ using } C = \frac{Q}{\omega_r R_s}$$

Energy stored in C = energy lost by q

$$U = \frac{q^2}{2C} = \frac{\omega_r R_s}{2Q} q^2 = \frac{V(0^+)}{2} q = k_{pm} q^2$$

with the **parasitic mode loss factor** $k_{pm} = \omega_r R_s/(2Q)$, given usually in [V/pC].

Capacitor discharges first through resistor

$$\dot{V}(0^+) = -\frac{\dot{q}}{C} = -\frac{I_R}{C} = -\frac{1}{C}\frac{V(0^+)}{R_s}$$
$$= -\frac{\omega_r^2 R_s}{Q^2} q = -\frac{2\omega_r k_{pm}}{Q} q.$$

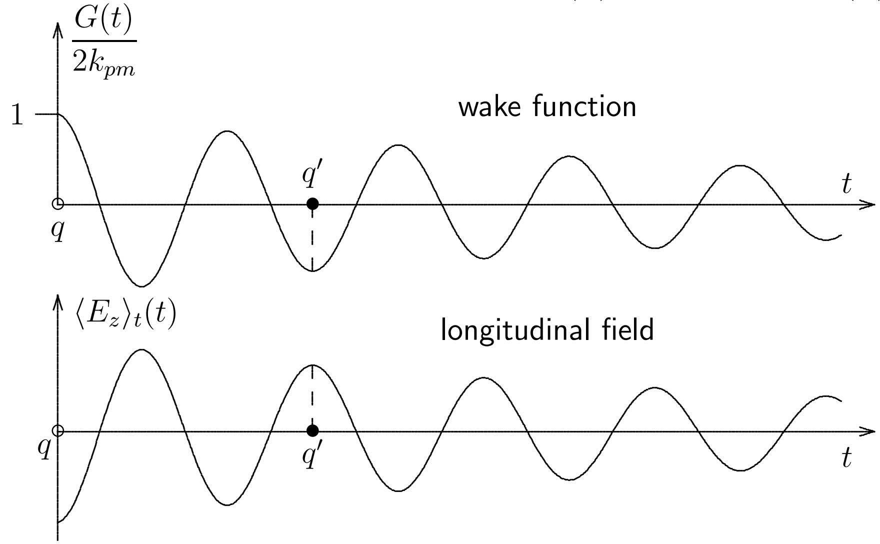
Initial conditions $V(0^+)$, $\dot{V}(0^+)$ give from

$$\text{general solution } V(t) = \mathrm{e}^{-\alpha \mathrm{t}}\left(\mathrm{A}\cos\left(\omega_\mathrm{r}\sqrt{1-\frac{1}{4\mathrm{Q}^2}}\,\mathrm{t}\right) + \mathrm{B}\sin\left(\omega_\mathrm{r}\sqrt{1-\frac{1}{4\mathrm{Q}^2}}\,\mathrm{t}\right)\right)$$

$$\text{pulse response } V(t) = 2qk_{pm}\mathrm{e}^{-\alpha \mathrm{t}}\left(\cos\left(\omega_\mathrm{r}\sqrt{1-\frac{1}{4\mathrm{Q}^2}}\,\mathrm{t}\right) - \frac{\sin\left(\omega_\mathrm{r}\sqrt{1-\frac{1}{4\mathrm{Q}^2}}\,\mathrm{t}\right)}{2\mathrm{Q}\sqrt{1-\frac{1}{4\mathrm{Q}^2}}}\right)$$

$$G(t)=\frac{V(t)}{q}=2k_{pm}\mathrm{e}^{-\alpha \mathrm{t}}\left(\cos\left(\omega_{\mathrm{r}}\sqrt{1-\frac{1}{4\mathrm{Q}^2}}\,\mathrm{t}\right)-\frac{\sin\left(\omega_{\mathrm{r}}\sqrt{1-\frac{1}{4\mathrm{Q}^2}}\,\mathrm{t}\right)}{2\mathrm{Q}\sqrt{1-\frac{1}{4\mathrm{Q}^2}}}\right),\ \omega_{\mathrm{r}}=\frac{1}{\sqrt{\mathrm{LC}}}$$

$G(t)$ is called **Green or wake function**.
$G(t)\approx 2k_{pm}\mathrm{e}^{-\alpha \mathrm{t}}\cos(\omega_{\mathrm{r}}\mathrm{t})$ for $\mathrm{Q}\gg 1$
This voltage induced by charge q at $t=0$ changes energy of a second charge q' traversing cavity at t by $U=-q'V(t)=-qq'G(t)$.



$G(t)$ is related to longitudinal field E_z by an integration following the particle with $v\approx c$ and taking momentary field value

$$V=Gq=-\int_{z_1}^{z_2}E_z(z,t)dz=-f_t\int_{z_1}^{z_2}E_z(z)dz.$$

with "transit time factor" f_t. We use $G(t)>0$ where energy is lost.
A particle inside a bunch of charge q and current $I(t)$ going through a cavity at time t sees the wake function created by all the particles passing at earlier times $t'<t$ resulting in a voltage

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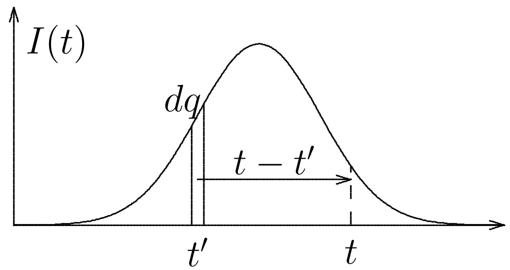
$$V(t)=\int_{-\infty}^{t}G(t')dq=\int_{-\infty}^{t}I(t')G(t')dt'=qW(t)$$

$W(t)=V(t)/q$ wake potential .



Impedance

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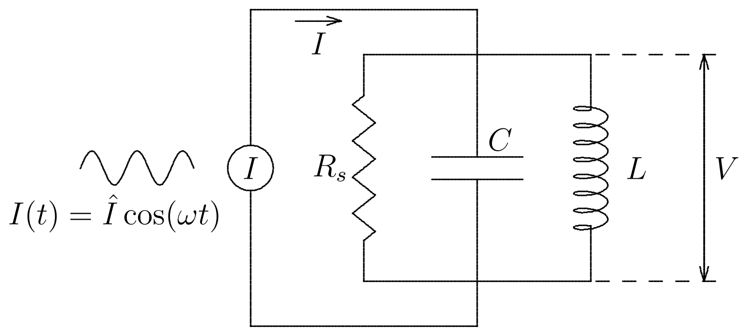


A **harmonic** excitation of circuit with current $I = \hat{I}\cos(\omega t)$ gives differential equation

$$\ddot{V} + \frac{\omega_r}{Q}\dot{V} + \omega_r^2 V = \frac{\omega_r R_s}{Q}\dot{I} = -\frac{\omega_r R_s}{Q}\hat{I}\omega\sin(\omega t).$$

Homogeneous solution damps leaving particular one $V(t) = A\cos(\omega t) + B\sin(\omega t)$. Put into diff-equation, separating cosine and sine

$$-(\omega^2 - \omega_r^2)A + \frac{\omega_r\omega}{Q}B = 0$$

$$(\omega^2 - \omega_r^2)B + \frac{\omega_r\omega}{Q}A = \frac{\omega_r\omega R_s}{Q}\hat{I}.$$

Induced voltage by the harmonic excitation

$$V(t) = \hat{I}R_s \frac{\cos(\omega t) + Q\frac{\omega^2-\omega_r^2}{\omega_r\omega}\sin(\omega t)}{1 + Q^2\left(\frac{\omega^2-\omega_r^2}{\omega_r\omega}\right)^2}$$

has a cosine term **in phase** with exciting current. It absorbs energy, is **resistive**. The sine term is **out of phase**, does not absorb energy, **reactive**. Ratio between voltage and current is **impedance** as **function of frequency** ω

$$Z_r(\omega) = R_s \frac{1}{1 + Q^2\left(\frac{\omega_r^2-\omega^2}{\omega_r\omega}\right)^2}$$

$$Z_i(\omega) = -R_s \frac{Q\frac{\omega^2-\omega_r^2}{\omega_r\omega}}{1 + Q^2\left(\frac{\omega^2-\omega_r^2}{\omega_r\omega}\right)^2}.$$

Resistive part $Z_r(\omega) \geq 0$, reactive part $Z_i(\omega)$ positive below, negative above ω_r.

Complex notation

We used a harmonic excitation of the form

$$I(t) = \hat{I}\cos(\omega t) = \hat{I}\frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

with $0 \leq \omega \leq \infty$.

It is convenient to use a complex notation

$$I(t) = \hat{I}e^{j\omega t} \quad \text{with} \quad -\infty \leq \omega \leq \infty$$

giving compact expressions. Using the differential equation

$$\ddot{V} + \frac{\omega_r}{Q}\dot{V} + \omega_r^2 V = \frac{\omega_r R_s}{Q}\dot{I}$$

with $I(t) = \hat{I}\exp(j\omega t)$ and seeking a solution $V(t) = V_0\exp(j\omega t)$, where V_0 is in general complex, one gets

$$\left(-\omega^2 + j\frac{\omega_r\omega}{Q} + \omega_r^2\right)V_0 e^{j\omega t} = j\frac{\omega_r\omega R_s}{Q}\hat{I}e^{j\omega t}.$$

The impedance, defined as the ratio V/I becomes

$$Z(\omega) = \frac{V_0}{\hat{I}} = \frac{R_s}{1 + jQ\left(\frac{\omega}{\omega_r} - \frac{\omega_r}{\omega}\right)}$$

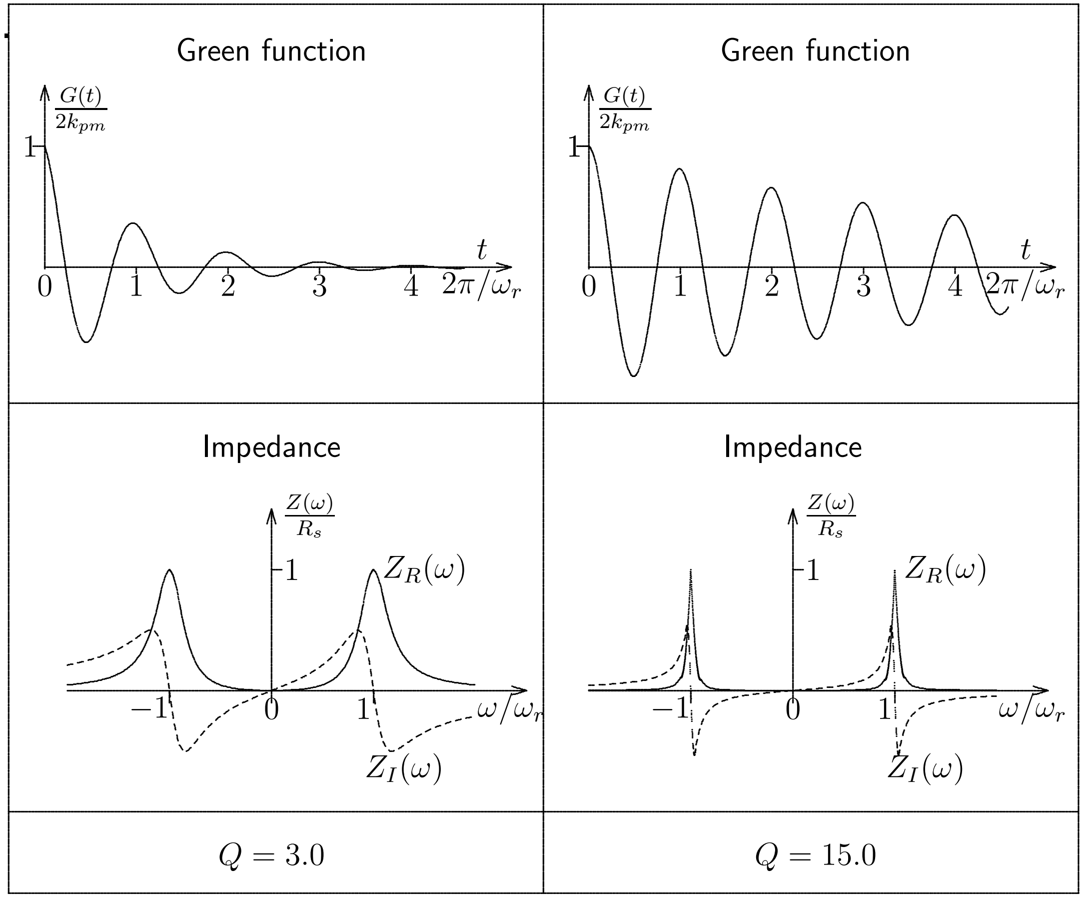
$$= R_s\frac{1 - jQ\frac{\omega^2-\omega_r^2}{\omega\omega_r}}{1 + Q^2\left(\frac{\omega^2-\omega_r^2}{\omega\omega_r}\right)^2} = Z_r + jZ_i$$

For $Q \gg 1$ the impedance is only large for $\omega \approx \omega_r$ or $|\omega - \omega_r|/\omega_r = |\Delta\omega|/\omega_r \ll 1$ and can be simplified

$$Z(\omega) \approx R_s\frac{1 - j2Q\frac{\Delta\omega}{\omega_r}}{1 + 4Q^2\left(\frac{\Delta\omega}{\omega_r}\right)^2}.$$

Caution: sometimes $I(t) = \hat{I}e^{-i\omega t}$ instead of $I(t) = \hat{I}e^{j\omega t}$ is used, this reverses the sign $Z_i(\omega)$.

Properties of Green functions and impedances



$$Z(\omega) = R_s \frac{1 - jQ\frac{\omega^2-\omega_r^2}{\omega\omega_r}}{1 + Q^2\left(\frac{\omega^2-\omega_r^2}{\omega\omega_r}\right)^2} = Z_r + jZ_i$$

The resonator impedance has some specific properties:

$$\text{at}\,\omega = \omega_r \;\rightarrow\; Z_r(\omega_r)\ \text{max.},\ \ Z_i(\omega_r) = 0$$

$$0 < \omega < \omega_r \;\rightarrow\; Z_i(\omega) > 0\ \text{(inductive)}$$

$$\omega > \omega_r \;\rightarrow\; Z_i(\omega) < 0\ \text{(capacitive)}$$

and any impedance or wake potential has the general properties

$$Z_r(\omega) = Z_r(-\omega)\ \ ,\ \ Z_i(\omega) = -Z_i(-\omega)$$

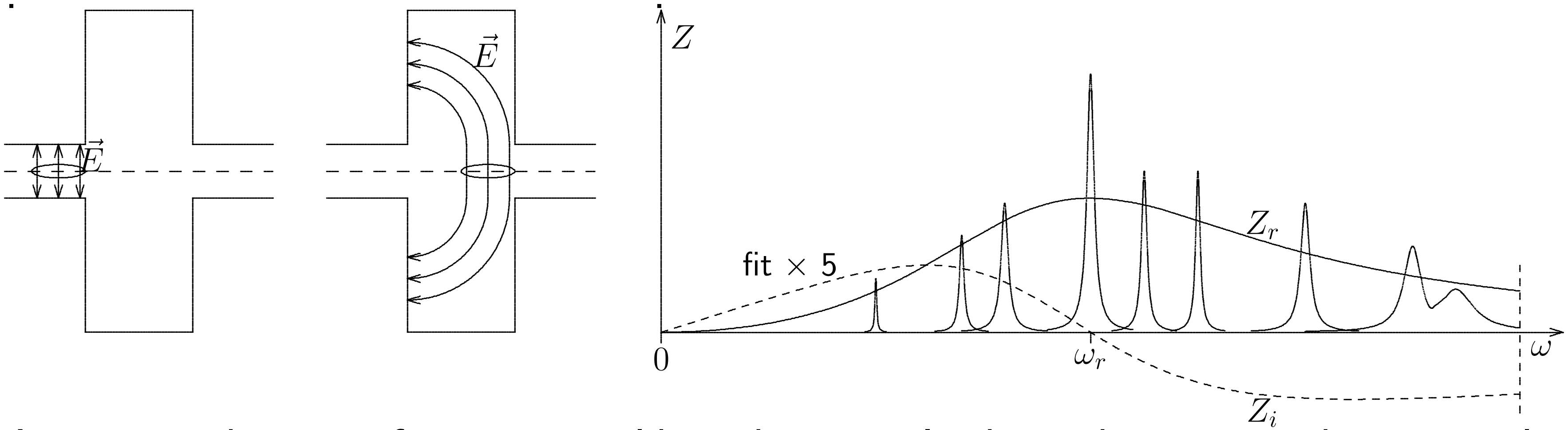
$$Z(\omega) = \int_{-\infty}^{\infty} G(t) \mathrm{e}^{-j\omega t} dt$$

$Z(\omega) \propto$ Fourier transform of $G(t)$

$$\text{for}\, t < 0 \;\rightarrow\; G(t) = 0,$$

no fields before particle arrives, $\beta \approx 1$.

Typical ring impedance



Aperture changes form cavity-like objects with $\omega_r,\ R_s$ and Q and impedance $Z(\omega)$ developed for $\omega < \omega_r$, where it is inductive

$$Z(\omega) = R_s \frac{1 - jQ\frac{\omega^2-\omega_r^2}{\omega\omega_r}}{1+\left(Q\frac{\omega^2-\omega_r^2}{\omega\omega_r}\right)^2} \approx j\frac{R_s\omega}{Q\omega_r} + ...$$

Sum impedance at $\omega \ll \omega_{rk}$ divided by mode number $n = \omega/\omega_0$ is with inductance L

$$\left|\frac{Z}{n}\right|_0 = \sum_k \frac{R_{sk}\omega_0}{Q_k\omega_{rk}} = L\omega_0 = L\frac{\beta c}{R}.$$

It depends on impedance per length, $\approx$ 15 Ω in older, 1 Ω in newer rings. The shunt impedances R_{sk} increase with ω up to cut-off frequency where wave propagation starts and become wider and smaller. A broad band resonator fit helps to characterize impedance giving $Z_r,\ Z_i,\ G(t)$ useful for single traversal effects. However, for multi-traversal instabilities narrow resonances at ω_{rk} must be used.

5) LONGITUDINAL DYNAMICS

A particle with momentum deviation Δp has different orbit length L, revolution time T_0 and frequency ω_0

$$\frac{\Delta L}{L} = \alpha_c \frac{\Delta p}{p} = \frac{\alpha_c}{\beta^2}\frac{\Delta E}{E}$$

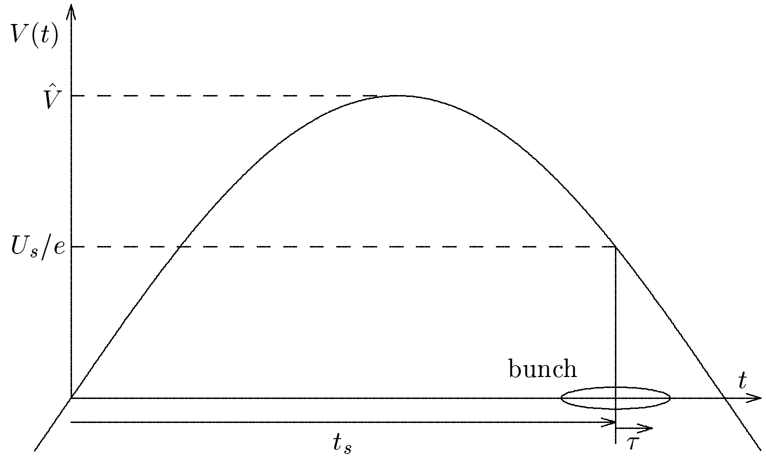
$$\frac{\Delta T}{T} = -\frac{\Delta \omega_0}{\omega_0} = \left(\alpha_c - \frac{1}{\gamma^2}\right)\frac{\Delta p}{p} = \eta_c \frac{\Delta p}{p}$$

with momentum compaction $\alpha_c = 1/\gamma_T^2$, slip factor η_c. At transition energy $m_0 c^2 \gamma_T$ the ω_0-dependence on Δp changes sign

$$E > E_T \rightarrow \frac{1}{\gamma^2} < \alpha_c \rightarrow \eta_c > 1, \ \frac{\Delta \omega_0}{\Delta E} < 0$$

$$E < E_T \rightarrow \frac{1}{\gamma^2} > \alpha_c \rightarrow \eta_c < 1, \ \frac{\Delta \omega_0}{\Delta E} > 0\,.$$

For $\gamma \gg 1 \rightarrow \Delta p/p \approx \Delta E/E = \epsilon$, $\eta_c \approx \alpha_c$.



RF-cavity of voltage $\hat{V}$, frequency $\omega_{\text{RF}} = h\omega_0$, SR energy loss U the energy gain or loss of a particle in one turn $\delta\epsilon = \delta E/E$ is

$$\delta E = e\hat{V}\sin(h\omega_0(t_s + \tau)) - U$$

t_s= synchronous arrival time at the cavity, τ= deviation from it, synchronous phase $\phi_s = h\omega_0 t_s$. For $h\omega_0\tau \ll 1$ we develop

$$\delta E = e\hat{V}\sin(\phi_s) + h\omega_0 e\hat{V}\cos\phi_s \tau - U.$$

For $\delta E/E \ll 1$ use smooth approximation $\dot{E} \approx \delta E/T_0$, $\dot{\tau} = \Delta T/T_0 = \eta_c \Delta E/E$

$$\dot{E} = \frac{\omega_0 e\hat{V}\sin\phi_s}{2\pi} + \frac{\omega_0^2 he\hat{V}\cos\phi_s}{2\pi}\tau - \frac{\omega_0}{2\pi}U.$$

Use $T_0 = 2\pi/\omega_0$, relative energy $\epsilon = \Delta E/E$

$$\dot{\epsilon} = \frac{\omega_0 e\hat{V}\sin\phi_s}{2\pi E} + \frac{\omega_0^2 he\hat{V}\cos\phi_s}{2\pi E}\tau - \frac{\omega_0}{2\pi}\frac{U}{E}.$$

Energy loss U may depend on ϵ and τ

$$U(\epsilon,\tau) \approx U_0 + \frac{\partial U}{\partial E}\Delta E + \frac{\partial U}{\partial t}\tau$$

giving for the derivative of the energy loss

$$\begin{aligned}\dot{\epsilon} &= \frac{\omega_0^2 he\hat{V}\cos\phi_s}{2\pi E}\tau - \frac{\omega_0}{2\pi}\frac{\partial U}{\partial E}\epsilon - \frac{\omega_0}{2\pi}\frac{\partial U}{\partial t}\tau \\ \dot{\tau} &= \eta_c \epsilon\end{aligned}$$

where we used that for synchronous particle $\epsilon = 0\,,\ \tau = 0$ we have $U_0 = e\hat{V}\sin\phi_s$

Combining these into a second order equation

$$\ddot{\tau} + \frac{\omega_0}{2\pi}\frac{\partial U}{\partial E}\dot{\tau} + \left(\omega_{s0}^2 + \frac{\omega_0\eta_c}{2\pi E}\frac{\partial U}{\partial t}\right)\tau = 0,$$

$$\omega_{s0}^2 = \frac{-\omega_0^2 h\eta_c e\hat{V}\cos\phi_s}{2\pi E}\ ,\ \alpha_s = \frac{1}{2}\frac{\omega_0}{2\pi}\frac{\partial U}{\partial E}$$

$$\omega_{s1}^2 = \omega_{s0}^2 - \alpha_s^2 + \frac{\omega_0\eta_c}{2\pi E}\frac{\partial U}{\partial t} \approx \omega_{s0}^2$$

$$\ddot{\tau} + 2\alpha_s\dot{\tau} + \omega_{s0}^2\tau = 0$$

$$\tau = \hat{\tau}e^{-\alpha_s t}\cos(\omega_{s1}t)\ ,\ \epsilon = \hat{\epsilon}e^{-\alpha_s t}\sin(\omega_{s1}t)$$

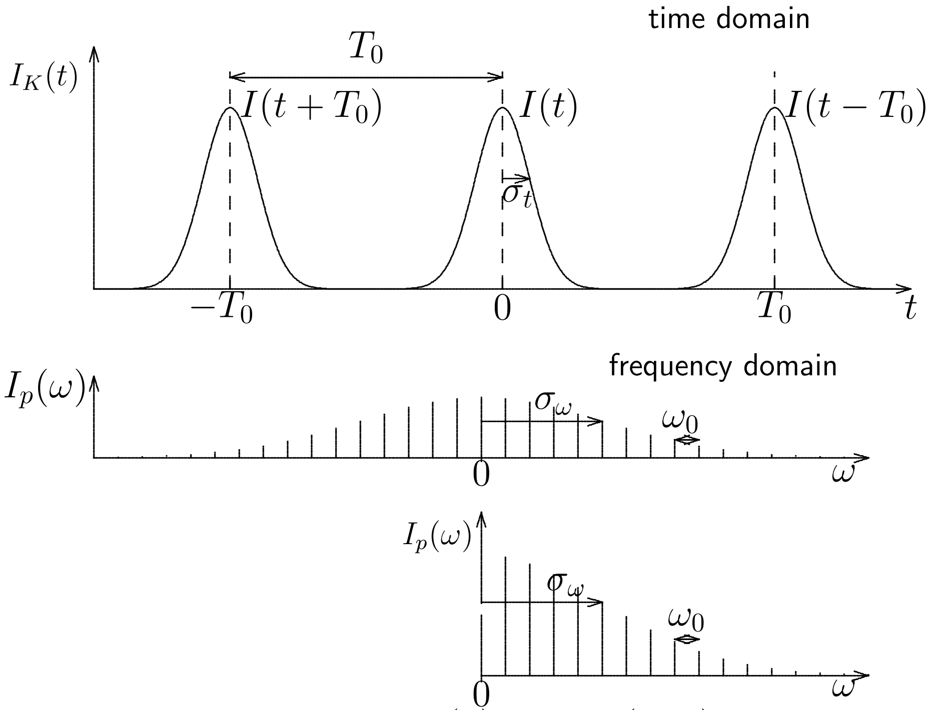
From $\dot{\tau} = \eta_c\epsilon$ we get $\hat{\epsilon} = \omega_{s0}\hat{\tau}/\eta_c$.
To get real ω_{s0} we need $\cos\phi_s \le 0$ above transition where $\eta_c > 0$ and vice versa.
To get a stable (decaying) solution we need an energy loss which increases with E

$$\alpha_s = \frac{\omega_0}{4\pi}\frac{\partial U}{\partial E} = \frac{\omega_0}{4\pi E}\frac{\partial U}{\partial \epsilon} > 0.$$

7) ROBINSON INSTABILITY

Stationary bunch

Spectrum



Symmetric bunch, $I(t) = I(-t)$, circulates with turns k of duration T_0, is a periodic current and expressed by a Fourier series

$$\tilde{I}(\omega) = \frac{1}{\sqrt{2\pi}} \int I(t)\cos(\omega t)dt, \ I(t) \qquad \text{single}$$

$$I_k(t) = \sum_{-\infty}^{\infty} I(t - kT_0) = \sum_{-\infty}^{\infty} I_p \mathrm{e}^{jp\omega_0 t} \quad \text{multi}$$

$$= I_0 + 2\sum_{p=1}^{\infty} I_p \cos(p\omega_0 t) \qquad \text{traversal}$$

$$I_p = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} I(t)\mathrm{e}^{jp\omega_0 t}dt$$

$$= \frac{1}{T_0}\int_0^{T_0} I(t)\cos(p\omega_0 t)dt = \frac{\omega_0}{\sqrt{2\pi}}\tilde{I}(p\omega_0).$$

With $I(t) = I(-t)$, real I_p, cosine terms only.
At low frequencies $I_p \approx I_0$
Gaussian bunch:

$$I(t) = \frac{q}{\sqrt{2\pi}\sigma_t}\mathrm{e}^{-\frac{t^2}{2\sigma_t^2}}, \ I_p = \frac{q}{T_0}e^{\frac{-p^2\omega_0^2}{2\sigma_\omega^2}}, \ \sigma_\omega = \frac{1}{\sigma_t}.$$

Voltage induced by a stationary bunch

Stationary bunch induces voltage in impedance $Z(\omega) = Z_r(\omega) + jZ_i(\omega)$

Use $Z(0) = 0$, combine positive/negative frequencies with $Z_r(-\omega) = Z_r(\omega)$, $Z_i(-\omega) = -Z_i(\omega)$

$$I_k(t) = \sum_{p'=-\infty}^{\infty} I_p \mathrm{e}^{jp'\omega_0 t} = I_0 + 2\sum_{p'=1}^{\infty} I_p \cos(p'\omega_0 t)$$

$$V_k(t) = \sum_{p=-\infty}^{\infty} Z(p\omega_0) I_p \mathrm{e}^{jp\omega_0 t} = 2\sum_{p=1}^{\infty} I_p \left[Z_r(p\omega_0)\cos(p\omega_0 t) - Z_i(p\omega_0)\sin(p\omega_0 t)\right]$$

Energy loss of a stationary bunch

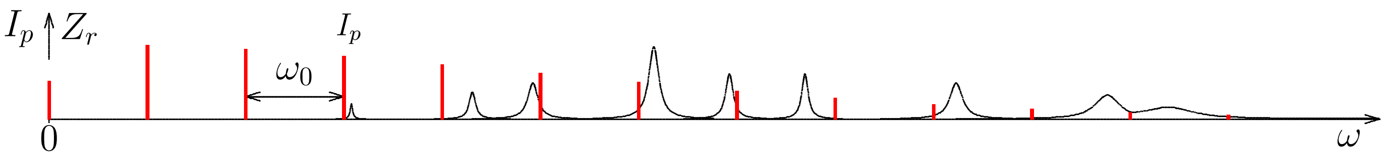
Energy lost by the whole bunch with N_b particles per turn in impedance $Z(\omega)$ is

$$W_b = \int_0^{T_0} I_k(t) V_k(t) dt$$

$$W_b = T_0 \sum_{p=-\infty}^{\infty} I_p^2 Z(p\omega_0) = 2T_0 \sum_1^{\infty} I_p^2 Z_r(p\omega_0)$$

has only Z_r. Loss $U = W_b/N_b$ per particle is

$$U = \frac{2T_0}{N_b} \sum_1^{\infty} I_p^2 Z_r(p\omega_0) = \frac{2e}{I_0} \sum_1^{\infty} I_p^2 Z_r(p\omega_0).$$



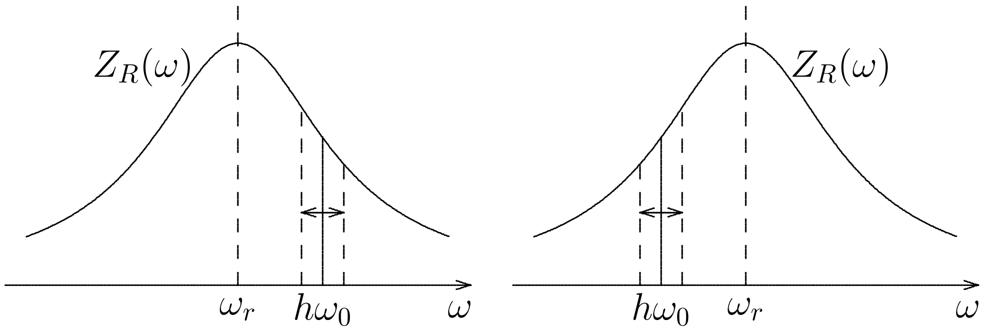
This contains integrals

$$\int_0^{T_0} \cos(p'\omega_0 t) \sin(p\omega_0 t) dt = 0.$$

$$\int_0^{T_0} \cos(p'\omega_0 t) \cos(p\omega_0 t) dt = \begin{cases} \frac{T_0}{2} & \text{for } p' = p \\ 0 & \text{for } p' \neq p \end{cases}$$

Robinson instability

Qualitative treatment

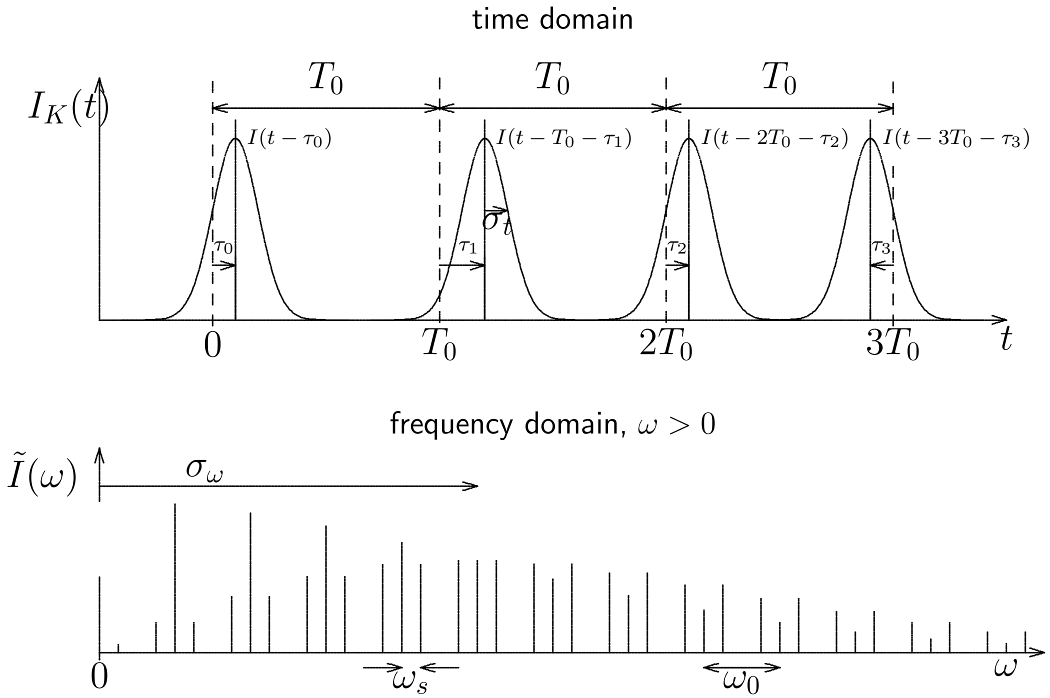


Important longitudinal instability of a bunch interacting with an narrow impedance, called **Robinson** instability. In a qualitative approach we take single bunch and a narrow-band cavity of resonance frequency ω_r and impedance $Z(\omega)$ taking only its resistive part Z_r. The revolution frequency ω_0 depends on energy deviation ΔE

$$\frac{\Delta\omega_0}{\omega_0} = -\eta_c \frac{\Delta E}{E}.$$

While the bunch is executing a coherent dipole mode oscillation $\epsilon(t) = \hat{\epsilon}\cos(\omega_s t)$ its energy and revolution frequency are modulated. **Above transition** ω_0 is **small** when the **energy is high** and ω_0 is **large** when the **energy is small**. If the cavity is tuned to a resonant frequency slightly smaller than the RF-frequency $\omega_r < p\omega_0$ the bunch sees a higher impedance and **loses more energy** when it has an **energy excess** and it **loses less energy** when it has a **lack of energy**. This leads to a **damping** of the oscillation. If $\omega_r > p\omega_0$ this is reversed and leads to an **instability**. Below transition energy the dependence of the revolution frequency is reversed which changes the stability criterion.

Oscillating bunch



Bunch executing synchrotron oscillation with $\omega_s = \omega_0 Q_s$ and amplitude $\hat{\tau}$ modulates passage time t_k at cavity in successive turns k

$$I_k(t) = \sum_{k=-\infty}^{\infty} I(t - kT_0 - \tau_k)$$

with $\tau_k = \hat{\tau}\cos(2\pi Q_s k) \approx \hat{\tau}\cos(\omega_s t)$

giving current without DC-part

$$I_k(t) = 2\sum_{\omega>0} I_p \cos(p\omega_0(t - \hat{\tau}\cos(\omega_s t))).$$

Develop for $p\omega_0\hat{\tau} \ll 1$

$\cos(p\omega_0\hat{\tau}) \approx 1$, $\sin(p\omega_0\hat{\tau}) \approx p\omega_0\hat{\tau}$

$$\begin{aligned} I_k(t) &\approx 2\sum_{\omega>0} I_p \left[\cos(p\omega_0 t) + p\omega_0\hat{\tau}\sin(p\omega_0 t)\cos(\omega_s t)\right] \\ &= 2\sum_{\omega>0} I_p \left[\cos(p\omega_0 t) + \frac{p\omega_0\hat{\tau}}{2}(\sin((p+Q_s)\omega_0 t) + \sin((p-Q_s)\omega_0 t))\right]. \end{aligned}$$

The modulation by the synchrotron oscillation results in sidebands in the spectrum. They are out of phase with respect to carriers and increase first with frequency $p\omega_0$.

Voltage induced by oscillating bunch

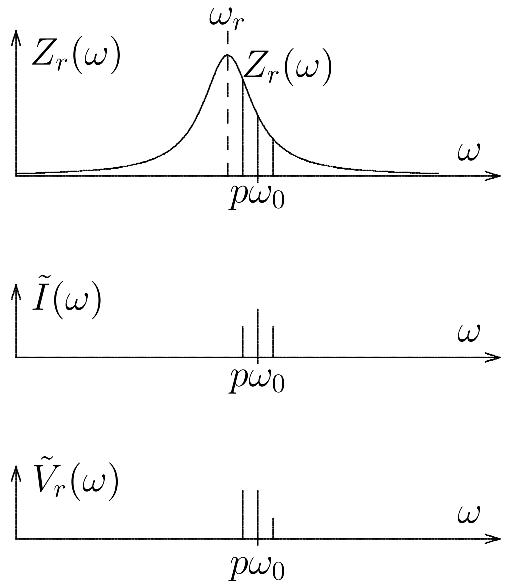
Abbreviate: $\omega_p^+ = (p+Q_s)\omega_0$, $\omega_p^- = (p-Q_s)\omega_0$

$$I_k(t) = 2\sum_{\omega>0} I_p \left[\cos(p\omega_0 t) + \frac{p\omega_0\hat{\tau}}{2}(\sin(\omega_p^+ t) + \sin(\omega_p^- t))\right].$$

We restrict on resistive impedance Z_r and get voltage

$$\begin{aligned} V_{kr}(t) &= 2\sum_{\omega>0}^{\infty} I_p \left[Z_r(p\omega_0)\cos(p\omega_0 t)\right. \\ &+ \left.\frac{p\omega_0\hat{\tau}}{2}\left(Z_r(\omega_p^+)\sin(\omega_p^+ t) + Z_r(\omega_p^-)\sin(\omega_p^- t)\right)\right] \end{aligned}$$

$$V_{kr}(t) = 2\sum_{\omega>0}^{\infty} I_p \left[Z_r(p\omega_0)\cos(p\omega_0 t)\right.$$
$$+\frac{p\omega_0\hat{\tau}}{2}\left[Z_r(\omega_p^+)\left(\sin(p\omega_0 t)\cos(\omega_s t) + \cos(p\omega_0 t)\sin(\omega_s t)\right)\right.$$
$$\left.\left. + Z_r(\omega_p^-)\left(\sin(p\omega_0 t)\cos(\omega_s t) - \cos(p\omega_0 t)\sin(\omega_s t)\right)\right]\right]$$



Synchr. motion, smoothed:

$$\tau_k = \hat{\tau}\cos(2\pi Q_s k) \;\rightarrow$$
$$\tau = \hat{\tau}\cos(\omega_s t)$$
$$\dot{\tau} = -\omega_s\hat{\tau}\sin(\omega_s t) = \eta_c\epsilon$$

$$\begin{aligned} V_{kr}(t) &= 2\sum_{\omega>0}^{\infty} I_p \left[Z_r(p\omega_0)\cos(p\omega_0 t)\right. \\ &+ \left.\frac{p\omega_0}{2}\left[Z_r(\omega_p^+)\left(\sin(p\omega_0 t)\tau - \cos(p\omega_0 t)\frac{\dot{\tau}}{\omega_s}\right) + Z_r(\omega_p^-)\left(\sin(p\omega_0 t)\tau + \cos(p\omega_0 t)\frac{\dot{\tau}}{\omega_s}\right)\right]\right] \end{aligned}$$

Energy exchange

Express factors differently, use $\dot\tau = \eta_c\epsilon$

$$I_K(t)=2\sum_{\omega>0} I_p\left[\cos(p'\omega_0 t) + p'\omega_0\sin(p'\omega_0 t)\tau\right]$$

$$V_{kr}(t)=2\sum_{\omega>0}^{\infty} I_p\left[Z_r(p\omega_0)\cos(p\omega_0 t) + \frac{p\omega_0}{2}\left[(Z_r(\omega_p^+)+Z_r(\omega_p^-))\sin(p\omega_0 t)\tau - (Z_r(\omega_p^+)-Z_r(\omega_p^-))\cos(p\omega_0 t)\frac{\eta\epsilon}{\omega_{s0}}\right]\right]$$

The energy per particle and turn exchanged between bunch and impedance

$$U(\tau,\dot\tau) = \frac{1}{N_b}\int_0^{T_0} I_K(t)V_K(t)dt,\ N_b = \frac{2\pi I_0}{e\omega_0}$$

Neglect higher terms in τ, ϵ, use integrals

$$\int_0^{T_0}\cos(p'\omega_0 t)\cos(p\omega_0 t)dt = \begin{cases}\frac{T_0}{2} & \text{if } p'=p\\ 0 & \text{if } p'\neq p\end{cases}$$

$$\int_0^{T_0}\cos(p'\omega_0 t)\sin(p\omega_0 t)dt = 0.$$

$$U = \frac{2e}{I_0}\sum_{\omega>0}^{\infty} I_p^2 Z_r(p\omega_0) - \frac{e}{I_0}\sum_{\omega>0}^{\infty} I_p^2 p\omega_0(Z_r(\omega_p^+)-Z_r(\omega_p^-))\frac{\eta\epsilon}{\omega_{s0}}$$

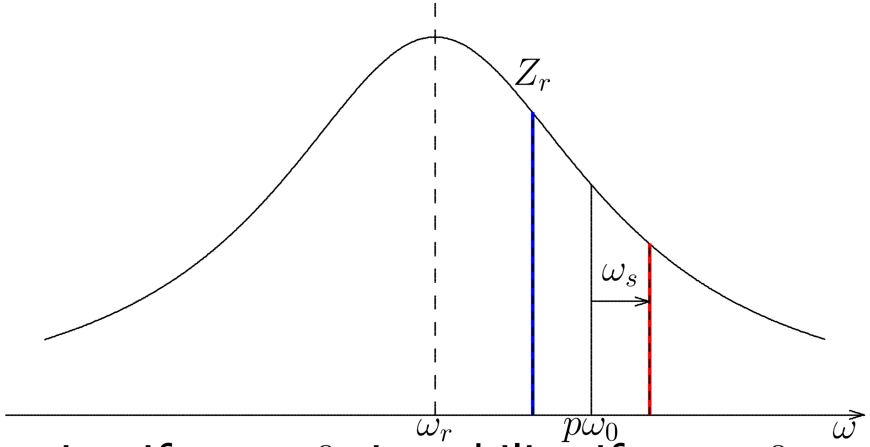
$$\frac{\partial U}{\partial\epsilon} = -\frac{e}{I_0}\sum_{\omega>0}^{\infty} I_p^2 p\omega_0(Z_r(\omega_p^+)-Z_r(\omega_p^-))\frac{\eta}{\omega_s}$$

Discussed stability of phase oscillation $\ddot\tau + 2\alpha_s\dot\tau + \omega_{s0}^2\tau = 0$, $\tau = \hat\tau e^{-\alpha_s t}\cos(\omega_{s1}t)$

$$\alpha_s = \frac{\omega_0}{4\pi E}\frac{dU}{d\epsilon} = \frac{-\omega_0^2\eta_c h e\hat V\cos\phi_s}{2\pi E}\,\frac{\sum I_p^2 p(Z_r(\omega_p^+)-Z_r(\omega_p^-)}{2I_0 h\hat V\cos\phi_s\omega_{s0}}.$$

$$\alpha_s = \frac{\omega_{s0}\sum p I_p^2(Z_r(\omega_p^+)-Z_r(\omega_p^-))}{2I_0 h\hat V\cos\phi_s} \quad \begin{matrix} >0 & \text{stable}\\ <0 & \text{unstable}\end{matrix}$$

Narrow impedance, only one harmonic p



Damping if $\alpha_s > 0$, instability if $\alpha_s < 0$

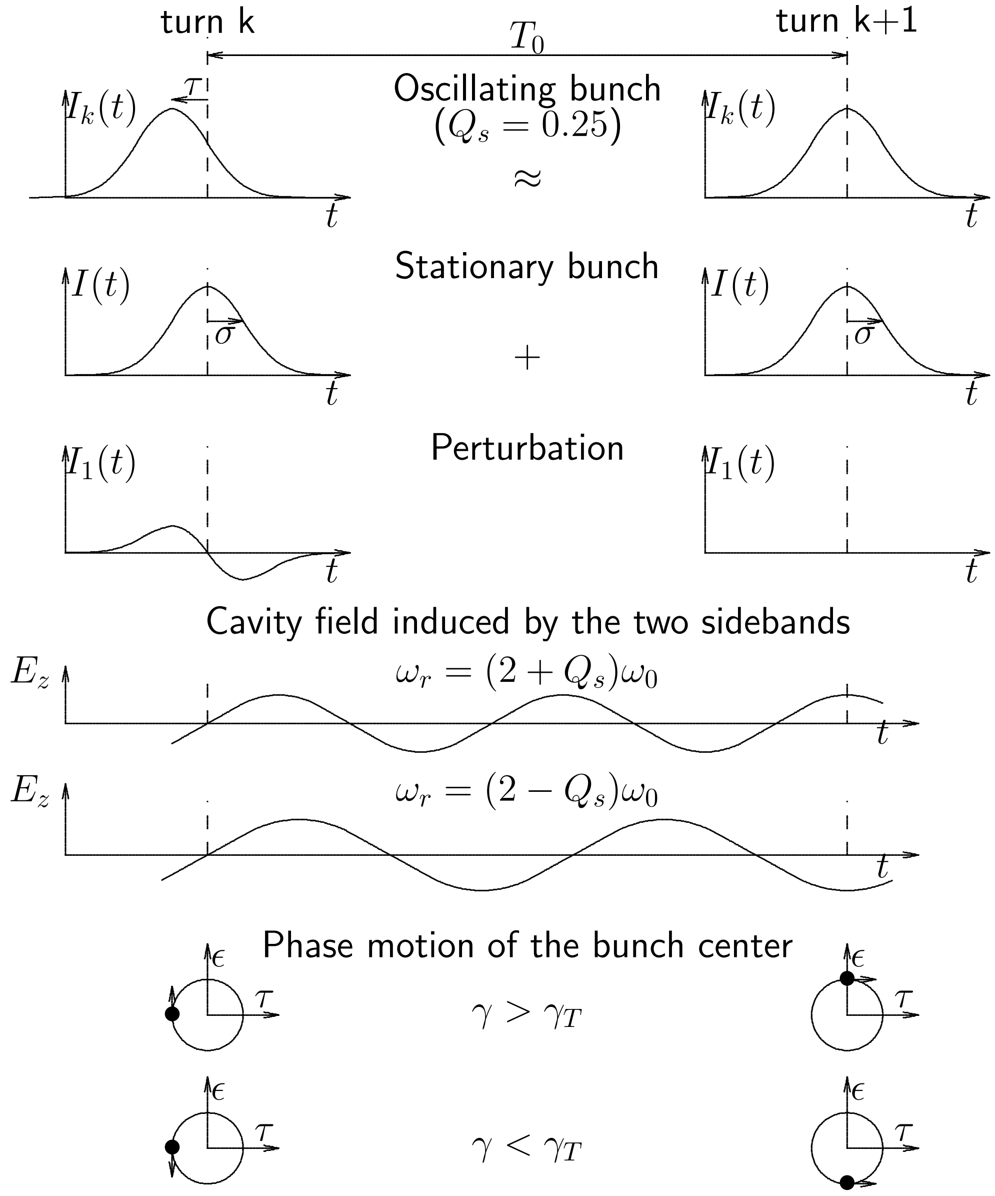
$$\epsilon = \hat{\epsilon} \mathrm{e}^{-\alpha_s t} \sin(\omega_s t)$$

$$\alpha_s = \frac{\omega_{s0} p I_p^2 (Z_r(\omega_p^+) - Z_r(\omega_p^-))}{2 I_0 h \hat{V} \cos\phi_s} > 0$$

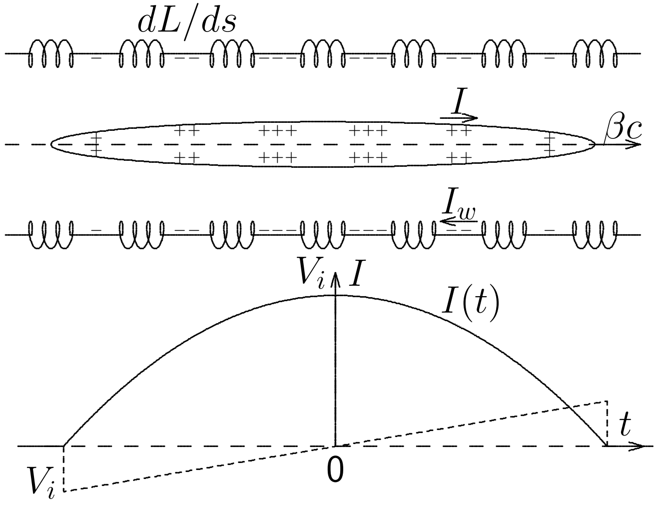
Above transition: $\cos\phi_s < 0$, stability if: $Z_r(\omega_p^-) > Z_r(\omega_p^+)$ Damping rate proportional to difference in Z_r between lower and upper sideband. Important narrow-band impedance = RF-cavity: $p = h,\ I_p \approx I_0$.

$$\frac{\alpha_s}{\omega_{s0}} \approx \frac{I_0 (Z_r(\omega_p^+) - Z_r(\omega_p^-))}{2\hat{V}\cos\phi_s} \propto \frac{\Delta\text{induced } V}{V_{RF} \text{ slope}}.$$

Qualitative understanding

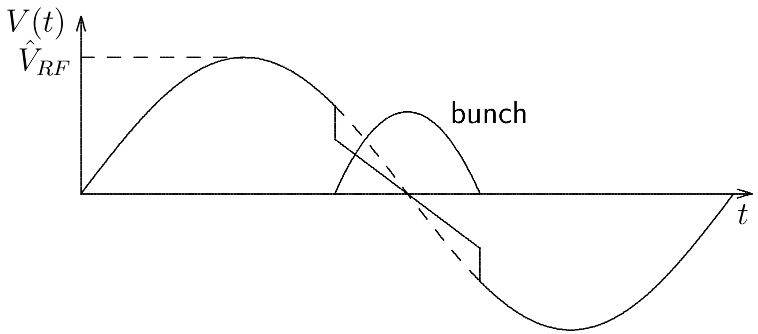


6) POTENTIAL WELL BUNCH LENGTHENING



$$E_z = -\frac{dL}{dz}\frac{dI_w}{dt} = \frac{dL}{dz}\frac{dI_b}{dt}$$

$$V = -\int E_z dz = -L\frac{dI_b}{dz}$$



We take a parabolic bunch form

$$I_b(\tau) = \hat{I}\left(1 - \frac{\tau^2}{\hat{\tau}^2}\right) = \frac{3\pi I_0}{2\omega_0\hat{\tau}}\left(1 - \frac{\tau^2}{\hat{\tau}^2}\right)$$

$$\frac{dI_b}{d\tau} = -\frac{3\pi I_0 \tau}{\omega_0 \hat{\tau}^3},\ I_0 = \langle I_b \rangle,$$

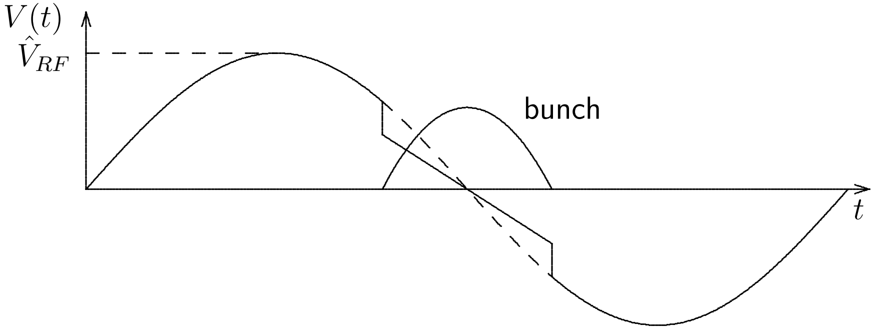
$$V = \hat{V}(\sin\phi_s + h\omega_0\cos\phi_s\tau) + \frac{3\pi I_0 L\tau}{\omega_0\hat{\tau}^3},\ L\omega_0 = \left|\frac{Z}{n}\right|$$

$$V = \hat{V}\left[\sin\phi_s + \cos\phi_s h\omega_0\left(1 + \frac{3\pi|Z/n|_0 I_0}{h\hat{V}\cos\phi_s(\omega_0\hat{\tau})^3}\right)\tau\right]$$

$$\omega_{s0}^2 = -\frac{\omega_0^2 h\eta_c e\hat{V}\cos\phi_s}{2\pi E}$$

$$\omega_s^2 = \omega_{s0}^2\left[1 + \frac{3\pi|Z/n|_0 I_0}{h\hat{V}_{RF}\cos\phi_s(\omega_0\hat{\tau})^3}\right]$$

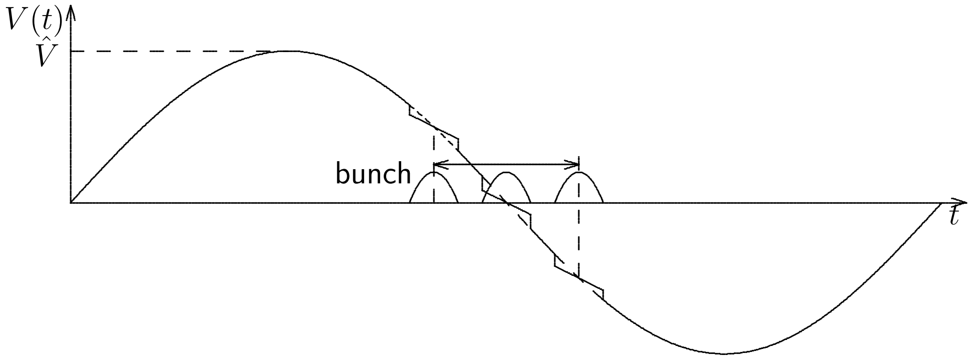
$$\frac{\Delta\omega_s}{\omega_0} = \frac{\omega_s - \omega_{s0}}{\omega_{s0}} \approx \frac{3\pi|Z/n|_0 I_0}{2h\hat{V}_{RF}\cos\phi_s(\omega_0\hat{\tau}_0)^3}$$



$$\frac{\omega_s^2}{\omega_{s0}^2} = 1 + \frac{3\pi |Z/n|_0 I_0}{h\hat{V}_{RF}\cos\phi_s (\omega_0\hat{\tau})^3}$$

$$\frac{\omega_s - \omega_{s0}}{\omega_{s0}} = \frac{\Delta\omega_s}{\omega_s} \approx \frac{3\pi |Z/n|_0 I_0}{2h\hat{V}\cos\phi_s (\omega_0\hat{\tau}_0)^3}$$

Only incoherent frequency of single particles is changed (reduced for $\gamma > \gamma_T$, increased for $\gamma < \gamma_T$), but not the coherent dipole (rigid bunch) mode. This separates the two.



Reduction of ω_s reduces longitudinal focusing and increases the bunch length

$$\hat{\tau} = \hat{\epsilon}\eta_c/\omega_s \ , \ \hat{\tau}^2 = \hat{\tau}\hat{\epsilon}\eta_c/\omega_s = \mathcal{E}_s\eta_c/\omega_s$$

rel. energy spread $\hat{\epsilon}$, long. emitt. $\mathcal{E}_s = \hat{\tau}\hat{\epsilon}$

Protons: $\mathcal{E}_s$= constant, $\tau \propto 1/\sqrt{\omega_s}$

small: $\dfrac{\Delta\hat{\tau}}{\hat{\tau}_0} \approx -\dfrac{\Delta\omega_s}{2\omega_{s0}} \approx -\dfrac{3\pi |Z/n|_0 I_0}{4h\hat{V}\cos\phi_s(\omega_0\hat{\tau}_0)^3}$,

or: $\left(\dfrac{\hat{\tau}}{\hat{\tau}_0}\right)^4 + \dfrac{3\pi |Z/n|_0 I_0}{h\hat{V}\cos\phi_s(\omega_0\hat{\tau}_0)^3}\left(\dfrac{\hat{\tau}}{\hat{\tau}_0}\right) - 1 = 0$

Electrons: $\hat{\epsilon}$= const. by syn. rad. $\hat{\tau} \propto 1/\omega_s$

small: $\dfrac{\Delta\hat{\tau}}{\hat{\tau}_0} \approx -\dfrac{\Delta\omega_s}{\omega_{s0}} \approx -\dfrac{3\pi |Z/n|_0 I_0}{2h\hat{V}\cos\phi_s(\omega_0\hat{\tau}_0)^3}$,

or: $\left(\dfrac{\hat{\tau}}{\hat{\tau}_0}\right)^3 - \dfrac{\hat{\tau}}{\hat{\tau}_0} + \dfrac{3\pi |Z/n|_0 I_0}{h\hat{V}\cos\phi_s(\omega_0\hat{\tau}_0)^3} = 0$