LANDAU DAMPING

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1) Introduction

Mechanism of Landau damping

A single oscillator with resonant frequency ω_r reacts to a pulse excitation with a free oscillation. A harmonic excitation with ω, after some transient, gives forced oscillation at the same frequency but a phase which depends on $\omega_r - \omega$. For $\omega = \omega_r$ the oscillation amplitude grows linear with time.

We take a set of oscillators having resonant frequencies ω_{rj} with distribution $f(\omega_r)$. A pulse excitation gives each oscillator the same initial velocity $\dot{x}(0)$ followed by a free oscillation with ω_{rj}. For impedances or beam observation the **center-of-mass motion** is relevant. Due to their different ω_{ri} the oscillating particles change phase with respect to each other and the center-of-mass motion is slowly reduced.

The **coherent** center-of-mass motion is 'damped' while the **incoherent** motion of the particles continues. This damping is faster the larger the spread of resonant frequencies. It differs from other damping mechanisms and the decay is usually not exponential.

For a harmonic excitation the phases of the individual oscillations are different and depend on $\omega - \omega_{rj}$ leading to some cancelation which reduces the amplitude of the center-of-mass motion.

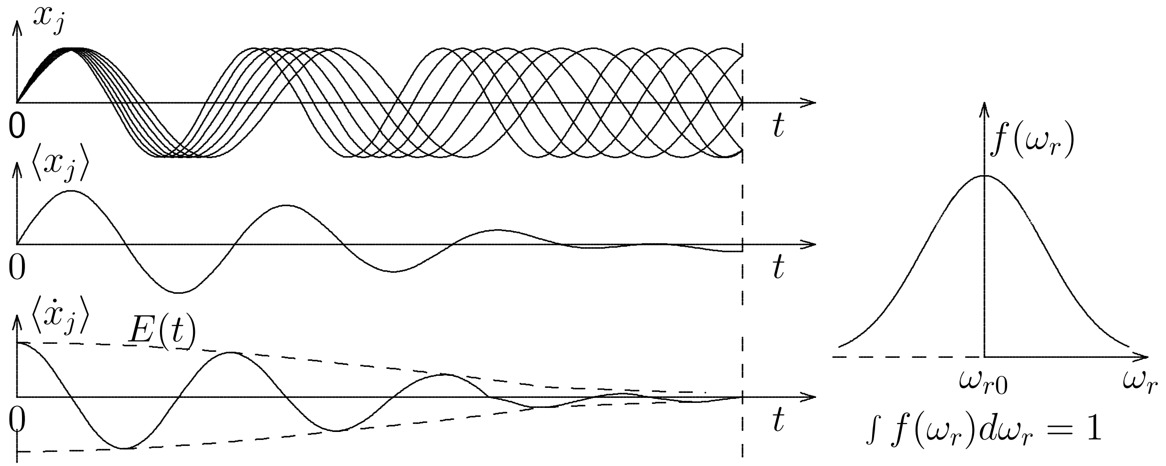
Treatment of Landau damping

Landau damping can be understood from different points of view. We treat it here in a manner close to beam observation and experiment.

The fields induced by the center-of-mass motion are modified by the beam surroundings (impedance) and act back on the beam. This can lead to an instability with a threshold determined by the beam response. Below this threshold the frequency spread eliminates any coherent motion at **infinitesimal small amplitudes**. Above, the voltage induced in the resistive part of the impedance leads to an increase of initial coherent motion and we have an instability.

The amount of Landau damping depends on the frequency distribution $f(\omega_r)$ or its derivative at the frequency ω of the instability. It can happen that the coherent (center-of-mass) motion has a different frequency than the individual particles and Landau damping becomes ineffective.

2) Response of an oscillator-set to excitation
Response to a pulse excitation



Set of oscillators j with ω_{rj} get at $t = 0$ kick with $\dot{x}_j(0^+) = \dot{x}_0$ and do free oscillations with different ω_{rj} and fixed amplitude $\hat{x}_j = \dot{x}_0 / \omega_{rj}$

$$\dot{x}_j(t) = \dot{x}_0 \cos(\omega_{rj} t)$$
$$x_j = \hat{x}_j \sin(\omega_{rj} t)$$

Response of single particle and center-of-mass

$$\dot{x}_j(t)\rangle = \dot{x}_0 \cos(\omega_r t), \;\; x_j = (\dot{x}_0/\omega_{rj}) \sin(\omega_{rj} t)$$

$\langle \dot{x}(t) \rangle = \dot{x}_0 \int f(\omega_r) \cos(\omega_r t) d\omega_r \propto$ inv. FT

narrow distribution $\Delta\omega_r = \omega_r - \omega_{r0} \ll \omega_{r0}$

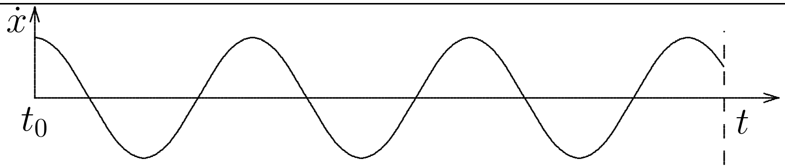
$$\langle \dot{x}(t) \rangle = \dot{x}_0 \int f(\omega_{r0} + \Delta\omega_r) \cos((\omega_{r0} + \Delta\omega_{r0})t) d\omega_r$$
$$g(t) = \langle \dot{x}(t) \rangle / \dot{x}_0 = \cos(\omega_{r0} t) I_1(t) + \sin(\omega_{r0}) I_2(t)$$
$$= \cos(\omega_{r0} t - \phi) E(t), \;\; (E(t) = \text{ envelope})$$

with inverse Fourier integrals

$$I_1(t) = \int f(\Delta\omega_r) \cos(\Delta\omega_r t) d\omega_r$$
$$I_2(t = -\int f(\Delta\omega_r)) \sin(\Delta\omega_r t) d\omega_r$$
$$g(t) \propto \mathcal{F}^{-1}_{\cos}(f(\Delta\omega_r)) \cos(\omega_{r0} t) - \mathcal{F}^{-1}_{\sin}(f(\Delta\omega_r)) \sin(\omega_{r0} t)$$
$$E(t) \; = \; \sqrt{I_1^2(t) + I_2^2(t)}$$

The center-of-mass velocity response $g(t)$ of frequency distribution $f(\Delta\omega_r)$ to a pulse excitation is proportional to its inverse Fourier transform times oscillation at ω_{r0}.

Single oscillator response to harmonic excitation



Velocity response of single oscillator with ω_r to a pulse excitation at a time t_0 is

$$\dot{x}(t) = \dot{x}_0 \cos(\omega_r(t - t_0)).$$

Use harmonic excitation at ω starting at t_1 and lasting to observation time t.

Done by small kicks with harmonic modulation

$$d\dot{x}_0 = \frac{d\hat{\dot{x}}}{dt_0}\cos(\omega t_0)dt_0 = G dt_0 = \hat{G}\cos(\omega t_0)dt_0$$

with acceleration $G(t)$. Velocity at time t is

$$\dot{x}(t) = \hat{G}\int_{t_1}^{t}\cos(\omega t_0)\cos(\omega_r(t - t_0))dt_0.$$

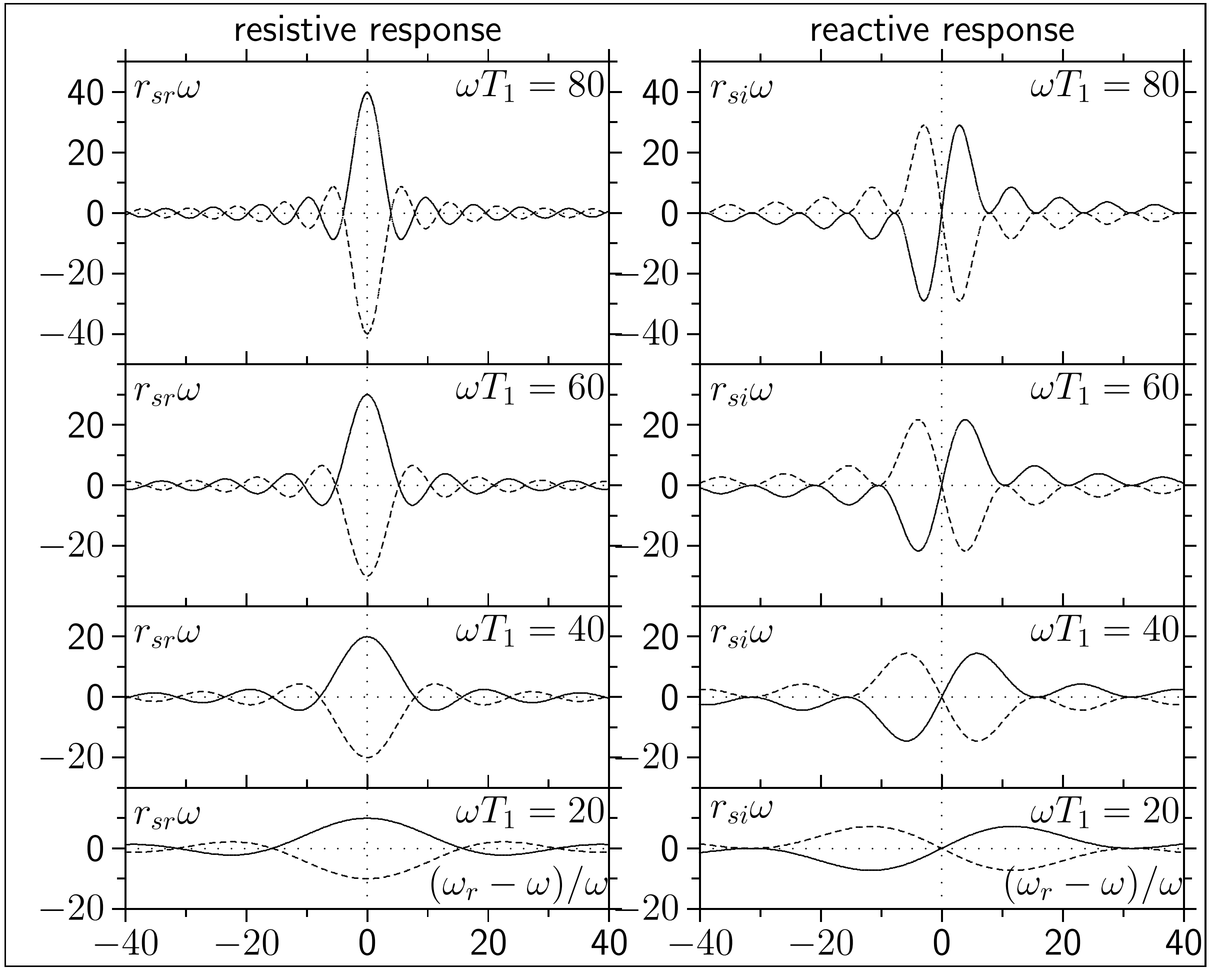
Call $T = t - t_0,\ T_1 = t - t_1$,
develop $\cos(\omega(t - T))$

$$\frac{\dot{x}(t)}{\hat{G}} = -\int_{T_1}^{0}\cos(\omega(t - T))\cos(\omega_r T)dT = \int_0^{T_1}(\cos(\omega t)\cos(\omega T) + \sin(\omega t)\sin\omega T))\cos(\omega_r T)dT,$$

$$= \frac{1}{2}\left[\cos(\omega t)\frac{\sin((\omega_r - \omega)T_1)}{\omega_r - \omega} - \sin(\omega t)\frac{1 - \cos((\omega_r - \omega)T_1)}{\omega_r - \omega}\right], \qquad (\text{ used } (\omega_r - \omega)/\omega_r \ll 1$$

$$= \cos(\omega t)r_{sr}(\omega_r) + \sin(\omega t)r_{si}(\omega_r).$$

Velocity and acceleration are in phase for the first, resistive, term, but out of phase for the second, reactive, term.



We plot the single oscillator response as a function of $\Delta\omega = \omega_r - \omega$ for different excitation times T_1 It becomes concentrated around $\omega_r = \omega$ where its resistive part has a maximum and the reactive one goes through zero.

The fast oscillation for large T_1 averages while integrating over $f(\omega_r)$ giving with $\int_{-\infty}^{\infty} dx \sin(ax)/x = \pi$

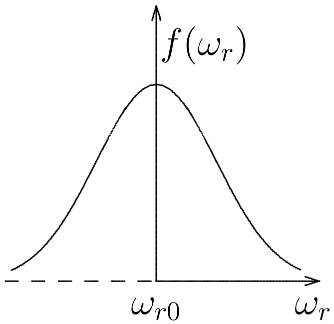
$$r_{sr} = \frac{\sin((\omega_r - \omega)T_1)}{2(\omega_r - \omega)} \approx \begin{cases} \infty, \omega = \omega_r \\ 0\ , \omega \neq \omega_r \end{cases}$$

$$\approx \frac{\pi}{2}\delta(\omega_r - \omega)$$

$$r_{si} = -\frac{1 - \cos((\omega_r - \omega)T_1)}{2(\omega_r - \omega)} \approx \begin{cases} 0 & , \omega = \omega_r \\ \frac{1}{2(\omega_r - \omega)} & , \omega \neq \omega_r \end{cases}$$

$$\dot{x} = \hat{G}\cos(\omega t) r_{sr}(\omega_r) + \sin(\omega t) r_{si}(\omega_r)\dot{x}(t) \approx \frac{\hat{G}}{2}\left[\cos(\omega t)\pi\delta(\omega_r - \omega) - \sin(\omega t)\left(\frac{1}{\omega_r - \omega}\right)_{\omega \neq \omega_r}\right]$$

Oscillator set response to harmonic excitation



$$f(\omega_r) = \frac{1}{N}\frac{dN}{d\omega_r}$$

$$\int_0^\infty f(\omega_r)d\omega_r = 1$$

The velocity of the center-of-mass motion is obtained by integrating the single oscillator response over the distribution

$$\dot{x}(t) \approx \frac{\hat{G}}{2}\left[\cos(\omega t)\pi\delta(\omega_r - \omega) - \sin(\omega t)\left(\frac{1}{\omega_r - \omega}\right)_{\omega\neq\omega_r}\right]$$

$$\begin{aligned}
\langle\dot{x}(t)\rangle &= \int_0^\infty \dot{x}(t,\omega_r)f(\omega_r)d\omega_r \\
\frac{\langle\dot{x}(t)\rangle}{\hat{G}} &= \frac{1}{2}\left(\cos(\omega t)\pi f(\omega) - \sin(\omega t)\mathsf{PV}\int_{-\infty}^{\infty}\frac{f(\omega_r)d\omega_r}{\omega_r - \omega}\right) \\
&= \cos(\omega t)r_r(\omega) + \sin(\omega t)r_i(\omega)
\end{aligned}$$

$$\mathsf{PV}\int\frac{f(\omega_r)}{\omega - \omega_r}d\omega_r = \lim_{\epsilon\to 0}\left[\int_{-\infty}^{\omega-\epsilon}\frac{f(\omega_r)}{\omega_r - \omega} + \int_{\omega+\epsilon}^{\infty}\frac{f(\omega_r)}{\omega_r - \omega}\right]d\omega_r$$

This response to harmonic excitation is called **transfer function**. Its resistive part is proportional to the distribution at ω and vanishes therfore if the excitation frequency lies outside the distribution $f(\omega_r)$.

Remember: ω_r is the resonant frequency of a particle in the distribution,
ω the frequency of excitation.

Short derivation using complex notation

$$\cos(\omega t) = \frac{\mathrm{e}^{j\omega t} + \mathrm{e}^{-j\omega t}}{2} \;\rightarrow \frac{\mathrm{e}^{j\omega t}}{2},\; -\infty \le \omega \le \infty$$

$$\ddot{x} + \omega_r^2 x = \frac{\hat{G}}{2}\mathrm{e}^{j\omega t}\;,\; (-\omega^2 + \omega_r^2)x = \frac{\hat{G}}{2}\mathrm{e}^{j\omega t}$$

$$x = \frac{\hat{G}\mathrm{e}^{j\omega t}}{2(\omega_r^2 - \omega^2)} = \frac{\hat{G}\mathrm{e}^{j\omega t}}{4\omega}\left(\frac{1}{\omega_r - \omega} - \frac{1}{\omega_r + \omega}\right)$$

$$\frac{\langle x\rangle_+}{\hat{G}} = \frac{\mathrm{e}^{j\omega t}}{4\omega}\int_0^\infty \frac{f(\omega_r)}{\omega_r - \omega}d\omega_r\;,\quad \frac{\langle \dot{x}\rangle_+}{\hat{G}} = j\omega\frac{\langle x\rangle_+}{\hat{G}}$$

$$\int_{-\infty}^{\infty}\frac{f(\omega_r)d\omega_r}{\omega_r - \omega} = \pm j\pi f(\omega) + \mathsf{PV}\!\int\frac{f(\omega_r)d\omega_r}{\omega_r - \omega}$$

Use complex notation with positive and negative frequencies.

Displacement response, single oscillator. For $\omega > 0$ only first and for $\omega < 0$ only second term is large. Taking the first and integrating over $f(\omega_r)$

Integration over pole gives PV (principle value) integral plus imaginary residue.

Resolve sign ambiguity by $\dot{x}(-\infty) = 0$

$$\text{for } \omega > 0\; \frac{\langle \dot{x}\rangle_+}{\hat{G}} = \frac{\mathrm{e}^{j\omega t}}{4}\left[\pi f(\omega) + j\mathsf{PV}\int_{-\infty}^{\infty}\frac{f(\omega_r)}{\omega_r - \omega}d\omega_r\right]$$

$$\text{for } \omega < 0\; \frac{\langle \dot{x}\rangle_-}{\hat{G}} = \frac{\mathrm{e}^{-j\omega t}}{4}\left[\pi f(\omega) - j\mathsf{PV}\int_{-\infty}^{\infty}\frac{f(\omega_r)}{\omega_r - \omega}d\omega_r\right]$$

$$\frac{\langle \dot{x}\rangle_+}{\hat{G}} + \frac{\langle \dot{x}\rangle_-}{\hat{G}} = \frac{1}{2}\left[\cos(\omega t)\pi f(\omega) - \sin(\omega t)\mathsf{PV}\int_{-\infty}^{\infty}\frac{f(\omega_r)}{\omega_r - \omega}d\omega_r\right],\ \text{(agrees with previous)}$$

Response for a Gaussian frequency distribution

$$f(\omega_r) = \frac{1}{\sqrt{2\pi}\sigma_\omega} \mathrm{e}^{-\frac{\Delta\omega_r^2}{2\sigma_\omega^2}}, \int_{-\infty}^{\infty} f(\Delta\omega_r) d\omega_r = 1$$

frequency distribution

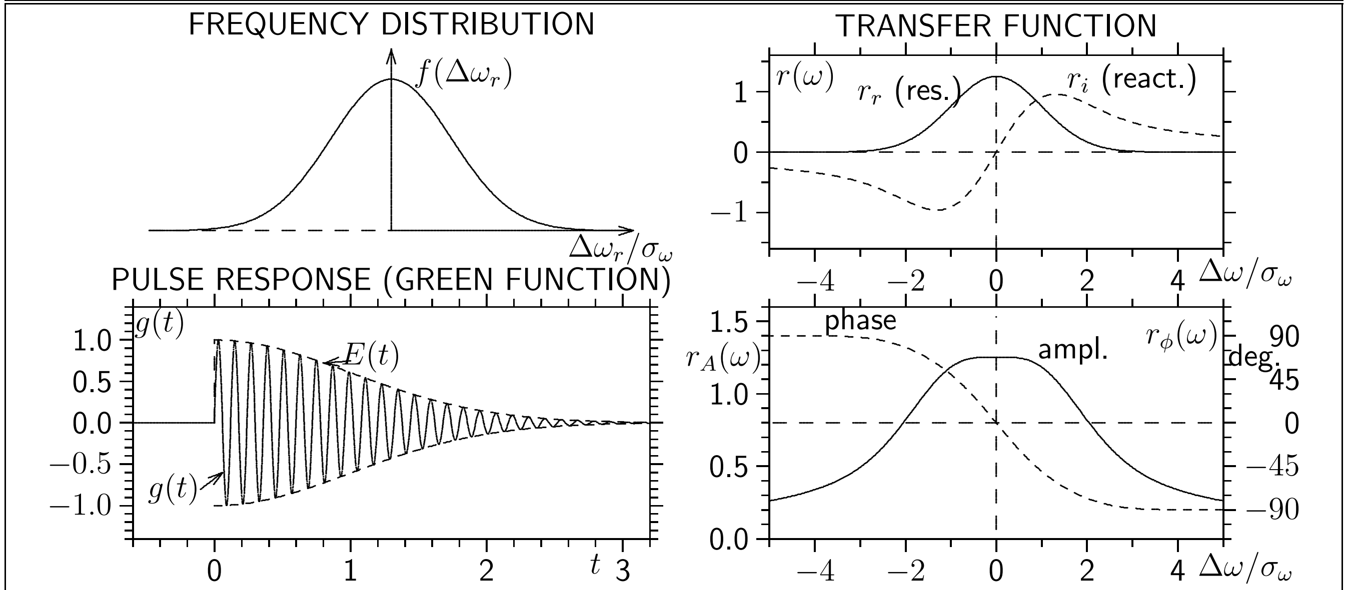
$$g(t) = \frac{\cos(\omega_r t)}{\sqrt{2\pi}\sigma_\omega} \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{\Delta\omega_r^2}{2\sigma_\omega^2}} \cos(\Delta\omega_r t) d\omega_r = \mathrm{e}^{-\frac{\sigma_\omega^2 t^2}{2}} \cos(\omega_{r0} t)$$

pulse response
$g(t) = E(t)\cos(\omega_{r0} t)$

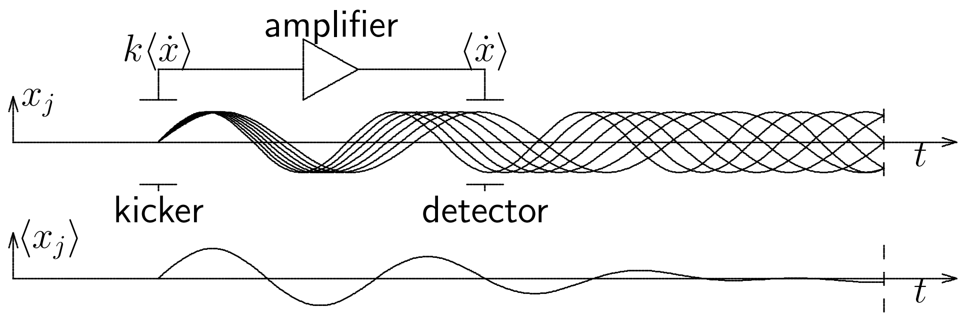
$$r_r(\omega) = \int_0^{\infty} \mathrm{e}^{-\frac{\Delta\omega_r^2}{2\sigma_\omega^2}} \cos(\Delta\omega t) dt = \frac{\pi}{\sqrt{2\pi}\sigma_\omega} \mathrm{e}^{-\Delta\omega^2/2\sigma_\omega^2}$$

$$r_i(\omega) = \int_0^{\infty} \mathrm{e}^{-\frac{\Delta\omega_r^2}{2\sigma_\omega^2}} \sin(\Delta\omega t) dt = \frac{\sqrt{2}}{\sigma_\omega} \mathrm{e}^{-\frac{\Delta\omega^2}{2\sigma_\omega^2}} \int_0^{\frac{\Delta\omega}{\sqrt{2}\sigma_\omega)}} \mathrm{e}^{t'^2} dt'$$

transfer function, from FT of $E(t)$, Dawson integral



Landau damping of oscillator set



Center-of-mass motion of oscillator set is measured by detector, amplified signal is fed to a kicker which produces an acceleration G_s in phase with velocity. Can lead to growing oscillation, i.e. negative feed-back system.

$$\frac{\langle\dot{x}\rangle}{G} = [r_r + jr_i] = \frac{1}{2}\left(\pi f(\omega) + jPV\int \frac{f(\Delta\omega_r)}{\omega_r - \omega}d\omega_r\right)$$

$$= \left(\frac{\pi}{\sqrt{2\pi}\sigma_\omega}\mathrm{e}^{-\Delta\omega^2/2\sigma_\omega^2} + j\frac{\sqrt{2}}{\sigma_\omega}\mathrm{e}^{-(\Delta\omega/\sigma_\omega)^2/2}\int_0^{\Delta\omega/(\sqrt{2}\sigma_\omega)}\mathrm{e}^{t'^2}dt'\right)$$

$$\langle\dot{x}\rangle = G\frac{\pi}{\sqrt{2\pi}\sigma_\omega}, \quad \text{for } \Delta\omega = 0$$

Center-of-mass velocity response to acceleration
$G = \hat{G}\exp(j\omega t)$

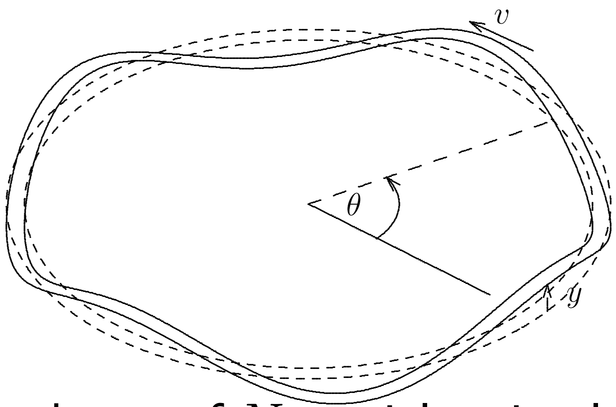
Take excitation at central frequency $\Delta\omega = 0$. $r_i = 0$

$$\langle\dot{x}\rangle = k\langle\dot{x}\rangle\frac{\pi}{\sqrt{2\pi}\sigma_\omega}$$

$$k < \sqrt{\frac{2}{\pi}}\sigma_\omega, \quad \text{stable.}$$

Replace external G by the one of feed-back $G_s = k\langle\dot{x}\rangle$, assume a gain k just sufficient to keep oscillation going.
This maximum gain k still giving stability is proportional to frequency spread. Landau damping suppresses an accidental coherent oscillation at infinitesimally small levels.

3) Transverse coasting beam instability
Oscillation modes



A coasting beam of N particles circulates with ω_0, current $I = eN\omega_0/(2\pi)$ in a ring of uniform focusing. Each particle executes a betatron oscillation of $Q\omega_0$

$$\theta_i = \theta_{0i} + \omega_0 t \quad , \quad y_i(t) = \hat{y}\cos(Q\omega_0(t - t_i)).$$

Depending on the phases $Q\omega_0 t_i$ between adjacent particles we have different modes. We choose a form as seen at fixed location θ

$$y(t) = \hat{y}\cos(n\theta - \omega t)\ ,\ y(0) = \hat{y}\cos(n\theta).$$

Frozen in time $t = 0$ gives closed wave with n periods. Following a particle $\theta_s(t) = \theta_0 + \omega_0 t$ gives betatron oscillation with frequency $Q\omega_0$.

$$\begin{aligned} y_s &= \hat{y}\cos(n\theta_0 - (\omega - n\omega_0)t) \\ &= \hat{y}\cos(n\theta_s - Q\omega_0 t) \end{aligned}$$

giving for the frequency ω seen by stationary observer

$$\omega = (n{+}Q)\omega_0 = \omega_\beta \ \text{ with } \ -\infty < n < \infty.$$

Divide modes into fast and slow waves according to sign of phase difference between adjacent particle

$$\begin{aligned} \omega_{\beta f} &= (n_f + Q)\omega_0\ ,\ n_f > -Q \\ \omega_{\beta s} &= (n_s - Q)\omega_0\ ,\ n_s > Q. \end{aligned}$$

Effect of momentum spread

Betatron frequencies of beam with nominal momentum:

$$\omega_{\beta f} = (n_f + Q)\omega_0 \;\; , \;\; \omega_{\beta s} = (n_s - Q)\omega_0,$$

for $\omega_{\beta f}$ close to $\omega_{\beta s}$, $n_f \neq n_s$. Through

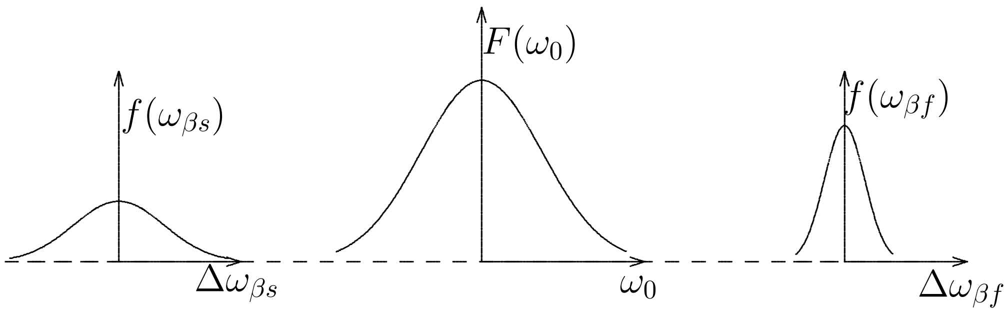
$$\frac{\Delta E}{E} = \beta^2 \frac{\Delta p}{p} = -\frac{\beta^2}{\eta_c}\frac{\Delta \omega_0}{\omega_0} \text{ , and } \;\Delta Q = Q' \frac{\Delta p}{p}$$

they are affected by a momentum deviation

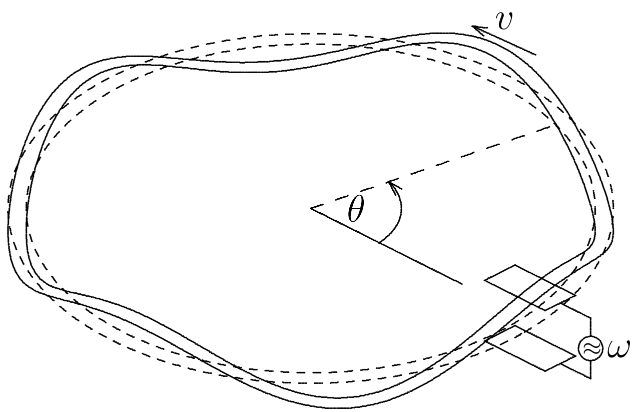
$$\Delta\omega_{\beta f} \; = \; (Q' - \eta_c(n_f + Q))\omega_0 \frac{\Delta p}{p}$$

$$\Delta\omega_{\beta s} \; = \; (Q' - \eta_c(n_s - Q))\omega_0 \frac{\Delta p}{p}.$$

resulting in two frequency distributions $f(\omega_{\beta f})$, $f(\omega_{\beta s})$.



Response of narrow particle string



Excite a ring of monoenergetic particles

$$\ddot{y} + \omega_0 Q^2 y = \hat{G}\cos(\omega t)$$

Seek solution $y(t) = \hat{y}\cos(n\theta - \omega)$, drive particles not at $\omega_0 Q$ but close to either the fast or slow wave $\omega_{\beta f} = (n_f + Q)\omega_0$ or $\omega_{\beta s} = (n_s - Q)\omega_0$, here $n_f = n_s = n$.

$$\left(-(n\omega_0 - \omega)^2 + Q^2\omega_0^2\right)\hat{y}\cos(n\theta - \omega t) = \hat{G}\cos(\omega t)$$

$$\hat{y} = \frac{\hat{G}}{\omega_0^2 Q^2 - (n\omega_0 - \omega)^2} = \frac{-\hat{G}}{(\omega - \omega_0(n+Q)(\omega - \omega_0(n-Q)}$$

$$= \frac{-\hat{G}}{(\omega - \omega_{\beta f})(\omega - \omega_{\beta s})} = \frac{\hat{G}}{2\omega_0 Q}\left(\frac{1}{\omega - \omega_{\beta s}} - \frac{1}{\omega - \omega_{\beta f}}\right).$$

$$\left(\frac{\hat{y}}{\hat{G}}\right)_f \approx -\frac{1}{2\omega_0 Q}\left(\frac{1}{\omega - \omega_{\beta f}}\right), \quad \left(\frac{\hat{y}}{\hat{G}}\right)_s \approx \frac{1}{2\omega_0 Q}\left(\frac{1}{\omega - \omega_{\beta s}}\right).$$

substituting into diff. equation

excite and observe at $\theta = 0$

gives response,
excite fast wave $\omega \approx \omega_{\beta f}$, first term much smaller than second and vice versa, responses have opposite sign, discussed later.

Response of the whole beam

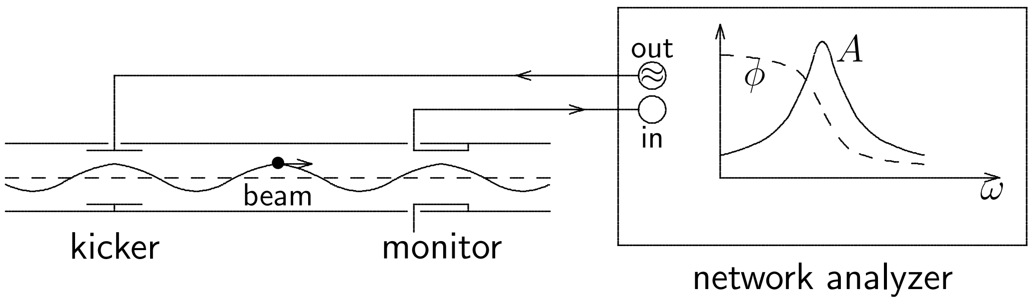
The whole beam has frequency distribution $f(\omega_{\beta f})$ and $f(\omega_{\beta s})$ Using $G(t) = \hat{G}\mathrm{e}j\omega t$ The center of mass responses in displacement and velocity are related $\langle\dot{y}\rangle = j\omega\langle y\rangle$

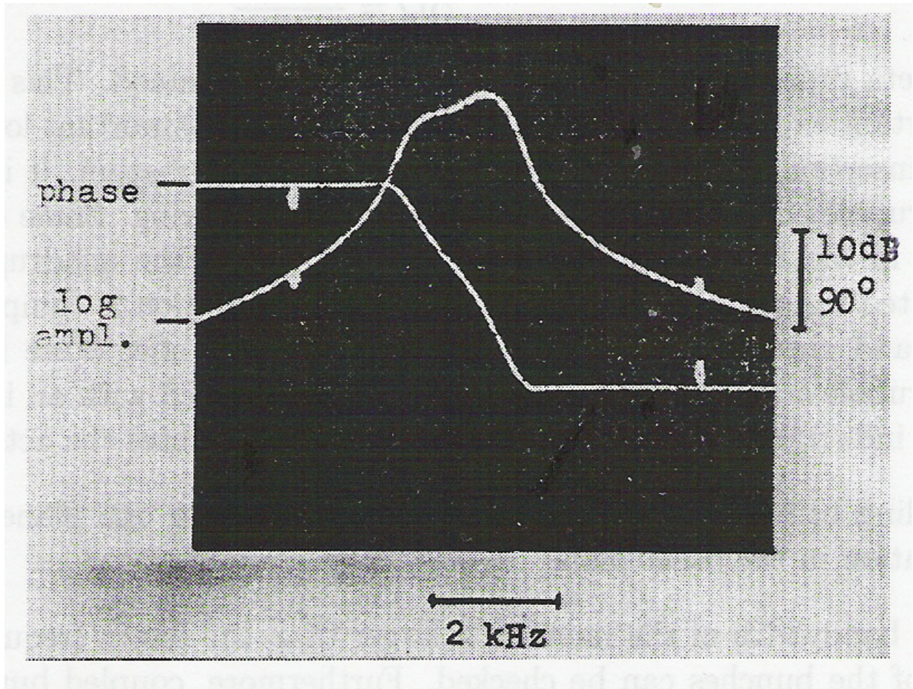
$$\begin{aligned}
\langle\hat{\dot{y}}\rangle_f &= -j\frac{\hat{G}\omega}{2Q\omega_0}\int\frac{f(\omega_{\beta f})}{\omega_{\beta f}-\omega}d\omega_{\beta f}) \\
&= -\frac{\hat{G}\omega}{2Q\omega_0}\left(\pi f(\omega) + jPV\int\frac{f(\omega_{\beta f})}{\omega_{\beta f}-\omega}\right)d\omega_{\beta f} \\
\langle\hat{\dot{y}}\rangle_s &= j\frac{\hat{G}\omega}{2Q\omega_0}\int\frac{f(\omega_{\beta s})}{\omega_{\beta s}-\omega}d\omega_{\beta s} \\
&= \frac{\hat{G}\omega}{2Q\omega_0}\left(\pi f(\omega) + jPV\int\frac{f(\omega_{\beta s})}{\omega_{\beta s}-\omega}\right)d\omega_{\beta s}
\end{aligned}$$

The term $\pi f(\omega)$ is real, exciting acceleration and responding velocity are in phase resulting in an absorption of energy and damping, called Landau damping. It is only present if the excitation frequency ω is within the frequency distribution of the individual particles. The second term is imaginary and gives the out-of-phase response being of less interest.

The spread in betatron frequencies is given by the momentum spread and the dependence of revolution frequency ω_0 and betatron tune Q on momentum deviation $\Delta p/p$. It is therefore determined by an **external parameter** which is not affected by the excitation of betatron oscillations.

Measuring the beam response

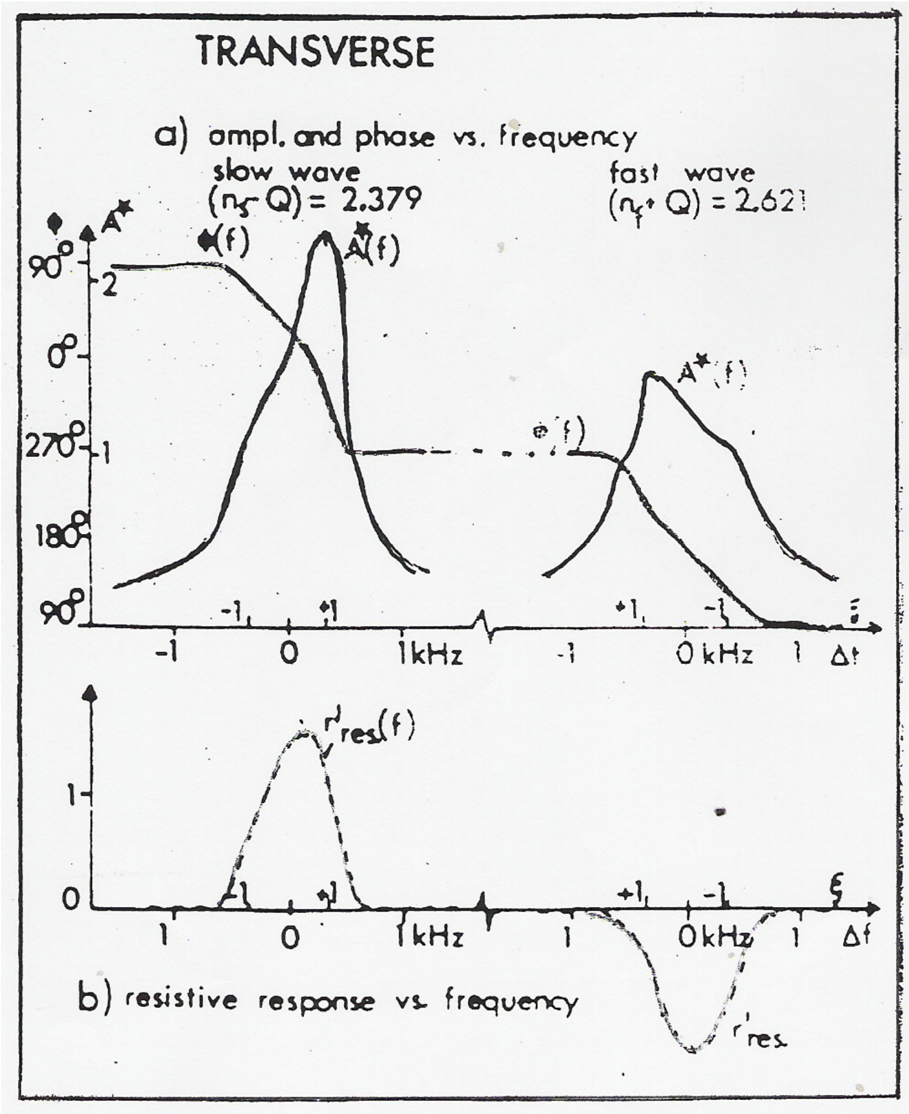




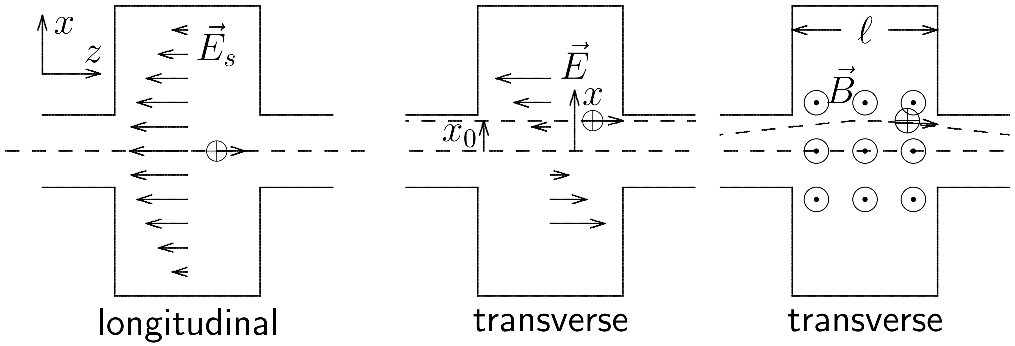
Vertical TF of an unbunched beam in the ISR

The center of mass displacement response can directly be measured with a network analyzer. Here, we derived the velocity response which is more transparent for understanding the resistive and reactive behavior of the beam. In measurements the displacement is observed and our equation have to be converted to analyze the results. Due to cable delays the real and imaginary part of the response are often mixed. It is easier to measure amplitude and phase response and correct the latter off-line.

Beam response measurement at upper and lower side-band



Transverse impedance



$$Z_T(\omega) = j\frac{\int \left(\vec{E}(\omega) + [\vec{v} \times \vec{B}(\omega)]\right)_T ds}{I x(\omega)} = -\frac{\omega \int \left(\vec{E}(\omega) + [\vec{v} \times \vec{B}(\omega)]\right)_T ds}{I \dot{x}(\omega)}$$

Response to applied G of fast and slow wave

$$\langle \dot{x} \rangle_f = -\frac{G\omega}{2Q\omega_0}\left(\pi f(\omega) + jPV \int \frac{f(\Delta\omega_r)}{\omega_r - \omega} d\omega_r\right)$$

$$\langle \dot{x} \rangle_s = \frac{G\omega}{2Q\omega_0}\left(\pi f(\omega) - jPV \int \frac{f(\Delta\omega_r)}{\omega_r - \omega} d\omega_r\right)$$

induced fields in Z_T give G_s averaged over $2\pi R$

$$G_s = \frac{e\oint \left(\vec{E}(\omega) + [\vec{\beta} \times \vec{B}(\omega)]\right)_T ds}{2\pi R m_0 \gamma} = \frac{-e Z_T I \langle \dot{x} \rangle}{\gamma m_0 2\pi R \omega}$$

Stability limit

$$\langle \dot{x} \rangle_s = -\frac{G\omega}{2Q\omega_0}\left(\pi f(\omega) - jPV\int \frac{f(\Delta\omega_r)}{\omega_r - \omega} d\omega_r\right)$$

$$= -\frac{G\omega}{2Q\omega_0}\frac{\pi}{\sqrt{2\pi}\sigma_\omega}\mathrm{e}^{-\frac{\Delta\omega^2}{2\sigma_\omega^2}}\left[1 + j\frac{2}{\sqrt{\pi}}\int_0^{\frac{\Delta\omega}{\sqrt{2}\sigma_\omega)}} \mathrm{e}^{t'^2}dt'\right]$$

$$Z_T = -\frac{\omega}{I\langle\dot{x}\rangle(\omega)}\oint\left(\vec{E}(\omega) + [\vec{\beta}\times\vec{B}(\omega)]\right)_T ds$$

$$\hat{G}_s = -\frac{eZ_T I\langle\dot{x}\rangle}{\gamma m_0 2\pi R\omega}$$

response of the slow wave

take Gaussian distribution

$$f(\omega_r) = \frac{1}{\sqrt{2\pi}\sigma_\omega}\mathrm{e}^{-\frac{\Delta\omega_r^2}{2\sigma_\omega^2}},\ \Delta\omega = \omega - \omega_r$$

voltage induced in Z_T gives acceleration G_s If $\hat{G}_s = \hat{G}$, self sustained oscillation without drive, threshold.

$$\langle \dot{x} \rangle_s = -\frac{G\omega}{2Q\omega_0}\frac{\pi}{\sqrt{2\pi}\sigma_\omega} = -\frac{ecZ_{Tr}I\langle\dot{x}\rangle}{4\pi Q m_0 c^2\gamma}$$

$$1 = \frac{ecZ_{Tr}I}{4\sqrt{2\pi}\sigma_\omega Q m_0 c^2\gamma},\quad Z_{Tr} \le \frac{4\sqrt{2\pi}\sigma_\omega Q m_0 c^2\gamma}{ecI}$$

use $\Delta\omega = 0$, get only real response

stability condition

Stability diagram

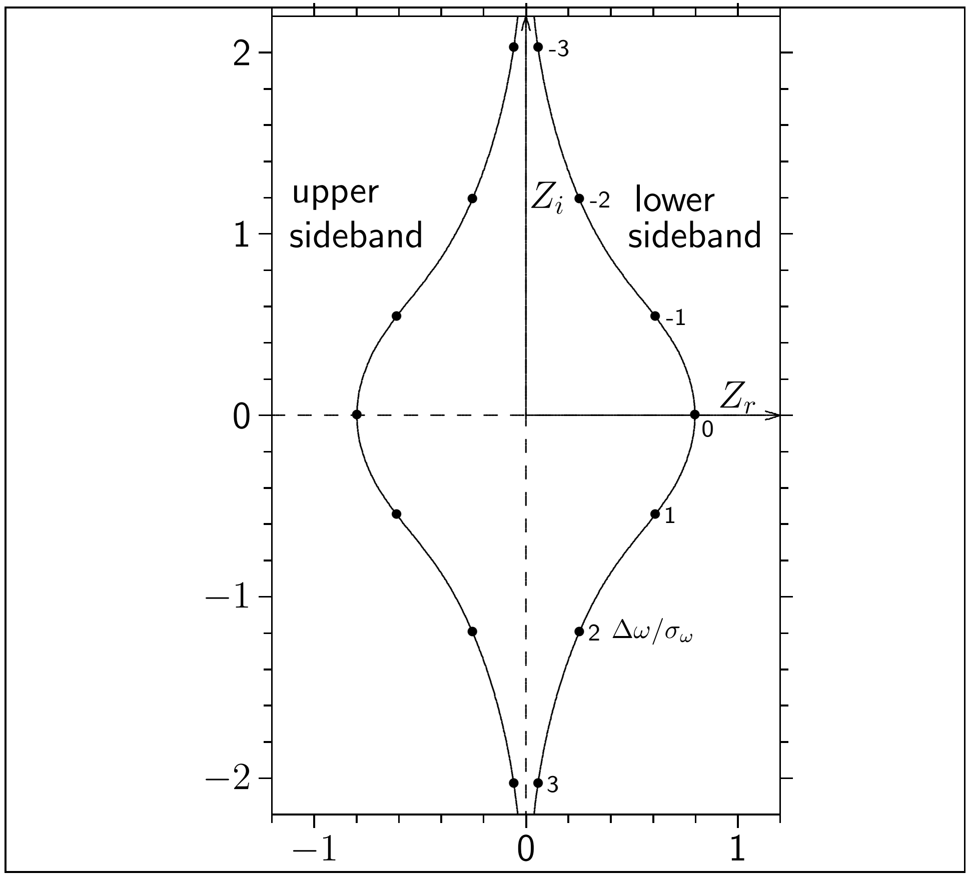
For the general complex response and impedance the stability condition can be expressed as a diagram by relating the beam parameter against the inverse response of the beam, i.e. inverse amplitude plotted against the negative phase, inverse Nyquist diagram.

□

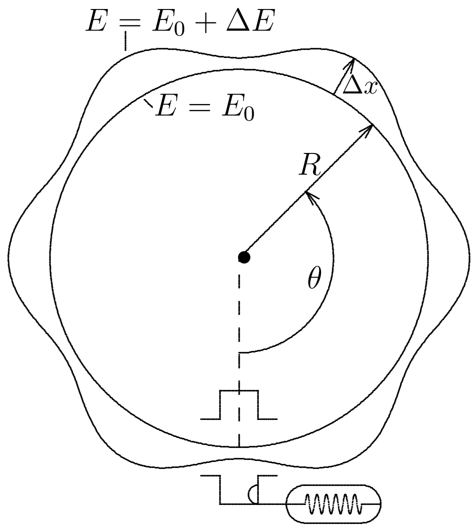
$$\text{slow wave:} \quad \frac{jecIZ_T(\omega)}{4\pi QE} \leq \frac{1}{\int \frac{f(\omega_{\beta s})}{\omega_{\beta s}-\omega} d\omega_{\beta s}}$$

$$\text{fast wave:} \quad \frac{jecIZ_T(\omega)}{4\pi QE} \geq \frac{-1}{\int \frac{f(\omega_{\beta f})}{\omega_{\beta f}-\omega} d\omega_{\beta f}}.$$

Relation between complex impedance and complex beam response to excitation. If impedances inside central curve we have stability, outside an instability. Curve itself represents threshold.



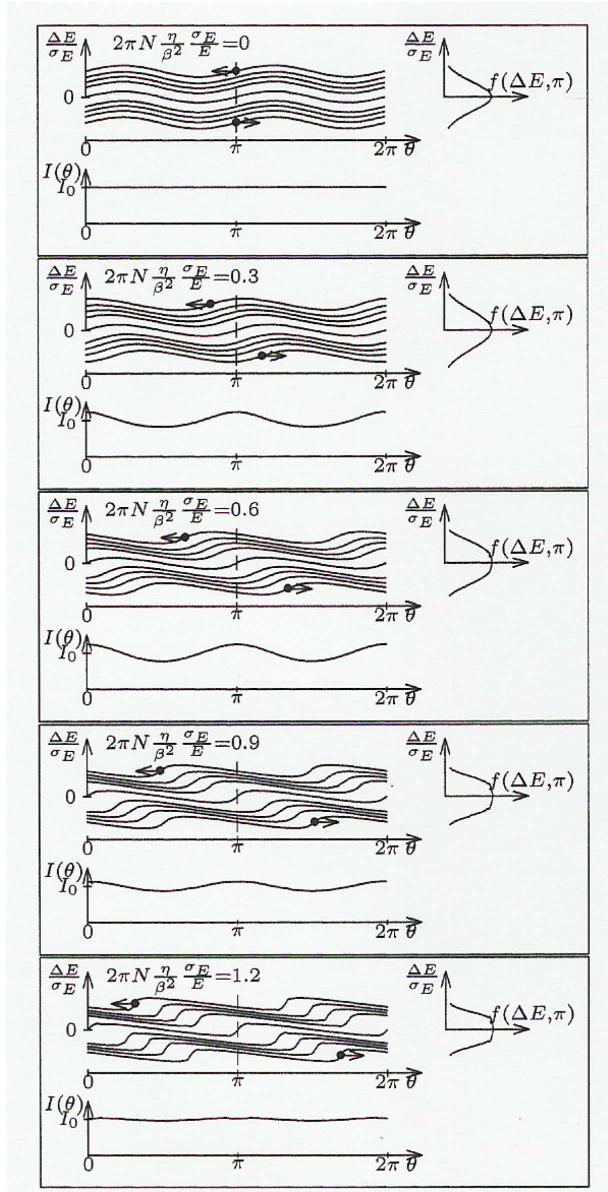
4) Longitudinal coasting beam instability
Dynamics



$$\Delta\omega_0/\omega_0 = -\frac{\eta_c}{\beta^2}\frac{\Delta E}{E}$$

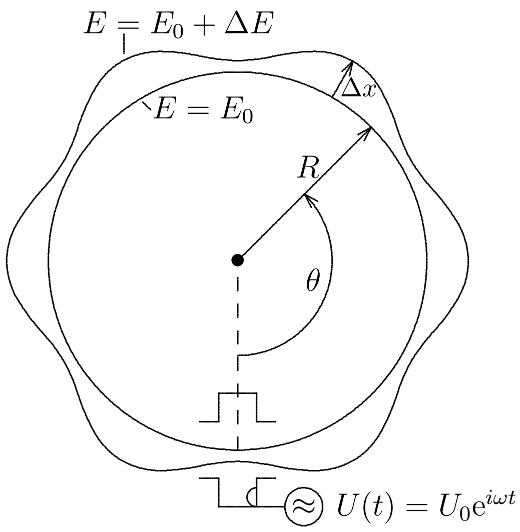
$$f_0(\Delta E) \;\rightarrow\; F_0(\Delta\omega_0)$$

At $t = 0$ we pulse excite mode



$$\begin{aligned}
\delta E &= \delta E_0 \cos(n\theta), \quad \text{at } t = 0\text{, pulse excite mode} \\
f(0^+) &= f_0(\Delta E + \delta E) \approx f_0(\Delta E) + \frac{df_0(\Delta E)}{dE}\delta E \\
&= f_0 + \frac{df_0}{dE}\delta E_0 \cos(n\theta), \\
f(t) &= f_0 + \frac{df_0}{dE}\cos(n\theta - \omega_r t) \\
\omega_r &= \omega_0 - \omega_0\eta_c\Delta E/E \\
I(t) &= Ne\omega_0 \textstyle\int f(\Delta E)dE = I_0 + I_1(t) \quad \text{current.}
\end{aligned}$$

Response to a harmonic excitation

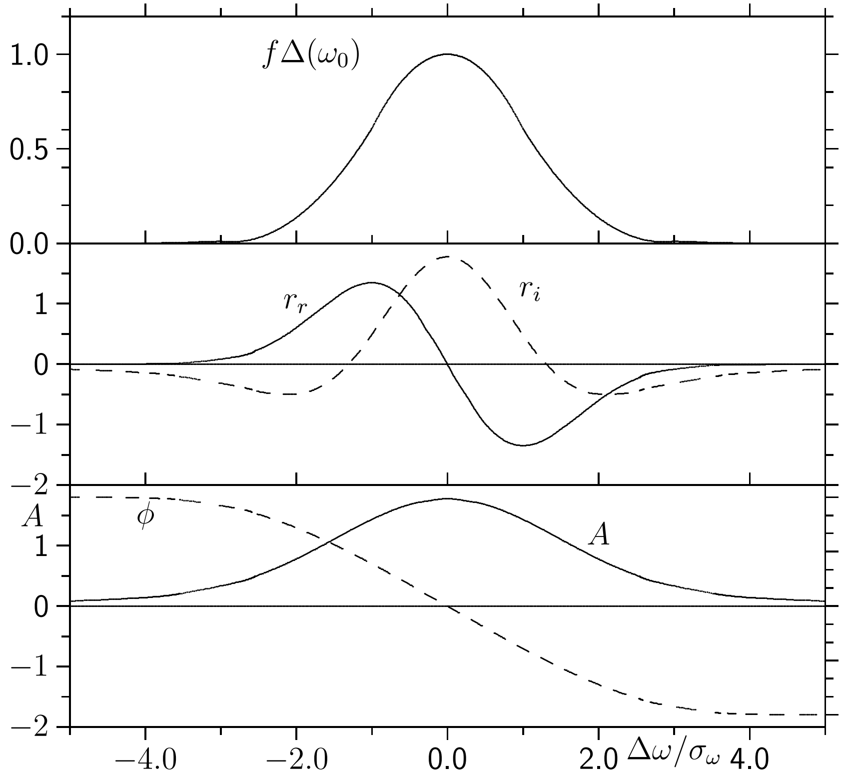


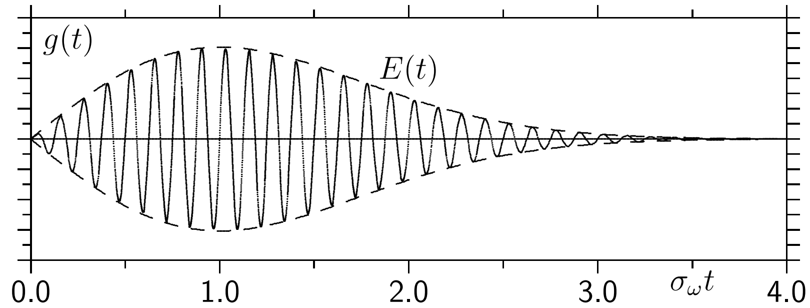
$$\frac{\Delta E}{E} = -\frac{1}{\eta_c}\frac{\Delta \omega_0}{\omega_0}, \quad \Delta\omega_0 = \omega_r - \omega_0$$

$$I(t) = Ne\omega_0 \int f(\Delta E) dE = I_0 + I_1(t)$$

$$I_1(t) \propto e^{i\omega t} = U(t)(r_r(\omega) + j r_i(\omega))$$

$$I_1(t) = \frac{-jNe^2\omega_0^3 U(t)}{2\pi\beta^2 E}\int \frac{dF_0/d\omega_0}{\omega - n\omega_0} d\omega_0$$

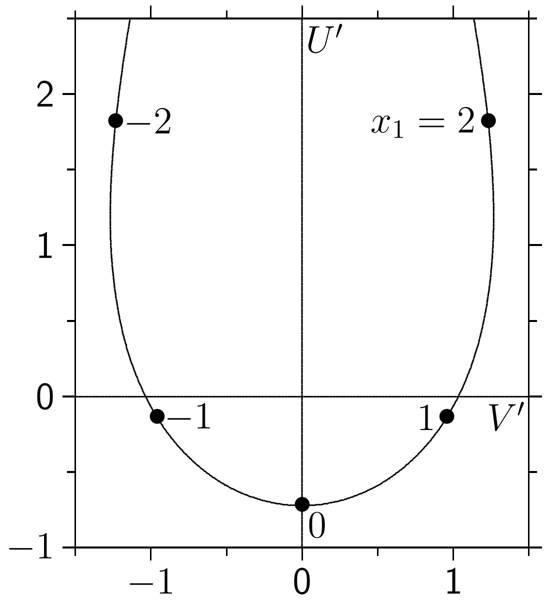




Longitudinal stability limit

$$I_1(t) = \frac{Ne^2\omega_0^3 U(t)}{2\pi\beta^2 E}\left(\pi\frac{dF_0}{d\omega_0}(\omega) - jPV\int\frac{dF_0(\omega_0)/dt}{\omega - n\omega_0}d\omega_0\right)$$
$$1 = \frac{N\mathrm{e}^2\omega_0^3\eta Z(\omega)}{2\pi\beta^2 E}\left(\frac{\pi dF_0}{d\omega_0}(\omega) - PVj\int\frac{dF_0(\omega_0)/dt}{\omega - n\omega_0}d\omega_0\right)$$
]

Response is perturbed current $I_1(t) = U(t)(r_r(\omega) + jr_i(\omega))$ Stability limit if $I_1(t)$ induces in $Z(\omega)$ just voltage used to excite beam $U(t) = I_1(t)Z(\omega)$

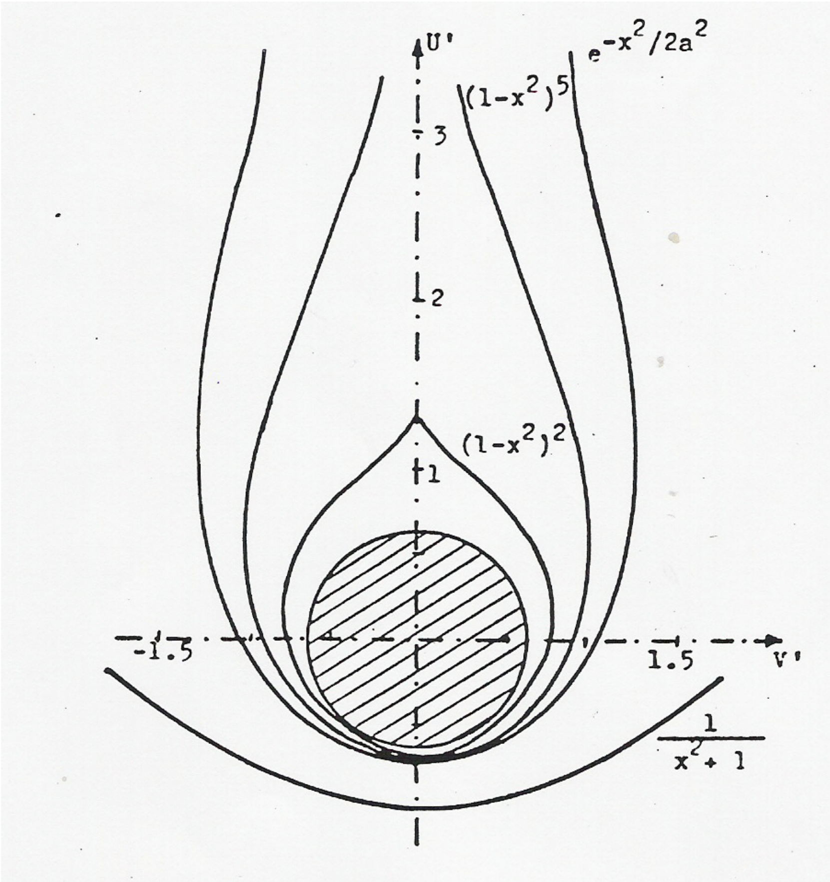


Mapping between complex impedance and complex response presented as stability diagram.
Separate beam and distribution form parameters.

$$I_0 = \frac{Ne\omega_0}{2pi}, \qquad \delta p = \text{half width half height}$$
$$S = \eta\omega_0\frac{\delta p}{p}\ \text{spread},\ x = \frac{\omega_r - \omega_0}{S},\ \ x_1 = \frac{\omega - n\omega_0}{nS}$$
$$g_0(x) = \frac{2\pi S F_0(\omega_r)}{N},\ \int g_0(x)dx = 1.$$

$$1 = -[V' + iU']\left[\pi\frac{dg_0}{dx}(x_1) - iPV\int\frac{\frac{df}{dx}}{x - x_1}dx\right],\ [V' + iU'] = \frac{eI_0 Z(\omega)/n}{2\pi\beta^2 E_0\eta(\Delta p/p)^2}$$

Longitudinal stability criterion (Keil-Schnell)



Stability diagrams, (A. Ruggiero, V. Vaccaro)

We separate effects due the distribution form and the ones due to beam and accelerator parameters and got the normalized stability diagram

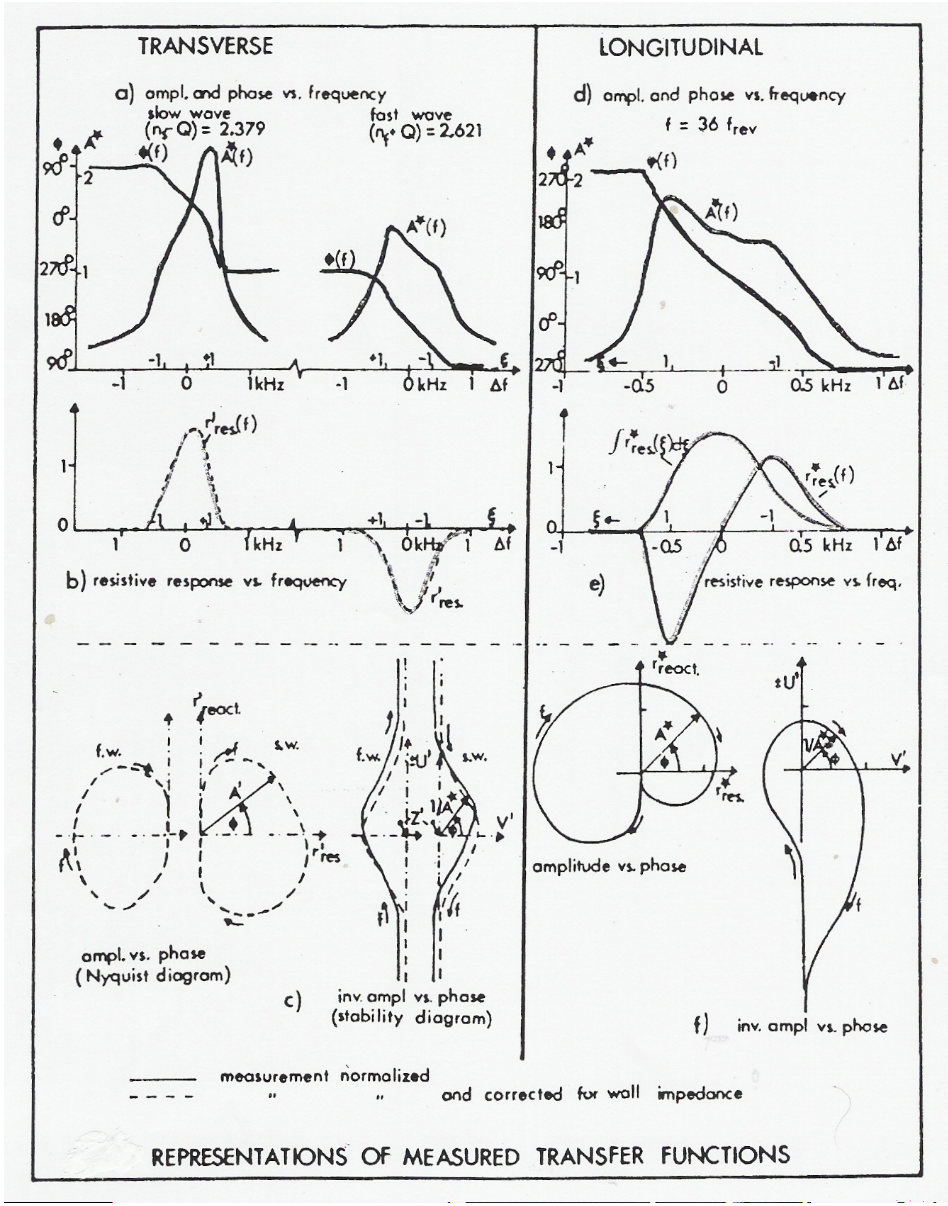
$$V' + iU' = \frac{eI_0 Z(\omega)/n}{2\pi\beta^2 E_0 \eta_c (\delta p/p)^2}$$

$$= -\left[\pi \frac{dg_0}{dx}(x_1) - iPV \int \frac{\frac{df}{dx}}{x - x_1} dx\right]^{-1}$$

Approximate these diagrams by circel of radius 0.6, get Keil-Schnell stability criterion

$$\left|\frac{Z}{n}\right| \leq \frac{2\pi\beta^2 E \eta_c (\delta p/p)^2}{eI_0}.$$

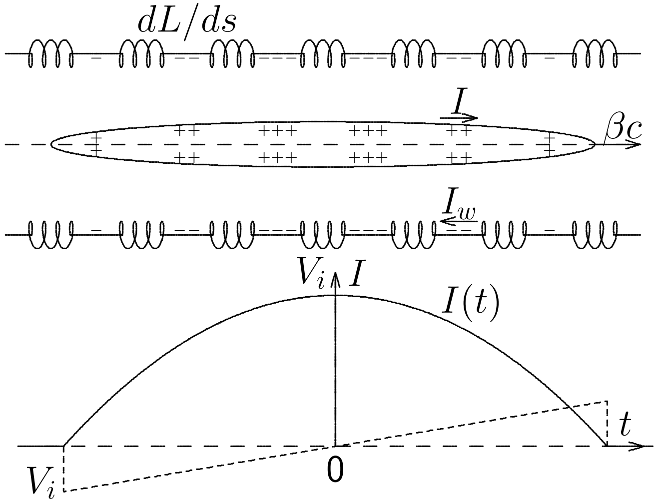
Measured coasting beam responses



Transverse: In each side-band the phase changes by π. The resistive response in is positive for the slow and negative for the fast wave.

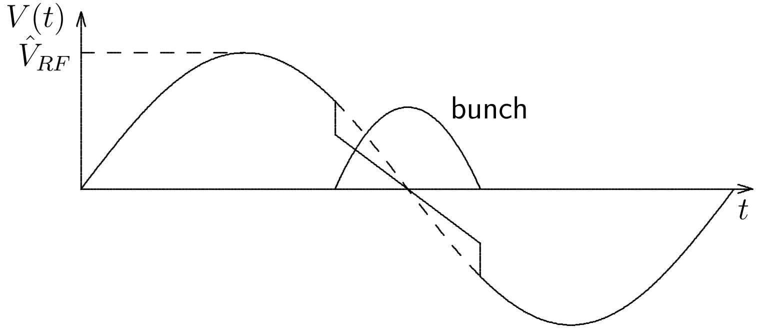
Longitudinal: Each revolution harmonics gives a 2π phase change.

6) POTENTIAL WELL BUNCH LENGTHENING



$$E_z = -\frac{dL}{dz}\frac{dI_w}{dt} = \frac{dL}{dz}\frac{dI_b}{dt}$$

$$V = -\int E_z dz = -L\frac{dI_b}{dz}$$



We take a parabolic bunch form

$$I_b(\tau) = \hat{I}\left(1-\frac{\tau^2}{\hat{\tau}^2}\right) = \frac{3\pi I_0}{2\omega_0\hat{\tau}}\left(1-\frac{\tau^2}{\hat{\tau}^2}\right)$$

$$\frac{dI_b}{d\tau} = -\frac{3\pi I_0\tau}{\omega_0\hat{\tau}^3},\ I_0 = \langle I_b\rangle,$$

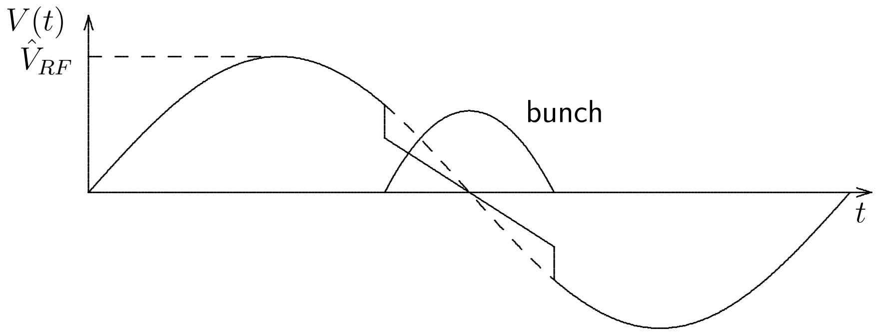
$$V = \hat{V}(\sin\phi_s + h\omega_0\cos\phi_s\tau) + \frac{3\pi I_0 L\tau}{\omega_0\hat{\tau}^3},\ L\omega_0 = \left|\frac{Z}{n}\right|$$

$$V = \hat{V}\left[\sin\phi_s + \cos\phi_s h\omega_0\left(1+\frac{3\pi|Z/n|_0 I_0}{h\hat{V}\cos\phi_s(\omega_0\hat{\tau})^3}\right)\tau\right]$$

$$\omega_{s0}^2 = -\frac{\omega_0^2 h\eta_c e\hat{V}\cos\phi_s}{2\pi E}$$

$$\omega_s^2 = \omega_{s0}^2\left[1+\frac{3\pi|Z/n|_0 I_0}{h\hat{V}_{RF}\cos\phi_s(\omega_0\hat{\tau})^3}\right]$$

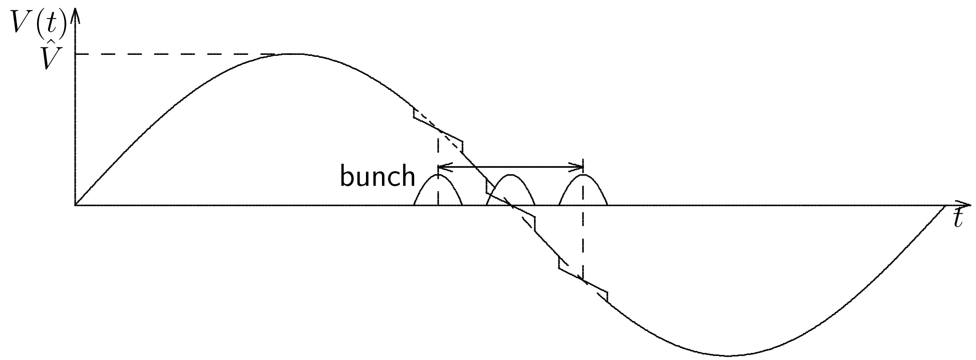
$$\frac{\Delta\omega_s}{\omega_0} = \frac{\omega_s-\omega_{s0}}{\omega_{s0}} \approx \frac{3\pi|Z/n|_0 I_0}{2h\hat{V}_{RF}\cos\phi_s(\omega_0\hat{\tau}_0)^3}$$



$$\frac{\omega_s^2}{\omega_{s0}^2} = 1 + \frac{3\pi |Z/n|_0 I_0}{h\hat{V}_{RF}\cos\phi_s(\omega_0\hat{\tau})^3}$$

$$\frac{\omega_s - \omega_{s0}}{\omega_{s0}} = \frac{\Delta\omega_s}{\omega_s} \approx \frac{3\pi |Z/n|_0 I_0}{2h\hat{V}\cos\phi_s(\omega_0\hat{\tau}_0)^3}$$

Only incoherent frequency of single particles is changed (reduced for $\gamma > \gamma_T$, increased for $\gamma < \gamma_T$), but not the coherent dipole (rigid bunch) mode. This separates the two.



Reduction of ω_s reduces longitudinal focusing and increases the bunch length

$$\hat{\tau} = \hat{\epsilon}\eta_c/\omega_s \ , \ \hat{\tau}^2 = \hat{\tau}\hat{\epsilon}\eta_c/\omega_s = \mathcal{E}_s\eta_c/\omega_s$$

rel. energy spread $\hat{\epsilon}$, long. emitt. $\mathcal{E}_s = \hat{\tau}\hat{\epsilon}$

Protons: $\mathcal{E}_s$= constant, $\tau \propto 1/\sqrt{\omega_s}$

small: $\dfrac{\Delta\hat{\tau}}{\hat{\tau}_0} \approx -\dfrac{\Delta\omega_s}{2\omega_{s0}} \approx -\dfrac{3\pi |Z/n|_0 I_0}{4h\hat{V}\cos\phi_s(\omega_0\hat{\tau}_0)^3}$,

or: $\left(\dfrac{\hat{\tau}}{\hat{\tau}_0}\right)^4 + \dfrac{3\pi |Z/n|_0 I_0}{h\hat{V}\cos\phi_s(\omega_0\hat{\tau}_0)^3}\left(\dfrac{\hat{\tau}}{\hat{\tau}_0}\right) - 1 = 0$

Electrons: $\hat{\epsilon}$= const. by syn. rad. $\hat{\tau} \propto 1/\omega_s$

small: $\dfrac{\Delta\hat{\tau}}{\hat{\tau}_0} \approx -\dfrac{\Delta\omega_s}{\omega_{s0}} \approx -\dfrac{3\pi |Z/n|_0 I_0}{2h\hat{V}\cos\phi_s(\omega_0\hat{\tau}_0)^3}$,

or: $\left(\dfrac{\hat{\tau}}{\hat{\tau}_0}\right)^3 - \dfrac{\hat{\tau}}{\hat{\tau}_0} + \dfrac{3\pi |Z/n|_0 I_0}{h\hat{V}\cos\phi_s(\omega_0\hat{\tau}_0)^3} = 0$

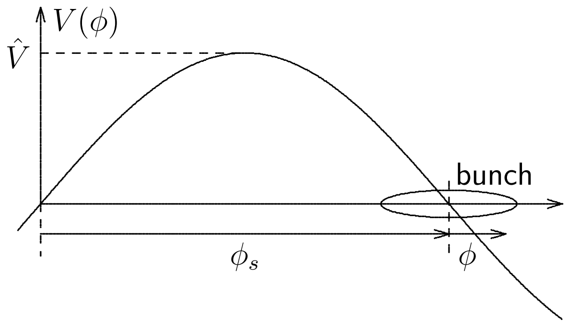
5) Bunched beams - Landau damping by non-linearities

In bunched beams the frequency spread is mostly due to non-linearities which make the oscillation frequency dependent on amplitude. The calculation is more complicated but usually only the amount of spread is of interest.
Example: synchrotron oscillation with parameters E_0, $T_0 = 2\pi/\omega_0$ and $\phi_s = h\omega_0 t_s = \pi$ with deviations $\epsilon = \Delta E/E_0 \approx \Delta p/p_0$ and $\phi = h\omega_0\tau$.
next approximation, developing $\sin\phi \approx \phi - \phi^3/6$, seeking solution of the form $\phi = \hat{\phi}\cos(\omega_s t)$

$$(-\omega_s^2 + \omega_{s0}^2)\hat{\phi}\cos(\omega_s t) - \hat{\phi}^3\omega_{s0}^2\cos^3(\omega_s t)/6 = 0$$

using $\cos^3 x = (3\cos x + \cos(3x))/4$, neglecting higher harmonics gives

$$\frac{\omega_s}{\omega_{s0}} = \sqrt{1 - \frac{1}{8}\hat{\phi}^2} \approx 1 - \frac{1}{16}\hat{\phi}^2,\ \frac{\Delta\omega_s}{\omega_{s0}} = -\frac{\hat{\phi}^2}{16}$$

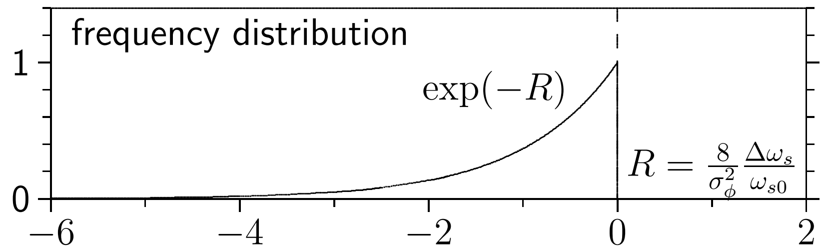


linear:

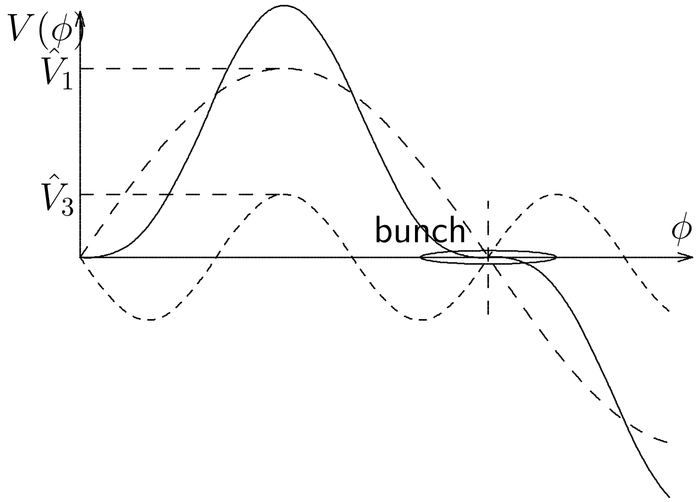
$$\ddot{\phi} + \omega_{s0}^2\sin\phi = 0\ ,\ \frac{\omega_{s0}^2}{\omega_0^2} = \frac{\eta_c h e\hat{V}}{2\pi E_0}$$

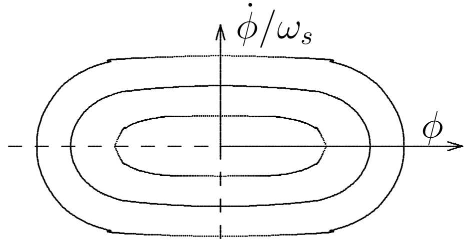
$$\phi \ll 1 \rightarrow \ddot{\phi} + \omega_{s0}^2\phi = 0 \text{ solution}$$

$$\epsilon = \hat{\epsilon}\cos(\omega_{s0}t)\ ,\ \tau = \hat{\tau}\sin(\omega_{s0}t)$$

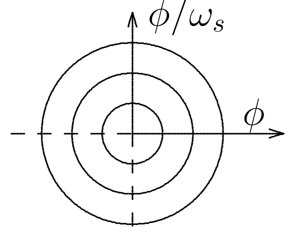


Increase Landau damping - double RF-system

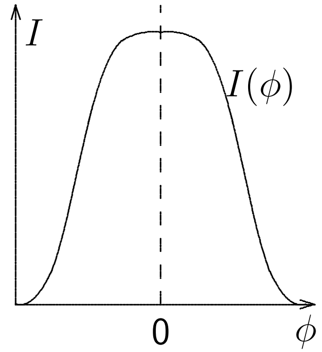




Double RF-system with n=3



Single RF-system



Two RF-systems ω_{RF} and $n\omega_{RF}$, $\phi_s \approx \pi$,

$$V(\phi) = -\hat{V}_1 \sin\phi + \hat{V}_n \sin(n\phi) \text{ with } \hat{V}_n / = \hat{V}_1/n$$

$$V(\phi) \approx \hat{V}_1 \left(-\left(\phi - \frac{\phi^3}{6}\right) + \frac{1}{n}\left(n\phi - \frac{n^3\phi^3}{6}\right)\right)$$

$$= -\hat{V}_1 \frac{n^2-1}{6}\phi^3$$

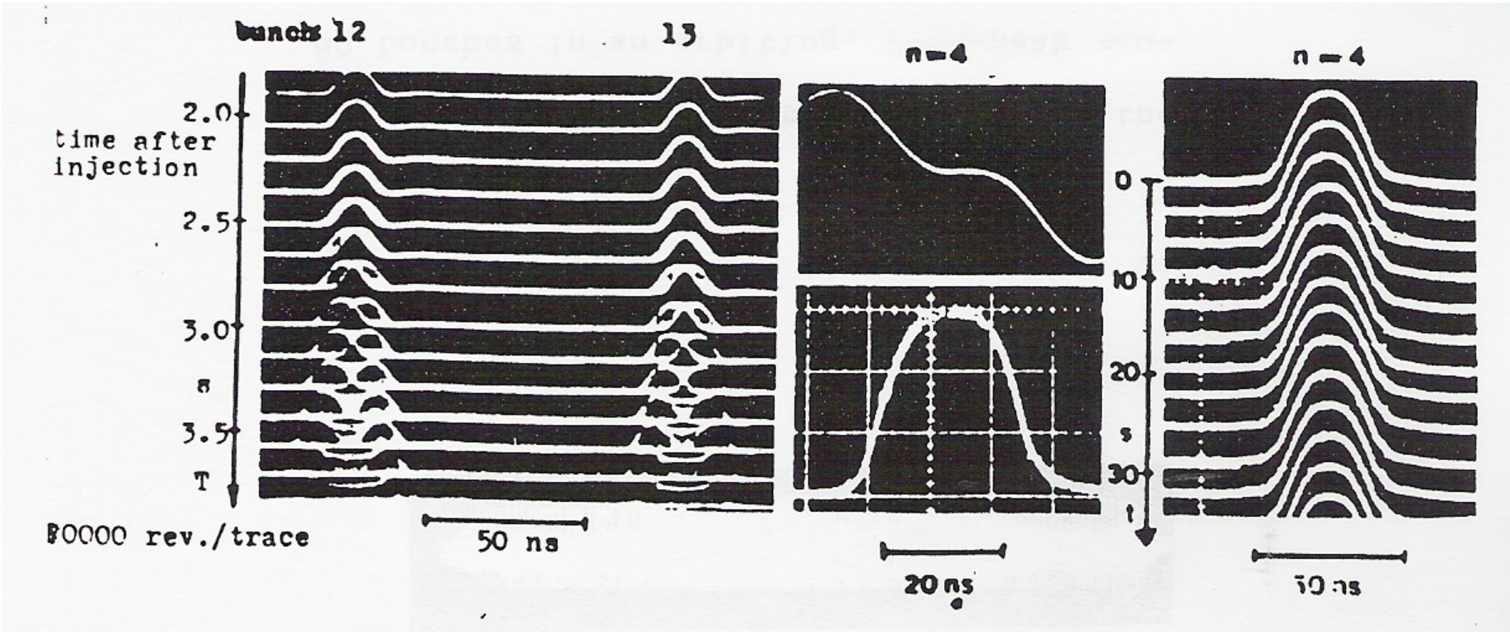
$$\ddot{\phi} + \omega_{s0}^2 \frac{n^2-1}{6}\phi^3 = 0 \;,\; \omega_s \propto \hat{\phi}$$

$$\frac{\dot{\phi}^2}{2} + \omega_{s0}^2 \frac{n^2-1}{24}\phi^4 = H = \text{constant}$$

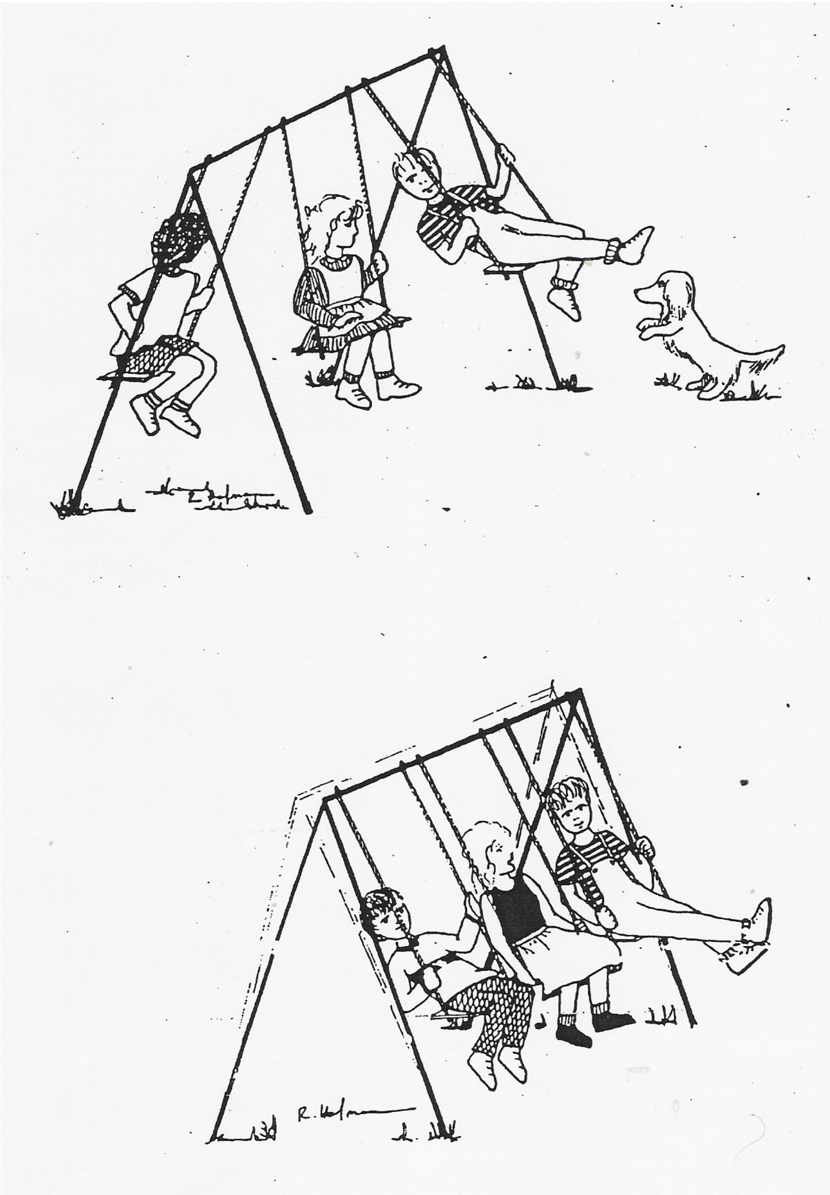
$$I(\phi) = \hat{I} \exp\left[-\omega_{s0}^2 \frac{n^2-1}{24\sigma_{\dot{\phi}}^2}\phi^4\right]$$

ω_{s0} and ω_s are synchrotron frequencies of basic and double RF-system. There strong amplitude dependence gives large spread and Landau damping. The flat voltage leads to a long bunch.

Landau damping by double RF



Frequency spread and shift



Bibliography

Landau damping can be presented in many different ways. We used here an approach close to beam observation and experiments involving pulse and harmonic excitation. Details of the mathematics involved can be found in books on filter theory. Furthermore, integrals involving residues and mapping used in the stability diagrams is treated in mathematical books on complex numbers and theory of functions.
In some of the references listed below Landau damping is presented from other points of view.

H.G. Hereward, "The elementary theory of Landau damping"; CERN 1965 - 20, PS-Division.

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D. Sagan, "On the physics of Landau damping"; Amer. J. Physics, 1984.

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